Research Article

Estimation of Parameters of Generalized Inverted Exponential Distribution for Progressive Type-II Censored Sample with Binomial Removals

Sanjay Kumar Singh, Umesh Singh, and Manoj Kumar

Department of Statistics and DST-CIMS, Banaras Hindu University, Varanasi 221005, India

Correspondence should be addressed to Manoj Kumar; manustats@gmail.com

We obtained the maximum likelihood and Bayes estimators of the parameters of the generalized inverted exponential distribution in case of the progressive type-II censoring scheme with binomial removals. Bayesian estimation procedure has been discussed under the consideration of the square error and general entropy loss functions while the model parameters follow the gamma prior distributions. The performances of the maximum likelihood and Bayes estimators are compared in terms of their risks through the simulation study. Further, we have also derived the expression of the expected experiment time to get a progressively censored sample with binomial removals, consisting of specified number of observations from generalized inverted exponential distribution. An illustrative example based on a real data set has also been given.

1. Introduction

The one parameter exponential distribution is the simplest and the most widely discussed distribution in the context of life testing. This distribution plays an important role in the development to the theory, that is, any new theory developed can be easily illustrated by the exponential distribution due its mathematical tractability; see Barlow and Proschan [1] and Leemis [2]. But its applicability is restricted to a constant hazard rate because hardly any item/system can be seen which has time independent hazard rate. Therefore, the number of generalizations of the exponential distribution has been proposed in earlier literature where the exponential distribution is not suitable to the real problem. For example, the gamma (sum of independent exponential variates) and Weibull (power transformed distribution) distributions are the most popular generalizations of the exponential distribution. Most of the generalizations of the exponential distribution possess the constant, nonincreasing, nondecreasing and bathtub hazard rates.

But in practical real problems, there may be a situation where the data shows the inverted bathtub hazard rate (initially increases and then decreases, i.e., unimodal). Let us take an example, in the course of study of breast cancer data, we observed that the mortality increases initially, reaches to a peak after some time, and then declines slowly, that is, associated hazard rate is inverted bathtub or particularly unimodal. For such types of data, another extension of the exponential distribution has been proposed in statistical literature. That is known as one parameter inverse exponential or one parameter inverted exponential distribution (IED) which possess the inverted bathtub hazard rate. Many authors have proposed the use of IED in survival analysis; see Lin et al. [3], and Singh et al. [4]. Abouammoh and Alshingiti [5] have proposed two parameters generalization of IED called as generalized inverted exponential distribution (GIED) and they have showed that GIED is better than IED for real data set on the basis of likelihood ratio test and *K-S* statistics. They have also discussed the maximum likelihood and least square methods for the estimation of the unknown parameters of GIED. Krishna and Kumar [6] have studied the reliability estimation based on progressive type-II censored sample under classical setup.

In life testing experiments, situations do arise that the units under study are lost or removed from the experiments while they are still alive; that is, generally we get censored data from the life testing experiments. The loss of units may occur due to time constraints, giving type-I censored data. In type-I censoring scheme, the number of observations is random as the experiment is terminated at a prespecified time. Sometimes, the experiment has to be terminated after a prefixed number of observations and the data, thus obtained, is referred as type-II censored data. Besides these, there are many uncontrolled causes, resulting to loss of intermediate observations; see Balakrishnan [7]. One such censoring procedure named as progressive type-II censoring scheme can be described notationally as follows. Let the lifetimes of n identical units/items be studied. At the first failure x_1, called first stage, r_1 units are removed from the remaining $(n-1)$ surviving units. At second failure x_2, called second stage r_2 units from remaining $n-2$, $-r_1$ units are removed, and so on, till mth failure is observed; that is, at mth stage all the remaining $(n - R_1 - \cdots - R_{m-1} - m + 1)$ units are removed. It may be mentioned here that the number of units r_i dropped out from the test at ith stage should be less than $n - m - \sum_{l=0}^{i-1} r_l$ in order to insure the availability of m observations. In many practical situations, r_i's may be random and cannot be predetermined, for example, in clinical trials. Considering r_i to be random, Yuen and Tse [8] have discussed progressive censoring scheme with binomial removals. They assumed that the number of random removals at the stage r_i is random and follows binomial distribution with probability p. It may be noted here that in clinical trials, the assumption that r_i's are less than $n - m - \sum_{l=0}^{i-1} r_l$ looks unrealistic, but, in life testing experiments, it should not pose any problem as it is used to decide the value of r_i's only. Thus, r_1 (at 1th stage) is to be considered to follow the binomial distribution with parameter $n-m$ and p, that is, binomial $(n-m, p)$, and in the same way, r_2 (at 2th stage) follows binomial $(n-m-r_1, p)$. In general, the number of units removed at ith stage r_i follows the binomial distribution with parameter $n - m - \sum_{l=0}^{i-1} r_l$ and p for $i = 1, 2, 3, \ldots, m-1$.

For further details on progressive censoring and its further development, readers may be referred to Balakrishnan [7]. The estimation of parameters of several lifetime distributions based on progressive censored samples has been studied by many authors; see Childs and Balakrishnan [9], Balakrishnan and Kannan [10], Mousa and Jaheen [11], and Ng et al. [12]. The progressive type-II censoring with binomial removal has been considered by Tse et al. [13] for Weibull distribution, Wu and Chang [14] for exponential distribution. Under the progressive type-II censoring with random removals, Wu and Chang [15], Yuen and Tse [8], and Singh et al. [16] developed the estimation problem for the Pareto distribution, Weibull distribution, and exponentiated Pareto distribution, respectively.

The objective of this paper is to obtain the MLEs and Bayes estimators of the unknown parameters of GIED under symmetric and asymmetric loss functions and compare the performances of the competing estimators. Further, we have also investigated the total experiment time of experiment

on the basis of numerical study. The rest of the paper has been organized in the following section. Section 2 provides a brief discussion about the progressive type-II censoring scheme with binomial removals. In the next section, we have obtained MLEs and Bayes estimators of the model parameters. The expression for the expected experiment time for progressive type-II censored data with binomial removals has been derived in Section 4. The algorithm for simulating the progressive type-II censored data with binomial removal has been described in Section 5. The comparison study of the MLEs and Bayes estimators has been given in Section 6. In Section 7, the methodology is illustrated through a real data set. Finally, the conclusions have been provided in the last section.

2. The Model

The GIED (λ, α) with cumulative density function (cdf) is expressed as follows

$$F(x) = 1 - \left(1 - e^{-\lambda/x}\right)^{\alpha}; \quad x \geq 0, \ \lambda > 0, \ \alpha > 0, \quad (1)$$

and the probability density function (pdf) is given by:

$$f(x) = \frac{\alpha \lambda}{x^2} e^{-\lambda/x} \left(1 - e^{-\lambda/x}\right)^{\alpha-1}; \quad x \geq 0, \ \lambda > 0, \ \alpha > 0. \quad (2)$$

The survival function of x is

$$S(x) = \left(1 - e^{-\lambda/x}\right)^{\alpha}; \quad x \geq 0, \ \alpha > 0, \ \lambda > 0. \quad (3)$$

Let $(X_1, R_1), (X_2, R_2), (X_3, R_3), \ldots, (X_m, R_m)$ be the progressive type-II censored sample, where R_i's are random removals. For fixed values of R_i's, say $R_1 = r_1, R_2 = r_2, R_3 = r_3, \ldots, R_m = r_m$, the conditional likelihood function can be written as

$$L(\alpha, \lambda; x \mid R = r) = c^* \prod_{i=1}^{m} f(x_i) \left[S(x_i)\right]^{r_i}, \quad (4)$$

where $c^* = n(n - r_1 - 1)(n - r_1 - r_2 - 2)(n - r_1 - r_2 - r_3 - 3) \cdots (n - r_1 - r_2 - r_3, \ldots, r_m - m + 1)$ and $0 \leq r_i \leq (n - m - r_1 - r_2 - r_3, \ldots, r_{i-1})$ for $i = 1, 2, 3, \ldots, m$ and $r_0 = 0$. Substituting (2) and (3) into (4), we get

$$L(\alpha, \lambda; x \mid R = r) = c^* \prod_{i=1}^{m} \frac{\alpha \lambda}{x_i^2} e^{-\lambda/x_i} \left(1 - e^{-\lambda/x_i}\right)^{\alpha-1}$$
$$\times \left\{\left(1 - e^{-\lambda/x_i}\right)^{\alpha}\right\}^{r_i}. \quad (5)$$

As mentioned earlier, individual units removed from the test at ith stage, $i = 1, 2, \ldots, m$ independent of each other and the probability of removal is same p for all. Therefore, the number R_i of the unit removed at ith stage $1 = 1, 2, 3, \ldots, m$ follows binomial distribution with parameters $(n - m - \sum_{l=1}^{i-1} r_l, p)$; that is,

$$P(R_1 = r_1; p) = \binom{n-m}{r_1} p^{r_1} (1-p)^{n-m-r_1}, \quad (6)$$

and for $i = 2, 3, \ldots, m - 1$,

$$P\left(R_i = r_i \mid R_{i-1} = r_{i-1}, \ldots R_1 = r_1\right)$$

$$= \binom{n - m - \sum_{l=0}^{i-1} r_l}{r_i} p^{r_i}(1 - p)^{n - m - \sum_{l=0}^{i-1} r_l}. \quad (7)$$

Now, we further assume that R_i's are independent of X_i's for all i. Then, the full likelihood function takes the following form:

$$L\left(\alpha, \lambda, p; x\right) = L\left(\alpha, \lambda; x \mid R = r\right) P\left(R = r; p\right), \quad (8)$$

where

$$P\left(R = r; p\right)$$

$$= P\left(R_1 = r_1\right) P\left(R_2 = r_2 \mid R_1 = r_1\right)$$

$$\times P\left(R_3 = r_3 \mid R_2 = r_2, R_1 = r_1\right)$$

$$\cdots P\left(R_{m-1} = r_{m-1} \mid R_{m-2} = r_{m-2}, \ldots R_1 = r_1\right). \quad (9)$$

Substituting (6) and (7) into (9), we get

$$P\left(R = r; p\right) = \frac{(n - m)! \, p^{\sum_{i=1}^{m-1} r_i}(1 - p)^{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i}}{\left(n - m - \sum_{l=1}^{i-1} r_l\right)! \prod_{i=1}^{m-1} r_i!}. \quad (10)$$

Now, using (5), (8), and (10), we can write the full likelihood function in the following form:

$$L\left(\alpha, \lambda, p; x\right) = A L_1\left(\alpha, \lambda\right) L_2\left(p\right), \quad (11)$$

where

$$A = \frac{c^*\,(n - m)!}{\left(n - m - \sum_{l=1}^{i-1} r_l\right)! \prod_{i=1}^{m-1} r_i!},$$

$$L_1\left(\alpha; \lambda\right) = \prod_{i=1}^{m} \frac{\alpha \lambda}{x_i^2} e^{-\lambda/x_i}\left(1 - e^{-\lambda/x_i}\right)^{\alpha-1}\left\{\left(1 - e^{-\lambda/x_i}\right)^{\alpha}\right\}^{r_i}, \quad (12)$$

$$L_2\left(p\right) = p^{\sum_{i=1}^{m-1} r_i}(1 - p)^{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i}. \quad (13)$$

3. Classical and Bayesian Estimation of Parameters

3.1. Maximum Likelihood Estimation. In this section, we have obtained the MLEs of the parameters α, λ, and p based on progressive type-II censored data with binomial removals. We observe from (11), (12), and (13) that likelihood function is multiplication of three terms, namely, A, L_1, and L_2. Out of these, A does not depend on the parameters α, λ, and p. L_1, does not involve p and it is function of α and λ only, whereas L_2 involves p only. Therefore, the MLEs of α and λ can be derived by maximizing L_1. Similarly, the MLE of p can be obtained by maximizing L_2.

Taking log of both sides of (12), we have

$$\ln L_1\left(\alpha; \lambda\right) = m \ln\left(\lambda\right) + m \ln\left(\alpha\right) - \lambda \sum_{i=1}^{m} \frac{1}{x_i}$$

$$+ \left(\alpha - 1\right) \sum_{i=1}^{m} \ln\left(1 - e^{-\lambda/x_i}\right) \quad (14)$$

$$+ \sum_{i=1}^{m} r_i \ln\left(1 - e^{-\lambda/x_i}\right)^{\alpha}.$$

Thus, MLEs of α and λ can be obtained by simultaneously solving the following nonlinear normal equations which are as follows:

$$\frac{\partial \ln L_1\left(\alpha\right)}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\left(r_i + 1\right) \ln\left(1 - e^{-\lambda/x_i}\right) = 0, \quad (15)$$

$$\frac{\partial \ln L_1\left(\lambda\right)}{\partial \lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m} \frac{1}{x_i}$$

$$+ \sum_{i=1}^{m} \left\{\alpha\left(r_i + 1\right) - 1\right\} \frac{e^{-\lambda/x_i}}{x_i\left(1 - e^{-\lambda/x_i}\right)} = 0. \quad (16)$$

From (15), we obtain the MLE of α as a function of λ, say $\widehat{\alpha}(\lambda)$, where

$$\widehat{\alpha}\left(\lambda\right) = \frac{-m}{\sum_{i=1}^{m}\left(r_i + 1\right) \ln\left(1 - e^{-\lambda/x_i}\right)}. \quad (17)$$

Putting $\widehat{\alpha}(\lambda)$ in (14), we obtain

$$\ln L_1\left(\widehat{\alpha}\left(\lambda\right); \lambda\right) = m \ln\left(\lambda\right) + m \ln\left(\widehat{\alpha}\left(\lambda\right)\right)$$

$$- \lambda \sum_{i=1}^{m} \frac{1}{x_i} + \left(\widehat{\alpha}\left(\lambda\right) - 1\right) \sum_{i=1}^{m} \ln\left(1 - e^{-\lambda/x_i}\right)$$

$$+ \sum_{i=1}^{m} r_i \ln\left(1 - e^{-\lambda/x_i}\right)^{\widehat{\alpha}(\lambda)}. \quad (18)$$

Therefore, the MLE of λ can be obtained by maximizing (18) with respect to λ. Once $\widehat{\lambda}$ is obtained, $\widehat{\alpha}$ can be obtained from (17) as $\widehat{\alpha}(\widehat{\lambda})$. Therefore, it reduces the two-dimensional problem to a one-dimensional problem which is relatively easier to solve with fixed point iteration method. For details about the fixed point iteration method, readers may refer Rao [17].

The log of $L_2(p)$ takes the following form:

$$\ln L_2\left(p\right) = \ln p \sum_{i=1}^{m-1} r_i + \ln\left(1 - p\right)$$

$$\times \left(\left(m - 1\right)\left(n - m\right) - \sum_{i=1}^{m-1} r_i\left(m - i\right)\right). \quad (19)$$

The first order derivative of $\ln L_2(p)$ with respect to p is

$$\frac{\partial \ln L_2\left(p\right)}{\partial p} = \frac{\sum_{i=1}^{m-1} r_i}{p} - \frac{\left(m - 1\right)\left(n - m\right) - \sum_{i=1}^{m-1} r_i\left(m - i\right)}{1 - p}. \quad (20)$$

Setting $(\partial \ln L_2(p))/\partial p = 0$ and solving, we get the MLE of p as

$$\widehat{p}_{\mathrm{mle}} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1} r_i (m-i-1)}. \tag{21}$$

3.2. Bayes Estimators. In order to obtain the Bayes estimators of the parameters α and λ based on progressively type-II censored data with binomial removals. We must assume that the parameters α and λ are random variables. Following Nassar and Eissa [18] and Kim et al. [19], we assume that these are independently distributed. The random variables α and λ have prior distribution with respective prior pdfs

$$g_1(\alpha) = \frac{a_1^{b_1} e^{-a_1 \alpha} \alpha^{b_1 - 1}}{\Gamma(b_1)}; \quad 0 < \alpha < \infty, \ a_1 > 0, \ b_1 > 0,$$

$$g_2(\lambda) = \frac{a_2^{b_2} e^{-a_2 \lambda} \lambda^{b_2 - 1}}{\Gamma(b_2)}; \quad 0 < \lambda < \infty, \ a_2 > 0, \ b_2 > 0,$$

$$\tag{22}$$

respectively. It may be noted that the gamma prior $g_1(\alpha)$ and $g_2(\lambda)$ are flexible enough to cover wide variety of the prior believes experimenter. Based on the assumptions stated above, the joint prior pdf of α and λ is

$$g(\alpha, \lambda) = g_1(\alpha) g_2(\lambda); \quad \alpha > 0, \ \lambda > 0. \tag{23}$$

Combining the priors given by (22) with likelihood given by (8), we can easily obtain joint posterior pdf of (α, λ) as

$$\pi(\alpha, \lambda \mid x, r) = \frac{J_1}{J_0}, \tag{24}$$

where

$$J_1 = \alpha^{(m+b_1-1)} \lambda^{(m+b_2-1)} e^{-(a_1 \alpha + a_2 \lambda)}$$

$$\times \left(\prod_{i=1}^{m} \frac{1}{x_i^2} e^{-\lambda/x_i} \left(1 - e^{-\lambda/x_i}\right)^{(r_i+1)\alpha - 1} \right), \tag{25}$$

$$J_0 = \iint_0^\infty J_1 \, d\alpha \, d\lambda.$$

Hence, the respective marginal posterior pdfs of α and λ are given by

$$\pi_1(\alpha \mid x, r) = \int_0^\infty \frac{J_1}{J_0} d\lambda,$$

$$\pi_2(\lambda \mid x, r) = \int_0^\infty \frac{J_1}{J_0} d\alpha. \tag{26}$$

Usually the Bayes estimators are obtained under square error loss function (SELF)

$$l_1(\phi, \widehat{\phi}) = \epsilon_1 (\phi - \widehat{\phi})^2; \quad \epsilon_1 > 0, \tag{27}$$

where $\widehat{\phi}$ is the estimate of the parameter ϕ and the Bayes estimator $\widehat{\phi}_S$ of ϕ comes out to be $E_\phi[\phi]$, where E_ϕ denotes the posterior expectation. However, this loss function is symmetric loss function and can only be justified if over estimation and under estimation of equal magnitude are of equal seriousness. But in practical situations, this may not be true. A number of asymmetric loss functions are available in statistical literature. Let us consider the general entropy loss function (GELF) proposed by Calabria and Pulcini [20] defined as follows:

$$l_2(\phi, \widehat{\phi}) = \epsilon_2 \left(\left(\frac{\widehat{\phi}}{\phi} \right)^\delta - \delta \ln \left(\frac{\widehat{\phi}}{\phi} \right) - 1 \right); \quad \epsilon_2 > 0. \tag{28}$$

The constant δ, involved in (28), is its shape parameter. It reflects departure from symmetry. When $\delta > 0$, then over estimation (i.e., positive error) causes more serious consequences than under estimation (i.e., negative error) and converse for $\delta < 0$. The Bayes estimator $\widehat{\phi}_G$ of ϕ under GELF is given by

$$\widehat{\phi}_G = \left[E_\phi \left(\phi^{-\delta} \right) \right]^{(-1/\delta)} \tag{29}$$

provided that the posterior expectation exits. It may be noted here that, for $\delta = -1$, the Bayes estimator under loss (27) coincides with the Bayes estimator under SELF l_1. Expressions for the Bayes estimators $\widehat{\alpha}_G$ and $\widehat{\lambda}_G$ for α and λ, respectively, under GELF can be given as

$$\widehat{\alpha}_G = \left[\int_0^\infty \alpha^{-\delta} \pi_1(\alpha \mid x, r) \, d\alpha \right]^{(-1/\delta)}, \tag{30}$$

$$\widehat{\lambda}_G = \left[\int_0^\infty \lambda^{-\delta} \pi_2(\lambda \mid x, r) \, d\lambda \right]^{(-1/\delta)}. \tag{31}$$

Substituting the posterior pdfs from (26) in (30) and (31), respectively, and then simplifying then, we get the Bayes estimators $\widehat{\alpha}_G$ and $\widehat{\lambda}_G$ of α and λ as follows:

$$\widehat{\alpha}_G = \left[\left(\int_0^\infty \lambda^{b_2+m-1} e^{-a_2\lambda} \int_0^\infty \alpha^{b_1+m-\delta-1} e^{-a_1\alpha} \right. \right.$$

$$\times \left(\prod_{i=1}^{m} \left(\frac{1}{x_i^2} \right) e^{-\lambda/x_i} \left(1 - e^{-\lambda/x_i}\right)^{(r_i+1)\alpha - 1} \right) d\alpha \, d\lambda \right)$$

$$\times \left(\int_0^\infty \lambda^{b_2+m-1} e^{-a_2\lambda} \int_0^\infty \alpha^{b_1+m-1} e^{-a_1\alpha} \right.$$

$$\times \left(\prod_{i=1}^{m} \left(\frac{1}{x_i^2} \right) e^{-\lambda/x_i} \right.$$

$$\left. \left. \left. \times \left(1 - e^{-\lambda/x_i}\right)^{(r_i+1)\alpha - 1} \right) d\alpha \, d\lambda \right)^{-1} \right]^{-1/\delta},$$

$$\hat{\lambda}_G = \left[\left(\int_0^\infty \lambda^{b_2+m-\delta-1} e^{-a_2\lambda} \int_0^\infty \alpha^{b_1+m-1} e^{-a_1\alpha} \right. \right.$$

$$\times \left(\prod_{i=1}^m \left(\frac{1}{x_i^2} \right) e^{-\lambda/x_i} \left(1 - e^{-\lambda/x_i} \right)^{(r_i+1)\alpha-1} \right) d\alpha\, d\lambda \right)$$

$$\times \left(\int_0^\infty \lambda^{b_2+m-1} e^{-a_2\lambda} \int_0^\infty \alpha^{b_1+m-1} e^{-a_1\alpha} \right.$$

$$\times \left(\prod_{i=1}^m \left(\frac{1}{x_i^2} \right) e^{-\lambda/x_i} \right.$$

$$\left. \left. \left. \times \left(1 - e^{-\lambda/x_i} \right)^{(r_i+1)\alpha-1} \right) d\alpha\, d\lambda \right)^{-1} \right]^{-1/\delta}. \tag{32}$$

It may be noted here that the integrals involved in the expressions for the Bayes estimators $\hat{\alpha}_G$ and $\hat{\lambda}_G$ cannot be obtained analytically and one needs numerical techniques for computations. We, therefore, have proposed to use Markov Chain Monte Carlo (MCMC) methods. In MCMC techniques, we considered the Metropolis-Hastings algorithms to generate samples from posterior distributions and these samples are used to compute Bayes estimates. The Gibbs is an algorithm for simulating from the full conditional posterior distributions while the metropolis-hastings algorithm generates samples from an arbitrary proposal distribution. The conditional posterior distributions of the parameters α and λ can be written as

$$\pi_1^* (\alpha \mid \lambda, x, r) \propto \alpha^{(m+b_1-1)} e^{-(a_1-\sum_{i=1}^m (r_i+1)\log(1-e^{-\lambda/x_i}))\alpha},$$

$$\pi_2^* (\lambda \mid \alpha, x, r) \propto \lambda^{(m+b_2-1)} e^{-a_2\lambda}$$

$$\times \prod_{i=1}^m e^{-\lambda/x_i} \left(1 - e^{-\lambda/x_i} \right)^{(r_i+1)\alpha-1}, \tag{33}$$

respectively. For the Bayes estimators, the following MCMC procedure is followed.

(I) Set the initial guess of α and λ say α_0 and λ_0.

(II) Set $i = 1$.

(III) Generate α_i from $\pi_1^*(\alpha \mid \lambda_{i-1}, x, r)$ and λ_i from $\pi_2^*(\lambda \mid \alpha_i, x, r)$.

(IV) Repeat steps (II)-(III), N times.

(V) Obtain the Bayes estimates of α and λ under GELF as

$$\hat{\alpha}_G = \left[E(\alpha^{-\delta} \mid \text{data}) \right]^{-1/\delta} = \left[\frac{1}{N-N_0} \sum_{i=1}^{N-N_0} \alpha_i^{-\delta} \right]^{-1/\delta},$$

$$\hat{\lambda}_G = \left[E(\lambda^{-\delta} \mid \text{data}) \right]^{-1/\delta} = \left[\frac{1}{N-N_0} \sum_{i=1}^{N-N_0} \lambda_i^{-\delta} \right]^{-1/\delta}, \tag{34}$$

where N_0 is the burn-in-period of Markov Chain. Substituting δ equal to -1 in step (V), we get Bayes estimates of α and λ under SELF.

(VI) To compute the HPD interval of α and λ, order the MCMC sample of α and λ (say $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_N$ as $\alpha_{[1]}, \alpha_{[2]}, \alpha_{[3]}, \ldots, \alpha_{[N]}$) and $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N$ as $\lambda_{[1]}, \lambda_{[2]}, \lambda_{[3]}, \ldots, \lambda_{[N]})$. Then, construct all the $100(1 - \psi)\%$ credible intervals of α and λ; say $((\alpha_{[1]}, \alpha_{[N(1-\psi)+1]}), \ldots, (\alpha_{[N\psi]}, \alpha_{[N]}))$ and $((\lambda_{[1]}, \lambda_{[N(1-\psi)+1]}), \ldots, (\lambda_{[N\psi]}, \lambda_{[N]}))$, respectively. Here, $[x]$ denotes the largest integer less than or equal to x. Then, the HPD interval of α and λ are that interval which has the shortest length.

4. Expected Experiment Times

In practical situations, an experimenter may be interested to know whether the test can be completed within a specified time. This information is important for an experimenter to choose an appropriate sampling plan because the time required to complete a test is directly related to cost. Under progressive censoring with a fixed number of removal, the time is given by X_m. Following Balakrishnan and Aggarwala [21], the expected value of X_m is given by

$$E[X_m \mid R] = C(r) \sum_{l_1=0}^{r_1} \sum_{l_2=0}^{r_2} \cdots \sum_{l_m=0}^{r_m} (-1)^B \frac{\binom{r_1}{l_1} \cdots \binom{r_m}{l_m}}{\prod_{i=1}^{m-1} h(l_i)}$$

$$\times \int_0^\infty x f(x) F^{h(l_m)-1}(x)\, dx, \tag{35}$$

where $B = \sum_{i=1}^m l_i$, $h(l_i) = l_1 + l_2 + \cdots + l_i + i$, $C(r) = n(n - r_1 - 1)(n - r_1 - r_2 - 2) \cdots [n - \sum_{i=1}^{m-1} (r_i + 1)]$ and i is the number of live units removed from experiment (number of failure units). Using the pdf and cdf of GIED, the equation will be

$$E[X_m \mid R] = C(r) \sum_{l_1=0}^{r_1} \sum_{l_2=0}^{r_2} \cdots \sum_{l_m=0}^{r_m} (-1)^B \frac{\binom{r_1}{l_1} \cdots \binom{r_m}{l_m}}{\prod_{i=1}^{m-1} h(l_i)} S_1, \tag{36}$$

where

$$S_1 = \int_0^\infty \frac{\alpha\lambda}{x} e^{-\lambda/x} \left(1 - e^{-\lambda/x} \right)^{\alpha-1}$$

$$\times \left[1 - \left(1 - e^{-\lambda/x} \right)^\alpha \right]^{h(l_m)-1} dx. \tag{37}$$

After simplification it reduces to

$$S_1 = \alpha\lambda \sum_{k=0}^{h(l_m)-1} (-1)^k \binom{h(l_m) - 1}{k}$$

$$\times \int_0^\infty \frac{e^{-\lambda/x}}{x} \left(1 - e^{-\lambda/x} \right)^{\alpha(k+1)-1} dx. \tag{38}$$

Putting this value in (36), the expected test time is given by

$$
E\left[X_m \mid R\right] = C\left(r\right) \sum_{l_1=0}^{r_1} \sum_{l_2=0}^{r_2} \cdots \sum_{l_m=0}^{r_m} (-1)^B \frac{\binom{r_1}{l_1} \cdots \binom{r_m}{l_m}}{\prod_{i=1}^{m-1} h\left(l_i\right)}
$$
$$
\times \sum_{k=0}^{h(l_m)-1} (-1)^k \binom{h\left(l_m\right)-1}{k} \tag{39}
$$
$$
\times \int_0^\infty \frac{e^{-\lambda/x}}{x} \left(1 - e^{-\lambda/x}\right)^{\alpha(k+1)-1} dx.
$$

The expected test time for progressively type-II censored data with binomial removals is evaluated by taking expectation on both sides (36) with respect to the R. That is,

$$
E\left[X_m\right] = E_R\left[E\left[X_m \mid R = r\right]\right]
$$
$$
= \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \cdots \sum_{r_{m-1}=0}^{g(r_{m-1})} P\left(R = r_i; p\right) E\left[X_m \mid R = r\right], \tag{40}
$$

where $g(r_i) = n - m - r_1 - \cdots - r_{i-1}$ and $P(R = r_i; p)$ is given in (10). For the expected time of a complete sampling, case with n test units is obtained by taking $m = n$ and $r_i = 0$ for all $i = 1, 2, \ldots, m$, in (39). We have

$$
E\left[X_n^*\right] = n\alpha\lambda \int_0^\infty \frac{e^{-\lambda/x}}{x} \left[1 - \left(1 - e^{-\lambda/x}\right)^\alpha\right]^{n-1}
$$
$$
\times \left(1 - e^{-\lambda/x}\right)^{\alpha-1} dx \tag{41}
$$

and the expected time of a type-II censoring is defined by the expected value of the mth failure time; then,

$$
E\left[X_m^*\right] = m\alpha\lambda \binom{n}{m} \int_0^\infty \frac{e^{-\lambda/x}}{x} \left[1 - \left(1 - e^{-\lambda/x}\right)^\alpha\right]^{m-1}
$$
$$
\times \left(1 - e^{-\lambda/x}\right)^{\alpha(n-m+1)-1} dx. \tag{42}
$$

The ratio of the expected experiment time (REET) denoted as δ_{REET} is computed between progressive type-II censored data with binomial removals (PT-II CBR) and the complete sampling. We define

$$
\delta_{\text{REET}} = \frac{E\left[X_m\right] \text{ under PT-II CBR}}{E\left[X_n^*\right] \text{ under complete sampling}}. \tag{43}
$$

It can be noted that δ_{REET} gives an important information in order to determine the shortest experiment time significantly, and if a much larger sample of n test units is used, the test is stopped, when mth failures are observed. But we are interested here, in order to consider various values of n, m and p numerically calculated, the expected experiment times under PT-II CBR and complete sample, which are derived in equations (40) and (41), respectively. The results are presented in Table 1. As mentioned earlier, analytical comparison of the expected test times under PT-II CBR and complete sampling test is very difficult. Hence, it is calculated for different values of n, m, and p. Here, we have considered $n = 6, 10$, and 15 and the choices of m are listed in Table 1. The various values of removal probability p considered are $p = 0.05, 0.1, 0.3, 0.5, 0.7$, and 0.9. The results thus obtained are summarized in Table 1. It is noted from the results that for a fixed value of effective sample size m, the values of δ_{REET} decrease as n increases. For fixed n, the values of δ_{REET} and expected termination time under PT-II CBR and complete sampling test increase as m increases. Moreover, from Table 1, the expected test time is influenced by the value of the removal probability p from effected sample size m. So, p is an important factor on the expected test time and when p is large, $(n-m)$ units are removed at the earlier stage of life test out of n units. Hence, this result gives an idea to the observed lifetimes much closer to tail of the failure time distribution. Thus, the expected test time of PT-II CBR is closed to that of complete sample. Figure 1 represents the ratio of the expected test time under PT-II CBR to the expected test time under complete sample versus n for $m = 8$ and different values of the removal probability p. Finally, we observed that form Figure 1, for larger values of p, the ratio is approaching to 1 quickly. Hence, up $p \leq 0.5$ is more significant for the reduction of expected test time. So, the expected termination time for binomial removal with $p = 0.5$ is to be taken for further calculation.

5. Algorithm for Sample Simulation under PT-II CBR

We need to simulate PT-II CBR from specified GIED and propose the use of the following algorithm.

  (I) Specify the value of n.
  (II) Specify the value of m.
  (III) Specify the value of parameters α, λ, and p.
  (IV) Generate random number r_i from $B(n-m-\sum_{l=0}^{i-1} r_l, p)$, for $i = 1, 2, 3, \ldots, m-1$.
  (V) Set r_m according to the following relation:
  (VI) $r_m = \begin{cases} n-m-\sum_{l=1}^{m-1} r_l, & \text{if } n-m-\sum_{l=1}^{m-1} r_l > 0, \\ 0, & \text{otherwise.} \end{cases}$
  (VII) Generate m independent $U(0,1)$ random variables $W_1, W_2, \ldots, W_m$.
  (VIII) For given values of the progressive type-II censoring scheme r_i $(i = 1, 2, \ldots, m)$
     set $V_i = W_i^{1/(i+r_m+\cdots+r_{m-i+1})}$ $(i = 1, 2, \ldots, m)$.
  (IX) Set $U_i = 1 - V_m V_{m-1}, \ldots, V_{m-i+1}$ $(i = 1, 2, \ldots, m)$, then $U_1, U_2, \ldots, U_m$ are PT-II CBR samples of size m from $U(0,1)$.
  (X) Finally, for given values of parameters α and λ, set $x_i = F^{-1}(U)$ $(i = 1, 2, \ldots, m)$. Then, $(x_1, x_2, \ldots, x_m)$ is the required PT-II CBR sample of size m from the GIED.

6. Simulation Studies

The estimators $\widehat{\alpha}_M$ and $\widehat{\lambda}_M$ denote the MLEs of the parameters α and λ, respectively, while $\widehat{\alpha}_S$, $\widehat{\lambda}_S$ and $\widehat{\alpha}_G$, $\widehat{\lambda}_G$ are

TABLE 1: Expected experiment time $E(X_m)$ and δ_{REET} (in the brackets) for $(\alpha, \lambda) = (2, 2)$ under PT-II CBR.

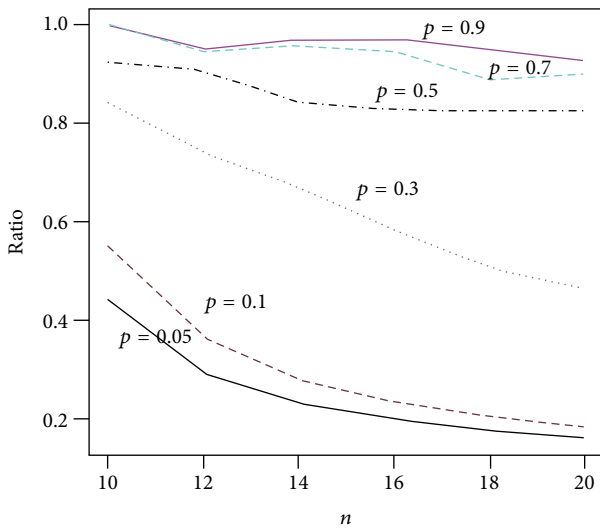| n	m	$p = 0.05$	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
	6	7.751048	7.751048	7.751048	7.751048	7.751048	7.751048
	5	3.892673	4.552206	5.648018	6.565731	6.5891	6.783896
		(0.50221)	(0.58730)	(0.72867)	(0.84707)	(0.85009)	(0.87522)
6	4	2.325743	2.562537	3.794699	4.684974	5.353976	5.466073
		(0.30005)	(0.33060)	(0.48957)	(0.60443)	(0.69074)	(0.70520)
	3	1.620274	1.681345	2.187986	3.042274	3.886906	4.45924
		(0.20903)	(0.21691)	(0.28228)	(0.39249)	(0.50146)	(0.57530)
	10	9.944451	9.944451	9.944451	9.944451	9.944451	9.944451
	9	6.127289	7.036948	9.479675	9.561345	9.576649	9.579844
		(0.61615)	(0.70762)	(0.95326)	(0.96147)	(0.96301)	(0.96333)
	8	4.060994	4.80158	7.879664	8.599637	9.252454	9.263861
		(0.40836)	(0.48284)	(0.79236)	(0.86476)	(0.93041)	(0.93156)
10	7	2.823925	3.398965	6.156561	7.549806	7.741783	8.483587
		(0.28396)	(0.34179)	(0.61909)	(0.75919)	(0.77850)	(0.85309)
	6	2.148269	2.457862	4.626875	6.63739	7.208771	7.344575
		(0.21603)	(0.24715)	(0.46527)	(0.66744)	(0.72490)	(0.73856)
	5	1.687802	1.846467	3.157211	4.95314	5.978016	6.629752
		(0.16972)	(0.18567)	(0.31748)	(0.49808)	(0.60114)	(0.66667)
	15	12.82014	12.82014	12.82014	12.82014	12.82014	12.82014
	14	8.630142	10.131632	12.066029	12.113158	12.19199	12.203057
		(0.67317)	(0.79029)	(0.94117)	(0.94485)	(0.95100)	(0.95186)
	13	6.077333	8.276934	11.045101	11.288722	11.463162	11.581711
		(0.47404)	(0.64561)	(0.86154)	(0.88054)	(0.89415)	(0.90339)
	12	4.516254	6.253768	10.623084	11.10239	11.475101	11.814146
		(0.352278)	(0.48780)	(0.82862)	(0.86601)	(0.89508)	(0.92153)
15	11	3.402202	4.837645	9.057676	10.012223	10.313773	10.388214
		(0.265379)	(0.37734)	(0.70651)	(0.78097)	(0.80449)	(0.81030)
	10	2.767468	3.713898	7.918575	9.783724	9.916002	9.924043
		(0.21586)	(0.28969)	(0.61766)	(0.76315)	(0.773470)	(0.77409)
	9	2.33351	2.91483	6.908012	9.04457	9.138829	9.148829
		(0.18201)	(0.22736)	(0.53884)	(0.70549)	(0.71285)	(0.71362)
	8	1.927279	2.392637	5.364918	7.933467	8.324875	8.794574
		(0.15033)	(0.186631)	(0.41847)	(0.618828)	(0.64935)	(0.68599)

FIGURE 1: δ_{REET} under PT-II CBR to δ_{REET} under complete sample.

the corresponding Bayes estimators under SELF and GELF, respectively. We compare the estimators obtained under GELF with corresponding Bayes estimators under SELF and MLEs. The comparisons are based on the simulated risks (average loss over sample space) under GELF. Here, $((\alpha_{L^c}\ \alpha_{U^c}), (\lambda_{L^c}\ \lambda_{U^c}))$ and $((\alpha_{L^h}\ \alpha_{U^h}), (\lambda_{L^h}\ \lambda_{U^h}))$ represent confidence interval (CI) and HPD of α and λ, respectively. Also (avg. L_{α^c}, avg. L_{λ^c}) and (avg. L_{α^h}, avg. L_{λ^h}) represent the average length of CI and HPD of α and λ, respectively. It may be mentioned here that the exact expressions for the risks cannot be obtained as estimators are not found in nice closed form. Therefore, the risks of the estimators are estimated on the basis of MonteCarlo simulation study of 5000 samples. It may be noted that the risks of the estimators will depend on values of n, m, p, λ, α, and δ. The choice of hyperparameters α and λ can be taken in such a way that if we consider any two independent information as prior mean and variance of α and λ, then, $(\mu_1 = b_1/a_1, \sigma_1 = b_1/a_1^2)$ and $(\mu_2 = b_2/a_2, \sigma_2 = b_2/a_2^2)$, respectively whereas μ_1 and μ_2 are considered as true values of the parameters α and λ for different confidence in terms of smaller, moderate, and larger variances, On the basis of this information, the hyper parameters of α and λ (say (b_1, a_1) and (b_2, a_2)) can be easily evaluated from this relation, $(a_1 = \mu_1/\sigma_1, b_1 = \mu_1^2/\sigma_1)$ and $(a_2 = \mu_2/\sigma_2, b_1 = \mu_2^2/\sigma_2)$, respectively.

In order to consider the variation in the values of m, δ, and ν, we have obtained the simulated risks for $m = 9, 12, 15,$

TABLE 2: Risks of estimators of α and λ under GELF for fixed value of $n = 15$, $m = 9$, $p = 0.5$, $a_1 = 0.00001$, $b_1 = 0.0001$, $a_2 = 0.00001$, and $b_2 = 0.0001$.

δ	$R_G(\widehat{\alpha}_M)$	$R_G(\widehat{\alpha}_S)$	$R_G(\widehat{\alpha}_G)$	$R_G(\widehat{\lambda}_M)$	$R_G(\widehat{\lambda}_S)$	$R_G(\widehat{\lambda}_G)$
−1.5	0.2227	0.2148	0.2147	0.1021	0.1011	0.1000
−0.75	0.0559	0.0529	0.0528	0.0252	0.0251	0.0250
−0.5	0.0268	0.0253	0.0251	0.0115	0.0114	0.0113
−0.25	0.0067	0.0063	0.0062	0.0031	0.0029	0.0027
0.25	0.0080	0.0073	0.0070	0.0031	0.0030	0.0027
0.5	0.0308	0.0282	0.0281	0.0115	0.0113	0.0112
0.75	0.0694	0.0630	0.0626	0.0275	0.0270	0.0261
1.5	0.3168	0.2800	0.2764	0.1178	0.1138	0.1134

TABLE 3: Risks of estimators of α and λ under SELF for fixed, $n = 15$, $\alpha = 2$, $\lambda = 2$, and $p = 0.5$.

(δ, v)	m	$R_S(\widehat{\alpha}_M)$	$R_S(\widehat{\alpha}_S)$	$R_S(\widehat{\alpha}_G)$	$R_S(\widehat{\lambda}_M)$	$R_S(\widehat{\lambda}_S)$	$R_S(\widehat{\lambda}_G)$
	9	1.7764	1.2074	1.2005	0.5226	0.4647	0.4641
(−0.25, 0.4)	12	1.4478	1.0763	1.0722	0.50122	0.4620	0.4600
	15	1.1742	0.9122	0.9093	0.3990	0.3984	0.3980
	9	1.7764	1.4362	1.4291	0.5226	0.5027	0.5021
(−0.25, 1)	12	1.4478	1.2430	1.2385	0.50122	0.4686	0.4681
	15	1.1742	0.9839	0.9809	0.3990	0.4334	0.4330
	9	1.7764	1.7631	1.7555	0.5226	0.5119	0.5107
(−0.25, 10)	12	1.4478	1.3656	1.3286	0.50122	0.4317	0.4308
	15	1.1742	1.2204	1.2200	0.3990	0.3867	0.3860
	9	1.7764	1.0960	1.0841	0.5226	0.4676	0.4665
(0.5, 0.4)	12	1.4478	1.0213	1.3286	0.50122	0.4187	0.4178
	15	1.1742	0.9169	0.9111	0.3990	0.3809	0.3802
	9	1.7764	1.5161	1.5010	0.5226	0.5414	0.5401
(0.5, 1)	12	1.4478	1.3290	1.3286	0.50122	0.5371	0.5359
	15	1.1742	1.1002	1.0936	0.3990	0.4365	0.4356
	9	1.7764	1.7352	1.7399	0.5226	0.5119	0.5107
(0.5, 10)	12	1.4478	1.3287	1.3286	0.50122	0.4317	0.4308
	15	1.1742	1.1708	1.1646	0.3990	0.3867	0.3860

$\lambda = 2 = \mu_2$ (say prior mean of λ), $\alpha = 2 = \mu_1$ (say prior mean of α), $\sigma_1 = (0.4, 1, 10)$ (say prior variance of α), $\sigma_2 = (0.4, 1, 10)$ (say prior variance of λ) and $\delta = (-0.25, 0.5)$ (say GELF loss parameter). Generating the samples of PT-II CBR as mentioned in Section 5, the simulated risks under SELF and GELF have been obtained for selected values of n, m, p, λ, α, and δ. The results are summarized in Tables 2–4. It is noted from Table 2 that for almost all the considered values of δ, the risks of the estimators $\widehat{\alpha}_G$ and $\widehat{\lambda}_G$ have minimum as compared to the considered competitive estimators, respectively, under GELF. To know the effect of variation in the value of other parameters on the risks of the estimators of α and λ, we arbitrarily fixed $\delta = 0.5$ (for the case when overestimation is more serious than underestimation) and $\delta = -0.25$ in reverse situation. Tables 3 and 4 represent the risks of estimators of α and λ under both losses in under and overestimation situations when the prior mean is same as the true value of the parameters $\alpha = 2$ and $\lambda = 2$ for smaller, moderate, and larger values of the prior variance of the parameters in order to consider the hyperparameters $(a_1 = a_2 = 5, b_1 = b_2 = 10)$,

$(a_1 = a_2 = 2, b_1 = b_2 = 4)$ and $(a_1 = a_2 = 1/5, b_1 = b_2 = 2/5)$, respectively. When the effective sample size m increases, the risks of all the estimators of α and λ under both losses decrease for $\delta = (-0.25, 0.5)$ and the simulated risks of $\widehat{\alpha}_G$ and $\widehat{\lambda}_G$ are smaller than those of $(\widehat{\alpha}_S, \widehat{\lambda}_S)$ and $(\widehat{\alpha}_M, \widehat{\lambda}_M)$ for all the considered cases, including those where under estimation is considered to be more serious than over estimation or viceversa. Under different prior variances along with variation of effective sample size m, the 95% HPD and CI intervals are obtained. In Table 5, it is observed that the average length of CI and HPD interval decreases when the effective sample size m increases and average length of HPD interval is always less than that of CI, which are represented in Figure 2 also.

7. Real Data Analysis

Here, we consider the real data set presented in Lawless [22] which represent the number of revolutions to failure for

TABLE 4: Risks of estimators of α and λ under GELF for fixed, $n = 15$, $\alpha = 2$, $\lambda = 2$, and $p = 0.5$.

(δ, v)	m	$R_G(\widehat{\alpha}_M)$	$R_G(\widehat{\alpha}_S)$	$R_G(\widehat{\alpha}_G)$	$R_G(\widehat{\lambda}_M)$	$R_G(\widehat{\lambda}_S)$	$R_G(\widehat{\lambda}_G)$
	9	0.031491	0.00581	0.00579	0.01332	0.00278	0.00276
$(-0.25, 0.4)$	12	0.02707	0.00504	0.00503	0.01257	0.00277	0.00267
	15	0.02279	0.00427	0.00426	0.01106	0.00240	0.00239
	9	0.03149	0.00634	0.00633	0.01332	0.00301	0.00204
$(-0.25, 1)$	12	0.02707	0.00547	0.00526	0.01257	0.00273	0.00270
	15	0.02279	0.00446	0.00445	0.01106	0.00250	0.00247
	9	0.03149	0.00704	0.00703	0.01332	0.00292	0.00182
$(-0.25, 10)$	12	0.02707	0.00552	0.00551	0.01257	0.00258	0.00247
	15	0.02279	0.00487	0.00486	0.01106	0.00237	0.002366
	9	0.03149	0.02357	0.02344	0.01332	0.01186	0.01096
$(0.5, 0.4)$	12	0.02707	0.02083	0.02074	0.01257	0.01057	0.01056
	15	0.02279	0.01902	0.01895	0.01106	0.00972	0.009686
	9	0.03149	0.02895	0.028800	0.01332	0.01301	0.01300
$(0.5, 1)$	12	0.02707	0.02513	0.02503	0.01257	0.01226	0.01225
	15	0.02279	0.02129	0.02122	0.01106	0.01083	0.01082
	9	0.03149	0.03124	0.03107	0.01332	0.01224	0.01223
$(0.5, 10)$	12	0.02707	0.02448	0.02438	0.01257	0.01053	0.01052
	15	0.02279	0.02184	0.02177	0.01106	0.00988	0.00987

TABLE 5: 95% CI and HPD for different sample m and prior variances v for fixed $n = 15$, $\alpha = 2$, and $\lambda = 2$.

v	m	α_{L^c}	α_{U^c}	avg. L_{α^c}	α_{L^h}	α_{U^h}	avg. L_{α^h}	λ_{L^c}	λ_{U^c}	avg. L_{λ^c}	λ_{L^h}	λ_{U^h}	avg. L_{λ^h}
	9	0.09344	5.3344	5.24096	1.9321	2.4080	0.4759	0.8449	3.5977	2.7528	1.8530	2.09834	0.24534
0.4	12	0.2799	4.7309	4.451	1.9411	2.4056	0.4645	0.9497	3.5490	2.5993	1.9610	2.2027	0.2417
	15	0.4641	4.4314	3.9673	1.9967	2.4505	0.4538	1.0034	3.4398	2.4364	1.9984	2.2311	0.2327
	9	0.09692	5.2768	5.17988	1.9774	2.5300	0.5526	0.8557	3.6465	2.7908	1.8733	2.1205	0.2472
1	12	0.2813	4.7557	4.4744	1.9867	2.5220	0.5353	0.9506	3.5485	2.5979	1.9917	2.2360	0.2443
	15	0.4691	4.4531	3.984	1.9967	2.5147	0.518	1.0110	3.4467	2.4357	1.9991	2.2341	0.2350
	9	0.0995	5.2702	5.1707	1.8770	2.5519	0.6749	0.8364	3.6046	2.7682	1.8671	2.113782	0.246682
10	12	0.2738	4.7471	4.4733	1.9431	2.5269	0.5838	0.9391	3.5221	2.583	1.9331	2.1699	0.2368
	15	0.4702	4.5050	4.0348	1.9954	2.5594	0.564	1.0185	3.4668	2.4483	1.9993	2.2334	0.2341

each of 23 ball bearing in a life test. But for the purpose of illustrating the method discussed in this paper, a PT-II CBR is generated from real data set, and we consider three schemes under PT-II CBR which are given in Table 6. Table 7 shows the MLEs and Bayes estimators of α and λ under SELF and GELF, CI/HPD interval based on complete real data set. For this real data set, Abouammoh and Alshingiti [5] indicated that the GIED provides satisfactory fit. This real data set was originally discussed by Lieblein and Zelen [23]. In Section 3, MCMC algorithm and mathematical treatments are given, for long run, we take noninformative prior for this purpose, and the value of the hyper parameters α and λ are taken as $(b_1 = 0.00001, a_1 = 0.0001)$ and $(b_2 = 0.00001, a_2 = 0.0001)$, respectively. Hence, on the basis of Table 6, we use noninformative prior under different degrees of censoring; the MLEs and the Bayes estimators and CI/HPD interval of α and λ under SELF and GELF for $\delta = \pm 0.5$ are presented in Tables 8 and 9, respectively. Hence, finally, the study of Tables 8 and 9 observed that the MLEs and Bayes estimators and

length of CI/HPD interval of α and λ decrease as degree of censoring decreases, respectively.

8. Conclusion

(1) On the basis of simulation study, we observed that the maximum likelihood and Bayes methods are used for estimating the parameters under GELF and SELF of GIED based on PT-II CBR. Bayes estimators have been obtained on the basis of MCMC method. These methods are applied to real data set based on the number of revolutions to failure for each of 23 ball bearing in a life test.

(2) It has been noticed, under consideration of different prior believes, from tables, that the estimated risks of estimators decrease as effective sample size increases and Bayes estimates have the smallest estimated risks as compared with their corresponding MLEs. Hence, the proposed estimators $(\widehat{\alpha}_G, \widehat{\lambda}_G)$ perform better than $(\widehat{\alpha}_S, \widehat{\lambda}_S)$ and $(\widehat{\alpha}_M, \widehat{\lambda}_M)$ for different degree of censoring, when under estimation is serious
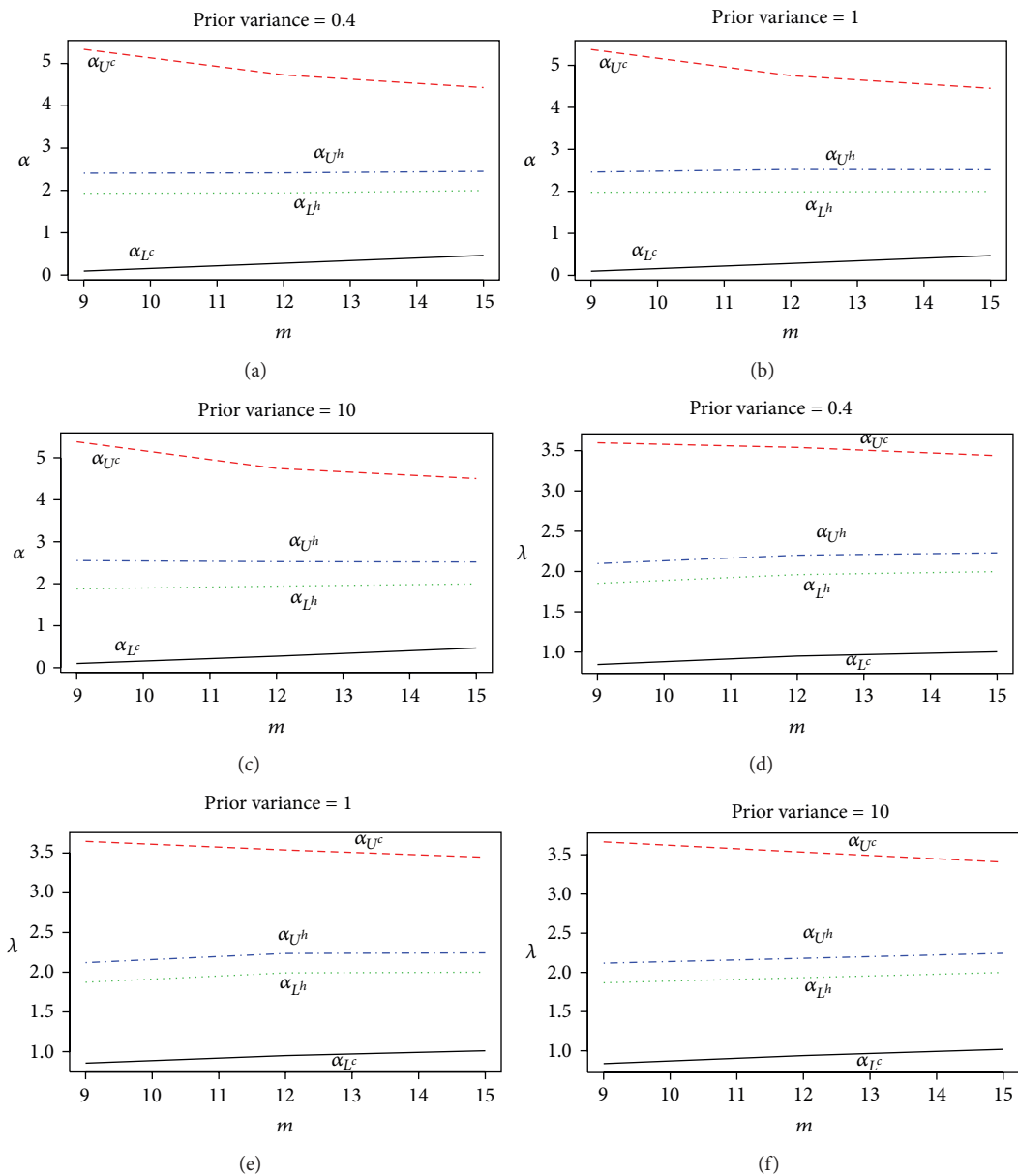
Figure 2: Average HPD and CI for different prior variances for fixed confidence coefficient.

Table 6: PT-II CBR under different censoring scheme ($S_{n:m}$) for fixed $n = 23$, $p = 0.5$.

$(S_{n:m})$	Samples											
$S_{23:12}$	17.88	48.4	68.64	68.88	84.12	93.12	98.64	105.12	105.84	127.92	128.04	173.4
	5	5	1	0	0	0	0	0	0	0	0	0
$S_{23:15}$	17.88	45.6	51.96	54.12	68.64	68.64	68.88	84.12	93.12	98.64	105.12	105.84
	4	2	0	2	0	0	0	0	0	0	0	0
	127.92	128.04	173.4									
	0	0	0									
$S_{23:18}$	17.88	45.6	48.4	51.84	51.96	54.12	55.56	68.64	68.64	68.88	84.12	93.12
	4	0	0	0	0	0	1	0	0	0	0	0
	98.64	105.12	105.84	127.92	128.04	173.4						
	0	0	0	0	0	0						

Table 7: Bayes and ML estimates based on real data set for $n = 23$, $p = 0.05$.

	MLE	Bayes estimates (MCMC)			Interval			
		SELF	GELF		95% CI		95% HPD	
			$\delta = -0.5$	$\delta = 0.5$	L^c	U^c	L^h	U^h
α	5.307588	5.270884	5.259487	5.276109	1.121988	9.493188	4.474905	6.08333
λ	129.9959	129.5706	129.3823	128.8193	77.7282	182.2635	110.5995	148.8837

Table 8: Bayes and ML estimates, CI and HPD intervals for α with fixed $n = 23$ and $p = 0.5$ under PT-II CBR.

$S_{n:m}$	MLE	Bayes estimates (MCMC)			Interval			
		SELF	GELF		95% CI		95% HPD	
			$\delta = -0.05$	$\delta = 0.05$	α_{L^c}	α_{U^c}	α_{L^h}	α_{U^h}
$S_{23:12}$	5.1029	5.0366	4.9872	4.9822	0.9933	11.9991	3.8315	6.2232
$S_{23:15}$	5.7702	5.6911	5.6898	5.6427	0.3036	11.8042	4.5576	6.8168
$S_{23:18}$	6.0633	5.9981	5.9582	5.9203	0.3357	11.7908	4.9370	7.1142

Table 9: Bayes and ML estimates, CI and HPD intervals for λ with fixed $n = 23$ and $p = 0.5$ under PT-II CBR.

$S_{n:m}$	MLE	Bayes estimates (MCMC)			Interval			
		SELF	GELF		95% CI		95% HPD	
			$\delta = -0.05$	$\delta = 0.05$	λ_{L^c}	λ_{U^c}	λ_{L^h}	λ_{U^h}
$S_{23:12}$	175.22	173.99	173.04	172.69	79.66	270.78	139.37	207.36
$S_{23:15}$	168.44	167.35	167.33	166.34	86.95	249.92	137.53	196.14
$S_{23:18}$	157.84	156.70	160.16	155.86	87.12	228.55	132.27	182.98

than over estimation and vice-versa. The CI/HPD interval is also obtained. We found the Bayes estimates are superior than those of the corresponding MLEs.

(3) We have obtained the expected test times under PT-II CBR and complete sampling to compare it. In Table 1, the numerical results indicate that the expected test time depends very much times on the values of removal probability. When the probability of removal is large, a slight reduction in the expected test time can be achieved only by increasing the total number of test units n.

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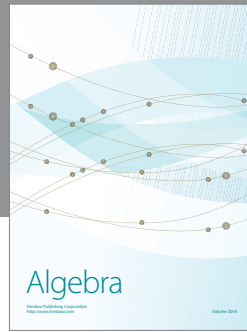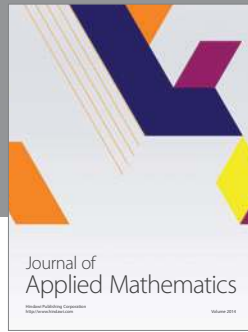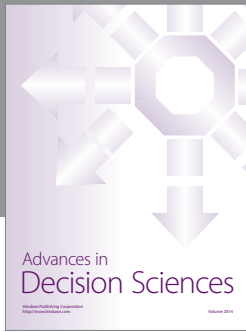
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