# ESTIMATION OF THE DIRECTIONAL PARAMETER OF THE OFFSET EXPONENTIAL AND NORMAL DISTRIBUTIONS IN THREE-DIMENSIONAL SPACE USING THE SAMPLE MEAN

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# Introduction

The problem originated from neutrino physics [1].

We consider a set of vectors in 3-dimensional Euclidean space $\mathbb{R}^3$.

We make a parametric assumption that this set is a sample of independent identically distributed variables, where a parameter of the distribution is a direction.

As our estimator we take the direction of the arithmetic mean of the sample. This allows a simpler mathematical treatment compared to other possible estimators, since the sum of variables corresponds to the convolution of their pdfs, and this can be calculated in a standard way using the Fourier transform.

Our goal is to find the distribution of the estimate in order to calculate confidence sets on the sphere, which we consider the precision of the estimator. We study both the exact case for finite samples and asymptotic cases, e.g. for number of events large.

Our parametric models are the exponential distribution, section 1, and the normal (Gaussian) distribution, section 2.

These results are new compared to previous studies. The author was unable to find directional results for the exponential distribution. In physical articles only the limiting case of large number of events is usually considered [1]. Mathematical literature on directional statistics usually deals with distributions on spheres [2], while in our case we have complete 3-dimensional information.

This work was written for those who are not necessarily statisticians or mathematicians but have met the problems treated here. Therefore the author attempted to use only the basic facts from mathematical undergraduate courses and introduced in detail more advanced notions when they were used. All the references used in this work can be found on the internet.

## Convolution of pdfs using the Fourier transform

The probability density function (pdf) $f(\mathbf{r})$ of the sum of two independent variables $\mathbf{r}_1 + \mathbf{r}_2$ in $\mathbb{R}^\mathrm{d}$ is given by the *convolution* of their pdfs:

$$f_{\mathbf{r}_1+\mathbf{r}_2}(\mathbf{r}) = (f_1 * f_2)(\mathbf{r}) = \int_{\mathbb{R}^\mathrm{d}} f_1(\mathbf{r}')f_2(\mathbf{r}-\mathbf{r}')\,\mathrm{d}\mathbf{r}'$$

We denote the *Fourier transform* [1] of a function $f(\mathbf{r})$ as

[1] this is similar to the *characteristic function* in probability theory, the latter is complex conjugate and without the factor $(2\pi)^{\frac{d}{2}}$

$$\hat{f}(\mathbf{p}) = \int_{\mathbb{R}^d} \frac{e^{-i\mathbf{p}\mathbf{r}}}{(2\pi)^{d/2}} f(\mathbf{r})\, \mathrm{d}^d\mathbf{r},$$

and the inverse Fourier transform of $f$ as $\tilde{f}$ (therefore $\hat{\tilde{f}}(x) \equiv f(x)$). With this definition the inverse Fourier transform operator is complex conjugate to the direct Fourier transform operator. Then

$$\begin{aligned}
\widehat{f * g}(\mathbf{p}) &= \int_{\mathbb{R}^d} \mathrm{d}^d\mathbf{r} \int_{\mathbb{R}^d} \frac{e^{-i\mathbf{p}\mathbf{r}}}{(2\pi)^{d/2}} f(\mathbf{r}')g(\mathbf{r}-\mathbf{r}')\, \mathrm{d}^d\mathbf{r}' \\
&= \int_{\mathbb{R}^d} \frac{e^{-i\mathbf{p}\mathbf{r}'}}{(2\pi)^{d/2}} f(\mathbf{r}')\, \mathrm{d}^d\mathbf{r}' \int_{\mathbb{R}^d} e^{-i\mathbf{p}(\mathbf{r}-\mathbf{r}')} g(\mathbf{r}-\mathbf{r}')\, \mathrm{d}^d\mathbf{r} \\
&= (2\pi)^{d/2} \hat{f}(\mathbf{p})\hat{g}(\mathbf{p}). \qquad (1)
\end{aligned}$$

This is a well-known property of the Fourier transform, that it maps the *convolution* of two pdfs to the *product* of their Fourier transforms.

Therefore the Fourier transform of the convolution of $n$ distributions $f$ is

$$\hat{f}_n(\mathbf{p}) = (\hat{f}(\mathbf{p}))^n (2\pi)^{\frac{(n-1)d}{2}} \qquad (2)$$

# 1 Exponential distribution

## 1.1 Introduction

Exponential distribution appeared in the author's studies connected to the problem in [1]. The observed pdf at large $x$ deviations was similar to $\sim e^{-\frac{x}{l}}$. To be spherically symmetric the pdf should be proportional to $\sim e^{-\frac{r}{l}}$. To calculate the normalisation factor, we take the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{\sqrt{x^2+y^2+z^2}}{l}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 4\pi\int_0^{\infty} r^2 e^{-\frac{r}{l}}\,\mathrm{d}r = 8\pi l^3$$

Hence the offset exponential probability density function in 3-dimensional space is

$$f_e(x,y,z|x_0,y_0,z_0) = \frac{1}{8\pi l^3} e^{-\frac{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}{l}} \tag{3}$$

**Fourier transform of $f_e$ and its convolutions**

In order to calculate the Fourier transform of $f_e$, we calculate the integral

$$\begin{aligned}\iiint_{\mathbb{R}^3} e^{-i\mathbf{pr}} e^{-\frac{\sqrt{(\mathbf{r}-\mathbf{r}_0)^2}}{l}}\,\mathrm{d}^3\mathbf{r} &= \iiint_{\mathbb{R}^3} e^{-i\mathbf{pr}_0} e^{-i\mathbf{pr}'} e^{-\frac{r'}{l}}\,\mathrm{d}^3\mathbf{r}' \\ &= 2\pi e^{-i\mathbf{pr}_0}\int_0^{\infty}\int_0^{\pi} e^{-ipr'\cos\theta-\frac{r'}{l}}\sin\theta\,\mathrm{d}\theta\, r'^2\,\mathrm{d}r',\end{aligned}$$

the inner integral on $\theta$

$$\int_{-1}^{1} e^{-ipr'\cos\theta}\,\mathrm{d}\cos\theta = \frac{1}{-ipr'}(e^{-ipr'}-e^{ipr'}) = \frac{1}{ipr'}(e^{ipr'}-e^{-ipr'}), \tag{4}$$

and the outer integral on $r'$ with one of the complex conjugate exponents

$$\int_0^{\infty} r' e^{ipr'-\frac{r'}{l}}\,\mathrm{d}r' = \left(r' = \frac{r}{\frac{1}{l}-ip}\right) = \frac{1}{(\frac{1}{l}-ip)^2}\int_0^{\infty} r e^{-r}\,\mathrm{d}r = \frac{1}{(\frac{1}{l}-ip)^2};$$

combining the two conjugate integrals, we obtain

$$\frac{1}{ip}\left(\frac{1}{(\frac{1}{l}-ip)^2} - c.c.\right) = \frac{(\frac{1}{l}+ip)^2-(\frac{1}{l}-ip)^2}{ip(\frac{1}{l}-ip)^2(\frac{1}{l}+ip)^2} = \frac{\frac{4ip}{l}}{ip(\frac{1}{l^2}+p^2)^2} = \frac{4}{l(\frac{1}{l^2}+p^2)^2},$$

therefore, taking into account the normalisation factor $\frac{1}{8\pi l^2}$ and the factor $(2\pi)^{-\frac{3}{2}}$ from the Fourier transform,

$$\hat{f}_e(\mathbf{p}) = \frac{1}{8\pi l^3} \frac{8\pi}{(2\pi)^{\frac{3}{2}} l(\frac{1}{l^2} + p^2)^2} e^{-i\mathbf{p}\mathbf{r}_0} = \frac{e^{-i\mathbf{p}\mathbf{r}_0}}{(2\pi)^{\frac{3}{2}}(1 + l^2p^2)^2} \tag{5}$$

From 2 and 5 we can learn the Fourier transform of the convolution of $n$ exponential distributions:

$$\hat{f}_n(\mathbf{p}) = \frac{e^{-i\mathbf{p}\mathbf{r}_0 n}}{(2\pi)^{\frac{3}{2}}(1 + l^2p^2)^{2n}}. \tag{6}$$

Thereby the convolution of $n$ exponential distributions is

$$f_n(\mathbf{r}) = \tilde{\hat{f}}_n(\mathbf{p}) = \iiint_{\mathbb{R}^3} \frac{e^{i\mathbf{p}\mathbf{r}}}{(2\pi)^{\frac{3}{2}}} \hat{f}_n(\mathbf{p})\,\mathrm{d}^3\mathbf{p} = \frac{1}{(2\pi)^3} \iiint_{\mathbb{R}^3} \frac{e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}_0 n)}}{(1 + l^2p^2)^{2n}}\,\mathrm{d}^3\mathbf{p},$$

choosing spherical coordinates with the $z$ axis along $\mathbf{r} - n\mathbf{r}_0$, the exponent becomes $e^{ip|\mathbf{r}-n\mathbf{r}_0|\cos\theta}$, and using 4 ,

$$\begin{aligned} f_n(\mathbf{r}) &= \frac{1}{(2\pi)^2} \int_0^\infty \frac{e^{ipr|\mathbf{r}-n\mathbf{r}_0|} - e^{-ipr|\mathbf{r}-n\mathbf{r}_0|}}{ip|\mathbf{r} - n\mathbf{r}_0|} \frac{1}{(1 + l^2p^2)^{2n}} p^2\,\mathrm{d}p = \\ &= \frac{1}{2\pi^2|\mathbf{r} - n\mathbf{r}_0|} \int_0^\infty \frac{p\sin(p|\mathbf{r} - n\mathbf{r}_0|)}{(1 + l^2p^2)^{2n}}\,\mathrm{d}p. \end{aligned}$$

Substituting inside the integral $p = \frac{x}{l}$,

$$f_n(\mathbf{r}) = \frac{1}{2\pi^2 l^2|\mathbf{r} - n\mathbf{r}_0|} \int_0^\infty \frac{x\sin(x\frac{|\mathbf{r}-n\mathbf{r}_0|}{l})}{(1 + x^2)^{2n}}\,\mathrm{d}x. \tag{7}$$

**Distribution of the sample mean $E_n$**

A statistic useful in practical applications is $\mathbf{r}_n = \frac{\mathbf{r}}{n}$ , the arithmetic mean of $\mathbf{r}$. We can calculate the probability density function $E_n(\mathbf{r}_n)$ of the random variable $\mathbf{r}_n$ and, using the conservation of probability under the change of variables $E_n(\mathbf{r}_n)\,\mathrm{d}^3\mathbf{r}_n = f_n(\mathbf{r})\,\mathrm{d}^3\mathbf{r}$ , we obtain $E_n(\mathbf{r}_n) = n^3 f_n(n\mathbf{r}_n)$

$$E_n(\mathbf{r}_n) = \frac{n^2}{2\pi^2 l^2|\mathbf{r}_n - \mathbf{r}_0|} \int_0^\infty \frac{x\sin(x\frac{n|\mathbf{r}_n-\mathbf{r}_0|}{l})}{(1 + x^2)^{2n}}\,\mathrm{d}x. \tag{8}$$

The integral can be calculated analytically using the formula 3.737(2) from [3] $[a > 0, \operatorname{Re}\beta > 0]$:

$$\int_0^\infty \frac{x\sin(ax)\,\mathrm{d}x}{(x^2+\beta^2)^{n+1}} = \begin{cases} \dfrac{\pi a e^{-a\beta}}{2^{2n}n!\beta^{2n-1}} \displaystyle\sum_{k=0}^{n-1} \frac{(2n-k-2)!(2a\beta)^k}{k!(n-k-1)!} \\ \dfrac{\pi}{2}e^{-a\beta} \qquad\qquad [n=0, \beta \geq 0] \end{cases} \tag{9}$$

Combining 8 and 9 , we obtain

$$E_n(\mathbf{r}_n) = \frac{n^3}{\pi l^3} \frac{e^{-\frac{n}{l}|\mathbf{r}_n - \mathbf{r}_0|}}{2^{4n-1}(2n-1)!} \sum_{k=0}^{2n-2} \frac{(4n-4-k)!(2\frac{n}{l}|\mathbf{r}_n - \mathbf{r}_0|)^k}{k!(2n-2-k)!} \tag{10}$$

In the case of $n = 1$ the sum in 10 is equal to 1 and we obtain 3.

## 1.2 Properties of $E_n$

In this subsection we study representations of $E_n$ other than 10 and its connection with hypergeometric functions.

**Calculation of the integral 9**

The integral 9 for $\alpha, \beta > 0$ can be easily reduced to that with $\beta = 1$:

$$\int_0^\infty \frac{x\sin(\alpha x)}{(x^2+\beta^2)^{n+1}}\mathrm{d}x \overset{x=\beta y}{=} \frac{1}{\beta^{2n}} \int_0^\infty \frac{y\sin(\alpha\beta y)}{(y^2+1)^{n+1}}\mathrm{d}y \tag{11}$$

$\boxed{n > 0}$ The integral 11 using integration by parts can be transformed to

$$\int_0^\infty \frac{x\sin(ax)}{(x^2+1)^{n+1}}\mathrm{d}x = -\frac{1}{2n}\sin(ax)\frac{1}{(1+x^2)^n}\Big|_0^\infty + \frac{a}{2n}\int_0^\infty \frac{\cos(ax)}{(x^2+1)^n}\mathrm{d}x =$$
$$= \frac{a}{2n}\int_0^\infty \frac{\cos(ax)}{(x^2+1)^n}\mathrm{d}x. \tag{12}$$

To compute this integral we apply complex analysis. We express

$$\int_0^\infty \frac{\cos(ax)}{(x^2+1)^n}\mathrm{d}x = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^\infty \frac{e^{iax}}{(x^2+1)^n}\mathrm{d}x, \tag{13}$$

the integrated function is meromorphic on whole complex plane and has poles at $\pm i$. We close the integration contour in the upper half-plane near infinity, where the integral is zero, and deform the contour into a small circle $\mathcal{C}$ of radius $r$ around the pole $+i$. We parameterise $z = i + re^{i\phi}$, then

$$\oint_{\mathcal{C}} \frac{e^{iaz}}{(z^2+1)^n}\mathrm{d}z = \int_0^{2\pi} \frac{e^{-a+iare^{i\phi}} ire^{i\phi}\mathrm{d}\phi}{(2ire^{i\phi}+r^2e^{2i\phi})^n} = \frac{e^{-a}}{2^n(ir)^{n-1}} \int_0^{2\pi} \frac{e^{iare^{i\phi}}e^{-i(n-1)\phi}\mathrm{d}\phi}{(1+\frac{r}{2i}e^{i\phi})^n} \tag{14}$$

In order to compute that integral, we decompose the integrand with respect to power series of $e^{i\phi}$ and use the identity

$$\int_0^{2\pi} e^{in\phi}\mathrm{d}\phi = \begin{cases} 0 & \text{if } n \neq 0, \\ 2\pi & \text{if } n = 0. \end{cases} \tag{15}$$

Since $r$ can be made sufficiently small, we can decompose the denominator using the binomial formula:

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k, \tag{16}$$

$$\binom{-n}{k} = \frac{(-n)(-n-1)\dots(-n-k+1)}{k!} = (-1)^k \binom{n+k-1}{k} \tag{17}$$

$$\begin{aligned}
&\int_0^{2\pi} \frac{e^{iare^{i\phi}}e^{-i(n-1)\phi}\mathrm{d}\phi}{(1+\frac{r}{2i}e^{i\phi})^n} \\
&= \int_0^{2\pi} \sum_{l=0}^{\infty} \frac{(iare^{i\phi})^l}{l!} \sum_{k=0}^{\infty} \binom{-n}{k} \left(\frac{r}{2i}e^{i\phi}\right)^k e^{-i(n-1)\phi}\mathrm{d}\phi = \\
&= 2\pi \sum_{k=0}^{n-1} \binom{-n}{k} \left(\frac{r}{2i}\right)^k \frac{(iar)^{n-1-k}}{(n-1-k)!} \\
&= 2\pi r^{n-1} \sum_{k=0}^{n-1} (-1)^k \binom{n+k-1}{k} \frac{i^{n-1-2k}a^{n-1-k}}{2^k(n-1-k)!} \\
&= \frac{2\pi(ir)^{n-1}}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n+k-1}{n-1} \frac{(2a)^{n-1-k}}{(n-1-k)!} \\
&= \frac{2\pi(ir)^{n-1}}{2^{n-1}} \sum_{k=0}^{n-1} \binom{2n-2-k}{n-1} \frac{(2a)^k}{k!}
\end{aligned} \tag{18}$$

Finally, we obtain

$$\int_0^{\infty} \frac{\cos(ax)}{(x^2+1)^n}\mathrm{d}x = \frac{\pi e^{-a}}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n-2-k}{n-1} \frac{(2a)^k}{k!} \tag{19}$$

$\boxed{n=0}$ As in the previous case, we transform

$$\int_0^\infty \frac{x\sin(ax)}{x^2+1}\mathrm{d}x = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^\infty \frac{xe^{iax}}{x^2+1}\mathrm{d}x \tag{20}$$

As earlier, we close the contour of integration in the upper half-plane, deform the contour to the pole $+i$ and parameterise the integration variable $z = i + re^{i\phi}$.

$$\int_0^{2\pi} \frac{(i+re^{i\phi})e^{-a+iare^{i\phi}}ire^{i\phi}}{2ire^{i\phi}+r^2e^{2i\phi}}\mathrm{d}\phi = e^{-a}\int_0^{2\pi}\frac{(i+re^{i\phi})e^{iare^{i\phi}}}{2+\frac{r}{i}e^{i\phi}}\mathrm{d}\phi \tag{21}$$

The integrand is analytic on $[0, 2\pi]$ and thus can be expressed as a series of non-negative powers of $e^{i\phi}$, from which only the zeroth term contributes to the integral and gives $\pi i$. Therefore

$$\int_0^\infty \frac{x\sin(ax)}{x^2+1}\mathrm{d}x = \frac{\pi}{2}e^{-a}.$$

### Proof that $E_n$ is properly normalised

The integral $\int_{\mathbb{R}^3} E_n(\mathbf{r}_n)\,\mathrm{d}^3\mathbf{r}_n$ is equal to 1, since $E_n$ is a properly normalised pdf. Here we calculate it explicitly using the formula 10.

After the parallel shift of $\mathbf{r}_n$ to $\mathbf{r}_0$, which doesn't affect the total integral, after changing to spherical coordinates and having integrated on the polar angles, we obtain the equality to be proved

$$\frac{n^3}{\pi l^3}4\pi\int_0^\infty r_n^2\,\mathrm{d}r_n\frac{e^{-\frac{n}{l}r_n}}{2^{4n-1}(2n-1)!}\sum_{k=0}^{2n-2}\frac{(4n-4-k)!(2\frac{n}{l}r_n)^k}{k!(2n-2-k)!} = 1.$$

Using the integral $\int_0^\infty x^n e^{-x}\,\mathrm{d}x = \Gamma(n+1) = n!$, this transforms to

$$\frac{1}{2^{4n-3}(2n-1)!}\sum_{k=0}^{2n-2}\frac{(4n-4-k)!2^k(k+2)!}{k!(2n-2-k)!} = 1. \tag{22}$$

This equality holds for $n = 1$. For $n = 2$ the left-hand side

$$\frac{1}{2^5 3!}\left(\frac{4!\,2!}{2!}+\frac{3!\,2\cdot 3!}{1!}+\frac{2!\,2^2\cdot 4!}{2!}\right) = \frac{1}{2^5}(4+12+16) = 1.$$

In the remaining of this subsubsection we prove the identity 22. Different techniques for calculation of closed forms of summations involving binomial coefficients can be found in [4]. Here we use the method of hypergeometric functions.

The Gaussian (or ordinary) hypergeometric function ${}_2F_1(a,b;c;z)$ is a special function represented by the hypergeometric series ([5], 7.2.1(1))

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+ \\ &+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+\dots \end{aligned}$$

This can be rewritten using the rising factorial or Pochhammer symbol

$$\begin{aligned} &(a)_0 = 1, \\ &(a)_n = a(a+1)\dots(a+n-1), \end{aligned} \tag{23}$$

then

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}. \tag{24}$$

In case when $a$ or $b$ is a negative integer, only a finite number of terms is non zero. Using Pochhammer symbol 23, we can express

$$\frac{n!}{(n-k)!} = n(n-1)\dots(n-k+1) = (-1)^k(-n)_k,$$

$$(n-k)! = \frac{n!}{(-1)^k(-n)_k} \tag{25}$$

$$(k+2)! = 1\cdot 2\cdot 3\dots(k+2) = 2\cdot(3)_k, \tag{26}$$

The sum in 22 can be rewritten as

$$\begin{aligned} \sum_{k=0}^{2n-2}\frac{(4n-4-k)!(k+2)!}{(2n-2-k)!}\frac{2^k}{k!} &\overset{(25,26)}{=} 2\frac{(4n-4)!}{(2n-2)!}\sum_{k=0}^{2n-2}\frac{(-2n+2)_k(3)_k}{(-4n+4)_k}\frac{2^k}{k!} = \\ &= 2\frac{(4n-4)!}{(2n-2)!}\,{}_2F_1(-2n+2,3;-4n+4;2) \end{aligned} \tag{27}$$

This hypergeometric function can be calculated using the formula 7.3.8(6) in [5]:

$${}_2F_1(-n,a;-2n;2) = 2^{2n}\frac{n!}{(2n)!}\left(\frac{a+1}{2}\right)_n. \tag{28}$$

Substituting 28 into 27, we obtain the original equality 22.

### 1.3 $\mathbf{CDF}_E(\cos\theta(\mathbf{r}_n))$

The original problem in [1] was to find the half of the opening angle of the cone whose origin is at (0,0,0) and the axis is the true direction, within which the given confidence level $cl$ (e.g. 68%) of events are contained.

When we work in spherical coordinates $(r, \phi, \theta)$, where $\theta$ is the angle to the true direction $\mathbf{r}_0$, we can compute the CDF as a function of $\theta$. Then the confidence interval $\theta_{cl}$ is $\mathrm{CDF}^{-1}(cl)$. This can be calculated numerically if we know the CDF.

In this subsection we calculate the cumulative distribution function of the polar angle $\theta$ of the statistic $\mathbf{r}_n$. For more compact formulae, we calculate the CDF as a function of $\cos\theta$.

Changing to spherical coordinates in 10 and integrating on $\phi$ and $r$,

$$\mathrm{CDF}_E(\cos\theta) = \frac{n^3}{l^3}\int_0^\infty r_n^2\,\mathrm{d}r_n \int_{\cos\theta}^1 \frac{e^{-\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta'}}}{2^{4n-2}(2n-1)!}\cdot \\ \cdot\sum_{k=0}^{2n-2}\frac{(4n-4-k)!(2\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta'})^k}{k!(2n-2-k)!}\,\mathrm{d}\cos\theta' \quad (29)$$

**Integral on $\cos\theta'$**

For shorter notation we define

$$\begin{pmatrix} a & = & r_n^2 + r_0^2 \\ b & = & 2r_nr_0 \\ c & = & \frac{n}{l} \end{pmatrix} \quad (30)$$

We assume $b \neq 0$, that is we exclude the point $r = 0$ from the integration. We calculate

$$\int_x^1 e^{-c\sqrt{a-bx'}}(c\sqrt{a-bx'})^k\mathrm{d}x' = \\ \begin{pmatrix} z = c\sqrt{a-bx'} \\ \mathrm{d}x' = -\frac{2z\mathrm{d}z}{bc^2} \end{pmatrix} = -\frac{2}{bc^2}\int_{c\sqrt{a-bx}}^{c\sqrt{a-b}} e^{-z}z^{k+1}\,\mathrm{d}z \quad (31)$$

$$\int x^k e^{-x}\,\mathrm{d}x = -x^k e^{-x} + k\int x^{k-1}e^{-x}\,\mathrm{d}x = -e^{-x}k!\sum_{i=0}^{k}\frac{x^i}{i!} \quad (32)$$

Combining 30, 31, 32

$$\int_{\cos\theta}^{1} e^{-\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta'}} \left(\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta'}\right)^k \mathrm{d}\cos\theta' =$$

$$= \frac{l^2}{n^2 r_n r_0} e^{-z}(k+1)! \sum_{i=0}^{k+1} \frac{z^i}{i!}\Bigg|_{\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta}}^{\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0}} =$$

$$= \frac{l^2(k+1)!}{n^2 r_0} \sum_{i=0}^{k+1} \frac{1}{i!}\Bigg($$

$$\frac{1}{r_n} e^{-\frac{n}{l}|r_n-r_0|} \left(\frac{n}{l}|r_n-r_0|\right)^i - \tag{33}$$

$$- \quad \frac{1}{r_n} e^{-\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta}} \left(\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta}\right)^i \tag{34}$$

$$\Bigg). \tag{35}$$

### Integral of 33 over $r_n$

Integrating 33, we multiply it by $r_n^2$ from the Jacobean, and expand the modulus

$$\int_0^\infty r_n e^{-\frac{n}{l}|r_n-r_0|} \left(\frac{n}{l}|r_n-r_0|\right)^i \mathrm{d}r_n = \tag{36}$$

$$\int_0^{r_0} r_n e^{-\frac{n}{l}(r_0-r_n)} \left(\frac{n}{l}(r_0-r_n)\right)^i \mathrm{d}r_n \tag{37}$$

$$+ \int_{r_0}^{\infty} r_n e^{-\frac{n}{l}(r_n-r_0)} \left(\frac{n}{l}(r_n-r_0)\right)^i \mathrm{d}r_n \tag{38}$$

The integral from 0 to $r_0$ is easily calculated using 32:

$$37 = \begin{pmatrix} \frac{n}{l}(r_0-r_n) = x, \\ r_n = r_0 - \frac{l}{n}x, \\ \mathrm{d}r_n = -\frac{l}{n}\mathrm{d}x \end{pmatrix} = -\int_{\frac{n}{l}r_0}^{0} \frac{l}{n}\left(r_0 - \frac{l}{n}x\right) e^{-x}x^i\,\mathrm{d}x \overset{(32)}{=}$$

$$= \frac{l}{n} r_0 e^{-x} i! \sum_{j=0}^{i} \frac{x^j}{j!}\Bigg|_{\frac{n}{l}r_0}^{0} - \frac{l^2}{n^2}\int_0^{\frac{n}{l}r_0} x^{i+1}e^{-x}\mathrm{d}x =$$

$$= \frac{l}{n} r_0 i! - \frac{l}{n} r_0 e^{-\frac{n}{l}r_0} i! \sum_{j=0}^{i} \frac{\left(\frac{n}{l}r_0\right)^j}{j!} - \frac{l^2}{n^2}\int_0^{\frac{n}{l}r_0} x^{i+1}e^{-x}\mathrm{d}x \tag{39}$$

We have retained the last integral, for it is useful in what follows.

The integral from $r_0$ to infinity is taken similarly:

$$38 = \frac{l}{n}\int_{r_0}^{\infty} e^{-\frac{n}{l}(r_n-r_0)}\left(\frac{n}{l}(r_n-r_0)\right)^{i+1} \mathrm{d}r_n$$
$$+ r_0\int_{r_0}^{\infty} e^{-\frac{n}{l}(r_n-r_0)}\left(\frac{n}{l}(r_n-r_0)\right)^{i} \mathrm{d}r_n = \left(\frac{l}{n}\right)^2 (i+1)! + \frac{l}{n}r_0 i! \quad (40)$$

Adding 39 and 40, we obtain

$$36 = 2\,i!\frac{l}{n}r_0 - \frac{l}{n}r_0 e^{-\frac{n}{l}r_0} i! \sum_{j=0}^{i}\frac{\left(\frac{n}{l}r_0\right)^j}{j!} + (i+1)!\frac{l^2}{n^2} - \frac{l^2}{n^2}\int_0^{\frac{n}{l}r_0} x^{i+1}e^{-x}\mathrm{d}x \quad (41)$$

In case of a large number of events $n$ or, more precisely, when $\frac{n}{l}r_0 \gg 1$, the exponent $e^{-\frac{n}{l}r_0}$ is much smaller than any power of $\frac{n}{l}r_0$, and

$$36 \simeq 2\,i!\,\frac{l}{n}\,r_0. \quad (42)$$

This corresponds to the case when most of the integral 36 is accumulated in a neighbourhood of $r_n = r_0$.

### Integral of 34 over $r_n$

In this subsubsection we calculate the integral

$$\int_0^{\infty} r_n e^{-\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta}}\left(\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta}\right)^i \mathrm{d}r_n. \quad (43)$$

We introduce

$$x = \frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta},$$

then we can rewrite

$$x^2\left(\frac{l}{n}\right)^2 = (r_n - r_0\cos\theta)^2 + r_0^2(1-\cos^2\theta)$$

Change from $r_n$ to $x$ is a change of coordinates if $r_n$ is uniquely defined through $x$ and vice versa. Therefore we should separately consider the regions $r_n \geq r_0\cos\theta$ and $r_n < r_0\cos\theta$. The point $r_n = r_0\cos\theta$ on a line of integration corresponds to the maximum of the pdf on that line (this is the nearest point on the line to the the mode of the distribution).

$\boxed{\boxed{\cos\theta \geq 0}}$

$\boxed{r_n \geq r_0 \cos\theta}$

$$r_n = r_0 \cos\theta + \sqrt{\frac{l^2}{n^2}x^2 - r_0^2(1-\cos^2\theta)}$$
$$\mathrm{d}r_n = \frac{\frac{l^2}{n^2}x\mathrm{d}x}{\sqrt{\frac{l^2}{n^2}x^2 - r_0^2(1-\cos^2\theta)}} = \frac{\frac{l}{n}x\mathrm{d}x}{\sqrt{x^2 - \frac{n^2}{l^2}r_0^2(1-\cos^2\theta)}}$$

$$\int_{r_0\cos\theta}^{\infty} r_n e^{-\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta}} \left(\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta}\right)^i \mathrm{d}r_n =$$
$$= \frac{l}{n}r_0\cos\theta \int_{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}^{\infty} \frac{x^{i+1}}{\sqrt{x^2 - \frac{n^2}{l^2}r_0^2(1-\cos^2\theta)}} e^{-x}\,\mathrm{d}x + \tag{44}$$
$$+ \frac{l^2}{n^2}\int_{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}^{\infty} x^{i+1}e^{-x}\,\mathrm{d}x \tag{45}$$

The integral 44 converges at the lower limit for $\cos\theta \neq 1$ because the integral $\int_a^b \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \int_a^b \frac{\mathrm{d}x}{\sqrt{(x-a)(x+a)}}$ converges. The integral

$$\int_a^{\infty} \frac{x^{i+1}e^{-x}\mathrm{d}x}{\sqrt{x^2-a^2}} \overset{(x=a\mathrm{ch}y)}{=} a^{i+1}\int_0^{\infty} \mathrm{ch}^{i+1}y\; e^{-a\mathrm{ch}y}\mathrm{d}y$$

can be expressed through the modified Bessel function $K_\nu$ (8.407 in [3]) using the formula 3.547(4) from [3]:

$$\int_0^{\infty} \exp\left(-\beta \mathrm{cosh}x\right)\cosh\left(\gamma x\right)\mathrm{d}x = K_\gamma\left(\beta\right) \qquad [\mathrm{Re}\,\beta > 0]$$

since $\mathrm{ch}^n x$ can be expressed as a sum of $\mathrm{ch}(kx)$ using 1.320(6) and 1.320(8) from [3].

$\boxed{0 \leq r_n \leq r_0\cos\theta}$

$$\sqrt{\frac{l^2}{n^2}x^2 - r_0^2(1-\cos^2\theta)} = r_0\cos\theta - r_n,$$
$$r_n = r_0\cos\theta - \sqrt{\frac{l^2}{n^2}x^2 - r_0^2(1-\cos^2\theta)}$$

The limits $r_n|_0^{r_0\cos\theta}$ are converted to $x|_{\frac{n}{l}r_0}^{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}$. The differential $\mathrm{d}r_n$ is the same as in the previous case $r_n \geq r_0\cos\theta$ except for the negative sign, which we omit changing the upper and the lower limits of the integration.

$$\int_0^{r_0\cos\theta} r_n e^{-\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta}}\left(\frac{n}{l}\sqrt{r_n^2+r_0^2-2r_nr_0\cos\theta}\right)^i \mathrm{d}r_n =$$

$$= \int_{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}^{\frac{n}{l}r_0}\left(r_0\cos\theta-\sqrt{\frac{l^2}{n^2}x^2-r_0^2(1-\cos^2\theta)}\right)\frac{e^{-x}\frac{l}{n}x^{i+1}\,\mathrm{d}x}{\sqrt{x^2-\frac{n^2}{l^2}r_0^2(1-\cos^2\theta)}} =$$

$$\begin{aligned} &= \frac{l}{n}r_0\cos\theta\int_{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}^{\frac{n}{l}r_0}\frac{x^{i+1}}{\sqrt{x^2-\frac{n^2}{l^2}r_0^2(1-\cos^2\theta)}}e^{-x}\,\mathrm{d}x \\ &-\frac{l^2}{n^2}\int_{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}^{\frac{n}{l}r_0}x^{i+1}e^{-x}\,\mathrm{d}x. \end{aligned} \tag{46}$$

Adding 44 and 45 to 46 gives

$$43 \overset{1>\cos\theta\geq 0}{=} \tag{47}$$

$$= \frac{l}{n}r_0\cos\theta\int_{\frac{n}{l}r_0}^{\infty}\frac{x^{i+1}}{\sqrt{x^2-\frac{n^2}{l^2}r_0^2(1-\cos^2\theta)}}e^{-x}\,\mathrm{d}x+\frac{l^2}{n^2}\int_{\frac{n}{l}r_0}^{\infty}x^{i+1}e^{-x}\,\mathrm{d}x+ \tag{48}$$

$$+2\frac{l}{n}r_0\cos\theta\int_{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}^{\frac{n}{l}r_0}\frac{x^{i+1}}{\sqrt{x^2-\frac{n^2}{l^2}r_0^2(1-\cos^2\theta)}}e^{-x}\,\mathrm{d}x. \tag{49}$$

$\boxed{\boxed{\cos\theta<0}}$ In this case $r_n$ is always bigger than $r_0\cos\theta$ and the integral becomes as in the case of 44, 45.

The lower limit is changed from $r_n = 0$ to $x = \frac{n}{l}r_0$,

$$43 \overset{\cos\theta<0}{=} \frac{l}{n}r_0\cos\theta\int_{\frac{n}{l}r_0}^{\infty}\frac{x^{i+1}}{\sqrt{x^2-\frac{n^2}{l^2}r_0^2(1-\cos^2\theta)}}e^{-x}\,\mathrm{d}x+\frac{l^2}{n^2}\int_{\frac{n}{l}r_0}^{\infty}x^{i+1}e^{-x}\,\mathrm{d}x. \tag{50}$$

Note that the only difference between 50 and 47 is 49.

We can combine the results for $\cos\theta < 0$ and for $\cos\theta \geq 0$ using the Heaviside step function:

$$\Theta(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases} \tag{51}$$

$$43 = \frac{l}{n} r_0 \cos\theta \int_{\frac{n}{l}r_0}^{\infty} \frac{x^{i+1}}{\sqrt{x^2 - \frac{n^2}{l^2} r_0^2 (1-\cos^2\theta)}} e^{-x}\,\mathrm{d}x + \frac{l^2}{n^2} \int_{\frac{n}{l}r_0}^{\infty} x^{i+1} e^{-x}\,\mathrm{d}x$$
$$+ 2\Theta(\cos\theta) \frac{l}{n} r_0 \cos\theta \int_{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}^{\frac{n}{l}r_0} \frac{x^{i+1}}{\sqrt{x^2 - \frac{n^2}{l^2} r_0^2 (1-\cos^2\theta)}} e^{-x}\,\mathrm{d}x \tag{52}$$

**CDF**$(\cos\theta(\mathbf{r}_n))$

Combining the calculations for the $\mathrm{CDF}(\cos\theta)$,

$$\mathrm{CDF}(\cos\theta) \overset{29,35}{=} \frac{n}{l}\frac{1}{r_0} \sum_{k=0}^{2n-2} \frac{(4n-4-k)!2^k(k+1)!}{k!(2n-2-k)!} \int_0^\infty r_n \mathrm{d}r_n \sum_{i=0}^{k+1} \frac{1}{i!} \Bigg($$
$$e^{-\frac{n}{l}|r_n - r_0|} \left(\frac{n}{l}|r_n - r_0|\right)^i - e^{-\frac{n}{l}\sqrt{r_n^2 + r_0^2 - 2r_n r_0 \cos\theta}} \left(\frac{n}{l}\sqrt{r_n^2 + r_0^2 - 2r_n r_0\cos\theta}\right)^i$$
$$\Bigg) \overset{41,52}{=} \frac{n}{l}\frac{1}{r_0} \sum_{k=0}^{2n-2} \frac{(4n-4-k)!2^k(k+1)}{(2n-2-k)!} \sum_{i=0}^{k+1} \frac{1}{i!} \Bigg($$
$$2\,i!\frac{l}{n} r_0 - \frac{l}{n} r_0 e^{-\frac{n}{l}r_0} i! \sum_{j=0}^{i} \frac{\left(\frac{n}{l}r_0\right)^j}{j!} + (i+1)!\frac{l^2}{n^2} - \frac{l^2}{n^2}\int_0^{\frac{n}{l}r_0} x^{i+1} e^{-x}\mathrm{d}x \tag{53}$$
$$-\frac{l}{n} r_0 \cos\theta \int_{\frac{n}{l}r_0}^{\infty} \frac{x^{i+1}}{\sqrt{x^2 - \frac{n^2}{l^2} r_0^2 (1-\cos^2\theta)}} e^{-x}\,\mathrm{d}x - \frac{l^2}{n^2} \int_{\frac{n}{l}r_0}^{\infty} x^{i+1} e^{-x}\,\mathrm{d}x \tag{54}$$
$$-2\Theta(\cos\theta) \frac{l}{n} r_0 \cos\theta \int_{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}^{\frac{n}{l}r_0} \frac{x^{i+1}}{\sqrt{x^2 - \frac{n^2}{l^2} r_0^2 (1-\cos^2\theta)}} e^{-x}\,\mathrm{d}x \Bigg)$$

The line 53 corresponds to 33, while 54 and below come from 34.

The last terms in 53 and 54 sum up to $-\frac{l^2}{n^2}\Gamma(i+2)$ and cancel with $(i+1)!\frac{l^2}{n^2}$ in 53. Thus we obtain the final answer:

$$\mathrm{CDF}_E(\cos\theta(\mathbf{r}_n,\mathbf{r}_0)) = \sum_{k=0}^{2n-2}\frac{(4n-4-k)!2^k(k+1)}{(2n-2-k)!}\Bigg(2(k+2)$$

$$- e^{-\frac{n}{l}r_0}\sum_{i=0}^{k+1}(k+2-i)\frac{\left(\frac{n}{l}r_0\right)^i}{i!} - \cos\theta\int_{\frac{n}{l}r_0}^{\infty}\frac{\sum_{i=0}^{k+1}\frac{x^{i+1}}{i!}}{\sqrt{x^2-\frac{n^2}{l^2}r_0^2(1-\cos^2\theta)}}e^{-x}\,\mathrm{d}x$$

$$- 2\Theta(\cos\theta)\cos\theta\int_{\frac{n}{l}r_0\sqrt{1-\cos^2\theta}}^{\frac{n}{l}r_0}\frac{\sum_{i=0}^{k+1}\frac{x^{i+1}}{i!}}{\sqrt{x^2-\frac{n^2}{l^2}r_0^2(1-\cos^2\theta)}}e^{-x}\,\mathrm{d}x\Bigg) \tag{55}$$

The first term in 55 is equal to $2^{4n-2}(2n-1)!$ due to 22.

## 1.4 Approximation of $E_n$ and $\theta(cl)$ for $n$ large

As in 1.2, we use

$$a_E = \frac{n}{l}|r_n - r_0| \tag{56}$$

for briefer notation. We also introduce [2]

$$F_n = \frac{n^3}{a}\int_0^{\infty}\frac{x\sin(ax)}{(1+x^2)^n}\mathrm{d}x, \tag{57}$$

then the pdf 8 can be expressed as

$$E_n = \frac{1}{16\pi^2 l^3}F_{2n}. \tag{58}$$

Let also

$$I_n = \int_0^{\infty}\frac{\cos(ax)}{(1+x^2)^n}\mathrm{d}x, \tag{59}$$

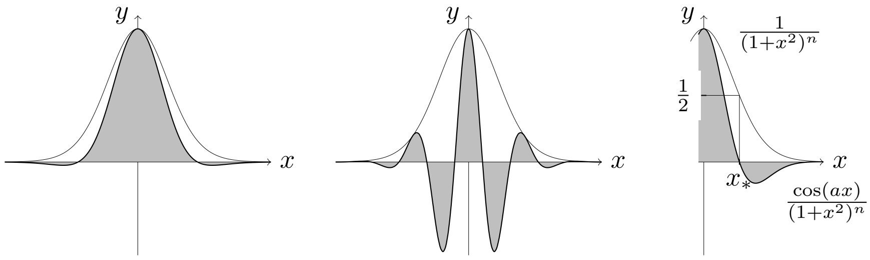
then due to 12

$$F_n = \frac{n^3}{2(n-1)}I_{n-1}. \tag{60}$$

Even though we have the exact formula 9 for $I_n$, we need to find a simple estimate for that when $n$ is large.

The function $(1+x^2)^{-n}$ has its maximum at $x=0$ and rapidly decreases to zero when $x$ and $n$ increase. Thus the integral 59 can accumulate most of

[2] the subscript E and the dependence F(a) are omitted throughout this section




(a) Smaller $a$ (b) Larger $a$ (c) $a_*$

Figure 1: $\frac{\cos(ax)}{(1+x^2)^n}$ for different $a$ values. The upper curve represents $\frac{1}{(1+x^2)^n}$.

its value near $x = 0$. For that to happen, $\cos(ax)$ should be positive when $(1 + x^2)^n$ is large enough.

Small $a$ (large $I_n(a)$) correspond to the maximum of the pdf, while large $a$ (small $I_n(a)$) correspond to the tail of the distribution. If we are interested in confidence levels neither too close to 1, nor too close to 0, we should explore the region where $I_n(a)$ takes intermediate values.

On figure 1 three different cases for the parameter $a$ are shown. On 1(a) $a$ is small, and the integral 59 is close to its maximum (as if $a = 0, \cos(ax) = 1$). On 1(b) $a$ is large, the $\cos(ax)$ oscillates fast and the integral 59 is small. To estimate the parameter of interest $a_*$, we define it such that the first zero of $\cos(ax)$ is when $(1+x^2)^{-n}$ is neither too large, nor too small. For an estimate we set it to $\frac{1}{2}$, fig. 1(c).

Solving $\frac{1}{(1+x^2)^n} = \frac{1}{2}$ gives $x_*^2 = 2^{\frac{1}{n}} - 1$ and

$$x_* \approx \frac{\sqrt{\ln(2)}}{\sqrt{n}}. \tag{61}$$

$a_*$ such that $\cos(a_* x_*) = 0$ is equal to $a_* = \frac{\pi}{2x_*} \approx \frac{\pi\sqrt{n}}{2\sqrt{\ln(2)}}$.

We can change 2 from the example to another number $A$ , then $a_*$ changes to

$$a_* \approx \frac{\pi\sqrt{n}}{2\sqrt{\ln(A)}}, \tag{62}$$

which depends very weakly on $A$ as $A$ grows. Therefore $a$ of interest for our problem is less than or of the order of $\sqrt{n}$.

The derivative w.r.t. $a$ of the integrand of $I_n$, $\left|\frac{-x\sin(ax)}{(1+x^2)^n}\right| < \frac{x}{(1+x^2)^n}$. The integral $\int_0^\infty \frac{x}{(1+x^2)^n} dx$ converges for $n > 1$, therefore, according to the well-known theorem [9], we can differentiate 60 on the parameter:

$$(I_n)'_a = \int_0^\infty \frac{-x\sin(ax)}{(1+x^2)^n}\mathrm{d}x \overset{12}{=} -\frac{a}{2(n-1)}I_{n-1}. \tag{63}$$

Now we derive a recursion formula for $I_n$.

$$I_n = \int_0^\infty \frac{(1+x^2)\cos(ax)}{(1+x^2)^{n+1}}\mathrm{d}x = I_{n+1} + \int_0^\infty \frac{x^2\cos(ax)}{(1+x^2)^{n+1}}\mathrm{d}x,$$

$$\int_0^\infty \frac{x^2\cos(ax)}{(1+x^2)^{n+1}}\mathrm{d}x = -\frac{1}{2n}\frac{1}{(1+x^2)^n}x\cos(ax)\Big|_0^\infty +$$

$$+\frac{1}{2n}\int_0^\infty \frac{\cos(ax)-ax\sin(ax)}{(1+x^2)^n}\mathrm{d}x = \frac{1}{2n}I_n - \frac{a^2}{4n(n-1)}I_{n-1},$$

therefore

$$I_{n+1} = \left(1-\frac{1}{2n}\right)I_n + \frac{a^2}{4n(n-1)}I_{n-1}. \tag{64}$$

$$I_{n+1} = I_n + O\left(n^{-1}\right)I_n, \text{ for } a \lesssim \sqrt{n}\ , \tag{65}$$

and substituting 65 into 63, we can solve the differential equation up to the terms of the order of $\frac{1}{n}$:

$$\frac{\mathrm{d}I_n(a)}{\mathrm{d}a} = -\frac{a}{2n}I_n\left(1+O\left(n^{-1}\right)\right),$$

$$I_n(a) = I_n(0)e^{-\frac{a^2}{4n}\left(1+O\left(n^{-1}\right)\right)} = I_n(0)e^{-\frac{a^2}{4n}}\left(1+O\left(n^{-1}\right)\right). \tag{66}$$

$I_n(0)$ can be found from the exact formula 19,

$$I_n(0) = \frac{\pi}{2^{2n-1}}\frac{(2n-2)!}{((n-1)!)^2}. \tag{67}$$

We can approximate $I_n(0)$ for large $n$ using Stirling's formula [6]

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+O\left(\frac{1}{n}\right)\right), \tag{68}$$

then

$$\frac{(2n-2)!}{((n-1)!)^2} \approx \frac{\sqrt{2}}{\sqrt{2\pi(n-1)}}\frac{(2n-2)^{2n-2}}{(n-1)^{2n-2}} = \frac{2^{2n-2}}{\sqrt{\pi}\sqrt{n-1}},$$

therefore

$$I_n(0) = \frac{\sqrt{\pi}}{2\sqrt{n}} + O(n^{-\frac{3}{2}}). \tag{69}$$

,

and

$$I_n \overset{66,69}{=} \frac{\sqrt{\pi}}{2\sqrt{n}} e^{-\frac{a^2}{4n}} \left(1 + O(n^{-1})\right), \tag{70}$$

$$F_n \overset{60,70}{=} \frac{n^{\frac{3}{2}}\sqrt{\pi}}{4} e^{-\frac{a^2}{4n}} \left(1 + O(n^{-1})\right), \tag{71}$$

$$E_n \overset{58,71}{=} \frac{(2n)^{\frac{3}{2}}}{64\pi^{\frac{3}{2}} l^3} e^{-\frac{a^2}{8n}} \left(1 + O(n^{-1})\right). \tag{72}$$

Substituting $a_E$ from 56, we express $E_n$ through the original parameters:

$$E_n(\mathbf{r}_n) = \frac{n^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}} 8l^3} e^{-\frac{n|r_n - r_0|^2}{8l^2}} (1 + O(n^{-1})), \tag{73}$$

and obtain in the leading order the normal distribution 77 with

$$\sigma_{E,n} = \frac{2l}{\sqrt{n}}, \tag{74}$$

which corresponds to the convolution of $n$ normal distributions with $\sigma_{E,1} = 2l$. In the limit of the number of events very large ($\sqrt{n}\frac{r_0}{2l} \gg 1$ and $\theta \ll 1$), one can apply the formula 100 to estimate a confidence interval $\theta$:

$$\theta_E \approx \frac{2l\sqrt{-2\ln(1 - cl)}}{r_0\sqrt{n}}. \tag{75}$$

# 2 Normal distribution

## 2.1 Introduction. Distribution of the sample mean $G_n$

A one-dimensional normal (or Gaussian) distribution has the pdf

$$g(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}. \tag{76}$$

With this definition the variance $\mathrm{E}[(x-x_0)^2] = \sigma^2$. For $d$ dimensions the spherically symmetric multivariate normal distribution is

$$g(\mathbf{r}) = \frac{1}{(2\pi\sigma^2)^{\frac{d}{2}}} e^{-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2\sigma^2}} \tag{77}$$

**The Fourier transform** of the multivariate normal distribution ()

$$\hat{g}(x) = \int_{\mathbb{R}^d} \frac{e^{-i\mathbf{pr}}}{(2\pi)^{\frac{d}{2}}} \frac{e^{-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2\sigma^2}}}{(2\pi\sigma^2)^{\frac{d}{2}}} \mathrm{d}^d\mathbf{r} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{e^{-\frac{(\mathbf{r}+i\mathbf{p}\sigma^2)^2}{2\sigma^2}}}{(2\pi\sigma^2)^{\frac{d}{2}}} e^{-\frac{\mathbf{p}^2\sigma^2}{2}} \mathrm{d}^d\mathbf{r} = \\ = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{\mathbf{p}^2\sigma^2}{2}} \tag{78}$$

This is similar to the normal distribution with the variance $\frac{1}{\sigma^2}$, except that it is not properly normalised, since the Fourier transform preserves the $L^2$-norm, but not the $L^1$-norm. For the Gaussian distribution the direct Fourier transform coincides with the inverse Fourier transform.

The Fourier transform of the convolution of $n$ $d$-dimensional Gaussian distributions (2)

$$\hat{g}_n(\mathbf{p}) = (2\pi)^{\frac{(n-1)d}{2}} \frac{1}{(2\pi)^{\frac{nd}{2}}} e^{-\frac{\mathbf{p}n\sigma^2}{2}} = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{\mathbf{p}^2 n\sigma^2}{2}}. \tag{79}$$

**The pdf of the convolution of $n$ Gaussian pdfs** , which corresponds to the sum of $n$ normally distributed variables, can be obtained by taking the inverse Fourier transform using (78) :

$$g_n(\mathbf{r}) = \frac{1}{(2\pi n\sigma^2)^{\frac{d}{2}}} e^{-\frac{\mathbf{r}^2}{2n\sigma^2}}. \tag{80}$$

This is again the normal distribution with the variance re-scaled to $n\sigma^2$. We use the average sum of Gaussian vectors $\mathbf{r}_n = \frac{\mathbf{r}}{n}$, and we shift the center of the distribution to $\mathbf{r}_0$; then the standard deviation becomes $\frac{\sigma}{\sqrt{n}}$:

$$G_n(\mathbf{r}_n) = \frac{n^{\frac{d}{2}}}{(2\pi\sigma^2)^{\frac{d}{2}}} e^{-\frac{n(\mathbf{r}_n-\mathbf{r}_0)^2}{2\sigma^2}} \tag{81}$$

## 2.2 $\mathrm{CDF}_G(\cos\theta(\mathbf{r}_n))$

In this subsection we work with $d = 3$. For calculation of integrals in spherical coordinates in arbitrary dimension one may consult [7].

$$\mathrm{CDF}(\cos\theta) = \frac{2\pi n^{\frac{3}{2}}}{(2\pi\sigma^2)^{\frac{3}{2}}} \int_0^\infty r^2 \mathrm{d}r \int_{\cos\theta}^1 e^{-\frac{n\left(r^2-2rr_0\cos\theta'+r_0^2\right)}{2\sigma^2}} \mathrm{d}\cos\theta',$$

the inner integral is taken easily in 3-dimensional space,

$$\int_{\cos\theta}^1 e^{\frac{2nrr_0\cos\theta'}{2\sigma^2}} \mathrm{d}\cos\theta' = \frac{\sigma^2}{nrr_0}\left(e^{\frac{2nrr_0}{2\sigma^2}} - e^{\frac{2nrr_0\cos\theta}{2\sigma^2}}\right),$$

therefore

$$\mathrm{CDF}(\cos\theta) = \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}r_0}\int_0^\infty r\left(e^{-\frac{n}{2\sigma^2}\left(r^2-2rr_0+r_0^2\right)} - e^{-\frac{n}{2\sigma^2}\left(r^2-2rr_0\cos\theta+r_0^2\right)}\right)\mathrm{d}r. \tag{82}$$

To calculate the first term in brackets, it is sufficient to calculate the second term and put $\cos\theta = 1$. We complete the square in the integral

$$\int_0^\infty re^{-\frac{n}{2\sigma^2}\left(r^2-2rr_0\cos\theta+r_0^2\right)}\mathrm{d}r = e^{-\frac{n}{2\sigma^2}r_0^2\left(1-\cos^2\theta\right)}\int_0^\infty re^{-\frac{n}{2\sigma^2}(r-r_0\cos\theta)^2}\mathrm{d}r, \tag{83}$$

then the latter integral we split into two ones with $r = (r - r_0\cos\theta) + r_0\cos\theta$. The first part is a total derivative with respect to $r - r_0\cos\theta = y$,

$$\int_0^\infty re^{-\frac{n}{2\sigma^2}(r-r_0\cos\theta)^2}\mathrm{d}r = \int_{-r_0\cos\theta}^\infty ye^{-\frac{ny^2}{2\sigma^2}}\mathrm{d}y + r_0\cos\theta\int_{-r_0\cos\theta}^\infty e^{-\frac{ny^2}{2\sigma^2}}\mathrm{d}y =$$
$$= \frac{\sigma^2}{n}e^{-\frac{nr_0^2\cos^2\theta}{2\sigma^2}} + r_0\cos\theta\int_{-r_0\cos\theta}^\infty e^{-\frac{ny^2}{2\sigma^2}}\mathrm{d}y.$$

The last term can be expressed through the *error function* ( [3], 8.250(1)):

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t, \tag{84}$$

$$\int_{-r_0\cos\theta}^{\infty} e^{-\frac{ny^2}{2\sigma^2}}\mathrm{d}y = \sqrt{\frac{2\sigma^2}{n}}\int_{-\frac{\sqrt{n}r_0}{\sqrt{2\sigma^2}}\cos\theta}^{\infty} e^{-t^2}\mathrm{d}t$$
$$= \sqrt{\frac{\pi\sigma^2}{2n}} + \sqrt{\frac{\pi\sigma^2}{2n}}\mathrm{erf}\left(\frac{\sqrt{n}r_0}{\sqrt{2\sigma^2}}\cos\theta\right) \tag{85}$$

Combining 83 and 85 into 82,

$$\mathrm{CDF}(\cos\theta) = \frac{\sqrt{n}}{\sqrt{2\pi\sigma^2}r_0}\left(\frac{\sigma^2}{n}e^{-\frac{nr_0^2}{2\sigma^2}} + r_0\sqrt{\frac{\pi\sigma^2}{2n}}\left(1+\mathrm{erf}\left(\frac{\sqrt{n}r_0}{\sqrt{2\sigma^2}}\right)\right) - \right.$$
$$\left. -e^{-\frac{n}{2\sigma^2}r_0^2\left(1-cos^2\theta\right)}\left(\frac{\sigma^2}{n}e^{-\frac{nr_0^2\cos^2\theta}{2\sigma^2}} + r_0\cos\theta\sqrt{\frac{\pi\sigma^2}{2n}}\left(1+\mathrm{erf}\left(\frac{\sqrt{n}r_0}{\sqrt{2\sigma^2}}\cos\theta\right)\right)\right)\right) \tag{86}$$

The first term cancels out, and we obtain the final result:

$$\mathrm{CDF}_G(\cos\theta(\mathbf{r}_n)) = \frac{1}{2}\left(1+\mathrm{erf}\left(\frac{\sqrt{n}r_0}{\sqrt{2}\sigma}\right)\right.$$
$$\left. - e^{-\frac{nr_0^2}{2\sigma^2}\left(1-cos^2\theta\right)}\cos\theta\left(1+\mathrm{erf}\left(\frac{\sqrt{n}r_0}{\sqrt{2}\sigma}\cos\theta\right)\right)\right) \tag{87}$$

Note that $\mathrm{CDF}_\mathrm{G}$ depends only on one combination of parameters $\sqrt{n}\frac{r_0}{\sigma}$.

## 2.3 Approximations of $\mathrm{CDF}_G(\cos\theta)$ and $\theta(cl)$

In this subsection we consider the behaviour of $\mathrm{CDF}_G(\cos\theta)$ for different values of $\cos\theta$ and parameters and the behaviour of confidence intervals ($\theta$ or $\cos\theta$) for given confidence levels $\mathrm{CDF}_G(\cos\theta) = cl$.

We introduce the parameter

$$a = \frac{\sqrt{n}r_0}{\sqrt{2}\sigma} \tag{88}$$

and express the CDF as

$$\mathrm{CDF}(\cos\theta) = \frac{1}{2}\left(1+\mathrm{erf}(a) - e^{-a^2\left(1-\cos^2\theta\right)}\cos\theta\left(1+\mathrm{erf}\left(a\cos\theta\right)\right)\right). \tag{89}$$

In what follows we work with 89, but keep in mind the expression of 88 through the original parameters of the distribution $n, r_0, \sigma$.

We are interested not only in limit cases, but even more in finite statistics samples. We group the terms according to their orders and keep lower order terms explicitly.

### $\theta$ close to 0, $n$ large

The asymptotic representation for the error function for large argument is ( [3], 8.254)

$$\operatorname{erf}(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \left( \sum_{k=0}^{n} (-1)^k \frac{(2k-1)!!}{(2z^2)^k} + O\left(|z|^{-2n-z}\right) \right) \tag{90}$$

(where $(-1)!! = 1$). $\operatorname{erf}(a)$ tends very rapidly to 1 as $a$ increases. Therefore the simplest approximation would be to substitute erf for 1 for large arguments. Thus for $a \gg 1$, $a \cos\theta \gg 1$

$$\operatorname{CDF}(\cos\theta) = 1 - e^{-a^2(1-\cos^2\theta)} \cos\theta + \alpha_1, \tag{91}$$

where

$$\alpha_1 = \frac{\operatorname{erf}(a) - 1}{2} - e^{-a^2(1-\cos^2\theta)} \cos\theta \frac{\operatorname{erf}(a\cos\theta) - 1}{2} = O\left(a^{-1} e^{-a^2}\right). \tag{92}$$

To express $\theta$ from 91 is more difficult, since the exponent power $a^2\left(1 - \cos^2\theta\right)$ can be arbitrary. We fix $\operatorname{CDF}(\cos\theta) = cl$, move the term with $\theta$ to the left side of 91, and take logarithm

$$\ln\cos\theta - a^2\left(1 - \cos^2\theta\right) = \ln\left(1 - cl + \alpha_1\right), \tag{93}$$

$$\frac{1}{2}\ln\left(1 - \sin^2\theta\right) - a^2\sin^2\theta = \ln\left(1 - cl\right) + \ln\left(1 + \frac{\alpha_1}{1 - cl}\right) \tag{94}$$

The equation 93 means that the results for lower $cl$ will be more precise than for $cl$ very close to 1, namely $1 - cl$ should be much more than $\alpha_1$. Lower $cl$ also corresponds to smaller $\theta$.

In order to solve 94 w.r.t. $\sin\theta$, we have to take a reasonable assumption $\sin^2\theta \ll 1$; we introduce

$$\beta_1 = \ln\left(1 - \sin^2\theta\right) + \sin^2\theta = O(\sin^4\theta), \tag{95}$$

we also rewrite the last term in 94 as

$$\alpha_2 = \ln\left(1 + \frac{\alpha_1}{1 - cl}\right) = O(\alpha_1) \tag{96}$$

Then from 94, 95, 96

$$\sin^2\theta = \frac{-\ln(1-cl)}{\frac{1}{2} + a^2} + \frac{\beta_1}{1 + 2a^2} - \frac{\alpha_2}{\frac{1}{2} + a^2} \tag{97}$$

The equation 94 can be solved with a better precision if we take into account more terms from the ln series (see e.g. [3] 1.511). Let

$$\beta_2 = \ln\left(1 - \sin^2\theta\right) + \sin^2\theta + \frac{1}{2}\sin^4\theta = O(\sin^6\theta), \tag{98}$$

then 94 transforms to a quadratic equation on $\sin^2\theta$

$$\begin{gathered}
\frac{1}{2}\beta_2 - \frac{1}{4}\sin^4\theta - \left(\frac{1}{2} + a^2\right)\sin^2\theta = \ln(1-cl) + \alpha_2, \\
\sin^4\theta + 2\left(1 + 2a^2\right)\sin^2\theta + 4\ln(1-cl) + 4\alpha_2 - 2\beta_2 = 0, \\
\sin^2\theta = -\left(1 + 2a^2\right) + \sqrt{(1+2a^2)^2 - 4\ln(1-cl) - 4\alpha_2 + 2\beta_2}
\end{gathered} \tag{99}$$

The most precise formula for big $a$ should be 99. For very big $a$ and small $\theta$ we can get a simpler expression from 97:

$$\theta \approx \frac{\sqrt{-\ln(1-cl)}}{a} \overset{88}{=} \frac{\sqrt{-2\ln(1-cl)}\sigma}{\sqrt{n}r_0} \tag{100}$$

### $\theta$ close to $\frac{\pi}{2}$

One can expect confidence intervals to be near $\frac{\pi}{2}$ when $a$ is neither too big nor too small. Therefore in this subsubsection we assume $a \sim 1$, so that $a\cos\theta \ll 1$.

The error function for small arguments can be approximated using integration of the exponent series in 84 term by term:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{3} + O\left(z^5\right)\right) \tag{101}$$

$$\text{CDF}(\cos\theta) = \frac{1}{2}(1+\text{erf}(a)) - \frac{e^{-a^2}}{2}\cos\theta(1+\frac{2}{\sqrt{\pi}}a\cos\theta+\gamma_1)$$
$$+\frac{e^{-a^2}}{2}\left(1-e^{a^2\cos^2\theta}\right)\cos\theta(1+\text{erf}(a\cos\theta)), \quad (102)$$
$$\text{where } \gamma_1 = \text{erf}(a\cos\theta) - \frac{2}{\sqrt{\pi}}a\cos\theta = O(a^3\cos^3\theta) \quad (103)$$

To find $\cos\theta(cl)$ we denote

$$\delta_1 = (e^{a^2\cos^2\theta}-1)(1+\text{erf}(a\cos\theta)) = O(\cos^2\theta), \quad (104)$$

then

$$\cos\theta\left(1+\frac{2}{\sqrt{\pi}}a\cos\theta\right)+\cos\theta(\gamma_1+\delta_1) = (1+\text{erf}(a)-2cl)e^{a^2} \quad (105)$$

In the leading order the solution $\cos\theta$ of 105 is the r.h.s. of 105. Therefore when we solve that equation up to $O(\cos^3\theta)$, we chose the '+' root:

$$\cos\theta = \frac{-1+\sqrt{1+\frac{8}{\sqrt{\pi}}ae^{a^2}(1+\text{erf}(a)-2cl)-\frac{8}{\sqrt{\pi}}a\cos\theta(\gamma_1+\delta_1)}}{\frac{4}{\sqrt{\pi}}a}. \quad (106)$$

**$\theta$ close to $\pi$**

$$\text{CDF}(\cos\theta) = \frac{1}{2}(1+\text{erf}(a)) - \frac{1}{2}e^{-a^2\sin^2\theta}\cos\theta(1+\text{erf}(a\cos\theta)) \quad (107)$$

The situation when $\theta$ is close to $\pi$ can appear when we have $a$ small and we are interested in large confidence levels (our precision is low, but still allows us to exclude a region near the pole $\theta=\pi$). In this subsubsection we don't take assumptions on $a$, but use a Taylor series expansion of $\text{erf}(z)$ at an arbitrary point:

$$\text{erf}(a+\Delta) = \text{erf}(a) + \frac{2}{\sqrt{\pi}}e^{-a^2}\Delta + O(\Delta^2). \quad (108)$$

Therefore

$$\operatorname{erf}(a\cos\theta) = \operatorname{erf}\left(-a\sqrt{1-\sin^2\theta}\right) = -\operatorname{erf}(a) + \frac{1}{\sqrt{\pi}} a e^{-a^2} \sin^2\theta + \varepsilon_1, \quad (109)$$

$$\varepsilon_1 = O(\sin^4\theta)$$

To find $\theta(cl)$ we solve the equation

$$\begin{aligned} &e^{-a^2\sin^2\theta}(-\cos\theta)(1+\operatorname{erf}(a\cos\theta)) = 2cl - 1 - \operatorname{erf}(a), \\ &-a^2\sin^2\theta + \frac{1}{2}\ln(1-\sin^2\theta) + \ln(1+\operatorname{erf}(a\cos\theta)) = \ln(2cl-1-\operatorname{erf}(a)) \end{aligned} \quad (110)$$

Using 109,

$$\begin{aligned} \ln(1+\operatorname{erf}(a\cos\theta)) &= \ln(1-\operatorname{erf}(a)) + \ln\left(1 + \frac{\frac{1}{\sqrt{\pi}} a e^{-a^2}\sin^2\theta + \varepsilon_1}{1-\operatorname{erf}(a)}\right) \\ &= \ln(1-\operatorname{erf}(a)) + \frac{a e^{-a^2}}{\sqrt{\pi}(1-\operatorname{erf}(a))}\sin^2\theta + \varepsilon_2, \quad (111) \end{aligned}$$

$$\varepsilon_2 = O(\sin^4\theta).$$

Using 95 and 111, we obtain

$$\begin{aligned} \sin^2\theta\left(-a^2 - \frac{1}{2} + \frac{a e^{-a^2}}{\sqrt{\pi}(1-\operatorname{erf}(a))}\right) = -\frac{\beta_1}{2} - \ln(1-\operatorname{erf}(a)) - \varepsilon_2 \\ + \ln(2cl-1-\operatorname{erf}(a)), \end{aligned}$$

$$\sin^2\theta = \left(a^2 + \frac{1}{2} - \frac{a e^{-a^2}}{\sqrt{\pi}(1-\operatorname{erf}(a))}\right)^{-1}\left(\ln\left(\frac{1-\operatorname{erf}(a)}{2cl-1-\operatorname{erf}(a)}\right) + \frac{\beta_1}{2} + \varepsilon_2\right). \quad (112)$$

The r.h.s. of 112 is positive, since the argument of the last ln is larger than 1: $1-\operatorname{erf}(a) > 2cl - 1 - \operatorname{erf}(a)$. However, the denominator of the logarithm's argument should also be positive,

$$cl > \frac{1+\operatorname{erf}(a)}{2}. \quad (113)$$

This means that if we want to exclude some percentage of the outcomes of $r_n$ with the directions near the pole, we should chose a confidence level which satisfies 113.

This is a necessary, but not a sufficient condition on $cl$. A more precise condition is that the r.h.s. of 112 is less than 1.

When $a$ is small we obtain $2\ln\left(\frac{1}{2cl-1}\right) < 1$, and $cl > \frac{1}{2}(1+e^{-1/2}) \approx 0.80$.

# Conclusions

In this work a new approach to solving the problem in [1] was proposed. The precision of the sample mean estimator was calculated analytically for the offset exponential and normal distributions both for a finite sample and for limiting cases.

Even though the original applied problem concerned the exponential distribution, the normal distribution was found to be also useful because of the central limit theorem [10]. It was shown explicitly how the distribution of the sample mean of the exponential pdf converges near the mode to the normal distribution.

While the normal distribution is tractable easier and has simpler formulae for the distribution of the sample mean and for the directional CDF, the exponential distribution has richer mathematical properties. While the distribution of the convolution of normal pdfs depends only on one combination of parameters, for the exponential distribution this is not the case. While the normal distribution is stable, the exponential one is not. Geometric techniques were used to deal with the limiting case of the exponential distribution. It was shown that the spherical projection of the sample mean of the exponential distribution has connections with hypergeometric functions and modified Bessel functions.

In this study we didn't concern other estimators, such as MLEs or the mean of the sample's projection on the sphere. Note that in [1] it was stated that the mean of unit vectors is a more precise estimator than the arithmetic sample mean. It might also be useful for mathematical applications to study the normal and exponential distributions in dimensions other than three.

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