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Estimation of the largest Lyapunov exponent in systems with impacts

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Abstract

The method of estimation of the largest Lyapunov exponent for mechanical systems with impacts using the properties of synchronization phenomenon is demonstrated. The presented method is based on the coupling of two identical dynamical systems and is tested on the classical Duffing oscillator with impacts. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

The estimation of Lyapunov exponents is one of the fundamental tasks in the studies of dynamical systems. Lyapunov exponents measure the exponential rates of divergence or convergence of nearby trajectories in state space. Theoretical studies of Oseledec [1] and numerical algorithm of Benettin et al. [2,3] and Wolf et al. [4] allow an easy estimation of the spectrum of Lyapunov exponents for smooth systems described by the known motion equation. If such equations for the system are not known or it is non-smooth, the estimation of Lyapunov exponents is not straightforward. Other methods are based on the reconstruction of attractor from time series [4].

Many real engineering systems can be considered as discontinuous ones. The typical example of such a system is the mechanical system with impacts. In recent years many papers describing the dynamical behavior of linear and non-linear mechanical systems with impacts have been published. However, the methods of the calculation of Lyapunov exponents for such systems have been proposed only in several papers [4,5,8].

In this paper author propose a method of estimation of the largest Lyapunov exponent using chaos synchronization [7] which is motivated by Fujisaka and Yamada's theoretical studies [6]. They have found the linear dependence between the synchronization value of the coupling coefficient of two identical dynamical systems and the largest value of Lyapunov exponent of such coupled systems. This condition of synchronization was formulated only for the case of the symmetrical negative feedback coupling between systems under consideration. The theoretical analysis (described in Section 2) generalizes the condition of synchronization also for the case of non-symmetrical coupling. As an example Fujisaka and Yamada have used the Lorenz system having smooth continuous character. The studies carried out by the author show that this approach can be applied for non-smooth dynamical systems as well, in particular for mechanical systems with impacts which have "strong" discontinuous nature.

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2. Theoretical assumptions of the method

Consider a dynamical system which is composed of two identical n-dimensional subsystems coupled by one-to-one negative feedback mechanism with a pair of coupling coefficients d_x and d_y. The first order differential equations describing such a system can be written as

$$\begin{aligned} \dot{x} &= f(x) + d_x(y - x), \\ \dot{y} &= f(y) + d_y(x - y), \end{aligned} \tag{1}$$

where $x, y \in \mathbb{R}^n$ and $d_{x,y} = [d_{x,y}; d_{x,y}; \ldots; d_{x,y}]^{\mathrm{T}} \in \mathbb{R}^n \geqslant 0$ is a coupling vector. Let's assume that for $d_x = d_y = 0$ each of the subsystems of system (1) evolves on an asymptotically stable chaotic attractor A.

In further analysis to simplify the notation and allow for the visualization the value $n = 3$ is taken and it is assumed that the evolution on the attractor A is characterized by one positive Lyapunov exponent.

Let us now introduce a new variable z representing the *state difference* between both subsystems during the time evolution. This variable is defined by the expression

$$z = x - y, \tag{2}$$

where $z \in \mathbb{R}^n$. The absolute value of *state difference*

$$\|\vec{z}\| = |x - y|, \tag{3}$$

describes the norm ($\|\vec{z}\| \in \mathbb{R}^1 \geqslant 0$) of the *state difference* vector

$$\vec{z} = \vec{x} - \vec{y}. \tag{4}$$

The image of the vector equation (Eq. (4)) in phase space is shown in Fig. 1. The *state difference* vector is fixed to the end of vector $\vec{y}$ and its end is in touch with the end of vector $\vec{x}$ (see Fig. 1). The time evolution of the *state difference* vector $\vec{z}$ is defined in z_i system of coordinates and can be determined as a sum of its $\vec{z}_i$ contributions in all directions of phase space

$$\vec{z} = \sum_{i=1}^{n} \vec{z}_i. \tag{5}$$

The origin of this system is fixed to the origin of vector $\vec{z}$. The axes of z_i-directions are parallel to the appropriate axes of phase space.

The condition of full synchronization state between coupled subsystems of the system (1) is given by the relation

$$\|\vec{z}\| = |x - y| = 0. \tag{6}$$

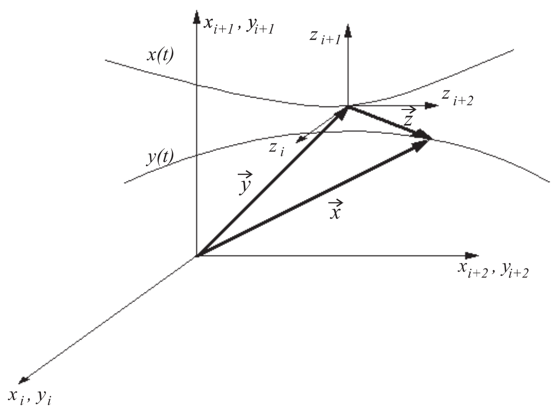

Fig. 1. The image of the vector equation (Eq. (4)) in phase space.

Substituting Eq. (2) in Eq. (1) we obtain

$$\begin{aligned} \dot{x} &= f(x) - d_x z, \\ \dot{y} &= f(y) + d_y z. \end{aligned} \tag{7}$$

The time evolution of the *state difference* is given by first time derivative of Eq. (2)

$$\dot{z} = \dot{x} - \dot{y}, \tag{8}$$

and can be written as a subtraction of both parts of Eq. (7)

$$\dot{z} = f(x) - f(y) - (d_x + d_y)z. \tag{9}$$

After putting the expression $y = x - z$ (arising from Eq. (2)) in Eq. (9) we can rewrite the system under consideration (Eq. (1)) in the following form:

$$\begin{aligned} \dot{x} &= f(x) - d_x z, \\ \dot{z} &= f(x,z) - (d_x + d_y)z. \end{aligned} \tag{10}$$

In Eqs. (9) and (10) the relation describing time evolution of *state difference* is given clearly. We can see that in the synchronized state ($z = 0$) the motion on the attractor of the analyzed system (Eqs. (1) or (10)) (in $2n$-dimensional phase space) reduces to the motion on the attractor of one of its identical subsystems (in n-dimensional phase space).

In further analysis two forms of Eqs. (1) and (10) are considered:
1. with zero coupling coefficients ($d_x = d_y = 0$) and non-zero system parameters;
2. with non-zero coupling coefficients and zero system parameters ($f(x) = f(y) = 0$).

2.1. Positive Lyapunov exponent effect

In the first of the above cases ($d_x = d_y = 0$) the equations describing the system under consideration assume the following forms:

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{y} &= f(y), \end{aligned} \tag{11}$$

or

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{z} &= f(x,z). \end{aligned} \tag{12}$$

The solutions of Eqs. (11) and (12) starting from different initial conditions ($x(0) \neq y(0)$ or $z(0) \neq 0$) represent two independent trajectories $x(t)$ and $y(t)$ on the attractor A (see Fig. 1). If the initial conditions are the same the evolution of both subsystems is identical and we have ideal synchronization ($x = y; z = 0$).

For small *state difference* vector, i.e. $\|\vec{z}\| \ll |A|$, where $|A| \in \mathbb{R}^1 \geqslant 0$ is the attractor size (maximum distance between two points on the attractor) in the phase space, we can assume that the distance between trajectories of the subsystems under consideration is given by the linearized equation resulting from the definition of Lyapunov exponent

$$\delta_i = |\delta_{i0}| \mathrm{e}^{\lambda_i t}, \tag{13}$$

where λ_i is a Lyapunov exponent, δ_i the norm of the contribution $\vec{\delta}_i$ of λ-*distance* vector $\vec{\delta}$ in the direction associated with i, the number of Lyapunov exponent and δ_{i0} is an initial distance in the same direction. The total λ-*distance* vector is a sum of the perpendicular contributions

$$\vec{\delta} = \sum_{i=1}^{n} \vec{\delta}_i. \tag{14}$$

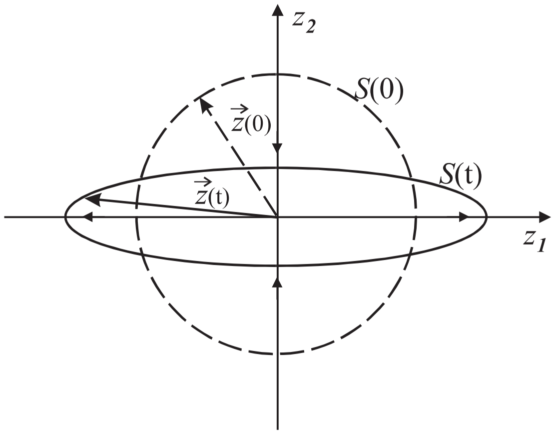

Fig. 2. Positive Lyapunov exponent effect (see description in paragraph 2.1).

The Lyapunov exponents are related to the expanding or contracting nature of different directions in phase space. The number of Lyapunov exponents is equal to the phase space dimension. For dissipative dynamical systems the sum of Lyapunov exponents is negative.

Let us now assume that the perpendicular axes of z_i-directions (*state difference* vector) are covered with the axes associated with λ-directions (λ*-distance* vector). In this case, for nearby orbits we have the equality of *state difference* vector and λ*-distance* vector

$$\vec{z} = \vec{\delta}. \tag{15}$$

Such a situation for the assumed 3-dimensional example is shown in Fig. 2. Axis z_1 represents the expanding direction of phase space connected with positive Lyapunov exponent ($\lambda_{\max} > 0$) and axis z_2 is associated with contracting direction ($\lambda_{\min} < 0$). The direction corresponding to the exponent $\lambda_i = 0$ (perpendicular to the plane of Fig. 2) is always tangential to phase trajectory. Such expanding and contracting effects cause that during the time evolution the sphere of initial conditions $S(0)$ (interrupted line) is deformed into ellipsoid $S(t)$ (solid line). Also the initial *state difference* vector $\vec{z}(0)$ increases at a rate proportional to the largest Lyapunov exponent value according to the relation

$$\|\vec{z}(t)\| = \|\vec{z}(0)\| \mathrm{e}^{\lambda^t_{\max}}. \tag{16}$$

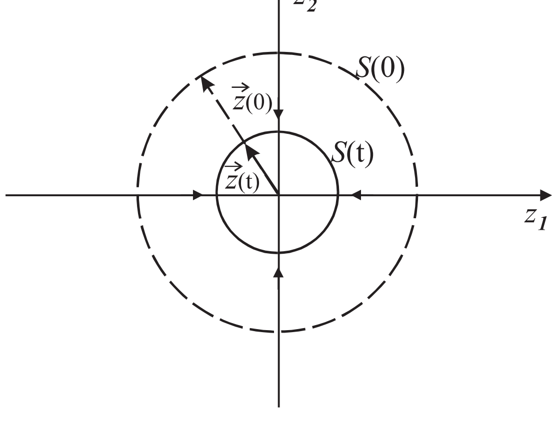

Fig. 3. Negative feedback coupling effect (see description in paragraph 2.2).

In agreement with previously taken assumptions the system described by Eq. (12) is an equivalent of the system given by Eq. (11). Hence, the largest Lyapunov exponent $\lambda_{\max}$ in Eq. (16) is characteristic for both identical subsystems of Eq. (11).

2.2. Negative feedback coupling effect

If we assume that the system parameters in Eq. (10) are equal to zero and coupling coefficients have non-zero values the system under consideration is given by following first time derivative equations:

$$\begin{aligned} \dot{x} &= -d_x z, \\ \dot{z} &= -(d_x + d_y)z. \end{aligned} \tag{17}$$

The solution of Eq. (17) in part describing the *state difference* is

$$z(t) = |z(0)|\mathrm{e}^{-(d_x+d_y)t}. \tag{18}$$

Hence, the norm of the *state difference* vector is given by similar to Eq. (18) relation

$$\|\vec{z}(t)\| = \|\vec{z}(0)\|\mathrm{e}^{-(d_x+d_y)t}. \tag{19}$$

Fig. 3 shows us the time evolution of the *state difference* vector and the sphere of initial conditions $S(0)$ in phase space of system (Eq. (17)). This picture shows us that the coupling has a convergential nature and causes the permanent decreasing of the *state difference* vector according to Eq. (19) at the rate proportional to the sum of coupling coefficients. From the same reason the volume of the sphere $S(t)$ diminishes during the time evolution but the sphere is not deformed because the system (Eq. (17)) has a linear character.

The above assumed (in paragraph 2.1) covering of the z_i-directions and λ_i-directions axes does not take place in reality. The spatial orientation of the coordinates system of directions associated with a given λ_i exponent varies in a complicated way during the evolution on the attractor. But in Fig. 3 it is shown that the negative feedback effect (connected with coupling) acts in all directions of the phase space with the same rate. Therefore, for full form of the system under consideration (Eq. (7) or Eq. (10)), with non-zero system and coupling parameters, we can observe the linear covering of the above described, independent effects:
1. exponential divergence of nearby trajectories associated with positive Lyapunov exponent;
2. exponential convergence due to the introduced coupling.

The first of these phenomena occurs only in direct neighborhood of the synchronized state where linear effects are dominant. The second one acts in the entire phase space. For that reason the described covering of both exponential effects (Eqs. (16) and (19)) takes place only nearby the synchronized state and is described by the relation

$$\|\vec{z}\| = \|\vec{z}(0)\|\mathrm{e}^{[\lambda_{\max}-(d_x+d_y)]t}. \tag{20}$$

The synchronization between the coupled subsystems of Eq. (1) is possible when the norm of *state difference* vector decreases to zero during the time evolution. It forces the fulfilling of the inequality

$$d_x + d_y > \lambda_{\max}. \tag{21}$$

The Eq. (21) gives us the linear dependence between the largest Lyapunov exponent and the coefficients of coupling. The fulfilling of inequality (Eq. (21)) is the condition of synchronization of two coupled identical dynamical systems (Eq. (1)).

3. Estimation procedure

The properties of chaos synchronization described in previous section allow author to propose a new method of the largest Lyapunov exponent estimation.

To simplify the procedure of estimation the unidirectional coupling in Eq. (1) has been assumed. In this case one of the coupling coefficients is equal to zero (say, $d_x = 0$, $d_y = d$) and Eq. (1) is reduced to the following form:

$$\begin{aligned}\dot{x} &= f(x),\\ \dot{y} &= f(y) + d(x - y).\end{aligned} \tag{22}$$

Now the condition of synchronization (Eq. (22)) is given by the inequality

$$d > \lambda_{\max}. \tag{23}$$

The smallest value of the coupling coefficient d, for which the synchronization takes place d_s is assumed to be equal to the maximum Lyapunov exponent

$$d_s = \lambda_{\max}. \tag{24}$$

To apply the method for any dynamical system it is necessary to build a double system with the coupling according to Eq. (22). The next step is a numerical research of the synchronization parameter d_s for such augmented system. If the tested coupling parameter d reaches the boundary value d_s then the largest Lyapunov exponent of the investigated system amounts to d_s.

The presented method can be successfully applied for non-smooth dynamical systems e.g. for mechanical systems with impacts. In such systems the discontinuous effect appears only at the instant of impact and causes the sudden jump of the *state difference* vector length (norm). However, the impact effect does not exert any influence on the phenomena in the phase space described in Section 2, shown in Figs. 2 and 3. Hence, the condition of synchronization for two identical, dynamical systems with impacts is also given by the above inequality (Eq. (23)).

The main problem which has been observed during practical applications of our method for the systems with impacts is a long time of transient motion (longer than for continuous systems) before the system under consideration achieves the synchronized state. This problem appears near the boundary value d_s.

To make faster achievement of searched d_s value possible, the idea called *elastic coupling* [9] has been applied.

4. Example

In this section we present the practical application of the method for mechanical system with impacts. As an example of such a system the classical Duffing oscillator with impacts has been used: (i) with single buffer (Fig. 4(a)), (ii) with two opposite buffers (Fig. 4(b)).

When there is no contact between the vibrating mass and the buffer, the motion of the system is described by well known Duffing equation

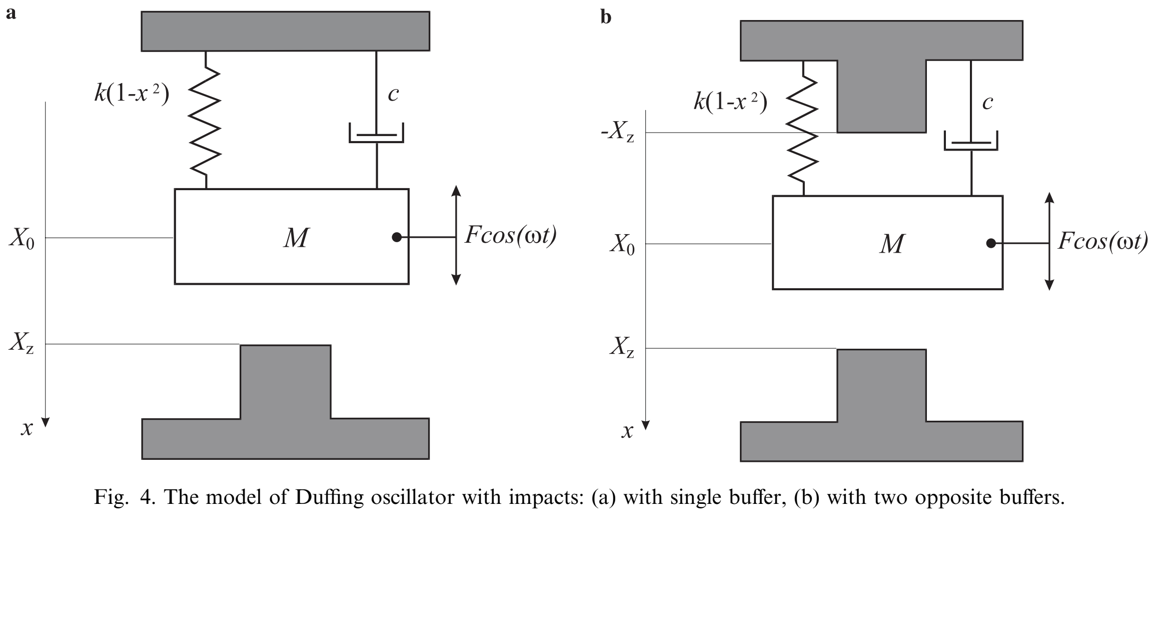

Fig. 4. The model of Duffing oscillator with impacts: (a) with single buffer, (b) with two opposite buffers.

$$m\ddot{x} - kx(1 - x^2) + c\dot{x} = F\cos(\omega t) \tag{25}$$

where m is the mass of the oscillator, $k(1-x^2)$ the non-linear stiffness, c the coefficient of viscous damping, F the amplitude of exciting force and, ω is the frequency of forcing.

Dividing Eq. (25) by mass m and introducing the substitutions: $x = x_1, \mathrm{d}x_1/\mathrm{d}t = x_2$, $a = k/m$, $h = c/m$, $p = F/m$ and taking impacts into consideration the equation describing Duffing oscillator with impacts can be written in the following form:

$$\begin{aligned} x_1 < X_0: \quad & \dot{x}_1 = x_2, \\ & \dot{x}_2 = p\cos(\omega t) + ax_1(1 - x_1^2) - hx_2, \\ x_1 \geqslant X_0: \quad & x_{2a} = -Rx_{2b}, \end{aligned} \tag{26}$$

for a single buffer, and

$$\begin{aligned} |x_1| < X_0: \quad & \dot{x}_1 = x_2, \\ & \dot{x}_2 = p\cos(\omega t) + ax_1(1 - x_1^2) - hx_2, \\ |x_1| \geqslant X_0: \quad & x_{2a} = -Rx_{2b}, \end{aligned} \tag{27}$$

for two opposite buffers.

In the above equations X_0 is a position of the buffer, R the coefficient of restitution, x_{2b} the velocity in a moment before impact and x_{2a} is the velocity just after impact.

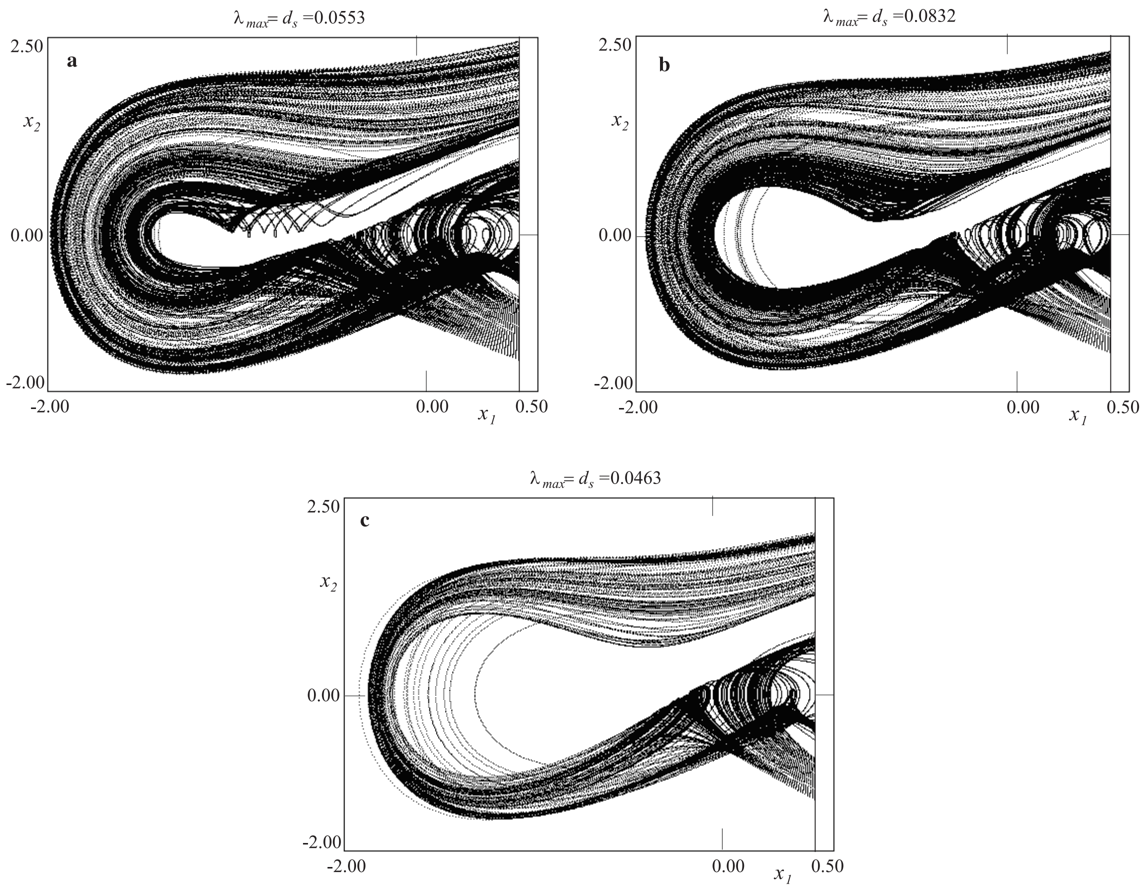

Fig. 5. The phase portraits showing the chaotic solutions of Duffing oscillator with impacts (single buffer-Eq. (26)) and the largest Lyapunov exponent $\lambda_{\max}$ associated with them: (a) $h = 0.05$, (b) $h = 0.10$, (c) $h = 0.20$; $a = 1.00$, $p = 1.00$, $\omega = 1.00$, $X_0 = 0.50$, $R = 0.65$.

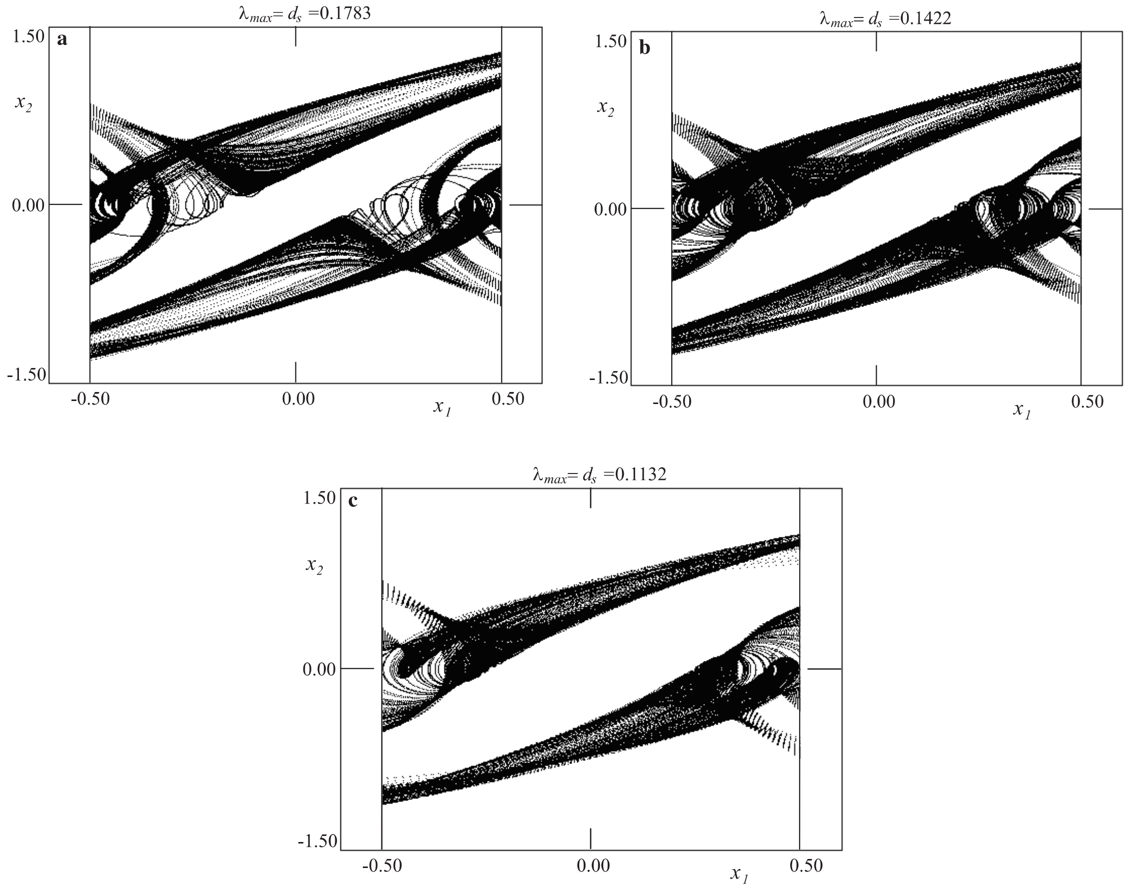

Fig. 6. The phase portraits showing the chaotic solutions of Duffing oscillator with impacts (two buffers Eq. (27)) and the largest Lyapunov exponent $\lambda_{\max}$ associated with them: (a) $h = 0.10$, (b) $h = 0.20$, (c) $h = 0.40$; $a = 1.00$, $p = 1.00$, $\omega = 1.00$, $X_0 = 0.50$, $R = 0.65$.

After putting the systems under consideration (Eqs. (26) and (27)) in Eq. (22) we obtain the augmented system for phase of motion between impacts ($x_1 < X_0$ or $|x_1| < X_0$) in the following form:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= p\cos(\omega t) + ax_1(1 - x_1^2) - hx_2, \\
\dot{y}_1 &= y_2 + d(x_1 - y_1), \\
\dot{y}_2 &= p\cos(\omega t) + ay_1(1 - y_1^2) - hy_2 + d(x_2 - y_2),
\end{aligned} \tag{28}$$

where p, a, h, ω are the above described system parameters and d is the coupling coefficient.

The expressions describing the conditions of impact ($x_1 \geqslant X_0$ or $|x_1| \geqslant X_0$) are the same as in Eqs. (26) and (27).

In the next step the largest value of Lyapunov exponent of systems under consideration has been determined for chosen values of parameters according to the way described in Section 3. The results of numerical calculations are presented in Figs. 5 and 6 which show the phase portraits and associated with them the largest Lyapunov exponents (on the top of picture) which have been obtained using the described synchronization method.

5. Conclusions

The presented method allows us to estimate the value of the largest Lyapunov exponent of the mechanical systems with impacts based on the properties of chaos synchronization. The described procedure can be also applied for other kinds of non-smooth dynamical systems e.g. mechanical systems with dry friction The developed method will be useful in quantifying, predicting and understanding chaos in non-smooth discontinuous systems for which the straightforward calculation of the Lyapunov exponents is not possible. The approach presented in this paper can be generalized to the higher dimensional ($n > 3$) systems. This method can be applied for both numerical and experimental estimation of the largest Lyapunov exponent. In the first case one has to know the equation of motion. In the experimental case two examples of the system have to be created and coupled together. In the mechanical systems it is impossible to realize the full form of coupling proposed in demonstrated method in real experiment. It is possible only on the way of numerical analysis.

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