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Research Article

Estimation of Population Median under Robust Measures of an Auxiliary Variable

Muhammad Irfan,[1] **Maria Javed**,[1] **Sandile C. Shongwe**,[2] **Muhammad Zohaib**,[1] and **Sajjad Haider Bhatti**[1]

[1]*Department of Statistics, Government College University, Faisalabad, Pakistan*
[2]*Department of Mathematical Statistics and Actuarial Science, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9301, South Africa*

Correspondence should be addressed to Muhammad Irfan; mirfan@gcuf.edu.pk

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In this paper, a generalized class of estimators for the estimation of population median are proposed under simple random sampling without replacement (SRSWOR) through robust measures of the auxiliary variable. Three robust measures, decile mean, Hodges–Lehmann estimator, and trimean of an auxiliary variable, are used. Mathematical properties of the proposed estimators such as bias, mean squared error (MSE), and minimum MSE are derived up to first order of approximation. We considered various real-life datasets and a simulation study to check the potentiality of the proposed estimators over the competitors. Robustness is also examined through a real dataset. Based on the fascinating results, the researchers are encouraged to use the proposed estimators for population median under SRSWOR.

1. Introduction

Extensive work has been done on the estimation of the population mean, proportion, variance, regression coefficient, and so forth; but very little attention has been made to propose the efficient estimators of the median. In many situations, researchers are often interested in dealing with variables such as income, expenditure, taxes, consumption, and production; and the latter variables have highly skewed distributions. In such situations, the median is considerably a more appropriate measure of location than the mean. The problem of estimation of median under simple random sampling scheme has been discussed by Gross [1], Sedransk and Meyer [2], and Smith and Sedransk [3]. Kuk and Mak [4] were the first authors to investigate the estimation of the median using auxiliary information. After Kuk and Mak's [4] estimator, Singh et al. [5], Aladag and Cingi [6], Solanki and Singh [7], Shabbir and Gupta [8], Baig et al. [9], and Shabbir et al. [10] have developed different estimators for estimating finite population median based on the known conventional measures of the auxiliary variable under different sampling schemes. A brief explanation of Kuk and Mak's [4] estimator is as described as follows.

Let Y and X be the study and the auxiliary variables selected from a finite population $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \ldots, \Theta_N\}$ of size "N" under simple random sampling without replacement (SRSWOR) subject to the constraint $n < N$. Further let Y_i and X_i, $i = (1, 2, \ldots, N)$&y_i and x_i, $i = (1, 2, \ldots, n)$ be the values of the ith units of the population and sample, respectively. Let M_Y and M_X be the population median of the study and auxiliary variables with the probability density functions given by $f_Y(M_Y)$ and $f_X(M_X)$, respectively. We further assume that $f_Y(M_Y)$ and $f_X(M_X)$ are positive.

Suppose that $y_{(1)} \leq y_{(2)} \leq y_{(3)} \leq \cdots \leq y_{(n)}$ are the y values of sample units in ascending order; furthermore, let s be the integer such that $Y_{(s)} \leq M_Y \leq Y_{(s+1)}$ and $p = s/n$ are the proportion of Y values in the sample which are less than or equal to M_Y. Kuk and Mak [4] considered a two-way classification (p_{ij}) as given in Table 1.

Table 1: Matrix of proportions.

	$Y \le M_Y$	$Y > M_Y$	Total
$X \le M_X$	p_{11}	p_{21}	$p_{\cdot 1}$
$X > M_X$	p_{12}	p_{22}	$p_{\cdot 2}$
Total	$p_{1\cdot}$	$p_{2\cdot}$	1

Suppose that $\widehat{M}_Y$ and $\widehat{M}_X$ are the sample estimators of M_Y and M_X; then the correlation coefficient between $(\widehat{M}_Y, \widehat{M}_X)$ is $\rho_{(\widehat{M}_Y, \widehat{M}_X)}(or)\rho_c(or)\rho = 4p_{11} - 1$ ranging from -1 to $+1$ as p_{11} increases from 0 to 0.5, where p_{11} is the proportion of units in the population with $X \le M_X$ and $Y \le M_Y$. Gross [1] proved that $\widehat{M}_Y$ is consistent and asymptotically normally distributed with mean M_Y and variance

$$\widehat{M}_Y = (1 - f)(4n)^{-1}[f_Y(M_Y)]^2, \qquad (1)$$

where $f = (n/N)$ is the sampling fraction.

Efficiency of the ratio, product, and regression type estimators are ambiguous in the presence of the extreme values/outlier(s) in the dataset. In our present study, the problem under consideration is to estimate the median for finite population and suggest some generalized classes of estimators by utilizing known robust measures of an auxiliary variable under SRSWOR. The novelty of this work is as follows:

(i) Robust measures (i.e., decile mean, Hodges–Lehmann estimator, and trimean) of an auxiliary variable are utilized for the first time to investigate the progressive estimation of the population median

(ii) A variety of estimators can be generated through the proposed generalized estimator

(iii) Robustness study is examined to check the performance of the proposed generalized estimator in the presence of outlier

The following relative error terms and notations are used to obtain the mathematical properties such as bias, mean squared error (MSE), and minimum MSE of various estimators: $e_0 = ((\widehat{M}_Y - M_Y)/M_Y), e_1 = ((\widehat{M}_X - M_X)/M_X)$ such that $E(e_0) = E(e_1) = 0$.

$$E\left(e_0^2\right) = \lambda C_{MY}^2,$$
$$E\left(e_1^2\right) = \lambda C_{MX}^2, \qquad (2)$$
$$E(e_o e_1) = \lambda C_{MYX} = \rho C_{MY} C_{MX},$$

where

$$C_{MY} = \frac{1}{M_Y f_Y(M_Y)} \text{ be the population coefficient of variation of } Y,$$

$$C_{MX} = \frac{1}{M_X f_X(M_X)} \text{ be the population coefficient of variation of } X,$$

$$\lambda = \frac{1}{4}\left(\frac{1}{n} - \frac{1}{N}\right) \text{ be the finite population correction (f.p.c) factor.} \qquad (3)$$

The rest of the article is organized in the following way. Section 2 gives comprehensive details of existing estimators for the population median. Section 3 proposes generalized classes of estimators for estimating population median using robust measures of an auxiliary variable. Bias, mean squared error (MSE), and minimum MSE of generalized classes of estimators are derived up to the first degree of approximation in the same section. Four real-life datasets and a simulation study are performed in Section 4 to check the potential of the new estimators as compared to the existing ones. Robustness of the proposed estimators is evaluated by carrying out a real-life dataset in Section 5. Section 6 contains the concluding remarks and some recommendations.

2. Existing Median Estimators

The major drawback of all the suggested estimators for estimating population median is that they are based on the usual conventional measures of an auxiliary variable. In this section, we discuss the usual and well-known estimators for estimating population median under SRSWOR as suggested by different authors.

Kuk and Mak [4] suggested a ratio-type estimator by assuming the known median of the X variable.

$$\widehat{M}_R = \widehat{M}_Y\left(\frac{M_X}{\widehat{M}_X}\right). \qquad (4)$$

The expression for mean square error of $\widehat{M}_R$ estimator is given as

$$\text{MSE}\left(\widehat{M}_R\right) \cong \lambda M_Y^2\left(C_{MY}^2 + C_{MX}^2 - 2C_{MYX}\right). \qquad (5)$$

The exponential ratio-type estimator for estimating median is given as

$$\widehat{M}_{\text{EX}} = \widehat{M}_Y \exp\left(\frac{M_X - \widehat{M}_X}{M_X + \widehat{M}_X}\right). \qquad (6)$$

The MSE of $\widehat{M}_{\text{EX}}$ up to the first degree of approximation is given by

$$\text{MSE}\left(\widehat{M}_{\text{EX}}\right) \cong \lambda M_Y^2\left(C_{MY}^2 + \frac{1}{4}C_{MX}^2 - C_{MYX}\right). \qquad (7)$$

Singh [11] developed an unbiased difference estimator which is given by

$$\widehat{M}_D = \widehat{M}_Y + d\left(M_X - \widehat{M}_X\right), \qquad (8)$$

where d is an unknown constant whose value needs to be determined.

Minimum MSE of $\widehat{M}_D$ up to the first degree of approximation is as follows:

$$\text{MSE}\left(\widehat{M}_D\right)_{\min} \cong \lambda M_Y^2 C_{MY}^2\left(1 - \rho^2\right). \qquad (9)$$

Remark 1. The MSE of $\widehat{M}_D$ is always smaller than the MSE of $\widehat{M}_Y, \widehat{M}_R$ and $\widehat{M}_{\text{EX}}$ if $\rho > 0$, $(C_{MX} - \rho C_{MX})^2 > 0$ and $(0.5 C_{MX} - \rho C_{MX})^2 > 0$, respectively.

Rao [12] and Gupta et al. [13], respectively, suggested three difference types of estimators for estimating median as

$$\widehat{M}_{D1} = d_1\widehat{M}_Y + d_2\left(M_X - \widehat{M}_X\right),$$

$$\widehat{M}_{D2} = \left\{d_3\widehat{M}_Y + d_4\left(M_X - \widehat{M}_X\right)\right\}\left(\frac{M_X}{\widehat{M}_X}\right), \tag{10}$$

$$\widehat{M}_{D3} = \left\{d_5\widehat{M}_Y + d_6\left(M_X - \widehat{M}_X\right)\right\}\left(\frac{M_X - \widehat{M}_X}{M_X + \widehat{M}_X}\right),$$

where $d_1, d_2, d_3, d_4, d_5,$ and d_6 are unknown constants.

The minimum MSE of $\widehat{M}_{D1}$ at optimum values of $d_{1\,(\mathrm{opt})} = (1/(1 + \lambda C_{MY}^2(1-\rho^2)))$ and $d_{2\,(\mathrm{opt})} = (M_Y/M_X)((\rho C_{MY}/C_{MX})/(1 + \lambda C_{MY}^2(1-\rho^2)))$ is given by

$$\mathrm{MSE}\left(\widehat{M}_{D1}\right)_{\min} \cong \frac{\lambda M_Y^2 C_{MY}^2\left(1-\rho^2\right)}{1 + \lambda C_{MY}^2\left(1-\rho^2\right)}. \tag{11}$$

The minimum MSE of $\widehat{M}_{D2}$ at optimum values of $d_{3\,(\mathrm{opt})} = (1/(1 - \lambda C_{MX}^2 + \lambda C_{MY}^2(1-\rho^2)))$ and $d_{4\,(\mathrm{opt})} = (M_Y/M_X)(1 + d_{3\,(\mathrm{opt})}(\rho C_{MY}/C_{MY}) - 2)$ is given by

$$\mathrm{MSE}\left(\widehat{M}_{D2}\right)_{\min} \cong M_Y^2\left[\frac{\left(1-\lambda C_{MX}^2\right)\lambda C_{MY}^2\left(1-\rho^2\right)}{\left(1-\lambda C_{MX}^2\right) + \lambda C_{MY}^2\left(1-\rho^2\right)}\right]. \tag{12}$$

The minimum MSE of $\widehat{M}_{D3}$ at optimum values of $d_{5\,(\mathrm{opt})} = ((1 - (\lambda C_{MX}^2/5))/1 + \lambda C_{MY}^2(1-\rho^2))$ and $d_{6\,(\mathrm{opt})} = (M_Y/M_X)\left\{(1/2) + d_{5\mathrm{opt}}((\rho C_{MY}/C_{MX}) - 1)\right\}$ is given by

$$\mathrm{MSE}\left(\widehat{M}_{D3}\right)_{\min} \cong M_Y^2\left[\frac{\lambda C_{MY}^2\left(1-\rho^2\right) - (1/64)\lambda^2 C_{MX}^4 - (1/4)\lambda^2 C_{MY}^2 C_{MX}^2\left(1-\rho^2\right)}{\left(1-\lambda C_{MX}^2\right) + \lambda C_{MY}^2\left(1-\rho^2\right)}\right]. \tag{13}$$

Shabbir and Gupta [8] suggested a generalized difference type estimator for the estimation of median as

$$\widehat{M}_{\mathrm{SG}} = \left[d_7\widehat{M}_Y + d_8\left(M_X - \widehat{M}_X\right)\right] \cdot \left[\left(\frac{aM_X + b}{a\widehat{M}_X + b}\right)^{\alpha_1} \exp\left\{\frac{\alpha_2 a\left(M_X - \widehat{M}_X\right)}{a\left\{(\gamma-1)M_X + \widehat{M}_X\right\} + 2b}\right\}\right], \tag{14}$$

where d_7 and d_8 are unknown constants whose values need to be determined, a and b are the known population parameters, and α_1, α_2 and γ are the scalar quantities.

Remark 2. By substitution of the scalar quantities as $\alpha_1 = b = 0, \alpha_2 = \gamma = a = 1$, equation (14) becomes

$$\widehat{M}_{\mathrm{SG}} = \left[d_7\widehat{M}_Y + d_8\left(M_X - \widehat{M}_X\right)\right]\left[\exp\left(\frac{M_X}{\widehat{M}_X} - 1\right)\right]. \tag{15}$$

The minimum MSE of $\widehat{M}_{\mathrm{SG}}$ at optimum values of $d_{7\,(\mathrm{opt})} = ((1 - (1/2)\lambda C_{MX}^2)/(1 + \lambda C_{MY}^2 (1-\rho^2)))$ and $d_{8\,(\mathrm{opt})} = (M_Y/M_X)[1 + d_{7\,(\mathrm{opt})}\{(\rho C_{MY}/C_{MX}) - 2\}]$ is given by

$$\mathrm{MSE}\left(\widehat{M}_{\mathrm{SG}}\right)_{\min} \cong M_Y^2\left[\frac{\lambda C_{MY}^2\left(1-\rho^2\right) - (1/4)\lambda^2 C_{MX}^4 - \lambda^2 C_{MY}^2 C_{MX}^2\left(1-\rho^2\right)}{1 + \lambda C_{MY}^2\left(1-\rho^2\right)}\right]. \tag{16}$$

3. Proposed Generalized Estimator

One eminent disadvantage of existing estimators/class of estimators is that they are typically based on conventional measures. Efficiency of the estimators is uncertain in the occurrence of the extreme values in the dataset. In this section, we define a generalized class of estimators for the estimation of population median using robust measures of an auxiliary variable with the linear combination of non-conventional measures: quartile deviation, midrange, interquartile range, and quartile average. We included three robust measures: decile mean suggested by Rana et al. [14], Hodges–Lehmann estimator suggested by Hettmansperger and McKean [15], and the trimean suggested by Wang et al. [16]. For more details of these robust measures, see the works of Irfan et al. [17, 18].

A generalized estimator for the estimation of population median is

$$T_{i(d)} = \widehat{M}_Y\left[\left\{m_1\left(\frac{\psi\widehat{M}_X + \delta}{\psi M_X + \delta}\right)^{\alpha_3}\exp\left(\frac{M_X - \widehat{M}_X}{M_X + \widehat{M}_X}\right)\right\} + \left\{m_2\left(\frac{\psi M_X + \delta}{\psi\widehat{M}_X + \delta}\right)^{\alpha_4}\right\}\right], \tag{17}$$

where m_1 and m_2 are suitably chosen constants, and α_i $(i = 3 \text{ and } 4)$ takes on the values $1, -1, 2, -2$ for designing new estimators. Note that ψ and δ may be any constant values or functions of the known robust measures as well as nonconventional measures associated with X variable.

Remark 3. Robust measures related to X are the following:

Trimean: $T_M = (Q_1 + 2Q_2 + Q_3)/4$
Hodges–Lehmann:
$H_L = \text{Median}((x_j + x_k)/2),\ 1 \le j \le k \le N$
Decile mean: $D_M = (\sum_{i=1}^{9} D_i/9)$

Remark 4. The nonconventional measures (i.e., interquartile range, midrange, quartile average, and quartile deviation) of an auxiliary variable can be defined as follows:

Interquartile range: $Q_R = Q_3 - Q_1$
Midrange: $M_R = ((x_{(1)} + x_{(N)})/2)$
Quartile average: $Q_A = (Q_3 + Q_1)/2$
Quartile deviation: $Q_D = ((Q_3 - Q_1)/2)$

Remark 5. By putting different values of α_i $(i = 3 \text{ and } 4)$ in equation (17), we get the following families of estimators:

(i) Put $\alpha_3 = 1$ and $\alpha_4 = 2$; proposed family of estimators reduces to

$$T^{\ominus}_{i(d)} = \widehat{M}_Y\left[\left\{m_1\left(\frac{\psi\widehat{M}_X + \delta}{\psi M_X + \delta}\right)\exp\left(\frac{M_X - \widehat{M}_X}{M_X + \widehat{M}_X}\right)\right\} + \left\{m_2\left(\frac{\psi M_X + \delta}{\psi\widehat{M}_X + \delta}\right)^2\right\}\right]. \quad (18)$$

(ii) Put $\alpha_3 = -1$ and $\alpha_4 = -1$; proposed family of estimators reduces to

$$T^{\oplus}_{i(d)} = \widehat{M}_Y\left[\left\{m_1\left(\frac{\psi M_X + \delta}{\psi\widehat{M}_X + \delta}\right)\exp\left(\frac{M_X - \widehat{M}_X}{M_X + \widehat{M}_X}\right)\right\} + \left\{m_2\left(\frac{\psi\widehat{M}_X + \delta}{\psi M_X + \delta}\right)\right\}\right]. \quad (19)$$

(iii) Put $\alpha_3 = -1$ and $\alpha_4 = -2$; proposed family of estimators reduces to

$$T^{\otimes}_{i(d)} = \widehat{M}_Y\left[\left\{m_1\left(\frac{\psi M_X + \delta}{\psi\widehat{M}_X + \delta}\right)\exp\left(\frac{M_X - \widehat{M}_X}{M_X + \widehat{M}_X}\right)\right\} + \left\{m_2\left(\frac{\psi\widehat{M}_X + \delta}{\psi M_X + \delta}\right)^2\right\}\right]. \quad (20)$$

(iv) Put $\alpha_3 = 2$ and $\alpha_4 = 2$; proposed family of estimators reduces to

$$T^{\circledast}_{i(d)} = \widehat{M}_Y\left[\left\{m_1\left(\frac{\psi\widehat{M}_X + \delta}{\psi M_X + \delta}\right)^2\exp\left(\frac{M_X - \widehat{M}_X}{M_X + \widehat{M}_X}\right)\right\} + \left\{m_2\left(\frac{\psi M_X + \delta}{\psi\widehat{M}_X + \delta}\right)^2\right\}\right]. \quad (21)$$

(v) Put $\alpha_3 = -2$ and $\alpha_4 = -1$; proposed family of estimators reduces to

$$T^{\circledcirc}_{i(d)} = \widehat{M}_Y\left[\left\{m_1\left(\frac{\psi M_X + \delta}{\psi\widehat{M}_X + \delta}\right)^2\exp\left(\frac{M_X - \widehat{M}_X}{M_X + \widehat{M}_X}\right)\right\} + \left\{m_2\left(\frac{\psi\widehat{M}_X + \delta}{\psi M_X + \delta}\right)\right\}\right]. \quad (22)$$

Remark 6. When we put robust measures of auxiliary variable with the linear combination of median, quartile deviation, midrange, interquartile range, and quartile average of an auxiliary variable in equation (17), we obtain different series of estimators such as $T^{\ominus}_{i(d)}, T^{\oplus}_{i(d)}, T^{\otimes}_{i(d)}, T^{\circledast}_{i(d)}$ and $T^{\circledcirc}_{i(d)}$. Some members of the class of estimator $T^{\circledcirc}_{i(d)}$ are presented in Table 2. Placing the same values of ψ and δ in $T^{\ominus}_{i(d)}, T^{\oplus}_{i(d)}, T^{\otimes}_{i(d)}$ and $T^{\circledast}_{i(d)}$, we obtain a number of estimators.

Remark 7. Putting appropriate constants or known conventional parameters of the auxiliary variable in place of ψ and δ in equation (17), we can get many optimal estimators. Conventional parameters associated with auxiliary variable X are variance, standard deviation, coefficient of variation, coefficient of skewness, coefficient of kurtosis, coefficient of correlation, and so forth.

3.1. Bias, MSE, and Minimum MSE of $T_{i(d)}$.

The suggested generalized class of estimators $T_{i(d)}$ in terms of e_0 and e_1 is expressed as follows:

$$T_{i(d)} = M_Y(1 + e_0)\left[\left\{m_1\left(1 + \frac{\psi M_X e_1}{\psi M_X + \delta}\right)^{\alpha_3} \cdot \exp\left(-\frac{e_1}{2}\left(1 + \frac{e_1}{2}\right)^{-1}\right)\right\} + \left\{m_2\left(1 + \frac{\psi M_X e_1}{\psi M_X + \delta}\right)^{-\alpha_4}\right\}\right]. \quad (23)$$

After some simplification of equation (23), we have

Table 2: Some members of $T^{\circledcirc}_{i(d)}$.

ψ	δ	i	Estimator $T^{\circledcirc}_{i(d)} = \widehat{M}_Y [\{m_1 ((\psi M_X + \delta)/(\psi \widehat{M}_X + \delta))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((\psi \widehat{M}_X + \delta)/(\psi M_X + \delta))\}]$
Q_A	T_M	1	$T^{\circledcirc}_{1(d)} = \widehat{M}_Y [\{m_1 ((Q_A M_X + T_M)/(Q_A M_X + T_M))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((Q_A \widehat{M}_X + T_M)/(Q_A M_X + T_M))\}]$
M_R	T_M	2	$T^{\circledcirc}_{2(d)} = \widehat{M}_Y [\{m_1 ((M_R M_X + T_M)/(M_R \widehat{M}_X + T_M))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((M_R \widehat{M}_X + T_M)/(M_R M_X + T_M))\}]$
H_L	T_M	3	$T^{\circledcirc}_{3(d)} = \widehat{M}_Y [\{m_1 ((H_L M_X + T_M)/(H_L \widehat{M}_X + T_M))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((H_L \widehat{M}_X + T_M)/(H_L M_X + T_M))\}]$
T_M	H_L	4	$T^{\circledcirc}_{4(d)} = \widehat{M}_Y [\{m_1 ((T_M M_X + H_L)/(T_M \widehat{M}_X + H_L))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((T \cdot M \widehat{M}_X + H_L)/(T \cdot M M_X + H_L))\}]$
Q_A	H_L	5	$T^{\circledcirc}_{5(d)} = \widehat{M}_Y [\{m_1 ((Q_A M_X + H_L)/(Q_A M_X + H_L))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((Q_A \widehat{M}_X + H_L)/(Q_A M_X + H_L))\}]$
H_L	Q_R	6	$T^{\circledcirc}_{6(d)} = \widehat{M}_Y [\{m_1 ((H_L M_X + Q_R)/(H_L \widehat{M}_X + Q_R))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((H_L \widehat{M}_X + Q_R)/(H_L M_X + Q_R))\}]$
M_R	H_L	7	$T^{\circledcirc}_{7(d)} = \widehat{M}_Y [\{m_1 ((M_R M_X + H_L)/(M_R \widehat{M}_X + H_L))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((M_R \widehat{M}_X + H_L)/(M_R M_X + H_L))\}]$
H_L	D_M	8	$T^{\circledcirc}_{8(d)} = \widehat{M}_Y [\{m_1 ((H_L M_X + D_M)/(H_L \widehat{M}_X + D_M))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((H_L \widehat{M}_X + D_M)/(H_L M_X + D_M))\}]$
D_M	T_M	9	$T^{\circledcirc}_{9(d)} = \widehat{M}_Y [\{m_1 ((D_M M_X + T_M)/(D_M \widehat{M}_X + T_M))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((D_M \widehat{M}_X + T_M)/(D_M M_X + T_M))\}]$
M_R	D_M	10	$T^{\circledcirc}_{10(d)} = \widehat{M}_Y [\{m_1 ((M_R M_X + D_M)/(M_R \widehat{M}_X + D_M))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((M_R \widehat{M}_X + D_M)/(M_R M_X + D_M))\}]$
Q_D	D_M	11	$T^{\circledcirc}_{11(d)} = \widehat{M}_Y [\{m_1 ((Q_D M_X + D_M)/(Q_D \widehat{M}_X + D_M))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((Q_D \widehat{M}_X + D_M)/(Q_D M_X + D_M))\}]$
D_M	Q_A	12	$T^{\circledcirc}_{12(d)} = \widehat{M}_Y [\{m_1 ((D_M M_X + Q_A)/(D_M \widehat{M}_X + Q_A))^2 \exp((M_X - \widehat{M}_X)/(M_X + \widehat{M}_X))\} + \{m_2 ((D_M \widehat{M}_X + Q_A)/(D_M M_X + Q_A))\}]$

$$T_{i(d)} = M_Y(1+e_0)\begin{bmatrix} m_1\left(1+\vartheta e_1\alpha_3+\frac{1}{2}\alpha_3(\alpha_3-1)(\vartheta e_1)^2\right)\exp\left(-\frac{e_1}{2}+\frac{e_1^2}{4}\right)+ \\ m_2\left(1-\vartheta e_1\alpha_4+\frac{1}{2}\alpha_4(\alpha_4+1)(\vartheta e_1)^2\right) \end{bmatrix}, \quad (24)$$

where $\vartheta = (\psi M_X/(\psi M_X+\delta))$.

Subtracting M_Y from both sides of equation (24), we get

$$\begin{aligned} T_{i(d)} - M_Y = M_Y\Bigg[& m_1\left(1-\frac{e_1}{2}+\frac{3e_1^2}{8}+\alpha_3\vartheta e_1-\alpha_3\vartheta\frac{e_1^2}{2}+\frac{\alpha_3(\alpha_3-1)\vartheta^2e_1^2}{2}+e_0-\frac{1}{2}e_0e_1+\alpha_3\vartheta e_0e_1\right) \\ & + m_2\left(1-\vartheta e_1\alpha_4+\frac{1}{2}\alpha_4(\alpha_4+1)\vartheta^2e_1^2+e_0-\alpha_4\vartheta e_0e_1\right)-1\Bigg]. \end{aligned} \quad (25)$$

The bias of the proposed estimators, $T_{i(d)}$, is defined as

$$\text{Bias}\left(T_{i(d)}\right) \cong E\left(T_{i(d)}-M_Y\right). \quad (26)$$

Taking expectations on both sides of equation (25), we get the bias of generalized class of estimators $T_{i(d)}$:

$$\text{Bias}\left(T_{i(d)}\right) = M_Y\begin{bmatrix} m_1\left(1+\frac{\lambda C_{MX}^2}{2}\left(\frac{3}{4}-\alpha_3\vartheta+\alpha_3(\alpha_3-1)\vartheta^2\right)+\lambda\rho C_{MY}C_{MX}\left(\alpha_3\vartheta-\frac{1}{2}\right)\right) \\ +m_2\left(1+\frac{\alpha_4(\alpha_4+1)\vartheta^2\lambda C_{MX}^2}{2}-\alpha_4\vartheta\lambda\rho C_{MY}C_{MX}\right)-1 \end{bmatrix}. \quad (27)$$

The MSE of the proposed estimators, $T_{i(d)}$, is defined as

$$\text{MSE}\left(T_{i(d)}\right) \cong E\left(T_{i(d)}-M_Y\right)^2. \quad (28)$$

Squaring both sides of equation (25), we have

$$\begin{aligned} \left(T_{i(d)}-M_Y\right)^2 = M_Y^2\Bigg[& m_1^2\left\{1-\frac{e_1}{2}+\frac{3e_1^2}{8}+\alpha_3\vartheta e_1-\alpha_3\vartheta\frac{e_1^2}{2}+\frac{\alpha_3(\alpha_3-1)\vartheta^2e_1^2}{2}+e_0-\frac{1}{2}e_0e_1+\alpha_3\vartheta e_0e_1\right\}^2 \\ & + m_2^2\left\{1-\vartheta e_1\alpha_4+\frac{\alpha_4(\alpha_4+1)\vartheta^2e_1^2}{2}+e_0-\alpha_4\vartheta e_0e_1\right\}^2+1 \\ & + 2m_1m_2\left(1-\frac{e_1}{2}+\frac{3e_1^2}{8}+\alpha_3\vartheta e_1-\alpha_3\vartheta\frac{e_1^2}{2}+\frac{\alpha_3(\alpha_3-1)\vartheta^2e_1^2}{2}+e_0-\frac{1}{2}e_0e_1+\alpha_3\vartheta e_0e_1\right) \\ & \cdot\left(1-\vartheta e_1\alpha_4+\frac{\alpha_4(\alpha_4+1)\vartheta^2e_1^2}{2}+e_0-\alpha_4\vartheta e_0e_1\right) \\ & - 2m\left(1-\frac{e_1}{2}+\frac{3e_1^2}{8}+\alpha_3\vartheta e_1-\alpha_3\vartheta\frac{e_1^2}{2}+\frac{\alpha_3(\alpha_3-1)\vartheta^2e_1^2}{2}+e_0-\frac{1}{2}e_0e_1+\alpha_3\vartheta e_0e_1\right) \\ & -2m_2\left(1-\vartheta e_1\alpha_4+\frac{\alpha_4(\alpha_4+1)\vartheta^2e_1^2}{2}+e_0-\alpha_4\vartheta e_0e_1\right)\Bigg]. \end{aligned} \quad (29)$$

Taking expectations on both sides of equation (29), we get the MSE of proposed estimators up to the first order of approximation as

$$\mathrm{MSE}\left(T_{i(d)}\right) = M_Y^2\left[1 + m_1^2 A_1 + m_2^2 A_2 + 2m_1 m_2 A_3 - 2m_1 A_4 - 2m_2 A_5\right], \tag{30}$$

where

$$\begin{aligned}
A_1 &= \left[1 + \lambda\left\{C_{MY}^2 + C_{MX}^2\left(1 + 2\alpha_3^2\vartheta^2 - \alpha_3\vartheta^2 - 2\alpha_3\vartheta\right) - 2pC_{MY}C_{MX}\left(1 - 2\alpha_3\vartheta\right)\right\}\right],\\
A_2 &= \left[1 + \lambda\left\{C_{MY}^2 + \vartheta^2 C_{MX}^2\left(2\alpha_4^2 + \alpha_4\right) - 4\alpha_4\vartheta p C_{MX}C_{MY}\right\}\right],\\
A_3 &= \left[1 + \lambda\left\{C_{MY}^2 - pC_{MY}C_{MX}\left(2\vartheta\left(\alpha_4 - \alpha_3\right) + 1\right) - C_{MX}^2\left(\alpha_3\alpha_4\vartheta^2 - \frac{3}{8} + \frac{\alpha_3\vartheta}{2} - \frac{\alpha_4\vartheta}{2} - \frac{\alpha_4\left(\alpha_4 + 1\right)\vartheta^2}{2} - \frac{1}{2}\alpha_3\left(\alpha_3 - 1\right)\vartheta^2\right)\right\}\right],\\
A_4 &= \left\{1 + \lambda p C_{MY}C_{MX}\left(\alpha_3\vartheta - \frac{1}{2}\right) + \lambda\frac{C_{MX}^2}{2}\left(\frac{3}{4} - \alpha_3\vartheta + \alpha_3\left(\alpha_3 - 1\right)\vartheta^2\right)\right\},\\
A_5 &= \left\{1 + \frac{1}{2}\alpha_4\left(\alpha_4 + 1\right)\vartheta^2\lambda C_{MX}^2 - \alpha_4\vartheta\lambda p C_{MY}C_{MX}\right\}.
\end{aligned} \tag{31}$$

Partially differentiating equation (30) with respect to m_1 and m_2 and equating them to zero, we get the optimal values of m_1 and m_2 as follows:

$$\begin{aligned}
m_{1(\mathrm{opt})} &= \frac{\left(A_2A_4 - A_3A_5\right)}{\left(A_1A_2 - A_3^2\right)},\\
m_{2(\mathrm{opt})} &= \frac{\left(A_1A_5 - A_3A_4\right)}{\left(A_1A_2 - A_3^2\right)}.
\end{aligned} \tag{32}$$

Placing these optimal values in equation (30), we obtained the minimum MSE as given by

$$\mathrm{MSE}\left(T_{i(d)}\right)_{\min} \cong M_Y^2\left[1 - \frac{\left(A_2A_4^2 + A_1A_5^2 - 2A_3A_4A_5\right)}{\left(A_1A_2 - A_3^2\right)}\right]. \tag{33}$$

4. Application

In this section, comparison of the $T_{i(d)}$ estimators with other existing estimators under study is given by using real-life application and simulated datasets.

4.1. Real-Life Application. We evaluated the performance of proposed class of estimators as compared to other competing estimators in terms of the MSE. For this purpose, we selected four real-life datasets:

Population 1: source: Singh [11].

Y = number of fish caught in the year 1995
X = number of fish caught by the marine recreation fishermen in the previous year 1994.

Population 2: source: Koyuncu and Kadilar [19].

Y = number of teacher's staff
X = number of enrolled students

Population 3: source: Singh [11].

Y = number of fish caught in the year 1995
X = number of fish caught by the marine recreation fishermen in the previous year 1993.

Population 4: source: Murthy [20].

Y = number of households
X = area in square miles

Table 3 presents the detailed descriptions of each of the abovementioned populations.

We calculated the MSE and minimum MSE of all the estimators, that is, $\widehat{M}_R, \widehat{M}_{\mathrm{EX}}, \widehat{M}_D, \widehat{M}_{D1}, \widehat{M}_{D2}, \widehat{M}_{D3}, \widehat{M}_{\mathrm{SG}}, T_{i(d)}^{\ominus}, T_{i(d)}^{\oplus}, T_{i(d)}^{\otimes}, T_{i(d)}^{\circledast}$, and $T_{i(d)}^{\circledcirc}$, for populations 1–4. Expressions for the MSE of all the existing and proposed estimators are given in Sections 2 and 3 in detail. All empirical results are summarized in Tables 4–9 and the important deductions are as follows:

(i) $\widehat{M}_{\mathrm{SG}}$ performs better than all existing estimators, that is, $\widehat{M}_R, \widehat{M}_{\mathrm{EX}}, \widehat{M}_D, \widehat{M}_{D1}, \widehat{M}_{D2}, \widehat{M}_{D3}$
(ii) All the proposed estimators, that is, $T_{i(d)}^{\ominus}, T_{i(d)}^{\oplus}, T_{i(d)}^{\otimes}, T_{i(d)}^{\circledast}$ and $T_{i(d)}^{\circledcirc}$, have minimum MSE as compared to all existing estimators
(iii) A deep insight of columns of $T_{i(d)}^{\circledcirc}$ provides the least MSE among all other classes of proposed estimators

4.2. Simulation Study. A Monte Carlo simulation study is conducted to assess the performance of the proposed

Table 3: Data statistics of four different populations.

Values	Population 1	Population 2	Population 3	Population 4
N	69	923	69	128
n	17	180	17	45
M_Y	2068	171	2068	686
M_X	2011	4123	2307	4.715
$f_Y(M_Y)$	0.00014	0.002676	0.00014	0.00092
$f_X(M_X)$	0.00014	0.00009409	0.00014	0.1154
ρ	0.151	0.855	0.314	0.468
$\beta_{1(X)}$	2.60	3.94	2.67	0.89
$\beta_{2(X)}$	7.61	18.72	7.61	0.40
M_R	19019.5	89530	17030	8.270
Q_D	1968	4141.5	1975	2.309
Q_A	2956	5870.5	3002	5.316
Q_R	3936	8283	3936	4.618
T_M	4043	7726	3777	5.385
H_L	3004	5502	2935	5.045
D_M	3853.4	7348.3	3615.2	5.378

Table 4: MSE and minimum MSE of the existing estimators.

Estimators	Population 1	Population 2	Population 3	Population 4
$\widehat{M}_R$	988372.76	58.627	746752.56	5361.578
$\widehat{M}_{EX}$	627420.21	53.068	524362.05	3377.321
$\widehat{M}_D$	552636.13	42.128	508766.02	3323.944
$\widehat{M}_{D1}$	489395.24	42.067	454675.78	3300.661
$\widehat{M}_{D2}$	480458.29	42.067	447982.61	3300.375
$\widehat{M}_{D3}$	471131.76	41.964	439763.44	3289.539
$\widehat{M}_{SG}$	402459.28	41.352	384146.79	3243.201

Table 5: Minimum MSE of estimators $T^{\ominus}_{i(d)}$.

Estimators	Population 1	Population 2	Population 3	Population 4
$T^{\ominus}_{1(d)}$	369076.3	40.289	359823.2	3234.103
$T^{\ominus}_{2(d)}$	368888.5	40.287	359714.2	3221.120
$T^{\ominus}_{3(d)}$	369072.8	40.289	359826.2	3235.205
$T^{\ominus}_{4(d)}$	368974.8	40.288	359772.6	3231.655
$T^{\ominus}_{5(d)}$	369019.2	40.289	359793.7	3232.035
$T^{\ominus}_{6(d)}$	369067.0	40.290	359831.9	3230.939
$T^{\ominus}_{7(d)}$	368879.0	40.287	359709.6	3219.404
$T^{\ominus}_{8(d)}$	369062.5	40.389	359820.4	3235.729
$T^{\ominus}_{9(d)}$	369024.6	40.289	359800.8	3233.736
$T^{\ominus}_{10(d)}$	368886.9	40.287	359713.2	3221.085
$T^{\ominus}_{11(d)}$	369172.1	40.290	359883.2	3261.507
$T^{\ominus}_{12(d)}$	368978.7	40.288	359778.2	3233.329

Table 6: Minimum MSE of estimators $T^{\oplus}_{i(d)}$.

Estimators	Population 1	Population 2	Population 3	Population 4
$T^{\oplus}_{1(d)}$	354935.7	40.653	358584.1	3216.740
$T^{\oplus}_{2(d)}$	354803.4	40.652	358509.7	3205.636
$T^{\oplus}_{3(d)}$	354933.2	40.653	358586.2	3218.205
$T^{\oplus}_{4(d)}$	354864.2	40.653	358549.6	3214.608
$T^{\oplus}_{5(d)}$	354895.5	40.653	358564.0	3214.955
$T^{\oplus}_{6(d)}$	354929.1	40.654	358590.1	3213.988
$T^{\oplus}_{7(d)}$	354797.1	40.654	358506.1	3204.203
$T^{\oplus}_{8(d)}$	354926.0	40.654	358582.2	3218.168
$T^{\oplus}_{9(d)}$	354899.2	40.653	358568.8	3216.419
$T^{\oplus}_{10(d)}$	354802.3	40.652	358509.0	3205.607
$T^{\oplus}_{11(d)}$	355003.2	40.654	358625.2	3242.254
$T^{\oplus}_{12(d)}$	354866.2	40.653	358553.4	3216.064

Table 7: Minimum MSE of estimators $T_{i(d)}^{\otimes}$.

Estimators	Population 1	Population 2	Population 3	Population 4
$T_{1(d)}^{\otimes}$	232289.5	39.237	283982.2	3142.874
$T_{2(d)}^{\otimes}$	232013.6	39.235	283834.3	3117.669
$T_{3(d)}^{\otimes}$	232284.3	39.238	284177.8	3146.160
$T_{4(d)}^{\otimes}$	232140.4	39.236	283913.2	3138.074
$T_{5(d)}^{\otimes}$	232205.6	39.237	283942.2	3138.857
$T_{6(d)}^{\otimes}$	232275.8	39.238	283948.6	3136.674
$T_{7(d)}^{\otimes}$	232000.6	39.235	283827.3	3114.381
$T_{8(d)}^{\otimes}$	232269.2	39.237	283978.4	3146.077
$T_{9(d)}^{\otimes}$	232213.5	39.237	283951.7	3142.153
$T_{10(d)}^{\otimes}$	232011.2	39.235	283833.0	3117.602
$T_{11(d)}^{\otimes}$	232430.3	39.238	284063.7	3198.588
$T_{12(d)}^{\otimes}$	232146.1	39.236	283921.2	3141.355

Table 8: Minimum MSE of estimators $T_{i(d)}^{\circledast}$.

Estimators	Population 1	Population 2	Population 3	Population 4
$T_{1(d)}^{\circledast}$	206469.7	37.512	256261.4	3140.691
$T_{2(d)}^{\circledast}$	206038.1	37.507	206023.5	3102.652
$T_{3(d)}^{\circledast}$	206987.6	37.513	256619.5	3145.516
$T_{4(d)}^{\circledast}$	206236.4	37.509	256151.0	3133.585
$T_{5(d)}^{\circledast}$	206338.5	37.511	256197.0	3134.749
$T_{6(d)}^{\circledast}$	206448.2	37.514	256280.4	3131.500
$T_{7(d)}^{\circledast}$	206017.7	37.514	256012.2	3097.565
$T_{8(d)}^{\circledast}$	206438.0	37.513	256255.3	3145.396
$T_{9(d)}^{\circledast}$	206350.8	37.511	256212.4	3139.628
$T_{10(d)}^{\circledast}$	206034.4	37.506	256021.3	3102.549
$T_{11(d)}^{\circledast}$	206689.6	37.515	256392.5	3217.482
$T_{12(d)}^{\circledast}$	206245.4	37.510	256163.2	3138.450

Table 9: Minimum MSE of estimators $T_{i(d)}^{\circledcirc}$.

Estimators	Population 1	Population 2	Population 3	Population 4
$T_{1(d)}^{\circledcirc}$	216068.8	36.883	254850.7	3116.231
$T_{2(d)}^{\circledcirc}$	215737.3	36.879	254667.0	3086.071
$T_{3(d)}^{\circledcirc}$	216062.6	36.884	255093.8	3120.166
$T_{4(d)}^{\circledcirc}$	215889.6	36.882	254765.5	3110.485
$T_{5(d)}^{\circledcirc}$	215968.0	36.883	254801.0	3111.423
$T_{6(d)}^{\circledcirc}$	216052.3	36.885	254865.4	3108.809
$T_{7(d)}^{\circledcirc}$	215721.6	36.879	254658.2	3082.140
$T_{8(d)}^{\circledcirc}$	216044.5	36.884	254846.0	3120.068
$T_{9(d)}^{\circledcirc}$	215977.5	36.883	254812.9	3115.368
$T_{10(d)}^{\circledcirc}$	215734.4	36.879	254665.3	3085.991
$T_{11(d)}^{\circledcirc}$	216238.0	36.886	254952.0	3183.047
$T_{12(d)}^{\circledcirc}$	215896.5	36.882	254774.9	3114.413

generalized estimators through a real population. We consider a real-life application of primary and secondary schools for 923 districts of Turkey in 2007, considering number of teachers as study variable and number of enrolled students as auxiliary variable (source: [19]). The following are some important measures of the dataset:

$$\begin{aligned}
\overline{Y} &= 436.4344,\\
\overline{X} &= 11440.498,\\
\rho_{yx} &= 0.855,\\
Q_D &= 4141.5,\\
Q_A &= 5870.5,\\
T_M &= 7726,\\
M_R &= 89530,\\
H_L &= 5502,\\
D_M &= 7348.3,\\
Q_R &= 8283.
\end{aligned} \tag{34}$$

The following steps are made to carry out the simulation study:

Step 1: select a SRSWOR of size n from the population of size N

Step 2: use sample data from step 1 to find the MSE of all the existing and proposed estimators

Step 3: perform 20,000 iterations to conduct step 1 and step 2

Step 4: get 20,000 values for MSE of all existing and proposed estimators

Step 5: take the average of 20,000 values obtained in step 4 to get the simulated MSE of each estimator

The following is revealed from Table 10:

(i) $\widehat{M}_{SG}$ performs better than all existing estimators, that is, $\widehat{M}_R, \widehat{M}_{EX}, \widehat{M}_D, \widehat{M}_{D1}, \widehat{M}_{D2}, \widehat{M}_{D3}$

(ii) Minimum MSE of all the proposed estimators is the least as compared to all the existing estimators under study

(iii) As sample size increases, there is a decrease in the minimum MSE of all the proposed estimators

It is concluded that our generalized estimator impeccably performs the best in the presence of extreme value(s).

5. Robustness of $T_{i(d)}$

In this section, robustness is examined to check the performance of the proposed generalized estimator as compared to other existing estimators under study. If the estimator performs efficiently in the presence of the extreme values, the estimator is called a robust estimator. For this purpose, we consider a real-life application taken from Punjab development statistics (PDS) for the year of 2012 [21]. For the deep study regarding robustness, different sample sizes are taken ($n = 6, 9$ and 12). The following are the important statistics of the data:

$$\begin{aligned}
Y &= \text{teaching staff in govt middle schools for boys and girls in 36 districts},\\
X &= \text{number of enrolled students in govt middle schools for boys and girls in 36 districts},\\
N &= 36,\\
M_Y &= 2033,\\
M_X &= 54559,\\
f_Y(M_Y) &= 0.0004219,\\
f_X(M_X) &= 0.0000141,\\
\rho &= 0.8888,\\
T_M &= 58995,\\
M_R &= 86569,\\
Q_D &= 17600.5,\\
Q_A &= 60775.5,\\
Q_R &= 35201,\\
H_L &= 61491.5,\\
D_M &= 58502.9.
\end{aligned} \tag{35}$$

Scatter plot confirms the presence of the extreme value in the dataset. Scatter plot can be seen in Figure 1. Therefore, we can access the robustness of the generalized estimator for this dataset. Numerical results based on the robustness study are reported in Table 11. It is revealed from Table 11 that the minimum MSE of all the proposed estimators is the least as compared to all the existing estimators under study. Moreover, as the sample size increases, the minimum MSE of all the proposed estimators decreases. Therefore, it is concluded that our proposed estimator performs impeccably in the presence of the extreme value(s).

Table 10: The MSE and minimum MSE of the existing and proposed estimators based on simulation study.

Estimators	Sample size		
	$n = 180$	$n = 200$	$n = 230$
$\widehat{M}_R$	58.936	51.433	42.951
$\widehat{M}_{\mathrm{EX}}$	53.386	46.773	38.902
$\widehat{M}_D$	41.995	36.778	30.654
$\widehat{M}_{D1}$	41.934	36.731	30.621
$\widehat{M}_{D2}$	41.933	36.730	30.621
$\widehat{M}_{D3}$	42.145	36.893	30.734
$\widehat{M}G_{PP}$	41.207	36.176	30.237
$T^{\ominus}_{1(d)}$	40.258	35.466	29.763
$T^{\ominus}_{2(d)}$	40.276	35.486	29.769
$T^{\ominus}_{3(d)}$	40.258	35.457	29.744
$T^{\ominus}_{4(d)}$	40.229	35.483	29.766
$T^{\ominus}_{5(d)}$	40.299	35.477	29.789
$T^{\ominus}_{6(d)}$	40.227	35.429	29.744
$T^{\ominus}_{7(d)}$	40.263	35.453	29.763
$T^{\ominus}_{8(d)}$	40.287	35.487	29.712
$T^{\ominus}_{9(d)}$	40.296	35.451	29.789
$T^{\ominus}_{10(d)}$	40.289	35.455	29.776
$T^{\ominus}_{11(d)}$	40.278	35.486	29.741
$T^{\ominus}_{12(d)}$	40.253	35.450	29.750
$T^{\oplus}_{1(d)}$	40.232	35.753	29.951
$T^{\oplus}_{2(d)}$	40.278	35.796	29.911
$T^{\oplus}_{3(d)}$	40.289	35.741	29.938
$T^{\oplus}_{4(d)}$	40.256	35.711	29.987
$T^{\oplus}_{5(d)}$	40.284	35.789	29.963
$T^{\oplus}_{6(d)}$	40.269	35.735	29.945
$T^{\oplus}_{7(d)}$	40.287	35.746	29.987
$T^{\oplus}_{8(d)}$	40.235	35.745	29.966
$T^{\oplus}_{9(d)}$	40.253	35.789	29.988
$T^{\oplus}_{10(d)}$	40.258	35.759	29.970
$T^{\oplus}_{11(d)}$	40.278	35.711	29.983
$T^{\oplus}_{12(d)}$	40.235	35.736	29.955
$T^{\otimes}_{1(d)}$	39.168	34.639	29.180
$T^{\otimes}_{2(d)}$	39.163	34.688	29.177
$T^{\otimes}_{3(d)}$	39.147	34.640	29.145
$T^{\otimes}_{4(d)}$	39.158	34.667	29.144
$T^{\otimes}_{5(d)}$	39.149	34.684	29.189
$T^{\otimes}_{6(d)}$	39.136	34.622	29.167
$T^{\otimes}_{7(d)}$	39.196	34.637	29.149
$T^{\otimes}_{8(d)}$	39.147	34.699	29.156
$T^{\otimes}_{9(d)}$	39.155	34.680	29.148
$T^{\otimes}_{10(d)}$	39.189	34.644	29.155
$T^{\otimes}_{11(d)}$	39.138	34.655	29.147
$T^{\otimes}_{12(d)}$	39.149	34.628	29.156
$T^{\circledast}_{1(d)}$	37.411	33.291	28.251
$T^{\circledast}_{2(d)}$	37.415	33.284	28.298
$T^{\circledast}_{3(d)}$	37.478	33.236	28.278
$T^{\circledast}_{4(d)}$	37.469	33.259	28.263
$T^{\circledast}_{5(d)}$	37.455	33.247	28.241
$T^{\circledast}_{6(d)}$	37.436	33.289	28.258
$T^{\circledast}_{7(d)}$	37.489	33.213	28.266
$T^{\circledast}_{8(d)}$	37.473	33.258	28.254
$T^{\circledast}_{9(d)}$	37.425	33.236	28.219
$T^{\circledast}_{10(d)}$	37.458	33.233	28.258
$T^{\circledast}_{11(d)}$	37.479	33.249	28.247
$T^{\circledast}_{12(d)}$	37.469	33.258	28.259
$T^{\circledcirc}_{1(d)}$	36.799	32.818	27.912
$T^{\circledcirc}_{2(d)}$	36.794	32.810	27.916
$T^{\circledcirc}_{3(d)}$	36.809	32.808	27.910
$T^{\circledcirc}_{4(d)}$	36.832	32.803	27.909
$T^{\circledcirc}_{5(d)}$	36.808	32.813	27.907

Table 10: Continued.

Estimators	Sample size		
	$n = 180$	$n = 200$	$n = 230$
$T^{\circledcirc}_{6(d)}$	36.780	32.803	27.922
$T^{\circledcirc}_{7(d)}$	36.809	32.812	27.907
$T^{\circledcirc}_{8(d)}$	36.805	32.822	27.916
$T^{\circledcirc}_{9(d)}$	36.789	32.805	27.919
$T^{\circledcirc}_{10(d)}$	36.802	32.801	27.907
$T^{\circledcirc}_{11(d)}$	36.813	32.817	27.919
$T^{\circledcirc}_{12(d)}$	36.799	32.811	27.909

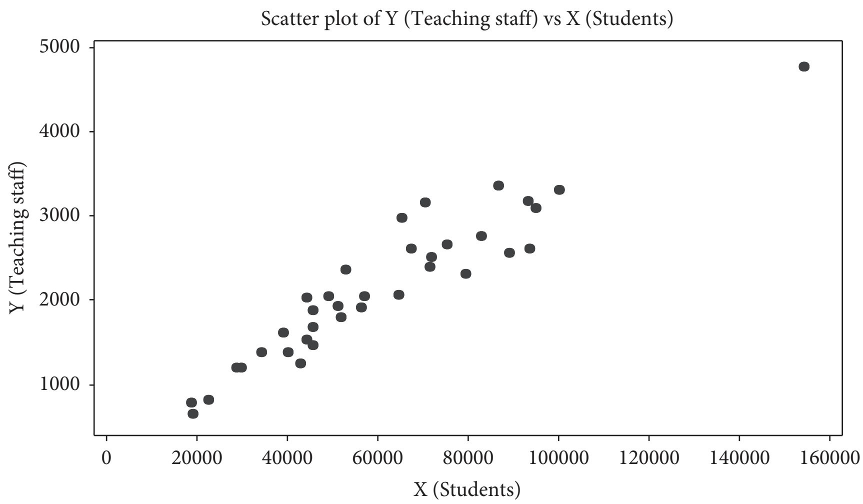

Figure 1: Scatter plot of teaching staff versus students.

Table 11: The MSE and minimum MSE of the existing and proposed estimators based on the robustness study.

Estimators	Sample size		
	$n = 6$	$n = 9$	$n = 12$
$\widehat{M}_R$	50949.190	30569.51	20379.68
$\widehat{M}_{EX}$	623884.21	37430.53	24953.68
$\widehat{M}_D$	40971.25	24582.75	16388.50
$\widehat{M}_{D1}$	40569.09	24437.40	16323.77
$\widehat{M}_{D2}$	40544.28	24432.13	16322.22
$\widehat{M}_{D3}$	39753.88	24142.77	16192.57
$\widehat{M}G_{PP}$	34666.67	22304.16	15373.79
$T^{\ominus}_{1(d)}$	27237.86	19454.23	14065.72
$T^{\ominus}_{2(d)}$	27236.56	19520.46	14065.67
$T^{\ominus}_{3(d)}$	27237.85	19454.19	14065.71
$T^{\ominus}_{4(d)}$	27237.93	19454.01	14065.73
$T^{\ominus}_{5(d)}$	27237.90	19454.12	14065.72
$T^{\ominus}_{6(d)}$	27237.45	19453.92	14065.65
$T^{\ominus}_{7(d)}$	27237.59	19507.05	14065.67
$T^{\ominus}_{8(d)}$	27237.84	19454.15	14065.71
$T^{\ominus}_{9(d)}$	27237.90	19454.10	14065.72
$T^{\ominus}_{10(d)}$	27237.56	19515.17	14065.66
$T^{\ominus}_{11(d)}$	27240.26	19454.23	14066.13
$T^{\ominus}_{12(d)}$	27237.93	19454.11	14065.73
$T^{\oplus}_{1(d)}$	30863.31	20815.27	14683.88
$T^{\oplus}_{2(d)}$	30863.14	20853.54	14683.85
$T^{\oplus}_{3(d)}$	30863.31	20815.25	14683.88
$T^{\oplus}_{4(d)}$	30863.35	20815.15	14683.89

Table 11: Continued.

Estimators	Sample size		
	$n = 6$	$n = 9$	$n = 12$
$T^{\oplus}_{5(d)}$	30863.34	20815.21	14683.89
$T^{\oplus}_{6(d)}$	30863.08	20815.10	14683.84
$T^{\oplus}_{7(d)}$	30863.16	20845.79	14683.86
$T^{\oplus}_{8(d)}$	30863.30	20815.23	14683.88
$T^{\oplus}_{9(d)}$	30863.34	20815.20	14683.89
$T^{\oplus}_{10(d)}$	30863.14	20850.48	14683.85
$T^{\oplus}_{11(d)}$	30864.70	20815.28	14684.12
$T^{\oplus}_{12(d)}$	30863.35	20815.20	14683.89
$T^{\otimes}_{1(d)}$	21617.19	17310.35	13083.88
$T^{\otimes}_{2(d)}$	21616.83	17392.15	13083.82
$T^{\otimes}_{3(d)}$	21617.18	17310.30	13083.88
$T^{\otimes}_{4(d)}$	21617.28	17310.08	13083.90
$T^{\otimes}_{5(d)}$	21617.24	17310.22	13083.89
$T^{\otimes}_{6(d)}$	21616.70	17309.97	13083.79
$T^{\otimes}_{7(d)}$	21616.87	17375.58	13083.82
$T^{\otimes}_{8(d)}$	21617.17	17310.25	13083.88
$T^{\otimes}_{9(d)}$	21617.24	17310.18	13083.89
$T^{\otimes}_{10(d)}$	21616.14	17385.61	13083.82
$T^{\otimes}_{11(d)}$	21620.11	17310.35	13084.40
$T^{\otimes}_{12(d)}$	21617.28	17310.20	13083.90
$T^{\circledast}_{1(d)}$	8921.61	12290.31	10740.24
$T^{\circledast}_{2(d)}$	8920.88	12461.16	10740.10
$T^{\circledast}_{3(d)}$	8921.58	12290.22	10740.23
$T^{\circledast}_{4(d)}$	8921.79	12289.76	10740.27
$T^{\circledast}_{5(d)}$	8921.71	12290.04	10740.26
$T^{\circledast}_{6(d)}$	8920.61	12289.52	10740.05
$T^{\circledast}_{7(d)}$	8920.95	12426.58	10740.12
$T^{\circledast}_{8(d)}$	8921.56	12290.12	10740.23
$T^{\circledast}_{9(d)}$	8921.71	12289.97	10740.26
$T^{\circledast}_{10(d)}$	8920.87	12447.52	10740.10
$T^{\circledast}_{11(d)}$	8927.54	12290.32	10741.34
$T^{\circledast}_{12(d)}$	8921.78	12290.00	10740.27
$T^{\circledcirc}_{1(d)}$	2994.91	9992.48	9678.24
$T^{\circledcirc}_{2(d)}$	2994.29	10135.55	9678.13
$T^{\circledcirc}_{3(d)}$	2994.88	9992.40	9678.23
$T^{\circledcirc}_{4(d)}$	2995.05	9992.02	9678.26
$T^{\circledcirc}_{5(d)}$	2994.99	9992.26	9678.25
$T^{\circledcirc}_{6(d)}$	2994.06	9992.82	9678.08
$T^{\circledcirc}_{7(d)}$	2994.36	10106.56	9678.14
$T^{\circledcirc}_{8(d)}$	2994.87	9992.32	9678.23
$T^{\circledcirc}_{9(d)}$	2994.98	9992.19	9678.25
$T^{\circledcirc}_{10(d)}$	2994.28	10124.12	9678.12
$T^{\circledcirc}_{11(d)}$	2999.87	9992.48	9679.15
$T^{\circledcirc}_{12(d)}$	2995.05	9992.22	9678.26

6. Concluding Remarks and Recommendations

We proposed the generalized classes of estimators for estimating population median under simple random sampling using robust measures of an auxiliary variable. Bias, mean squared error, and minimum mean squared error of the proposed generalized classes are derived up to the first degree of approximation. Four real-life datasets are used to check the numerical performance of the new estimators. A simulation study through a real dataset is also conducted to assess the potential of suggested classes of estimators. Robustness is also examined through a real dataset. On the basis of numerical findings, it is concluded that the new generalized classes can generate optimum estimators. Therefore, use of the proposed generalized class is recommended for future applications.

The possible extensions of this work are to estimate the following: (1) finite population median under other sampling designs like stratified random sampling, double sampling, rank set sampling, and so forth; (2) other unknown finite population parameters including mean, variance, and proportions; and (3) population median in the presence of nonsampling errors.

Data Availability

The data used to support the findings of this study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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