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ESTIMATION OF STEADY STATE PROBABILITY DISTRIBUTION OF SYSTEM SIZE IN M/M/1 QUEUE

Jayakrishna Udupa H.

ABSTRACT

Recently Choudhury and Borthakur (2008) obtained classical and Bayesian estimators of performance measures based on a random sample of size n from a geometric distribution with mean $\rho/(1-\rho)$, which is the steady state probability distribution of system size. Here we obtain classical estimators, the Maximum Likelihood (ML) and Uniformly Minimum Variance Unbiased (UMVU) estimators, and Bayesian estimators of P_k, k=0,1,2,.... relative to beta prior distribution and Weighted Squared Error Loss (WSEL) function as well as relative to Standard Two-Sided Power (STSP) distribution and squared error loss / WSEL functions. Bayes, ML and UMVU estimates of the probability that the server is idle are also computed. Also Consistent Asymptotic Normality (CAN) for P_k, k=0,1,2,.... are examined.

I. Introduction

Problems of estimation have been the interest of queuing researchers since the mid 1950's. Early researchers in classical estimation problems of queuing parameters

* *Department of Statistics, Christ University, Hosur Road, Bangalore – 560029, Phone: 080 – 40129335, Email: jayakrishna.udupa@christuniversity.edu*

include Clarke (1957), Wolf (1965) and Cox (1965) while that for Bayesian estimation problems include Muddapur (1972), Armero (1985) and Basawa et al. (1981). In the past few decades there has been a surge in research related to Bayesian inference for queuing parameters. Research in the last decade includes Armero and Conesa (1998, 2000), Conti (1998), Armero and Bayarri(1999), Butler and Huzurbazaar (2000), Zheng and Seila (2000), and Keissler and Lund (2009). Estimation in M/M/1 queue was studied by Srinivas et al. (2009) using the Imbedded Markov Chain (IMC) analysis of M/G/1 queue. They obtained Maximum Likelihood (ML) and Uniformly Minimum Variance Unbiased (UMVU) estimators of steady state measures of system performance. Sharma and Kumar (1999) obtained ML estimators of the same measures as well as Bayesian estimators relative to Squared Error Loss (SEL) function and beta prior distribution based on a sample of size K, from K independent M/M/1 queues, from geometric distribution

$$p(x) = (1-\rho)\rho^{x}, \qquad x = 0,1,2,... \tag{1.1}$$

which is the steady state distribution of system size and is denoted by p_x. Recently Choudhury and Borthakur (2008) derived the Bayes estimators of the traffic intensity parameter relative to SEL and beta / truncated uniform priors, based on a random sample from (1.1) using the ergodicity property of IMC to argue that the sample observations are independent. We assume throughout that there exists a random sample of observations $\underset{\sim}{X} = X_1, X_2, \ldots, X_n$ on X with geometric p.m.f in (1.1).

In this paper we intend to obtain the classical and Bayesian estimators of the steady state probability distribution of system size given by

$$p_k = (1-\rho)\rho^{k}, \; k = 0,1,2,\ldots$$

which is the same as the distribution in (1.1). The ML and UMVU estimators are derived in section 2 and 3 respectively. Section 4 deals with the CAN for p_k. Bayesian estimators of p_k relative to beta prior and loss functions SEL and Weighted Squared Error Loss (WSEL) are derived in section 5. In contrast, in section 6, the Bayes estimators of p_k relative to Standard Two-Sided Power (STSP) distribution and SEL / WSEL functions are obtained.

2. Maximum Likelihood Estimation

The likelihood of the observed sample drawn from (1.1) is given by

$$L(\rho;\underset{\sim}{x}) = (1-\rho)^{n}\rho^{n\bar{X}}$$

Now differentiating the log of the likelihood function with respect to ρ and equating it to zero gives the MLE of ρ as

$$\hat{\rho}=\frac{\overline{X}}{\overline{X}+1}$$

By using the invariance property of MLE, we thus obtain the MLE of steady state probability distribution as

$$\hat{p}_k=\left(1-\frac{\overline{X}}{\overline{X}+1}\right)\left(\frac{\overline{X}}{\overline{X}+1}\right)^k \quad , \; k=0,1,2,\ldots$$

3. Best Unbiased Estimation

The distribution in (1.1) is a member of the Power Series family of distributions (P.S.D) with p.m.f

$$f(x|\theta)=\frac{a(x)\theta^x}{c(\theta)}, \quad x\in T'$$

where T' is the support of X. Further $T=\sum_1^n X_i$ is a Complete Sufficient Statistic (C.S.S) for P.S.D. It is well known (See Lehmann (1983, P. 96 and Singh (1979)) that UMVU estimator of probabilities of P.S.D based on a sample of size n from P.S.D is given by

$$\tilde{f}(x|\theta)=\frac{a(x)A(t-x,n-1)}{A(t,n)} \quad , x\in T' ,$$

where $A(t, n)$ is the coefficient of θ^t in the expansion of $\{c(\theta)\}^n$. Using this for our case of estimation of geometric probabilities p_k based on a sample of size n from (1.1), we have the UMVU estimators of steady state probabilities given by

$$\tilde{p}_k=\frac{\binom{n+t-k-2}{t-k}}{\binom{n+t-1}{t}} \quad , t\geq k(\geq 0), n>1$$

as $a(x)=1$ and $A(t,n)=\binom{n+t-1}{t}$.

4. CAN Property

The Consistent Asymptotic Normality (CAN) property is examined for p_k following Kale (2005). To accomplish this we first obtain the consistent estimators. Now using the method of moments we can obtain the consistent estimator of ρ as the solution of moment equation $\overline{X}_n = \frac{\rho}{1-\rho}$. For this the condition $\frac{d\mu}{d\rho} = \frac{1}{(1-\rho)^2} \neq 0$ and is continuous for $\rho \in (0,1)$ is satisfied, which is sufficient for μ^{-1} to exist. The solution of moment equation is $\rho^* = \frac{\overline{X}_n}{\overline{X}_n+1}$ and is a consistent estimator of ρ. Further, to obtain CAN estimator of $L = \frac{\rho}{1-\rho}$ we note that by Central Limit Theorem (CLT)

$$\overline{X}_n \sim AN\left(L, \frac{L}{n(1-\rho)}\right)$$

Thus, by definition, $\overline{X}_n$ is CAN for L as $\overline{X}_n \xrightarrow{P} L$ by Weak law of Large Numbers (WLLN). As $\rho = \frac{L}{L+1}$ and $\frac{dg(L)}{dL} \neq 0$ and is continuous, by invariance property of CAN estimators, $\frac{\overline{X}_n}{\overline{X}_n+1}$ is CAN for ρ, with

$$\frac{\overline{X}_n}{\overline{X}_n+1} \sim AN\left(\rho, \frac{L}{n(1-\rho)}\left(\frac{1}{L+1}\right)^4\right)$$

As $p_k = (1-\rho)\rho^k = g(\rho)$ and $\frac{dg(\rho)}{d\rho} \neq 0$ and is continuous, by invariance property of CAN estimators

$$\left(1-\frac{\overline{X}_n}{\overline{X}_n+1}\right)\left(\frac{\overline{X}_n}{\overline{X}_n+1}\right)^k$$

is CAN for $(1-\rho)\rho^k$ with

$$\left(1-\frac{\overline{X}_n}{\overline{X}_n+1}\right)\left(\frac{\overline{X}_n}{\overline{X}_n+1}\right)^k \sim AN\left((1-\rho)\rho^k, \frac{\rho^k[k(1-\rho)-\rho]}{n(1-\rho)^2}\right).$$

5. Bayes Estimation Relative to Beta Prior

The parametric space of p_k is $\Theta=\{p_k: 0<p_k<1,\ k=0,1,2,\ldots\}$. The Bayes estimators of p_k relative to *SEL* function given by

$$L(\hat{p}_k, p_k)=(\hat{p}_k-p_k)^2, \quad k=0,1,2,\ldots \tag{5.1}$$

where $\hat{p}_k$ is an estimate of p_k, and beta prior distribution for ρ, denoted by $Be_1\ (\alpha,\beta)$, given by

$$\pi(\rho\mid\alpha,\beta)=\frac{1}{B(\alpha,\beta)}\rho^{\alpha-1}(1-\rho)^{\beta-1}, \quad 0<\rho<1, \tag{5.2}$$

with the parameter space of the prior parameters being $\{(\alpha,\beta):\alpha\geq 0,\beta\geq 0\}$, can be easily derived and turns out to be

$$\hat{p}_{k_{(B)}}=\frac{B(n\overline{X}+\alpha+k, n+\beta+1)}{B(n\overline{X}+\alpha, n+\beta)}, \quad k=0,1,2,\ldots$$

The Bayes estimator of p_k relative to WSEL function given by

$$L(\hat{p}_k, p_k)=w(\rho)(\hat{p}_k-p_k)^2, \quad k=0,1,2,\ldots \tag{5.3}$$

and prior (5.2) turns out to be

$$\hat{p}_{k(B)}=\frac{B(n\overline{X}+\alpha+k-1, n+\beta+3)}{B(n\overline{X}+\alpha-1, n+\beta+2)}, \quad k=0,1,2,\ldots$$

where the weight function $w(\rho)=\dfrac{(1-\rho)^2}{\rho}$.

A random sample of size 10 was simulated 10000 times from the distribution (1.1) and the Bayes estimates of p_0 (probability that the server is idle) relative to beta prior distribution $(\alpha = 2, \beta = 5)$ (5.2) and loss function (5.3) were obtained and is given in the following table.

Table1. Bayes (WSEL), ML, UMVU estimates of P_0

ρ	MLE	UMVUE	$\hat{p}_{0_{(B)}}$
0.19	0.81103	0.794351	0.797
0.39	0.610128	0.584795	0.669555
0.59	0.409668	0.384451	0.50883
0.79	0.210261	0.193299	0.300566
0.99	0.0099914	0.0090013	0.016834

6. Bayes Estimation Relative to STSP Prior

The STSP distribution was introduced by Van Dorp and Kotz (2002) as a distribution which models the difference between maturity interest rates of two successive time periods (month). This distribution is used as the prior distribution of ρ and is given by

$$\pi(\rho \mid \theta, \alpha) = \begin{cases} \alpha\left(\dfrac{\rho}{\theta}\right)^{\alpha-1}, & 0 < \rho \le \theta \\ \alpha\left(\dfrac{1-\rho}{1-\theta}\right)^{\alpha-1}, & \theta \le \rho < 1 \end{cases} \tag{6.1}$$

The posterior distribution of ρ, given the data, is given by

$$\pi(\rho \mid \theta, \alpha) = \begin{cases} \dfrac{\rho^{n\bar{x}+\alpha-1}(1-\rho)^n}{\int_0^\theta \rho^{n\bar{x}+\alpha-1}(1-\rho)^n}, & 0 < \rho \le \theta \\ \dfrac{\rho^{n\bar{x}}(1-\rho)^{n+\alpha-1}}{\int_0^\theta \rho^{n\bar{x}}(1-\rho)^{n+\alpha-1}}, & \theta < \rho \le 1 \end{cases}$$

Thus the Bayes estimator of p_k, relative to Squared Error Loss (SEL) in (5.1), is given by

$$\hat{p}_{k_{(B)}} = \begin{cases} \dfrac{B(0,\theta,n\overline{X}+\alpha+k,n+2)}{B(0,\theta,n\overline{X}+\alpha,n+1)}, & 0<\rho\le\theta \\ \dfrac{B(\theta,1,n\overline{X}+k+1,n+\alpha+1)}{B(\theta,1,n\overline{X}+1,n+\alpha)}, & \theta\le\rho<1 \end{cases}$$

for $k = 0,1,2,\ldots$ and $B(p, q, r, s)$ is the generalized incomplete beta function defined by $\int_p^q \rho^{r-1}(1-\rho)^{s-1}d\rho$

The Bayes estimator of p_k, relative to WSEL function (5.3) and STSP prior in (6.1), is given by

$$\hat{p}_{k_{(B)}} = \begin{cases} \dfrac{B(0,\theta,n\overline{X}+\alpha+k-1,n+4)}{B(0,\theta,n\overline{X}+\alpha-1,n+3)}, & 0<\rho\le\theta \\ \dfrac{B(\theta,1,n\overline{X}+k,n+\alpha+3)}{B(\theta,1,n\overline{X},n+\alpha+2)}, & \theta\le\rho<1 \end{cases}$$

A random sample of size 10 was simulated 10000 times from the distribution (1.1) and the Bayes estimates of p_0 (probability that the server is idle) relative to STSP prior distribution (6.1) and loss function (5.3) was obtained and is presented in the following table.

Table 2. Bayes (WSEL), ML, UMVU estimates of P_0

θ/ρ	0.19	0.39	0.59	0.79	0.99
0.1	0.812544	0.685398	0.492784	0.271528	0.0139323
0.2	0.868666	0.664285	0.492735	0.271528	0.0139323
0.3	0.827506	0.610774	0.490143	0.271528	0.0139323
0.4	0.805688	0.698192	0.470981	0.271528	0.0139323
0.5	0.797908	0.658887	0.423153	0.271436	0.0139323
0.6	0.796264	0.641683	0.499428	0.267626	0.0139323
0.7	0.796088	0.637869	0.4657	0.239382	0.0139323

Table 2. Contd.

θ/ρ	0.19	0.39	0.59	0.79	0.99
0.8	0.796081	0.637572	0.457899	0.27189	0.0139323
0.9	0.796081	0.637567	0.457586	0.252315	0.0139323
MLE	0.81103	0.610128	0.409668	0.210261	0.00999141
UMVUE	0.794351	0.584795	0.384451	0.193299	0.0090013

7. Conclusion

The Bayes, ML and UMVU estimators of p_k were obtained and computed for p_0 following the research in Srinivas and Jayakrishna (2009), where the interest was in expected system size. We are aware of the fact that computations do not reveal about the performance of estimators. Also Bayes estimation was restricted to SEL and WSEL loss functions. However, Bayes estimation relative to other loss functions as well as comparison of estimators is under consideration and may possibly be part of another communication.

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