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ESTIMATION OF STOCHASTIC SYSTEMS: ARBITRARY SYSTEM PROCESS WITH ADDITIVE WHITE NOISE OBSERVATION ERRORS[1]

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1. Introduction. The principal result of this paper, stated in Theorem 3, is a form of the Bayes theorem which is required for the solution of many problems in the control and estimation of stochastic systems. Although the original motivation for the problem treated here is in the field of control, it is more convenient to formulate it in terms of estimation. Its application to control will be discussed in a later paper.

We shall be concerned with the estimation of a "system process" $x(t)$, $0 \leqq t \leqq T$ which we assume to be defined as a stochastic process $x(t, \eta)$ on a known probability space $(\Omega_X, \mathcal{B}_X, P_X)$, $(\eta \in \Omega_X)$. It is further assumed that the system process cannot be observed directly. Instead we have available an "observation process" $z(\tau)$ which is given by

$$z(\tau) = \int_0^\tau x(u)\, du + w(\tau), \qquad 0 \leqq \tau \leqq T, \tag{1.1}$$

where $w(\tau)$ is a standard Wiener process independent of the system process. Our available data is $z(\tau)$, $0 \leqq \tau \leqq t$, for t fixed in the interval $0 \leqq t \leqq T$, and using this data we wish to estimate some functional of the system process $x(\tau)$, $0 \leqq \tau \leqq T$,

$$G[x(\tau, \eta); 0 \leqq \tau \leqq T]. \tag{1.2}$$

It will be assumed that the resulting function $g(\eta)$ defined on $(\Omega_X, \mathcal{B}_X, P_X)$ by

$$g(\eta) = G[x(\tau, \eta); 0 \leqq \tau \leqq T] \tag{1.3}$$

is integrable.

The system process, or more precisely, the space Ω_X on which it is defined corresponds to the parameter space in the usual Bayes approach to the theory of estimation. Thus the probability P_X is the *a priori* distribution for the unknown parameter; the process $z(\tau)$, $0 \leqq \tau \leqq t$, is the observed random variable and we wish to estimate the function $g(\eta)$ defined on the parameter space.

We shall assume a squared error loss function. Hence we wish to find an estimate $\delta(z(\tau), 0 \leqq \tau \leqq t)$ which minimizes

$$E(g - \delta)^2. \tag{1.4}$$

To the memory of Norbert Wiener.

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It is well known that this is accomplished by letting

$$\delta = E[g \mid z(\tau), 0 \leqq \tau \leqq t]. \tag{1.5}$$

Our task then is to compute this conditional expectation. In Theorem 3 a formula is given for (1.5) where, in addition to the usual measurability and integrability assumptions, it is assumed only that the system process is square integrable almost surely

$$\int_0^T [x(t, \eta)]^2 \, dt < \infty \qquad \text{a.s.} \qquad P_X . \tag{1.6}$$

It should be noted that by the proper selection of the function g, this result can be used to solve the smoothing problem, the filtering problem and the estimation problem in addition to many others. For smoothing let

$$g(\eta) = x(s, \eta) \tag{1.7}$$

where $0 \leqq s < t$, for filtering

$$g(\eta) = x(t, \eta) \tag{1.8}$$

and for prediction

$$g(\eta) = x(s, \eta) \tag{1.9}$$

where $t < s \leqq T$. It may be noted that by letting

$$g_a(\eta) = L(\eta, a) \tag{1.10}$$

the conditional expectation (1.5) becomes the *a posteriori* Bayes risk for the loss function $L(\eta, a)$ where a belongs to an action space. Thus the conditional expectations of the form (1.5) are those required to solve the general Bayes decision problem.

The formula provided for the conditional expectation (4.32) of Theorem 3 is useful in applications only in the case that t is fixed. If the data is coming in continuously and we require an estimate which is being continuously revised to take into account the new data, then this formula, while valid, is not practical since the estimate at time $t + \Delta$ must be completely recomputed using all the past data. The value of the estimate at time t is of no use in computing the estimate at time $t + \Delta$. The practical method of computing an estimate which depends continuously on time is by the use of a stochastic differential equation. Under additional assumptions on the system process, the formula presented in this paper (4.32) can be used to obtain such a differential equation. This work will be given in a later paper.

Certain generalizations of this problem considered here can easily be handled by the methods of this paper. For example, both the system and the observation processes may be vector-valued. The observation equation (1.1) may be replaced by

$$z(t) = \int_0^t h(\tau, x(\tau)) \, d\tau + w(\tau), \qquad 0 \leqq \tau \leqq T, \tag{1.11}$$

where h satisfies appropriate regularity conditions. However, as each of these generalizations introduces complications in notation and technique, it was deemed best at this stage of the investigation to treat the simplest case which includes what we consider to be the essential difficulties inherent in the problem.

Further generalizations of the observation equation have been considered and will be presented later. For example, the case

$$dz(t) = h(\tau, x(\tau))\, d\tau + \sigma(\tau, x(\tau))\, dw(\tau) \tag{1.12}$$

can be solved by these methods. The success of generalization in this direction depends on the existence of results of the Cameron and Graves type quoted in Lemma 2.

One essential property of the estimation problem that is omitted here is the possibility of control in the distribution of the system process. This property is difficult to formulate rigorously and since it is not considered here, no attempt at such a formulation will be made. However, since the motivation for the work is the desire to obtain results valid for a "controlled" process, some comment is essential. Heuristically, in a "controlled" system at any given time t the distribution of the future of the system process ($x(\tau)$ for $\tau \geqq t$) is permitted to depend on the past of the observation process ($z(\tau)$ for $0 \leqq \tau < t$).

The formula in Theorem 3 was obtained earlier under the assumption that the system process is constant in time

$$x(t) = x \tag{1.13}$$

and x is a random variable with a finite number of states. It is presented along with the stochastic differential equation satisfied by (1.5) in an interesting and fundamental paper by W. M. Wonham [5]. This paper is, to the knowledge of the authors, the only rigorous work on this aspect of the estimation problem. In fact, the generally heuristic nature of the literature in this area justifies in our opinion what might appear to be an excessive attention to technical detail in the following treatment.

Theorem 1 in Section 1 states a general form of the Bayes theorem. In Theorem 2 the result is extended to the case in which conditional densities exist. In Section 3 the probability structure of the problem defined by (1.1) and (1.5) is presented in detail, and some lemmas which will be required later are proved. In Section 4, the main result of the paper (Theorem 3), is stated and proved. Theorem 3 is cast in a form which is convenient in the derivation of the stochastic differential equation to be presented in a later paper. Another form of Theorem 3, given as a Corollary in Section 5, is more appropriate for the estimation problem. Its use in a Monte Carlo computation procedure is also discussed in the last section.

2. General Bayes theorems. In Theorem 1 we consider an arbitrary random variable g, measurable with respect to a sub σ-field $\mathcal{A}_X$, and compute its conditional expectation with respect to another sub σ-field $\mathcal{A}_Z$. In Theorem 2, the same conditional expectation is computed under the further assumption of existence of conditional densities.

THEOREM 1. *On the probability space*

$$(\Omega, \mathcal{A}, P) \tag{2.1}$$

let $g(\omega)$ *be an integrable random variable measurable with respect to a sub* σ*-field* $\mathcal{A}_X$ *and let* $Q(A, \omega)$ *be a version of the conditional probability*

$$Q(A, \omega) = E(I_A \mid \mathcal{A}_X) \quad \text{a.s.} \tag{2.2}$$

for $A \,\varepsilon\, \mathcal{A}_Z \subset \mathcal{A}$. *Then* φ_g, *defined by*

$$\varphi_g(A) = \int g(\omega)Q(A, \omega)P(d\omega) \tag{2.3}$$

for $A \,\varepsilon\, \mathcal{A}_Z$ *is a finite signed measure on* $(\Omega, \mathcal{A}_Z)$; *it is absolutely continuous with respect to* P_Z, *the restriction of* P *to* $\mathcal{A}_Z$; *and its Radon-Nikodym derivative satisfies*

$$E(g \mid \mathcal{A}_Z) = d\varphi_g/dP_Z \quad \text{a.s.} \quad P_Z. \tag{2.4}$$

PROOF. From the integrability of g and the properties of conditional probabilities it is easily verified that φ_g is a finite signed measure.

Since the conditional expectation $E(g \mid \mathcal{A}_Z)$ is $\mathcal{A}_Z$-measurable, in order to verify that $E(g \mid \mathcal{A}_Z)$ is a.s. the Radon-Nikodym derivative in (2.4) it suffices to show that

$$E[I_A E(g \mid \mathcal{A}_Z)] = \varphi_g(A), \qquad A \,\varepsilon\, \mathcal{A}_Z. \tag{2.5}$$

For $A \,\varepsilon\, \mathcal{A}_Z$, $I_A(\omega)$ is $\mathcal{A}_Z$-measurable, so that

$$I_A E(g \mid \mathcal{A}_Z) = E(gI_A \mid \mathcal{A}_Z) \quad \text{a.s.} \tag{2.6}$$

Taking expectations, we have

$$E[I_A E(g \mid \mathcal{A}_Z)] = E[E(gI_A \mid \mathcal{A}_Z)] = E(gI_A). \tag{2.7}$$

Since g is $\mathcal{A}_X$-measurable

$$E(gI_A \mid \mathcal{A}_X) = gE(I_A \mid \mathcal{A}_X) \quad \text{a.s.} \tag{2.8}$$

Thus from (2.2), (2.3) and (2.8), we have

$$E(gI_A) = E[E(gI_A \mid \mathcal{A}_X)] = \int g(\omega)Q(A, \omega)P(d\omega) = \varphi_g(A). \tag{2.9}$$

The result (2.5) then follows from (2.7) and (2.9).

It is well-known that the conditional expectations $Q(A, \omega)$ defined by (2.2) need not be measures in A for ω fixed. Following Loève [2] (p. 137) we shall say that the conditional probabilities $Q(A, \omega)$ are *regular* provided $Q(A, \omega)$ is a probability measure in A for each fixed ω. This assumption will be required in

THEOREM 2. *Let the following conditions be satisfied:*

(i) *the conditional probabilities* $Q(A, \omega)$ *in* (2.2) *are regular,*

(ii) *the* σ*-field* $\mathcal{A}_Z$ *is generated by a countable family of sets, and*

(iii) *there exists a measure* λ *defined on* $(\Omega, \mathcal{A}_Z)$ *such that* $Q(A, \omega)$ *is absolutely continuous with respect to* λ *for* $\omega \,\varepsilon\, \Omega'$ *where* $P(\Omega') = 1$.

Then, it follows that

(iv) P_Z *is absolutely continuous with respect to* λ,

(v) *there exists a function* $q(\xi, \omega)$ *which is measurable on* $(\Omega \times \Omega, \mathfrak{A}_Z \times \mathfrak{A}_X)$ *and satisfies*

$$q(\xi, \omega) = (dQ/d\lambda)(\cdot, \omega)(\xi) \quad \text{a.e.} \quad \lambda \times P, \tag{2.10}$$

(vi)

$$0 < \int q(\xi, \omega)P(d\omega) < \infty \quad \text{a.s.} \quad P_Z \quad \textit{and} \tag{2.11}$$

(vii) *for* g *integrable and* $\mathfrak{A}_X$*-measurable*

$$E(g \mid \mathfrak{A}_Z) = \frac{\int g(\omega)q(\xi, \omega)P(d\omega)}{\int q(\xi, \omega)P(d\omega)} \quad \text{a.s.} \quad P_Z. \tag{2.12}$$

Proof. In Doob [2] (Example 2.7 of the Supplement, p. 616) the existence of a jointly measurable density $q(\xi, \omega)$ is shown in the case that $\mathfrak{A}_X = \mathfrak{A}_Z$ is generated by a countable family of sets. A very slight modification of the argument given there establishes the existence of the function $q(\xi, \omega)$ satisfying (2.10) for our case. The details of this argument will be omitted.

From the definition of $Q(A, \omega)$ in (2.2), for $A \in \mathfrak{A}_Z$

$$P_Z(A) = E(I_A) = E[E(I_A \mid \mathfrak{A}_X)] = \int Q(A, \omega)P(d\omega). \tag{2.13}$$

Thus, from the absolute continuity of $Q(A, \omega)$ with respect to λ assumed in condition (iii), it follows from (2.13) that $P_Z(A)$ and from (2.3) that φ_g are absolutely continuous with respect to λ. From Loève [4] (p. 141, Example 21)

$$d\varphi_g/d\lambda = (d\varphi_g/dP_Z) \cdot (dP_Z/d\lambda) \quad \text{a.e.} \quad \lambda. \tag{2.14}$$

Let

$$A_0 = \{\omega \mid (dP_Z/d\lambda)(\omega) = 0\}. \tag{2.15}$$

Then

$$\int_{A_0} (dP_Z/d\lambda)(\omega)\lambda(d\omega) = P_Z(A_0) = 0 \tag{2.16}$$

and hence

$$0 < dP_Z/d\lambda < \infty \quad \text{a.s.} \quad P_Z. \tag{2.17}$$

Finiteness follows from the finiteness of P_Z. From (2.14)

$$(d\varphi_g/dP_Z)(\omega) = (d\varphi_g/d\lambda)(\omega)/(dP_Z/d\lambda)(\omega) \quad \text{for} \quad \omega \notin A_0 \cup B_0 \tag{2.18}$$

where B_0 is set on which (2.14) does not hold so that $\lambda(B_0) = 0$. Since P_Z is absolutely continuous with respect to λ, $P_Z(B_0) = 0$ and hence

$$d\varphi_g/dP_Z = (d\varphi_g/d\lambda)/(dP_Z/d\lambda) \quad \text{a.s.} \quad P_Z. \tag{2.19}$$

It remains to show that

$$(2.20) \qquad (d\varphi_g/d\lambda)(\xi) = \int g(\omega)q(\xi, \omega)P(d\omega) \qquad \text{a.s.} \qquad P_Z$$

and

$$(2.21) \qquad (dP_Z/d\lambda)(\xi) = \int q(\xi, \omega)P(d\omega) \qquad \text{a.s.} \qquad P_Z .$$

From (2.3) and (2.10) of the theorem, for $A \in \mathcal{A}_Z$

$$(2.22) \qquad \varphi_g(A) = \int g(\omega)Q(A, \omega)P(d\omega) = \int g(\omega)[\int_A q(\xi, \omega)\lambda(d\xi)]P(d\omega).$$

Applying the Fubini theorem (see, for example, Loève [4], p. 136) on the product space $(\Omega \times \Omega, \mathcal{A}_Z \times \mathcal{A}_X, \lambda \times P_X)$ to (2.22), it follows that

$$(2.23) \qquad \varphi_g(A) = \int_A [\int g(\omega)q(\xi, \omega)P(d\omega)]\lambda(d\xi)$$

and that [ ] in (2.23) is an $\mathcal{A}_Z$-measurable function of ξ. Thus the expression [ ] in (2.23) is the Radon-Nikodym derivative of φ_g with respect to λ and (2.20) follows. The result (2.21) follows by the same argument for $g(\omega) \equiv 1$. Conclusion (vi) of the theorem follows from (2.17) and (2.21), and (vii) follows from (2.19), (2.20) and (2.21).

3. Function space formulation. Let $R^{[0,t]}$ be the space of all real-valued functions $z(\tau)$ for $0 \leqq \tau \leqq t$, let $\mathcal{B}_R^{[0,t]}$ be the product σ-field in $R^{[0,t]}$ defined in the usual manner, and let $C[0, t]$ be the space of real-valued continuous functions on $[0, t]$.

Define measurable spaces $(W, \mathcal{B}_W)$ and $(Z_t, \mathcal{B}_{Z_t})$ as follows:

$$(3.1) \qquad \begin{aligned} W &= C[0, T], & \mathcal{B}_W &= W \cap \mathcal{B}_R^{[0,T]}, \\ Z_t &= C[0, t], & \mathcal{B}_{Z_t} &= Z_t \cap \mathcal{B}_R^{[0,t]}, \end{aligned}$$

where $0 < t \leqq T$.

It will be assumed that a Wiener measure P_W is defined on $(W, \mathcal{B}_W)$ and that a probability space $(\Omega_X, \mathcal{B}_X, P_X)$ is also given. Elements of Ω_X and W will be denoted by η and w respectively. The probability space to which Theorem 2 will be applied is the product space defined by

$$(3.2) \qquad (\Omega, \mathcal{A}, P) = (\Omega_X \times W, \mathcal{B}_X \times \mathcal{B}_W, P_X \times P_W).$$

The σ-field $\mathcal{A}_X$ is induced by the projection transformation

$$(3.3) \qquad \Phi: (\Omega, \mathcal{A}) \to (\Omega_X, \mathcal{B}_X)$$

defined by

$$(3.4) \qquad \Phi(\eta, w) = \eta.$$

Thus

$$(3.5) \qquad \mathcal{A}_X = \Phi^{-1}(\mathcal{B}_X)$$

consists of the cylinder sets in $\Omega_X \times W$ with bases in $\mathcal{B}_X$.

We shall use the notation (3.3) extensively throughout the paper to indicate that Φ is a measurable transformation from the measurable space $(\Omega, \mathcal{A})$ into the measurable space $(\Omega_X, \mathcal{B}_X)$; that is, Φ is a single-valued point transformation from Ω into Ω_X for which $\Phi^{-1}(\mathcal{B}_X) \subset \mathcal{A}$.

It will be assumed that a real-valued stochastic process $x(u, \eta)$, $0 \leqq u \leqq T$, $\eta \varepsilon \Omega_X$, called the system process, is defined on $(\Omega_X, \mathcal{B}_X, P_X)$. The σ-field $\mathcal{A}_Z$ is induced by the transformation H to be defined in (3.8) and (3.9). The measurability of H is demonstrated in Lemma 1.

Since t will remain fixed throughout this section, we shall drop the subscript t, following the convention $Z = Z_t$ and $\mathcal{B}_Z = \mathcal{B}_{Z_t}$. It may be noted that in Lemma 1 the transformations h, H and Ψ all depend on t, but that this is not reflected in the notation at this point.

LEMMA 1. *If $x(u, \eta)$ is a (jointly) measurable process, then the transformations*

$$h: (\Omega_X, \mathcal{B}_X) \to (Z, \mathcal{B}_Z) \tag{3.6}$$

and

$$H: (\Omega, \mathcal{A}) \to (Z, \mathcal{B}_Z) \tag{3.7}$$

defined by

$$\begin{aligned}[h(\eta)](\tau) &= \int_0^\tau x(u, \eta)\,du \quad \text{for} \quad 0 \leqq \tau \leqq t \quad \text{if} \quad \int_0^t [x(u, \eta)]^2\,du < \infty \\ &= 0 \quad\quad\quad\quad\quad\quad \text{for} \quad 0 \leqq \tau \leqq t \quad \text{if} \quad \int_0^t [x(u, \eta)]^2\,du = \infty\end{aligned} \tag{3.8}$$

and

$$H(\eta, w)(\tau) = h(\eta)(\tau) + w(\tau), \qquad (0 \leqq \tau \leqq t) \tag{3.9}$$

are measurable and

$$H(\Omega) = Z. \tag{3.10}$$

PROOF. The process $x(u, \eta)$ is assumed to be jointly measurable on $([0, T] \times \Omega_X, \mathcal{B}_{[0,T]} \times \mathcal{B}_X, \mu_{[0,T]} \times P_X)$ where $\mathcal{B}_{[0,T]}$ is the Borel σ-field and $\mu_{[0,T]}$ is Lebesgue measure on the interval $[0, T]$. Thus $x^+(u, \eta)$ and $x^-(u, \eta)$ are also jointly measurable and by the Fubini theorem for positive functions, $\int_0^\tau x^+(u, \eta)\,du$ and $\int_0^\tau x^-(u, \eta)\,du$ are $\mathcal{B}_X$-measurable in η for τ fixed. Similarly, $\int_0^t [x(u, \eta)]^2\,du$ is $\mathcal{B}_X$-measurable and hence

$$C = \{\eta: \int_0^t [x(u, \eta)]^2\,du < \infty\} \;\varepsilon\; \mathcal{B}_X. \tag{3.11}$$

Define

$$\begin{aligned}h^+(\tau, \eta) &= \int_0^\tau x^+(u, \eta)\,du \quad \text{if} \quad \eta \varepsilon C \\ &= 0 \quad\quad\quad\quad\quad\quad \text{if} \quad \eta \not\varepsilon C; \\ h^-(\tau, \eta) &= \int_0^\tau x^-(u, \eta)\,du \quad \text{if} \quad \eta \varepsilon C \\ &= 0 \quad\quad\quad\quad\quad\quad \text{if} \quad \eta \not\varepsilon C.\end{aligned} \tag{3.12}$$

These functions are $\mathcal{B}_X$-measurable and finite for τ fixed $(0 \leqq \tau \leqq t)$ and hence $h(\tau, \eta)$ defined by

$$(3.13) \qquad h(\tau, \eta) = h^+(\tau, \eta) - h^-(\tau, \eta)$$

is finite and $\mathcal{B}_X$-measurable for τ fixed $(0 \leqq \tau \leqq t)$. It follows from (3.12) and (3.13) that

$$(3.14) \qquad h: (\Omega_X, \mathcal{B}_X) \to (R^{[0,t]}, \mathcal{B}_R^{[0,t]})$$

given by

$$(3.15) \qquad [h(\eta)](\tau) = h(\eta, \tau)$$

satisfies (3.8) of the theorem. Since h is $\mathcal{B}_X$-measurable coordinate-wise, it is measurable with respect to the product σ-field $\mathcal{B}_R^{[0,t]}$. From (3.8) $h(\eta, \tau)$ is clearly a continuous function of τ for $\eta \varepsilon \Omega_X$ fixed. Thus

$$(3.16) \qquad h(\Omega_X) \subset C[0, t] = Z$$

and h in (3.6) is measurable with respect to $\mathcal{B}_Z = C[0, t] \cap \mathcal{B}_R^{[0,t]}$. The transformation

$$(3.17) \qquad \Psi: (W, \mathcal{B}_W) \to (Z, \mathcal{B}_Z)$$

which restricts functions $w(\tau)$ for $0 \leqq \tau \leqq T$ to the range $0 \leqq \tau \leqq t$,

$$(3.18) \qquad [\Psi(w)](\tau) = w(\tau), \qquad 0 \leqq \tau \leqq t,$$

is clearly measurable.

Finally, defining

$$(3.19) \qquad H(\eta, w) = h(\eta) + \Psi(w),$$

it is easily seen that H is a measurable transformation from $(\Omega_X \times W, \mathcal{B}_X \times \mathcal{B}_W)$ to $(Z, \mathcal{B}_Z)$. Let z be an arbitrary element in Z and let $\eta \varepsilon \Omega_X$ be fixed. Then

$$(3.20) \qquad h(\eta) \varepsilon Z$$

and

$$(3.21) \qquad z_0 = z - h(\eta) \varepsilon Z$$

since both are continuous functions on $[0, t]$. Define

$$(3.22) \qquad \begin{aligned} w_0(\tau) &= z_0(\tau), & 0 \leqq \tau \leqq t, \\ &= z(t), & t \leqq \tau \leqq T. \end{aligned}$$

Then from (3.18)

$$(3.23) \qquad \Psi(w_0) = z_0$$

and

$$(3.24) \qquad H(\eta, w_0) = h(\eta) + \Psi(w_0) = h(\eta) + z_0 = z.$$

Thus

$$z \varepsilon H(\Omega) \tag{3.25}$$

and (3.10) follows. This concludes the proof of Lemma 1.

We shall write $\omega = (\eta, w)$ and $H(\omega)$ for $H(\eta, w)$. If the system process x is jointly measurable, it follows from the preceding lemma that

$$\mathcal{A}_Z = H^{-1}\mathcal{B}_Z \tag{3.26}$$

is a sub σ-field of $\mathcal{A}$. We recall from the definition of P_W and Ψ (3.18) that $P_W\Psi^{-1}$ defined by

$$P_W\Psi^{-1}(B) = P_W[\Psi^{-1}(B)], \qquad B \varepsilon \mathcal{B}_Z, \tag{3.27}$$

is a standard Wiener measure on $(Z, \mathcal{B}_Z)$. Let z_0 be a fixed element of Z and let P_{z_0} be the probability on $\mathcal{B}_Z$ given by

$$P_{z_0}(B) = P_W[w:\Psi(w) + z_0 \varepsilon B] \quad \text{for} \quad B \varepsilon \mathcal{B}_Z. \tag{3.28}$$

We shall require the following result due to Cameron and Graves [1] (Theorem 1, p. 162).

LEMMA 2. *If $z_0 \varepsilon Z$ is an absolutely continuous function of τ on* [0, t],

$$z_0(\tau) = \int_0^\tau x_0(u)\, du \qquad (0 \leqq \tau \leqq t) \tag{3.29}$$

where

$$\int_0^t [x_0(u)]^2\, du < \infty, \tag{3.30}$$

then P_{z_0} is absolutely continuous with respect to $P_W\Psi^{-1}$ and

$$\begin{aligned}(dP_{z_0}/dP_W\Psi^{-1})(z) \\ = \exp\{\textstyle\int_0^t x_0(\tau)\, dz(\tau) - \frac{1}{2}\int_0^t [x_0(\tau)]^2\, d\tau\} \qquad \text{a.s.} \qquad P_W\Psi^{-1}.\end{aligned} \tag{3.31}$$

It is understood that the first integral in the exponential in (3.31) is to be replaced by zero for those values of z for which the integral does not exist and hence that the Radon-Nikodym derivative is defined for all values of $z \varepsilon Z$.

LEMMA 3. *For $A \varepsilon \mathcal{A}_Z$ and $\omega \varepsilon \Omega$, define*

$$Q(A, \omega) = P_{h[\Phi(\omega)]}(HA) \tag{3.32}$$

where $P_{z_0}(B)$ is defined by (3.28), *h and H by* (3.6)–(3.9), *and Φ is the projection* (3.4). *Then $Q(A, \omega)$ is a regular conditional probability measure for P_Z given $\mathcal{A}_X$.*

PROOF. From the definitions of P_{z_0}, H and Ψ

$$P_{h(\eta)}(HA) = P_W[w:\Psi(w) + h(\eta) \varepsilon HA] = P_W[w:H(\eta, w) \varepsilon HA]. \tag{3.33}$$

Since from Lemma 1 (3.10) H is onto,

$$P_W[w:H(\eta, w) \varepsilon HA] = P_W[w: (\eta, w) \varepsilon A] = P_W(A_\eta) \tag{3.34}$$

where A_η is the section of A at η. Thus from (3.32), (3.33) and (3.34),

$$Q(A, \omega) = P_W(A_{\Phi(\omega)}). \tag{3.35}$$

It follows that $Q(A, \omega)$ is a measure in A for ω and hence η fixed. By the Fubini theorem, for $A \varepsilon \mathcal{B}_X \times \mathcal{B}_W$

$$P(A) = \int_{\Omega_X} P_W(A_\eta)\,(d\eta) \tag{3.36}$$

where $P_W(A_\eta)$ is a $\mathcal{B}_X$-measurable function of η. Since Φ is $\mathcal{A}_X$-measurable, it follows from (3.35) that $Q(A, \omega)$ is $\mathcal{A}_X$-measurable for $A \varepsilon \mathcal{A}_Z$ fixed. For $C \varepsilon \mathcal{A}_X$, since C is a cylinder set, there exists $B \varepsilon \mathcal{B}_X$ such that

$$C = B \times W. \tag{3.37}$$

Then for $A \varepsilon \mathcal{A}_Z$ from (3.36)

$$\int_C I_A(\omega) P(d\omega) = P(A \cap C) = \int_{\Omega_X} P_W[(A \cap C)_\eta] P_X(d\eta). \tag{3.38}$$

From (3.37)

$$P_W[(A \cap C)_\eta] = P_W(A_\eta) I_B(\eta), \tag{3.39}$$

and hence (3.38) may be written as

$$\begin{aligned} \int_C I_A(\omega) P(d\omega) &= \int_B P_W(A_\eta) P_X(d\eta) = \int_W \int_B P_W(A_\eta) P_X(d\eta) P_W(dw) \\ &= \int_W \int_B P_W(A_{\Phi(\omega)}) P(d\omega) = \int_C Q(A, \omega) P(d\omega). \end{aligned} \tag{3.40}$$

From (3.40) it follows that

$$Q(A, \omega) = E[I_A(\omega) \mid \mathcal{A}_X](\omega) \qquad \text{a.s.} \tag{3.41}$$

Hence $Q(A, \omega)$ is the required regular conditional probability measure.

4. Main theorem. Under the assumption that the system process $x(\tau, \eta)$ is jointly measurable and square integrable a.s., Theorem 2 will be applied to the probability space defined by (3.2) with $\mathcal{A}_X$ given by (3.5) and $\mathcal{A}_Z$ by (3.26). In order to do this the conditions (i)-(iii) of Theorem 2 must be verified. First, according to Lemma 3, $Q(A, \omega)$ given by (3.32) is a regular conditional probability measure for P_Z given $\mathcal{A}_X$, and hence (i) is satisfied. Condition (ii) is checked by noting that the σ-field $\mathcal{A}_Z$ is generated by the countable class of sets

$$\{H^{-1}(B_{t_0,a,b}) \mid t_0, a, b \quad \text{rational}\} \tag{4.1}$$

where

$$B_{t_0,a,b} = \{z \mid z \varepsilon Z, a < z(t_0) \leq b\}. \tag{4.2}$$

In condition (iii), the measure λ will be defined by

$$\lambda(A) = P_W \Psi^{-1}(HA), \qquad A \varepsilon \mathcal{A}_Z, \tag{4.3}$$

where $P_W\Psi^{-1}$ is defined by (3.27). Then, according to Lemma 2, P_{z_0} is absolutely continuous with respect to $P_W\Psi^{-1}$ provided z_0 satisfies (3.29) and (3.30). For $A \varepsilon \mathcal{A}_Z$ such that $\lambda(A) = 0$, from (4.3) clearly $P_W\Psi^{-1}(B) = 0$ where $B = HA$ and hence $P_{z_0}(B) = P_{z_0}(HA) = 0$. Thus from the definition of $Q(A, \omega)$ given by (3.32) it is clear that $Q(A, \omega)$ is absolutely continuous with respect to λ for all ω such that $h(\Phi(\omega))$ satisfies (3.29) and (3.30). Referring to the definitions of h and Φ given by (3.8) and (3.4) we see that

$$h[\Phi(\eta, w)](\tau)$$

$$(4.4) \qquad \begin{aligned} &= \int_0^\tau x(u, \eta)\,du \qquad \text{for } 0 \leqq \tau \leqq t \text{ if } \int_0^t [x(u\ \eta)]^2\,du < \infty \\ &= 0 \qquad \text{for } 0 \leqq \tau \leqq t \text{ if } \int_0^t [x(u, \eta)]^2\,du = \infty. \end{aligned}$$

Thus, since $x(u, \eta)$ is assumed to be square integrable a.s., it follows that

$$(4.5) \quad h[\Phi(\eta, w)](\tau) = \int_0^\tau x(u, \eta)\,du \qquad (0 \leqq \tau \leqq t) \qquad \text{a.s.} \qquad P_X \times P_W$$

where the exceptional (η, w) set does not, of course, depend on τ. Hence $h[\Phi(\omega)]$ has absolutely continuous sample functions a.s. and by assumption

$$(4.6) \qquad \int_0^t [x(u, \eta)]^2\,du < \infty \qquad \text{a.s.}$$

Thus (3.29) and (3.30) are satisfied a.s., and condition (iii) of Theorem 2 is seen to hold.

A formula for the Radon-Nikodym derivative on the right side of (2.10) of Theorem 2 can also be deduced from Lemma 2. For $\omega = (\eta, w)$ fixed and such that (4.5) and (4.6) are satisfied, by Lemma 2

$$(4.7) \qquad (dP_{h(\eta)}/dP_W\Psi^{-1})(z) = f_\eta(z) \qquad \text{a.s.} \qquad P_W\Psi^{-1}$$

where

$$(4.8) \qquad f_\eta(z) = \exp\{\int_0^t x(u, \eta)\,dz(u) - \tfrac{1}{2}\int_0^t [x(u, \eta)]^2\,du\}.$$

Define

$$(4.9) \qquad q_\omega(\xi) = f_{\Phi(\omega)}(H\xi) \quad \text{for} \quad \xi \,\varepsilon\, \Omega.$$

The function $f_{\Phi(\omega)}(z)$ is measurable on $(Z, \mathcal{B}_Z)$ for ω fixed as above since it is a R-N derivative. Thus $q_\omega(\xi)$ is measurable on $(\Omega, \mathfrak{A}_Z)$ since by Lemma 1 H is a measurable transformation from $(\Omega, \mathfrak{A}_Z)$ to $(Z, \mathcal{B}_Z)$. For $A \,\varepsilon\, \mathfrak{A}_Z$

$$(4.10) \quad \int_A q_\omega(\xi)\lambda(d\xi) = \int_A f_\omega(H\xi)P_W\Psi^{-1}H(d\xi) = \int_{HA} f_\omega(z)P_W\Psi^{-1}(dz),$$

(see, for example, Lehmann [3], Lemma 2, p. 38). From (4.7)

$$(4.11) \qquad \int_{HA} f_{\Phi(\omega)}P_W\Psi^{-1}(dz) = P_{h[\Phi(\omega)]}(HA),$$

and thus by definition of $Q(A, \omega)$ (3.32),

$$(4.12) \qquad \int_A q_\omega(\xi)\lambda(d\xi) = Q(A, \omega).$$

It follows that

$$(4.13) \qquad (d/d\lambda)Q(\cdot, \omega)(\xi) = q_\omega(\xi) \qquad \text{a.s.} \qquad \lambda$$

for $\omega \,\varepsilon\, \Omega'$, where Ω' is the set on which (4.5) and (4.6) hold and hence $P(\Omega') = 1$.

Now from Theorem 2 there exists a function $q(\xi, \omega)$ measurable on

$(\Omega \times \Omega, \mathcal{A}_Z \times \mathcal{A}_X)$ which satisfies

$$q(\xi, \omega) = q_\omega(\xi) \qquad \text{a.e.} \qquad \lambda \times P. \tag{4.14}$$

The conditional expectation given $\mathcal{A}_Z$ of every $\mathcal{A}_X$-measurable and integrable random variable g is given by (2.12) of Theorem 2.

To facilitate interpretation and application it is desirable to recast the right hand side of (2.12) in a more convenient form. For this purpose we introduce the probability space

$$(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P}) = (\Omega_X, \mathcal{B}_X, P_X) \times (\tilde{\Omega}_X, \tilde{\mathcal{B}}_X, \tilde{P}_X) \times (W, \mathcal{B}_W, P_W) \tag{4.15}$$

where the spaces $(\Omega_X, \mathcal{B}_X, P_X)$ and $(\tilde{\Omega}_X, \tilde{\mathcal{B}}_X, \tilde{P}_X)$ are identical.

We shall denote elements of $\tilde{\Omega}$ by ϖ (see, (4.41)) and expectations with respect to the probability space (4.15) by $\tilde{E}$. On the space $(\tilde{\Omega}, \tilde{\mathcal{A}})$, define the projections

$$\tilde{\Phi}_1 : (\tilde{\Omega}, \tilde{\mathcal{A}}) \to (\Omega_X, \mathcal{B}_X) \tag{4.16}$$

and

$$\tilde{\Phi}_2 : (\tilde{\Omega}, \tilde{\mathcal{A}}) \to (\Omega, \mathcal{A}) \equiv (\tilde{\Omega}_X \times W, \tilde{\mathcal{B}}_X \times \mathcal{B}_X) \tag{4.17}$$

by

$$\tilde{\Phi}_1(\eta, \tilde{\eta}, w) = \eta \tag{4.18}$$

and

$$\tilde{\Phi}_2(\eta, \tilde{\eta}, w) = (\tilde{\eta}, w). \tag{4.19}$$

From (4.17) letting

$$P = \tilde{P}_X \times P_W \tag{4.20}$$

it is easily seen that

$$\begin{aligned} \tilde{P}(\tilde{\Phi}_1^{-1} A) &= P_X(A) \quad \text{if} \quad A \in \mathcal{B}_X \quad \text{and} \\ \tilde{P}(\tilde{\Phi}_2^{-1} A) &= P(A) \quad \text{for} \quad A \in \mathcal{A}. \end{aligned} \tag{4.21}$$

Let $g(\eta)$ be an integrable random variable on $(\Omega_X, \mathcal{B}_X, P_X)$. Then $g\Phi(\omega) = g[\Phi(\eta, w)] = g(\eta)$ and $g\tilde{\Phi}_1(\tilde{\omega}) = g[\tilde{\Phi}_1(\eta, \tilde{\eta}, w)] = g(\eta)$ are integrable random variables on $(\Omega, \mathcal{A}, P)$ and $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})$ respectively. The conditional expectation $E(g\Phi \mid \mathcal{A}_Z)$ is an $\mathcal{A}_Z$ measurable function on $(\Omega, \mathcal{A})$. Since $\mathcal{A}_Z$ is induced by the transformation H (3.26), there exists a $\mathcal{B}_Z$-measurable function $F(z)$ on Z such that

$$E(g\Phi \mid \mathcal{A}_Z)(\omega) = F[H(\omega)] \tag{4.22}$$

(see Lehmann [3], Lemma 1, p. 37). To denote this function $F(z)$, we will use the more suggestive notation

$$E(g\Phi \mid H, \mathcal{B}_Z)(z) = F(z) \tag{4.23}$$

where $F(z)$ satisfies (4.22) and hence

$$E(g\Phi \mid \mathcal{A}_Z)(\omega) = E(g\Phi \mid H^{-1}\mathcal{B}_Z)(\omega) = E(g\Phi \mid H, \mathcal{B}_Z)(H(\omega)). \tag{4.24}$$

The σ-field $\tilde{\mathcal{A}}_Z$ in $(\tilde{\Omega}, \tilde{\mathcal{A}})$ is defined by

$$\tilde{\mathcal{A}}_Z = (H\tilde{\Phi}_2)^{-1}(\mathcal{B}_Z) = \tilde{\Phi}_2^{-1}(\mathcal{A}_Z). \tag{4.25}$$

Following the notation outlined above

$$\tilde{E}(G \mid \tilde{\mathcal{A}}_Z)(\varpi) = \tilde{E}[G \mid (H\tilde{\Phi}_2)^{-1}(\tilde{\mathcal{B}}_Z)](\varpi) = \tilde{E}(G \mid H\tilde{\Phi}_2, \mathcal{B}_Z)(H\tilde{\Phi}_2(\omega)) \tag{4.26}$$

where G is an arbitrary integrable (or non-negative) random variable on $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})$.

On the space $(Z, \mathcal{B}_Z)$ denote by $P_Z H^{-1}$ the probability measure

$$(P_Z H^{-1})(B) = P_Z(H^{-1}B) \quad \text{where} \quad B \in \mathcal{B}_Z. \tag{4.27}$$

Since the following theorem is to be applied in other connections, the dependence on t will be explicitly displayed.

THEOREM 3. *Let $x(\tau, \eta)$, $0 \leqq \tau \leqq t$, $\eta \in \Omega_X$, be a jointly measurable process such that*

$$\int_0^t [x(u, \eta)]^2 \, du < \infty \quad \text{a.s.} \quad P_X. \tag{4.28}$$

Then there exists a function $\gamma_Z(\varpi)$ measurable on $(\tilde{\Omega}, \tilde{\mathcal{A}})$ such that

$$\gamma_t(\eta, \tilde{\eta}, w) = \exp\Big[\int_0^t x(u, \eta)\, dw(u) + \int_0^t x(u, \tilde{\eta})x(u, \eta)\, du - \tfrac{1}{2}\int_0^t [x(u, \eta)]^2 \, du \quad \text{a.s.} \quad \tilde{P}, \tag{4.29}$$

$$0 < \tilde{E}(\gamma_t \mid H\tilde{\Phi}_2, \mathcal{B}_Z)(z) < \infty \quad \text{a.s.} \quad P_Z H^{-1}, \tag{4.30}$$

and

$$E(g\Phi \mid H, \mathcal{B}_Z)(z) = \tilde{E}(g\tilde{\Phi}_1 \cdot \gamma_t \mid H\tilde{\Phi}_2, \mathcal{B}_Z)(z) / \tilde{E}(\gamma_t \mid H\tilde{\Phi}_2, \mathcal{B}_Z)(z) \quad \text{a.s.} \quad P_Z H^{-1} \tag{4.31}$$

for all integrable random variables g on $(\Omega_X, \mathcal{B}_X, P_X)$.

PROOF. Consider the projection

$$\theta: (\Omega, \mathcal{A}_Z) \times (\Omega, \mathcal{A}_X) \rightarrow (\Omega_X, \mathcal{B}_X) \times (\Omega, \mathcal{A}_Z) \tag{4.32}$$

defined by

$$\theta(\xi, \omega) = (\Phi(\omega), \xi). \tag{4.33}$$

Observe that the range space of θ is $\Omega_X \times \Omega = \tilde{\Omega}$ where Ω is taken to be $\tilde{\Omega}_X \times W$, and $\mathcal{B}_X \times \mathcal{A}_Z \subset \tilde{\mathcal{A}}$. Further, θ is a measurable transformation with

$$\theta^{-1}(\mathcal{B}_X \times \mathcal{A}_Z) = \mathcal{A}_Z \times \mathcal{A}_X. \tag{4.34}$$

Since $q(\xi, \omega)$ is $\mathcal{A}_Z \times \mathcal{A}_X$ measurable, there exists $\gamma_t(\varpi)$ measurable on $(\Omega_X \times \Omega,$

$\mathcal{B}_X \times \mathcal{A}_Z)$ such that

$$q(\xi, \omega) = \gamma_t(\theta(\xi, \omega)) \tag{4.35}$$

(Lehmann [3], Lemma 1, p. 37).

It is easily seen that

$$\tilde{P} = P_X \times P_Z = (P_Z \times P)\theta^{-1} \quad \text{on} \quad \mathcal{B}_X \times \mathcal{A}_Z \tag{4.36}$$

where it will be remembered that P_Z is the restriction of P to $\mathcal{A}_Z$.

It suffices to take g non-negative and P_X-integrable on Ω_X. Since $q(\xi, \omega)$ is $\mathcal{A}_Z \times \mathcal{A}_X$ measurable, by the Fubini theorem

$$\int_\Omega g\Phi(\omega)q(\xi, \omega)P(d\omega) \tag{4.37}$$

is $\mathcal{A}_Z$ measurable in ξ. From (4.25) $\tilde{\Phi}_2$ is a measurable transformation from $(\tilde{\Omega}, \tilde{\mathcal{A}}_Z)$ to $(\Omega, \mathcal{A}_Z)$ and hence

$$\int g\Phi(\omega)q(\tilde{\Phi}_2(\tilde{\omega}), \omega)P(d\omega) \tag{4.38}$$

is an $\tilde{\mathcal{A}}_Z$ measurable function on $\tilde{\Omega}$. If $A \in \tilde{\mathcal{A}}_Z$, then there exists $B \in \mathcal{A}_Z$ such that $A = \Omega_X \times B$. Thus, again from the Fubini theorem and (4.36) (remembering that $\tilde{\mathcal{A}}_Z \subset \mathcal{B}_X \times \mathcal{A}_Z$)

$$\begin{aligned} &\int_A [\int_\Omega g\Phi(\omega)q(\tilde{\Phi}_2(\varpi), \omega)P(d\omega)]\tilde{P}(d\varpi) \\ &\qquad = \int_B \int_{\Omega_X} [\int_\Omega g\Phi(\omega)q(\xi, \omega)P(d\omega)]P_X(d\eta)P_Z(d\xi) \\ &\qquad = \int_B [\int_\Omega g\Phi(\omega)q(\xi, \omega)P(d\omega)]P_Z(d\xi). \end{aligned} \tag{4.39}$$

From the definitions of the transformations Φ, $\tilde{\Phi}_1$, $\tilde{\Phi}_2$ and θ,

$$g\Phi(\omega) = g\tilde{\Phi}_1(\varpi) = g\tilde{\Phi}_1(\theta(\xi, \omega)) = g(\eta) \tag{4.40}$$

where

$$\varpi = (\eta, \tilde{\eta}, w), \qquad \omega = (\eta, w'), \qquad \xi = \tilde{\Phi}_2(\varpi) = (\tilde{\eta}, w). \tag{4.41}$$

Thus from (4.35) and (4.36), the right hand side of (4.39) can be written as

$$\int_B \int_\Omega g\tilde{\Phi}_1[\theta(\xi, \omega)]\gamma_t[\theta(\xi, \omega)]P(d\omega)P_Z(d\xi) = \int_A g\tilde{\Phi}_1(\tilde{\omega})\, \gamma_t(\varpi)\tilde{P}(d\varpi) \tag{4.42}$$

since $\theta(B \times \Omega) = \Omega_X \times B = A$. It follows from (4.39) and (4.42) that

$$\tilde{E}(g\tilde{\Phi}_1 \cdot \gamma_t \mid \tilde{\mathcal{A}}_Z)(\tilde{\omega}) = \int_\Omega g\Phi(\omega)q[\tilde{\Phi}_2(\tilde{\omega}), \omega]P(d\omega) \qquad \text{a.s.} \quad \tilde{P}_Z. \tag{4.43}$$

Here $\tilde{P}_Z$ denotes the restriction of $\tilde{P}$ to $\tilde{\mathcal{A}}_Z$. In Theorem 2, if ξ is replaced by $\tilde{\Phi}_2(\tilde{\omega})$, the equations (2.11) and (2.12) hold a.s. on $(\tilde{\Omega}, \tilde{\mathcal{A}}_Z, \tilde{P}_Z)$. Thus from (4.43) we have

$$0 < \tilde{E}[\gamma_t \mid \tilde{\mathcal{A}}_Z](\varpi) < \infty \qquad \text{a.s.} \quad \tilde{P}_Z \tag{4.44}$$

and

$$E(g \mid \mathcal{A}_Z)(\tilde{\Phi}_2(\tilde{\omega})) = \tilde{E}(g\tilde{\Phi}_1 \cdot \gamma_t \mid \tilde{\mathcal{A}}_Z)(\varpi)/\tilde{E}(\gamma_t \mid \tilde{\mathcal{A}}_Z)(\tilde{\omega}) \qquad \text{a.s.} \quad \tilde{P}_Z. \tag{4.45}$$

From the notational convention defined in (4.24) and (4.26), we can write (4.44) and (4.45) as

$$(4.46)\qquad 0 < \tilde{E}(\gamma_t \mid H\tilde{\Phi}_2,\ \mathcal{B}_Z)(H\tilde{\Phi}_2(\varpi)) < \infty \qquad \text{a.s.} \qquad \tilde{P}_Z$$

and

$$(4.47)\qquad E(g \mid H,\ \mathcal{B}_Z)(H\tilde{\Phi}_2(\varpi)) = \tilde{E}(g\tilde{\Phi}_1 \cdot \gamma_t \mid H\tilde{\Phi}_2,\ \mathcal{B}_Z)(H\tilde{\Phi}_2(\tilde{\omega})) \\ [\tilde{E}(\gamma_t \mid H\tilde{\Phi}_2,\ \mathcal{B}_Z)(H\tilde{\Phi}_2(\tilde{\omega}))]^{-1} \qquad \text{a.s.} \qquad \tilde{P}_Z .$$

These equations will be used to show that (4.30) and (4.31) of the theorem hold a.s. $P_Z H^{-1}$. Let N be the set on which (4.30) or (4.31) does not hold. Since the functions involved are all $\mathcal{B}_Z$ measurable, $N \,\varepsilon\, \mathcal{B}_Z$. For $\varpi \,\varepsilon\, (H\tilde{\Phi}_2)^{-1}N$, (4.46) or (4.47) is violated, and hence from (4.46) and (4.47)

$$(4.48)\qquad \tilde{P}_Z[(H\tilde{\Phi}_2)^{-1}N] = 0.$$

From (4.21) and (4.27)

$$(4.49)\qquad 0 = \tilde{P}_Z(\tilde{\Phi}_2^{-1}H^{-1}N) = P_Z(H^{-1}N) = P_Z H^{-1}(N).$$

Thus (4.30) and (4.31) hold a.s. $P_Z H^{-1}$.

It remains to show that (4.29) holds. From (4.14) and (4.9)

$$(4.50)\qquad q(\xi,\ \omega) = f_{\Phi(\omega)}(H\xi) \qquad \text{a.e.} \qquad \lambda \times P.$$

Thus from (4.35) and (4.33)

$$(4.51)\qquad \gamma_t(\Phi(\omega),\ \xi) = f_{\Phi(\omega)}(H\xi) \qquad \text{a.e.} \qquad \lambda \times P$$

where $f_\eta(z)$ satisfies (4.8). Let $\tilde{\Omega}'$ be the subset of $\tilde{\Omega}$ on which

$$(4.52)\qquad \int_0^t [x(u,\ \eta)]^2\, du < \infty, \qquad \int_0^t [x(u,\ \tilde{\eta})]^2\, du < \infty, \qquad \text{and} \\ \int_0^t x(u,\ \eta)\, dw(u)$$

exists and is finite. From the form of the probability space $\tilde{\Omega}$ (4.15) and assumption (4.28), it follows that $\tilde{P}(\tilde{\Omega}') = 1$. For $\varpi \,\varepsilon\, \tilde{\Omega}'$ it is easily seen from the definition of H (3.8) and (3.9) that

$$(4.53)\qquad f_{\tilde{\Phi}_1(\tilde{\omega})}[H\tilde{\Phi}_2(\varpi)] \\ = \exp\left[\int_0^t x(u,\ \eta)\, dw(u) + \int_0^t x(u,\ \eta)x(u,\ \tilde{\eta})\, du - \tfrac{1}{2}\int_0^t [x(u,\ \eta)]^2\, du\right].$$

Let $\tilde{N}$ be the set in $\tilde{\Omega}$ on which

$$(4.54)\qquad \gamma_t(\eta,\ \tilde{\eta},\ w) \neq f_{\tilde{\Phi}_1(\tilde{\omega})}[H\tilde{\Phi}_2(\varpi)].$$

Since (4.51) is violated for $(\xi,\ w) \,\varepsilon\, \theta^{-1}(\tilde{N})$, there must exist a set $M \,\varepsilon\, \mathcal{A}_Z \times \mathcal{A}_X$ such that

$$(4.55)\qquad \theta^{-1}(\tilde{N}) \subset M$$

and

$$(4.56)\qquad (\lambda \times P)(M) = 0.$$

From (4.34), there must be a set $\tilde{M} \varepsilon \mathcal{B}_X \times \mathcal{A}_Z$ such that

$$M = \theta^{-1}(\tilde{M}). \tag{4.57}$$

Since θ is onto, it follows from (4.55) and (4.57) that

$$\tilde{N} \subset \tilde{M}. \tag{4.58}$$

It was shown earlier that P_Z is absolutely continuous with respect to λ. Thus from (4.56) it can be shown that

$$(P_Z \times P)(\theta^{-1}\tilde{M}) = 0 \tag{4.59}$$

and hence from (4.36)

$$\tilde{P}(\tilde{M}) = 0. \tag{4.60}$$

From (4.53), (4.54), (4.58) and (4.60) it follows that (4.29) of the Theorem holds a.s. P.

5. Discussion. A more explicit form of Theorem 3 is given in the following corollary.

COROLLARY. *Let $x(\tau, \eta)$, $0 \leqq \tau \leqq t$, $\eta \varepsilon \Omega_X$, be a jointly measurable process such that*

$$\int_0^t [x(u, \eta)]^2 du < \infty \qquad \text{a.s.} \qquad P_X. \tag{5.1}$$

Then

$$0 < \int \{\exp [\int_0^t x(u, \eta)\, dz(u) - \tfrac{1}{2}\int_0^t [x(u, \eta)]^2 du]\} P_X(d\eta) < \infty \qquad \text{a.s.} \qquad P_Z H^{-1} \tag{5.2}$$

and

$$\begin{aligned} E(g\Phi \mid H, \mathcal{B}_Z)(z) = \{&\int [g(\eta) \exp \{\int_0^t x(u, \eta)\, dz(u) - \tfrac{1}{2}\int_0^t [x(u, \eta)]^2 du\}] \\ &\cdot P_X(d\eta)\}\{\int [\exp [\int_0^t x(u, \eta)\, dz(u) - \tfrac{1}{2}\int_0^t [x(u, \eta)]^2 du]] \\ &\cdot P_X(d\eta)\}^{-1} \qquad \text{a.s.} \qquad P_Z H^{-1} \end{aligned} \tag{5.3}$$

The integrals in (5.3), taken over Ω_X are well defined since the expressions [ ] are $\mathcal{B}_X$ measurable a.s. P_X for $z \varepsilon \Omega_Z'$ where $P_Z H^{-1}(\Omega_Z') = 1$.

PROOF. From (4.36) $\gamma_t(\tilde{\omega})$ is measurable on $(\Omega_X \times \Omega, \mathcal{B}_X \times \mathcal{A}_Z)$ where $\tilde{\omega} = (\eta, \tilde{\eta}, w)$, $\eta \varepsilon \Omega_X$ and $(\tilde{\eta}, w) \varepsilon \Omega$. From (4.37) and the Fubini theorem, it can easily be shown that

$$\tilde{E}[g\tilde{\Phi}_1 \cdot \gamma_t \mid \tilde{\mathcal{A}}_Z](\eta, \tilde{\eta}, w) = \int g(\eta)\gamma_t'(\eta, \tilde{\eta}, w) P_X(d\eta) \qquad \text{a.s.} \qquad \tilde{P}_Z. \tag{5.4}$$

From (4.55), (4.59) and (4.61)

$$\gamma_t(\eta, \tilde{\eta}, w) = f_\eta(H(\tilde{\eta}, w)) \qquad \text{a.s.} \qquad \tilde{P} \tag{5.5}$$

where $f_\eta(z)$ is given by (4.9). Again by the Fubini theorem and (4.37) for $(\tilde{\eta}, w) \varepsilon \Omega'$ where $P_Z(\Omega') = 1$

$$\gamma_t(\eta, \tilde{\eta}, w) = f_\eta(H(\tilde{\eta}, w)) \qquad \text{a.s.} \qquad P_X. \tag{5.6}$$

The exceptional set here may depend on $(\tilde{\eta}, w)$.

From (5.4) and (5.6)

$$\tilde{E}[g\tilde{\Phi}_1 \cdot \gamma_t \mid \tilde{\mathfrak{A}}_Z](\tilde{\omega}) = \int g(\eta) f_\eta (H\tilde{\Phi}_2(\tilde{\omega})) P_X(d\eta) \quad \text{a.s.} \quad \tilde{P}_Z \tag{5.7}$$

where integration on the right side of (5.7) is understood to be with respect to the completion of the measure P_X. That is, the right side of (5.7) is $\tilde{\mathfrak{A}}_Z$ measurable in $(\tilde{\eta}, w)$ a.s. P_Z. By definition (4.27)

$$\tilde{E}(g\tilde{\Phi}_1 \cdot \gamma_t \mid H\tilde{\Phi}_2, \mathfrak{B}_Z)(H\tilde{\Phi}_2(\tilde{\omega})) = \tilde{E}(g\tilde{\Phi}_1 \cdot \gamma_t \mid (H\tilde{\Phi}_2)^{-1}\mathfrak{B}_Z)(\tilde{\omega}). \tag{5.8}$$

Since $(H\tilde{\Phi}_2)^{-1}\mathfrak{B}_Z = \tilde{\mathfrak{A}}_Z$, from (5.7), using an argument similar to that in the proof of Theorem 3 it can be shown that

$$\int g(\eta) f_\eta(z) P_Z(d\eta) = \tilde{E}(g\tilde{\Phi}_1 \cdot \gamma_t \mid H\tilde{\Phi}_2, \mathfrak{B}_Z)(z) \quad \text{a.s.} \quad P_Z H^{-1}. \tag{5.9}$$

The corollary then follows from Theorem 3, (5.9) and (4.9). From (5.6) the function $f_\eta(z)$ is $\mathfrak{B}_X$ measurable a.s. P_X for $z \in H(\Omega') = \Omega_Z'$ where $P_Z H^{-1}(\Omega_Z') = P_Z(\Omega') = 1$.

Formula (5.3) of the corollary can be used to find Monte Carlo approximations to the desired estimates. Suppose we wish to estimate a functional

$$G(x(\tau); 0 \leqq \tau \leqq T) \tag{5.10}$$

defined on the system process. For example, the form of this functional required for the smoothing, filtering, and prediction problems are given by (1.7), (1.8) and (1.9). We will assume that a sample of system processes is available. Thus the functions

$$x_n(\tau), \qquad 0 \leqq \tau \leqq T, \tag{5.11}$$

are independent for $n = 1, 2, \cdots, N$ and as random variables with values in function space, they have the distribution induced by P_X. Let

$$g_n = G(x_n(\tau), 0 \leqq \tau \leqq T). \tag{5.12}$$

It will be assumed that the process

$$z(\tau), \qquad 0 \leqq \tau \leqq t, \tag{5.13}$$

has been observed. Then we can approximate the "best" estimate of G by

$$\delta(z(\tau), 0 \leqq \tau \leqq t) = E[G \mid z(\tau), 0 \leqq \tau \leqq t] \cong N^{-1} \textstyle\sum_{n=1}^{N} g_n f_n / N^{-1} \sum f_n \tag{5.14}$$

where

$$f_n = \exp\{\textstyle\int_0^t x_n(u)\, dz(u) - \frac{1}{2}\int_0^t [x_n(u)]^2\, du\}. \tag{5.15}$$

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