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Estimation of Unobserved Expected Monthly Inflation Using Kalman Filtering

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Hamilton developed a technique for estimating financial market expectations of inflation based on the observed time-series properties of interest rates and inflation. The technique is based on a state-space representation derived from an underlying vector autoregressive process of the expected real interest rate and the expected inflation rate on lagged expectations and lagged values of the observed Treasury bill rate and the actual inflation rate. This article extends this work in two ways. First, we use monthly data, since the quarterly data used by Hamilton may obscure many interesting movements, especially for determining the role of inflationary expectations in stock price movements, and this is one of our primary interests. Second, we employ an alternative method developed by Burmeister and Wall for estimating the parameters of the model, and this method leads to a different identification proof. Both approaches share the use of the Kalman filter to estimate the unobserved variables, in this case, expected rates of inflation.

KEY WORDS: Expected inflation; Unobserved variables; Kalman filter; State space.

1. INTRODUCTION

Hamilton (1985) developed a technique for estimating financial market expectations of inflation based on the observed time series properties of interest rates and inflation. The technique is based on a state-space representation derived from an underlying vector autoregressive process of the expected real interest rate and the expected inflation rate on lagged expectations and lagged values of the observed Treasury bill rate and the actual inflation rate. Hamilton applied this technique to postwar quarterly data to generate historical estimates of the rates of inflation anticipated by bond markets. [The formulation is quite general and includes previous work on expected inflation as special cases in which most of the parameters are restricted to zero values; see Hamilton (1985) for details.]

This article extends this work in two ways. First, we use monthly data, since quarterly data may obscure many interesting movements, especially for determining the role of inflationary expectations in stock price movements, and this is one of our primary interests. Second, we employ an alternative method developed by Burmeister and Wall (1982, 1984a,b) for estimating the parameters of the model, and this method leads to a different identification proof. Both approaches share the use of the Kalman filter to estimate the unobserved variables, in this case, expected rates of inflation. [The use of Kalman filtering techniques to estimate unobserved variables in economics was introduced by Burmeister and Wall (1982).]

The estimated monthly expected inflation series exhibits the properties one would hope for, as did Hamilton's quarterly series; it is unbiased, rational, and efficient. In particular, we found no noncontemporaneous economic variables in the relevant information set that help explain the difference between the actual inflation series and our monthly estimated expected inflation series. The standard error of agents forecasting inflation one month ahead is approximately .3% at a monthly rate.

Monte Carlo simulations were conducted to ascertain how much of the total econometric uncertainty (as defined in Sec. 6) in our estimated inflation series is due to noise in the structural model and how much is due to parameter uncertainty. We found that the total econometric uncertainty is about .16% per month; about .13% of this is due to structural noise and about .09% is due to parameter uncertainty $[(.16)^2 \doteq (.13)^2 + (.09)^2]$. These numbers are roughly what one would expect given Hamilton's quarterly estimates.

One important application of our estimated expected inflation series is presented. We found that both the error in expected inflation for period t and the change in expected inflation between periods $t + 1$ and t are statistically significant macroeconomic factors in the context of the Ross (1976) Arbitrage Pricing Theory (APT) for explaining security returns in period t.

The presentation proceeds as follows: The model is explained in Section 2, and in Section 3 a state-space representation is derived. Identification is established in Section 4, with parameter estimates given in Section 5. In Section

6 we discuss our estimated expected inflation series, and tests of this series are given in Section 7. Finally, an application of our expected inflation series to the Arbitrage Pricing Theory is presented in Section 8, and the article concludes in Section 9 with a brief discussion of further research suggested by our work.

2. THE MODEL

We introduce the following notation:

$p(t)$ = the price level in month t as measured by the implicit deflator for nondurable consumption;

$\pi(t)$ = the rate of inflation between months t and $t - 1$ in % $\equiv \{[p(t) - p(t - 1)]/p(t - 1)\} \times 100$;

$\pi^e(t)$ = expected rate of inflation based on the available information at time t $\equiv E[\pi(t)|I(t)]$, where $I(t)$ is the information set available to economic agents at the beginning of month t;

$i(t)$ = nominal return on Treasury bills expiring at the end of month t in %, assumed to be known at the beginning of month t;

$r^e(t)$ = expected (ex ante) real interest rate $\equiv i(t) - \pi^e(t)$;

$r(t)$ = actual (ex post) real interest rate $\equiv i(t) - \pi(t)$;

$e(t)$ = the inflation forecast error made by agents $\equiv \pi(t) - \pi^e(t)$.

[One must be cautious to distinguish between the information set of the economic agents when they form the expectation $\pi^e(t)$ and the information set available to the econometrician who wishes to obtain an estimate of $\pi^e(t)$. Intuitively, the Kalman filter produces the "best" estimate of $\pi^e(t)$ (given either data over the whole sample or at least data up to and including time t) that is consistent with the model. In other words, given a model of how expectations and actual realizations interact, it produces the "best" estimate of unobserved expectations consistent with the other variables having their observed realizations.]

We note at the outset that there is a conceptual problem with the timing of rates because of the way in which the price level is measured. Price indices are constructed for month t by using data over the entire month, and hence they may be taken to measure the price level approximately at the middle of month t. Consequently, the rate of inflation $\pi(t)$ as defined measures the percentage rate of change in the price level from approximately the middle of month $t - 1$ to the middle of month t. The nominal interest rate $i(t)$, however, is measured from the beginning to end of month t.

Fortunately, however, with monthly data we found that our empirical results are robust with respect to 1-month timing changes in the inflation rate or nominal interest rate. Most important, when $i(t)$ is replaced by $i(t - 1)$ with everything else unchanged, there is essentially no alteration in our estimated expected inflation series. The empirical results reported subsequently match $i(t)$ with $\pi(t)$; hence the expected real rate of interest in month t is defined as the nominal rate of interest during month t minus the expected inflation rate from approximately the middle of month $t - 1$ to the middle of month t. Accordingly, if we take the index t to mean the beginning of month t, we assume that the information set contains

$$I(t) = \{\pi(t - 1), \pi(t - 2), \ldots, \pi^e(t), \pi^e(t - 1), \ldots, i(t), i(t - 1), i(t - 2), \ldots\},$$

and it could include other information as well. Even though $p(t - 1)$ and

$$\pi(t - 1) \equiv [p(t - 1) - p(t - 2)/p(t - 2)$$

are not announced until sometime during month t, we assume that agents at the beginning of month t know the price level for month $t - 1$ and are not surprised by subsequent announcements.

Following Hamilton (1985), we postulate that the following vector autoregressive process characterizes the evolution of the expected real interest rate and the expected inflation rate:

$$\begin{aligned} r^e(t) = {} & k_1 + \phi_1 r^e(t - 1) + \phi_2 r^e(t - 2) \\ & + \phi_3 r^e(t - 3) + \phi_4 r^e(t - 4) + \psi_1 \pi^e(t - 1) \\ & + \psi_2 \pi^e(t - 2) + \psi_3 \pi^e(t - 3) + \psi_4 \pi^e(t - 4) \\ & + \xi_1 \pi(t - 1) + \xi_2 \pi(t - 2) + \xi_3 \pi(t - 3) \\ & + \xi_4 \pi(t - 4) + \varepsilon_1(t) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \pi^e(t + 1) = {} & k_2 + \alpha_1 r^e(t) + \alpha_2 r^e(t - 1) + \alpha_3 r^e(t - 2) \\ & + \alpha_4 r^e(t - 3) + \beta_1 \pi^e(t) + \beta_2 \pi^e(t - 1) \\ & + \beta_3 \pi^e(t - 2) + \beta_4 \pi^e(t - 3) + \gamma_1 \pi(t) \\ & + \gamma_2 \pi(t - 1) + \gamma_3 \pi(t - 2) \\ & + \gamma_4 \pi(t - 3) + \varepsilon_2(t). \end{aligned} \tag{2}$$

In what follows, we assume that $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are uncorrelated with $r^e(t - j)$, $\pi^e(t - j)$, and $\pi(t - j)$ for all $j \geq 1$; thus the autoregressive dynamics of real interest rates and expectations of inflation are assumed to be stable and sufficiently simple to admit a low-order vector autoregressive representation of the form of (1) and (2). [For economic motivation of a simple autoregressive process for real interest rates, see Fama and Gibbons (1982) and Litterman and Weiss (1985).] The key innovation of Hamilton's technique, however, is that it does *not* require $\varepsilon_1(t)$ and $\varepsilon_2(t)$ to be uncorrelated with lagged values of other variables that may be known to agents but not to the econometrician. Thus Equation (2) is not the rule used by agents to forecast inflation, but simply corresponds to the statistical projection of those forecasts $\pi^e(t + 1)$ on a strict subset of the variables by which they are actually determined. The central idea is to rely on the hypotheses (1) and (2) of stable, simple dynamics, and use ex post data on $i(t+1)$, $\pi(t+1)$ to draw inferences about the unobserved ex ante magnitude $\pi^e(t+1)$, as in a standard signal–noise extraction problem. Agents face a forecasting problem in *formulating* $\pi^e(t+1)$, for which $i(t+1)$ and $\pi(t+1)$ are not known. The econometrician, by contrast, faces a signal problem of *inferring* $\pi^e(t+1)$, and for this latter purpose, use of $\pi(t+1)$ and $i(t+1)$ may be helpful. Of course, one uncovers only an inaccurate estimate of agents' forecasts $\pi^e(t+1)$ by this method, since the actual model used by agents in forecasting is presumed to be unavailable to the econometrician. Hamilton (1985) showed that standard filtering algorithms can be used to produce estimates of the degree of uncertainty associated with this problem of inference, which we refer to as "filter uncertainty" in the empirical results presented here.

In addition to assuming that the dynamics of real interest rates are stable and relatively simple, we also assume that markets are efficient; that is, all available information is used by agents to form their expectations. As discussed in detail in Hamilton (1985), this efficiency assumption imposes orthogonality conditions on (1) and (2) that imply that

$$E\{[\varepsilon_1(t), \varepsilon_2(t), e(t)]'[\varepsilon_1(\tau), \varepsilon_2(\tau), e(\tau)]\} = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_e^3 \end{bmatrix} \quad \text{if } t = \tau$$
$$= 0 \quad \text{if } t \neq \tau.$$

For some of the Monte Carlo simulations reported subsequently, we also take $[\varepsilon_1(t), \varepsilon_2(t), e(t)]$ to be trivariate normal.

As observed by Hamilton (1985), Fama and Gibbons (1982, 1984) employed a model that is a special case of Equation (1) with $\phi_1 = 1$ and all other parameters set to zero, and Gessler (1981) estimated ϕ_1 with all other parameters set to zero. Similarly, we allowed for a more complex dynamic evolution of expected real interest and inflation rates than that in earlier work by Mishkin (1981).

Indeed, the specification based on Equations (1) and (2) (presented in Sec. 3) entails 29 parameters in a nonlinear estimation problem, a task that pushed our available computing capacity near its limit. Thus further generality would be difficult to achieve at this time.

3. THE STATE–SPACE REPRESENTATION

Substitution of $i(t) - \pi^e(t) = r^e(t)$ into (2) and (1), respectively, gives us the state equation and the first output equation, as follows:

$$\begin{aligned}\pi^e(t+1) = {} & (\beta_1 - \alpha_1)\pi^e(t) + (\beta_2 - \alpha_2)\pi^e(t-1) \\ & + (\beta_3 - \alpha_3)\pi^e(t-2) + (\beta_4 - \alpha_4)\pi^e(t-3) \\ & + \alpha_1 i(t) + \alpha_2 i(t-1) + \alpha_3 i(t-2) \\ & + \alpha_4 i(t-3) + \gamma_1\pi(t) + \gamma_2\pi(t-1) \\ & + \gamma_3\pi(t-2) + \gamma_4\pi(t-3) + k_2 + \varepsilon_2(t) \qquad (3)\end{aligned}$$

and

$$\begin{aligned}i(t) = {} & \pi^e(t) + (\psi_1 - \phi_1)\pi^e(t-1) \\ & + (\psi_2 - \phi_2)\pi^e(t-2) + (\psi_3 - \phi_3)\pi^e(t-3) \\ & + (\psi_4 - \phi_4)\pi^e(t-4) + \phi_1 i(t-1) \\ & + \phi_2 i(t-2) + \phi_3 i(t-3) + \phi_4 i(t-4) \\ & + \xi_1\pi(t-1) + \xi_2\pi(t-2) + \xi_3\pi(t-3) \\ & + \xi_4\pi(t-4) + k_1 + \varepsilon_1(t). \qquad (4)\end{aligned}$$

The second output equation is simply

$$\pi(t) = \pi^e(t) + e(t), \qquad (5)$$

which states that the difference between actual and expected inflation is the forecast error $e(t)$. Our state-space application requires the assumption that $e(t)$ in Equation (5) be zero mean and white noise, so it is here that we are implicitly imposing the assumption of rational expectations. Subsequent tests on the estimated series $\hat{e}(t)$ will confirm that this specification is consistent with the data.

Hamilton (1985) showed that Equations (3), (4), and (5) constitute a state-space representation of the vector autoregressive model (1) and (2), as follows:

$$\begin{aligned}x(t+1) &= Fx(t) + Gz(t) + w(t), \\ y(t) &= Hx(t) + Dz(t) + v(t), \qquad (6)\end{aligned}$$

where

$$x(t) = [\pi^e(t), \pi^e(t-1), \pi^e(t-2), \pi^e(t-3), \pi^e(t-4)]',$$

$$\begin{aligned}z(t) = {} & [i(t), i(t-1), i(t-2), i(t-3), i(t-4), \\ & \pi(t), \pi(t-1), \pi(t-2), \pi(t-3), \pi(t-4), 1]',\end{aligned}$$

$$y(t) = [i(t), \pi(t)]',$$

$$w(t) = [\varepsilon_2(t), 0, 0, 0, 0]',$$

$$v(t) = [\varepsilon_1(t), e(t)]',$$

$$F = \begin{bmatrix} \beta_1 - \alpha_1 & \beta_2 - \alpha_2 & \beta_3 - \alpha_3 & \beta_4 - \alpha_4 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 0 & k_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & \psi_1 - \phi_1 & \psi_2 - \phi_2 & \psi_3 - \phi_3 & \psi_4 - \phi_4 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$D = \begin{bmatrix} 0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 & 0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & k_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note from the above that the last four state equations are simply $\pi^e(t) = \pi^e(t)$, $\pi^e(t-1) = \pi^e(t-1)$, $\pi^e(t-2) = \pi^e(t-2)$, and $\pi^e(t-3) = \pi^e(t-3)$.

4. IDENTIFICATION

There are 29 parameters to be estimated from the state-space specification (6): $\alpha_1, \ldots, \alpha_4; \beta_1, \ldots, \beta_4; \gamma_1, \ldots, \gamma_4; \phi_1, \ldots, \phi_4; \psi_1, \ldots, \psi_4; \xi_1, \ldots, \xi_4; k_1, k_2$; and σ_e, σ_{ε_1}, and σ_{ε_2}. We must verify that the cross-equation restrictions implicit in (6) from the underlying representations (1) and (2) are sufficient to identify these 29 parameters. In Hamilton (1985) the unknown parameters in (1) and (2) were identified and then estimated by maximum likelihood methods based on the implicit restrictions that they impose on the vector autoregressive moving average (ARMA) representation for $i(t)$, $\pi(t)$. Here, however, we show that identification can be established directly from the state-space representation (6). Thus we follow the approach of Wall (1984) and seek parameterizations that are unique in terms of their effect on the first and second moments of the observed dependent variables. This is exactly the approach taken by Hannan (1969, 1971, 1976), Kohn (1979), and others for vector ARMA model representations. The details are different, of course, because the characterization of the equivalence classes for state-space representations is different from that for vector ARMA representations.

More specifically, if we let R denote the variance–covariance (var–cov) matrix for $v(t)$ and Q denote the var–cov matrix for $w(t)$, then the first two moments of $y(t)$ are given by

$$E\{y(t)\} = \mu(t)$$

and

$$E\{[y(t) - E\{y(t)\}][y(s) - E\{y(s)\}]'\} = \Gamma(t, s),$$

where

$$\mu(t) = HF^tE\{x(0)\} + \sum_{i=0}^{t-1} F^{t-1-i}Gz(i) + Dz(t)$$

and

$$\begin{aligned}\Gamma(t, s) &= HF^{t-s}\Delta(s)H' && \text{if } t > s\\ &= HP(t)H' + R && \text{if } t = s\\ &= HP(t)(F^{t-s})'H' && \text{if } t < s.\end{aligned}$$

Here we employ $\Omega(t)$ to denote the var–cov matrix for the state $x(t)$; it evolves according to

$$\Omega(t + 1) = F\Omega(t)F' + Q, \qquad (7)$$

with $\Omega(0) = E\{x(0) - E\{x(0)\}][x(0) - E\{x(0)\}]'\}$. By focusing on the expressions for $\mu(t)$ and $\Gamma(t, s)$, we can ascertain how observational equivalence arises and thereby understand the basic cause of any identification problems in our state-space model.

We will first restrict ourselves to state-space representations that are minimal [i.e., completely controllable with respect to $z(t)$ and $w(t)$ and completely observable]. [This is completely analogous to the left-prime condition imposed on vector ARMA models for the purpose of establishing identification; see Hannan (1969, 1971), Kohn (1979), or Rosenbrock (1970).] Then we know that all observationally equivalent structures are related through an $n \times n$ nonsingular matrix T, as follows:

$$F_2 = T^{-1}F_1T;\quad G_2 = T^{-1}G_1;\quad Q_2 = T^{-1}Q_1(T^{-1})';$$

$$H_2 = H_1T;\quad D_2 = D_1;\quad R_2 = R_1.$$

Under such a relationship, the model structure specified by F_2, G_2, Q_2, H_2, D_2, and R_2 is indistinguishable from that given by F_1, G_1, Q_1, H_1, D_1, and R_1, using only first- and second-moment information. In fact, the $n \times n$ matrix T defines a family of equivalence classes in the space of all state-space models of dimension n. To obtain an identified model, we must select a structure (parameterization) that selects one and only one member from each equivalence class. Thus it is sufficient for identification that enough a priori structure in F, G, Q, H, D, and R has been specified to preclude any but the trivial transformation $T = I$.

In Appendix A, we show that the specification (6) admits only $T = I$ and hence is identified.

5. ESTIMATION OF PARAMETERS

The parameter and state variable estimations were carried out using standard Kalman-filtering techniques as described in Burmeister and Wall (1982, 1984a,b) and the references cited therein. To make this article self-contained, however, Appendix B, which describes the Kalman-filtering algorithm we used, is included.

Our sample of $p(t)$ and $i(t)$ covered January 1964–May 1983; because of lagged variables, the estimation was carried out over a sample from July 1964 to May 1983, containing 239 monthly observations. An initial guess of the 29-element parameter vector, θ, was made, and the initial var–cov matrix of the state vector, $x(t)$, was $20I_5$. Similarly, we set $\pi^e(-\tau) = \pi(-\tau)$ for $\tau = 1, \ldots, 4$; that is, out-of-sample expected inflation rates were set equal to actual values. With these initial conditions, the value of the negative log-likelihood function after one iteration was -611.17, and after 36 iterations it was -621.173. At this point, however, the gradients were between -1.4 and $+1.5$, too large to be acceptable. We then reinitialized the system with the value of $\hat{\theta}$ obtained from 36 iterations, but with the var–cov matrix for $x(t)$ again set at $20I_5$. Convergence was obtained after another 32 iterations, with the final gradients bounded by $-.00044$ and $+.00058$ and with a final value of the negative log-likelihood function equal to -621.178. The final $\hat{\theta}$ estimates and their standard errors are reported in Table 1. The Durbin–Watson statistics for the two output equations, (5) and (6), are 2.005 and 2.008, respectively. (Of course, these statistics must be interpreted with care because of the presence of lagged variables.)

On observing the pattern of significant parameters in Table 1 and referring to Equations (1) and (2), it becomes immediately evident that both the expected real interest rate,

Table 1. Estimated Parameter Values and Standard Errors (SE)

Parameter	Value	SE	Parameter	Value	SE
α_1	.23192	.44520	ψ_1	.38766	.48763
α_2	−.77227	.58946	ψ_2	−.25976	.58121
α_3	1.4758	.55312*	ψ_3	−.93821	.49609
α_4	−.99328	.28952*	ψ_4	.90838	.28423*
β_1	.37327	.45798	ξ_1	−.34229	.18519*
β_2	.12408	.55324	ξ_2	.22746	.24515
β_3	1.0941	.48001*	ξ_3	.044058	.12864
β_4	−.88155	.26457*	ξ_4	−.029353	.11214
γ_1	.39605	.19150*	k_1	−.021115	.025813
γ_2	−.18591	.23459	k_2	.055672	.025023*
γ_3	−.019142	.11849			
γ_4	−.013813	.10590			
ϕ_1	.66293	.44700	σ_{ε_2}	.044201	.034483
ϕ_2	.66047	.58373	σ_e	.30410	.029041*
ϕ_3	−1.3512	.58677*	σ_{σ_1}	.066367	.021898*
ϕ_4	1.0228	.31502*			

*Significant at the 95% level.

$r^e(t)$, and the expected rate of inflation, $\pi^e(t+1)$, appear to be influenced more significantly by lags in their expectations of 3 and 4 months than by 1- and 2-month lags. By contrast, actual inflation is significant in both equations at only its shortest lag.

Although one might be tempted to infer that the most recent 2 months of expectations are not relevant for forming current expectations, this conclusion is probably wrong. High parameter correlations, several exceeding .95 in absolute value, prevent us from distinguishing the separate influence of many parameters, and it is wrong to conclude that variables associated with statistically insignificant parameter values are economically unimportant. Given the multicollinearity problems inherent in monthly data, we must be satisfied with estimating an economically reasonable and useful series for expected inflation. [We note, however, that there may be more significant parameters in the state-space representation (6). For example, the H_{13} element of the matrix H is $\psi_2 - \phi_2$, and the standard error for $\psi_2 - \phi_2 = -.92023$ is .2510, indicating a high level of significance for H_{13}. Moreover, the forthcoming discussion associated with Table 2 reveals that the total parameter uncertainty is much less than the uncertainty due to additive errors.] Moreover, a focus on parameter values is especially uninteresting in view of the fact that (1) and (2) only represent the statistical projections of $r^e(t)$ and $\pi^e(t+1)$ on a strict subset of the relevant economic variables.

6. ESTIMATION OF THE STATE VARIABLE (EXPECTED INFLATION)

Given that we have estimated the true parameter vector $\hat{\theta}$, a straightforward application of the Kalman filter provides optimal (maximum likelihood) estimates of the unobserved state variable $x_1(t) \equiv \pi^e(t)$; this procedure is sketched in Appendix B.

The true θ is unknown, however, and we have only the estimate $\hat{\theta}$ and the associated var–cov matrix $\hat{\Sigma}$. Adopting the (asymptotic) Bayesian perspective that θ is itself a random variable distributed $N(\hat{\theta}, \hat{\Sigma})$, Hamilton (1985) showed that our estimate $\hat{\pi}^e(t)$ of the true expectations $\pi^e(t)$ has a variance given by

$$\text{var}[\hat{\pi}^e(t)] = E_\theta \, \text{var}[\hat{\pi}^e(t) \mid \theta] + \text{var}_\theta E[\hat{\pi}^e(t) \mid \theta]. \quad (8)$$

The first term on the right side of (8) is the expected value of the variance of $\hat{\pi}^e(t)$ associated with different draws of θ from the distribution $N(\hat{\theta}, \hat{\Sigma})$. For each draw of θ, we have that the filter uncertainty is

$$\text{var}[\hat{\pi}^e(t) \mid \theta] = P_{11}(t),$$

where $P_{11}(t)$ denotes the leading element in the var–cov matrix $P(t)$ defined in Appendix B. This filter certainly arises because the true model used by agents to forecast inflation is unknown to the econometrician. Similarly, for each draw of θ, the expected value of $\hat{\pi}^e(t \mid \theta)$ is simply our estimate of the first state variable $\hat{x}_1(t)$. Thus by calculating the variance of $\hat{x}_1(t)$ estimates for different draws of θ, we can calculate the parameter uncertainty equal to the second term on the right side of (8). The total econometric uncertainty in our estimate $\hat{\pi}^e(t)$ is the sum of the filter uncertainty and the parameter uncertainty.

In Table 2 we report the results of using the Kalman filter to estimate $\hat{\pi}^e(t)$, using the $\hat{\theta}$ values reported in Table 1. In Table 3 we report the results from 200 Monte Carlo simulations for the Kalman filter estimates, with θ drawn from $N(\hat{\theta}, \hat{\Sigma})$.

A first examination of our results reveals that our estimated expected inflation series is plausible:

1. The mean of the forecast error $\hat{e}(t) = \pi(t) - \hat{\pi}^e(t)$ is $-.0002$, sufficiently close to zero to be consistent with rational expectations.
2. The error $\hat{e}(t)$ is not autocorrelated ($\rho_{\hat{e}(t),\hat{e}(t-1)} = -.0056$), again consistent with rational expectations. The complete correlogram is given in Figure 1.
3. The variance of the $\hat{\pi}^e(t)$ is smaller than that of $\pi(t)$, but it is more autocorrelated; these results are consistent with the economically intuitive notion that expectations are smoother than realizations.
4. The forecast error made by agents is $e(t) = \pi(t) - \pi^e(t)$, and our estimate of the forecast error standard error from Table 1 is .3041 and from Table 2 is $(.0769)^{1/2} = .2773$. The larger value, however, is the maximum likelihood estimate, and thus 95% of the 1-month-ahead forecasts of inflation were incorrect by no more than .596% (= 1.96 × .3041) or 59.6 basis points.
5. The standard deviations of the total monthly econometric errors (due to both filter and parameter uncertainty) in $\hat{\pi}^e(t)$, our estimate of the actual expectations $\pi^e(t)$, are given in column 4 of Table 4. A "worst" case is approximately .27%, and "best" and "typical" cases are .13% and .15%, respectively.

In the next section we turn to a more careful examination of our estimated expected inflation series.

7. TESTS OF THE ESTIMATED EXPECTED INFLATION SERIES

Acceptance of the estimated expected inflation series, $\hat{\pi}^e(t)$, as economically reasonable requires further scrutiny.

Table 2. Results From the Kalman Filter Estimation, Using the Parameter Values in Table 1

Month/Year	$i(t)$	$\pi(t)$	$\hat{\pi}^e(t)$	$\hat{r}^e(t)$	$\hat{e}(t)$	Month/Year	$i(t)$	$\pi(t)$	$\hat{\pi}^e(t)$	$\hat{r}^e(t)$	$\hat{e}(t)$
July 1964	.300	.132	.132	.1684	.0005	June 1971	.370	.520	.320	.0499	.2002
Aug. 1964	.280	.000	.023	.2568	−.0232	July 1971	.400	.104	.291	.1085	−.1880
Sept. 1964	.280	.264	.218	.0615	.0454	Aug. 1971	.470	.517	.323	.1468	.1939
Oct. 1964	.290	.000	.072	.2179	−.0721	Sept. 1971	.370	.103	.336	.0344	−.2327
Nov. 1964	.290	.132	.190	.1003	−.0581	Oct. 1971	.370	.308	.296	.0739	.0122
Dec. 1964	.310	.131	.119	.1913	.0127	Nov. 1971	.370	.205	.412	−.0417	−.2068
Jan. 1965	.280	.000	.169	.1113	−.1687	Dec. 1971	.370	.511	.280	.0900	.2312
Feb. 1965	.300	.000	.105	.1947	−.1053	Jan. 1972	.290	.102	.368	−.0777	−.2660
March 1965	.360	.394	.270	.0903	.1240	Feb. 1972	.250	.610	.269	−.0193	.3405
April 1965	.310	.261	.194	.1165	.0679	March 1972	.270	.000	.410	−.1399	−.4099
May 1965	.310	.522	.348	−.0380	.1735	April 1972	.290	.101	.158	.1320	−.0570
June 1965	.350	.908	.470	−.1199	.4380	May 1972	.300	.303	.319	−.0186	−.0159
July 1965	.310	.000	.380	−.0703	−.3803	June 1972	.290	.101	.188	.1018	−.0876
Aug. 1965	.330	−.257	.224	.1059	−.4812	July 1972	.310	.302	.348	−.0377	−.0462
Sept. 1965	.310	.000	.205	.1054	−.2046	Aug. 1972	.290	.301	.261	.0293	.0399
Oct. 1965	.310	.258	.194	.1163	.0640	Sept. 1972	.340	.699	.446	−.1062	.2531
Nov. 1965	.350	.257	.322	.0284	−.0645	Oct. 1972	.400	.397	.434	−.0341	−.0373
Dec. 1965	.330	.385	.279	.0506	.1052	Nov. 1972	.370	.395	.423	−.0527	−.0274
Jan. 1966	.380	.383	.386	−.0061	−.0030	Dec. 1972	.370	.394	.420	−.0500	−.0263
Feb. 1966	.350	.763	.420	−.0701	.3433	Jan. 1973	.440	.784	.563	−.1230	.2213
March 1966	.380	.379	.469	−.0891	−.0903	Feb. 1973	.410	.876	.574	−.1636	.3019
April 1966	.340	.377	.428	−.0878	−.0504	March 1973	.460	1.254	.737	−.2766	.5174
May 1966	.410	.000	.370	.0402	−.3698	April 1973	.520	1.048	.845	−.3250	.2030
June 1966	.380	.125	.316	.0643	−.1904	May 1973	.510	.848	.762	−.2520	.0863
July 1966	.350	.000	.229	.1212	−.2288	June 1973	.510	.841	.793	−.2832	.0479
Aug. 1966	.410	.751	.430	−.0196	.3213	July 1973	.640	.185	.723	−.0829	−.5375
Sept. 1966	.400	.373	.412	−.0117	−.0390	Aug. 1973	.700	2.313	.929	−.2285	1.3845
Oct. 1966	.450	.248	.405	.0446	−.1579	Sept. 1973	.680	.090	.902	−.2217	−.8113
Nov. 1966	.400	.124	.350	.0496	−.2269	Oct. 1973	.650	1.084	.800	−.1503	.2837
Dec. 1966	.400	.000	.273	.1265	−.2735	Nov. 1973	.560	1.340	.976	−.4159	.3641
Jan. 1967	.430	.123	.324	.1057	−.2010	Dec. 1973	.640	1.146	.950	−.3100	.1960
Feb. 1967	.360	−.123	.172	.1876	−.2956	Jan. 1974	.630	1.482	1.069	−.4390	.4130
March 1967	.390	.000	.216	.1736	−.2164	Feb. 1974	.580	1.890	1.019	−.4388	.8712
April 1967	.320	.000	.165	.1548	−.1652	March 1974	.560	1.433	1.241	−.6813	.1917
May 1967	.330	.370	.204	.1260	.1659	April 1974	.750	.582	1.004	−.2542	−.4223
June 1967	.270	.491	.313	−.0429	.1785	May 1974	.750	1.074	.974	−.2242	.0998
July 1967	.310	.245	.269	.0406	−.0249	June 1974	.600	.491	.706	−.1061	−.2155
Aug. 1967	.310	.366	.340	−.0300	.0259	July 1974	.700	.407	.889	−.1887	−.4819
Sept. 1967	.320	.122	.217	.1028	−.0957	Aug. 1974	.600	1.297	.764	−.1639	.5331
Oct. 1967	.390	.243	.339	.0508	−.0965	Sept. 1974	.810	1.120	.973	−.1634	.1466
Nov. 1967	.360	.363	.261	.0989	.1021	Oct. 1974	.510	.712	.777	−.2674	−.0654
Dec. 1967	.330	.362	.362	−.0318	.0001	Nov. 1974	.540	.786	.709	−.1691	.0764
Jan. 1968	.400	.481	.412	−.0121	.0687	Dec. 1974	.700	.624	.950	−.2504	−.3269
Feb. 1968	.390	.359	.420	−.0299	−.0610	Jan. 1975	.580	.155	.379	.2011	−.2240
March 1968	.380	.238	.333	.0474	−.0942	Feb. 1975	.430	.232	.514	−.0837	−.2817
April 1968	.430	.476	.457	−.0267	.0189	March 1975	.410	.077	.366	.0441	−.2887
May 1968	.450	.473	.419	.0311	.0545	April 1975	.440	.308	.449	−.0087	−.1403
June 1968	.430	.236	.427	.0031	−.1913	May 1975	.440	.308	.259	.1811	.0486
July 1968	.480	.235	.394	.0857	−.1593	June 1975	.410	.920	.469	−.0590	.4505
Aug. 1968	.420	.352	.402	.0183	−.0500	July 1975	.480	1.443	.626	−.1463	.8167
Sept. 1968	.430	.467	.391	.0390	.0763	Aug. 1975	.480	.449	.704	−.2243	−.2552
Oct. 1968	.440	.465	.502	−.0620	−.0369	Sept. 1975	.530	.000	.416	.1137	−.4163
Nov. 1968	.420	.347	.372	.0479	−.0249	Oct. 1975	.560	.671	.552	.0077	.1183
Dec. 1968	.430	.346	.450	−.0202	−.1042	Nov. 1975	.410	.222	.359	.0509	−.1370
Jan. 1969	.530	.345	.407	.1233	−.0619	Dec. 1975	.480	.517	.548	−.0676	−.0306
Feb. 1969	.460	.115	.370	.0902	−.2553	Jan. 1976	.470	.147	.435	.0355	−.2875
March 1969	.460	.458	.330	.1303	.1280	Feb. 1976	.340	−.514	.160	.1804	−.6732
April 1969	.530	.570	.556	−.0265	.0130	March 1976	.400	−.148	.172	.2283	−.3192
May 1969	.480	.453	.385	.0950	.0680	April 1976	.420	.148	.188	.2319	−.0404
June 1969	.510	.676	.548	−.0383	.1281	May 1976	.370	.590	.205	.1647	.3847
July 1969	.530	.336	.483	.0468	−.1473	June 1976	.430	.220	.365	.0649	−.1452
Aug. 1969	.500	.335	.440	.0603	−.1049	July 1976	.470	.293	.306	.1638	−.0136
Sept. 1969	.620	.556	.496	.1240	.0602	Aug. 1976	.420	.511	.333	.0871	.1777
Oct. 1969	.600	.221	.470	.1302	−.2486	Sept. 1976	.440	.363	.378	.0620	−.0152
Nov. 1969	.520	.773	.414	.1056	.3582	Oct. 1976	.410	.362	.398	.0118	−.0367
Dec. 1969	.640	.548	.662	−.0225	−.1149	Nov. 1976	.400	.216	.317	.0832	−.1007
Jan. 1970	.600	.436	.475	.1248	−.0395	Dec. 1976	.400	.216	.347	.0526	−.1317
Feb. 1970	.620	.434	.497	.1233	−.0629	Jan. 1977	.360	.430	.293	.0672	.1376
March 1970	.570	.000	.415	.1547	−.4153	Feb. 1977	.350	.857	.452	−.1019	.4052
April 1970	.500	.540	.363	.1367	.1767	March 1977	.380	.283	.439	−.0594	−.1561
May 1970	.530	.322	.462	.0684	−.1394	April 1977	.380	.494	.408	−.0282	.0862
June 1970	.580	.000	.325	.2545	−.3255	May 1977	.370	.422	.402	−.0321	.0195
July 1970	.520	.321	.281	.2393	.0405	June 1977	.400	.560	.458	−.0584	.1014
Aug. 1970	.530	.107	.281	.2492	−.1741	July 1977	.420	.278	.442	−.0218	−.1634
Sept. 1970	.540	.426	.356	.1842	.0706	Aug. 1977	.440	.486	.427	.0126	.0584
Oct. 1970	.460	.425	.302	.1580	.1266	Sept. 1977	.430	.000	.395	.0348	−.3952
Nov. 1970	.460	.211	.349	.1108	−.1378	Oct. 1977	.490	.207	.348	.1416	−.1412
Dec. 1970	.420	.317	.298	.1225	.0190	Nov. 1977	.500	.758	.472	.0281	.2862
Jan. 1971	.380	−.105	.207	.1735	−.3117	Dec. 1977	.490	.274	.434	.0559	−.1605
Feb. 1971	.330	.105	.160	.1695	−.0552	Jan. 1978	.490	.682	.525	−.0348	.1573
March 1971	.300	.316	.190	.1099	.1254	Feb. 1978	.460	.339	.475	−.0146	−.1358
April 1971	.280	.419	.241	.0395	.1788	March 1978	.530	.608	.521	.0092	.0869
May 1971	.290	.313	.258	.0321	.0553	April 1978	.540	1.208	.632	−.0921	.5759

Table 2 (continued)

Month/Year	$i(t)$	$\pi(t)$	$\hat{\pi}^e(t)$	$\hat{r}^e(t)$	$\hat{e}(t)$	Month/Year	$i(t)$	$\pi(t)$	$\hat{\pi}^e(t)$	$\hat{r}^e(t)$	$\hat{e}(t)$
May 1978	.510	1.061	.700	−.1901	.3609	June 1981	1.350	.099	.445	.9047	−.3461
June 1978	.540	.787	.764	−.2245	.0229	July 1981	1.240	.595	.486	.7537	.1083
July 1978	.560	.456	.659	−.0985	−.2028	Aug. 1981	1.280	.246	.412	.8682	−.1655
Aug. 1978	.560	.389	.563	−.0032	−.1743	Sept. 1981	1.240	.590	.627	.6131	−.0372
Sept. 1978	.620	.452	.564	.0558	−.1123	Oct. 1981	1.210	.049	.311	.8990	−.2621
Oct. 1978	.680	.900	.642	.0379	.2576	Nov. 1981	1.070	.244	.399	.6710	−.1549
Nov. 1978	.700	.637	.664	.0360	−.0271	Dec. 1981	.870	.244	.129	.7407	.1142
Dec. 1978	.780	.443	.663	.1170	−.2200	Jan. 1982	.800	.972	.421	.3792	.5510
Jan. 1979	.770	1.386	.764	.0059	.6219	Feb. 1982	.920	−.433	.172	.7475	−.6056
Feb. 1979	.730	1.305	.886	−.1563	.4187	March 1982	.980	−.145	.036	.9443	−.1807
March 1979	.810	.798	.936	−.1257	−.1382	April 1982	1.130	−.242	−.071	1.2007	−.1713
April 1979	.800	1.096	.858	−.0581	.2379	May 1982	1.060	.340	.084	.9758	.2554
May 1979	.820	1.023	.861	−.0413	.1617	June 1982	.960	1.161	.288	.7319	.9329
June 1979	.810	1.073	.904	−.0940	.1690	July 1982	1.050	.239	.520	.5301	−.2809
July 1979	.770	.767	.819	−.0489	−.0524	Aug. 1982	.760	−.143	.052	.7085	−.1946
Aug. 1979	.770	.878	.821	−.0508	.0569	Sept. 1982	.510	.191	.041	.4688	.1498
Sept. 1979	.830	.870	.808	.0222	.0623	Oct. 1982	.590	.477	.194	.3964	.2830
Oct. 1979	.870	.805	.781	.0890	.0241	Nov. 1982	.630	−.047	.054	.5762	−.1013
Nov. 1979	.990	.742	.778	.2123	−.0361	Dec. 1982	.670	−.285	−.156	.8257	−.1291
Dec. 1979	.950	.736	.751	.1986	−.0153	Jan. 1983	.690	.143	−.029	.7190	.1718
Jan. 1980	.800	1.068	.722	.0776	.3456	Feb. 1983	.620	−.285	−.100	.7200	−.1852
Feb. 1980	.890	.834	.907	−.0167	−.0724	March 1983	.630	.286	.044	.5859	.2419
March 1980	1.210	1.489	.993	.2167	.4957	April 1983	.710	.665	.202	.5080	.4634
April 1980	1.260	.435	.841	.4192	−.4060	May 1983	.690	.567	.249	.4407	.3173
May 1980	.810	.541	.504	.3059	.0370	June 1983	.670	−.047	.192	.4780	−.2389
June 1980	.610	.592	.734	−.1244	−.1424	July 1983	.740	.470	.266	.4739	.2036
July 1980	.530	.589	.654	−.1241	−.0655	Aug. 1983	.760	.327	.275	.4850	.0523
Aug. 1980	.640	1.117	.576	.0641	.5411	Sept. 1983	.760	.140	.232	.5280	−.0922
Sept. 1980	.750	1.105	.634	.1162	.4712	Oct. 1983	.780	.000	.240	.5396	−.2404
Oct. 1980	.950	.520	.575	.3748	−.0549	Nov. 1983	.700	−.279	.098	.6019	−.3773
Nov. 1980	.960	.776	.624	.3361	.1525	Dec. 1983	.730	.420	.181	.5489	.2389
Dec. 1980	1.310	.822	.765	.5446	.0564	Jan. 1984	.760	1.022	.393	.3673	.6293
Jan. 1981	1.040	.764	.762	.2778	.0019	Feb. 1984	.710	−.184	.261	.4486	−.4454
Feb. 1981	1.070	.708	.652	.4182	.0560	March 1984	.730	.323	.247	.4832	.0758
March 1981	1.210	1.004	1.112	.0977	−.1083	April 1984	.810	−.230	.225	.5853	−.4544
April 1981	1.080	.000	.422	.6578	−.4222	May 1984	.780	−.276	.038	.7423	−.3139
May 1981	1.150	.199	.658	.4916	−.4596						

NOTE: The means of $i(t)$, $\pi(t)$, $\hat{\pi}^e(t)$, and $\hat{e}(t)$ are .561, .442, .442, and −.0002, respectively. The variances of $i(t)$, $\pi(t)$, $\hat{\pi}^e(t)$, and $\hat{e}(t)$ are .059, .169, .063, and .0769, respectively. The autocorrelations of $i(t)$, $\pi(t)$, $\hat{\pi}^e t$, and $\hat{e}(t)$ are .920, .456, .847, and −.0053, respectively.

In particular, we wish to test whether the expectations $r^e(t)$ and $\pi^e(t)$, generated by the underlying vector autoregressive processes (1) and (2), are stable for our $\hat{\theta}$ estimates reported in Table 1. This test requires that complex nonlinear combinations of the parameter estimates yield characteristic roots that are all stable.

We define the following polynomials in the lag operator L:

$$\alpha(L) = .23192L - .77227L^2 + 1.4758L^3 - .99328L^4,$$

$$\beta(L) = .37327L + .12408L^2 + 1.0941L^3 - .88155L^4,$$

$$\gamma(L) = .39605L - .18591L^2 - .019142L^3 - .013813L^4,$$

$$\phi(L) = .66293L + .66047L^2 - 1.3512L^3 + 1.0228L^4,$$

$$\psi(L) = .38766L - .25976L^2 - .93821L^3 + .90838L^4,$$

and

$$\xi(L) = -.34229L + .22746L^2 + .044058L^3 + .029353L^4.$$

We then substitute $\pi(t) = \pi^e(t) + e(t)$ into (1) and (2) so that both are expressed only in terms of current and lagged values of $r^e(t)$ and $\pi^e(t)$, plus error terms. The basic expectation dynamics can then be expressed as

$$\begin{bmatrix} 1 - \phi(L) & -[\psi(L) + \xi(L)] \\ -\alpha(L) & 1 - [\beta(L) + \gamma(L)] \end{bmatrix} \begin{bmatrix} r^e(t) \\ \pi^e(t) \end{bmatrix} = \text{forcing terms}. \quad (9)$$

Stochastic stability of (9) necessitates that the characteristic polynomial

$$[1 - \phi(z)][1 - \beta(z) - \gamma(z)] - \alpha(z)[\psi(z) + \xi(z)] = 0 \quad (10)$$

have roots outside the unit circle. For the parameter values reported in Table 1, stability of the joint expectations process requires that the roots of

$$1 - 1.4319z - .09953z^2 + .7862z^3 - .37972z^4 + .1715z^5 - .0950z^6 + .0372z^7 + 0.156z^8 = 0 \quad (11)$$

all exceed 1 in absolute value. These roots, reported in Table 5, are all stable.

A similar stability test involves the state equation (3). Substituting $\pi^e(t) = \pi(t) + e(t)$ into (3), we derive the dynamics for $\pi^e(t)$, where now $i(t)$ is taken as an exogenous forcing function, as follows:

$$[1 - \beta(L) + \alpha(L) - \gamma(L)]\pi^e(t) = \text{forcing terms}. \quad (12)$$

Stochastic stability of (12) requires that the characteristic roots of the polynomial

$$1 - \beta(L) + \alpha(L) - \gamma(L) = 0 \quad (13)$$

all exceed 1 in absolute value. The roots of (13) are reported in Table 6, and all are stable.

Another crucial test for our estimated expected inflation

Table 3. Results From 200 Monte Carlo Simulations for the Kalman Filter Estimates

Month	$\sum_{\theta} \pi^e(t)$ / 200	Parameter uncertainty	Filter uncertainty	Standard deviation of the econometric uncertainty	Month	$\sum_{\theta} \pi^e(t)$ / 200	Parameter uncertainty	Filter uncertainty	Standard deviation of the econometric uncertainty
July 1964	.131639	.000003	.093637	.306005	Jan. 1971	.201393	.004055	.015780	.140836
Aug. 1964	.021405	.000378	.083571	.289740	Feb. 1971	.175674	.003046	.015780	.137207
Sept. 1964	.232814	.000515	.052197	.229591	March 1971	.191211	.003342	.015779	.138279
Oct. 1964	.077440	.000757	.043253	.209785	April 1971	.251434	.003312	.015779	.138171
Nov. 1964	.191361	.001071	.032278	.182619	May 1971	.257377	.003187	.015779	.137716
Dec. 1964	.123722	.000838	.028977	.172642	June 1971	.333310	.003252	.015678	.137952
Jan. 1965	.164278	.001674	.024237	.160968	July 1971	.282151	.003048	.015778	.137209
Feb. 1965	.104940	.001163	.022951	.155287	Aug. 1971	.337999	.003767	.015778	.139805
March 1965	.270470	.001532	.020879	.149701	Sept. 1971	.319762	.003745	.015778	.139724
April 1965	.195437	.001726	.020199	.148073	Oct. 1971	.304107	.005399	.015778	.145521
May 1965	.349573	.001884	.019047	.144678	Nov. 1971	.400258	.005121	.015777	.144564
June 1965	.480042	.007103	.018659	.160506	Dec. 1971	.297176	.004788	.015777	.143406
July 1965	.365508	.006414	.017970	.156154	Jan. 1972	.352425	.005047	.015777	.144304
Aug. 1965	.225979	.008591	.017733	.162246	Feb. 1972	.288755	.006210	.015777	.148279
Sept. 1965	.201613	.004603	.017320	.148066	March 1972	.380208	.007563	.015777	.152773
Oct. 1965	.191907	.004423	.017161	.146914	April 1972	.177620	.010629	.015777	.162498
Nov. 1965	.313475	.002992	.016895	.141022	May 1972	.304721	.007025	.015776	.151001
Dec. 1965	.282130	.001712	.016789	.136021	June 1972	.193158	.008225	.015776	.154923
Jan. 1966	.377563	.002056	.016608	.136615	July 1972	.330960	.004719	.015776	.143160
Feb. 1966	.429543	.003742	.016535	.142397	Aug. 1972	.269266	.004156	.015776	.141180
March 1966	.452222	.002697	.016409	.138225	Sept. 1972	.436570	.004427	.015776	.142136
April 1966	.437072	.002627	.016355	.137774	Oct. 1972	.437714	.003180	.015776	.137680
May 1966	.352250	.004407	.016266	.143781	Nov. 1972	.414894	.002997	.015776	.137015
June 1966	.326944	.004951	.016225	.145521	Dec. 1972	.425417	.002600	.015776	.135559
July 1966	.213174	.004017	.016161	.142049	Jan. 1973	.558694	.002813	.015776	.136340
Aug. 1966	.446566	.005407	.016130	.146754	Feb. 1973	.584902	.004825	.015776	.143528
Sept. 1966	.391919	.005349	.016083	.146398	March 1973	.743305	.006794	.015776	.150231
Oct. 1966	.413802	.006631	.016058	.150629	April 1973	.849875	.007149	.015775	.151409
Nov. 1966	.338381	.003454	.016023	.139559	May 1973	.765967	.005798	.015775	.146879
Dec. 1966	.272176	.008007	.016003	.154953	June 1973	.801272	.004718	.015775	.143156
Jan. 1967	.315040	.004078	.015976	.141613	July 1973	.708857	.010361	.015775	.161667
Feb. 1967	.172661	.007680	.015961	.153757	Aug. 1973	.983362	.039160	.105775	.234382
March 1967	.209102	.004310	.015940	.142304	Sept. 1973	.850679	.028443	.015775	.210282
April 1967	.168221	.004908	.015927	.144345	Oct. 1973	.847152	.025253	.015775	.202553
May 1967	.201652	.004456	.015912	.142715	Nov. 1973	.961991	.031770	.015775	.218049
June 1967	.321342	.005057	.015901	.144769	Dec. 1973	.965882	.017226	.015775	.181663
July 1967	.260209	.004817	.015889	.143894	Jan. 1974	1.078973	.027448	.015775	.207902
Aug. 1967	.354807	.003911	.015880	.140679	Feb. 1974	1.055253	.019452	.015775	.187687
Sept. 1967	.209501	.003533	.015870	.139294	March 1974	1.228695	.026159	.015775	.204779
Oct. 1967	.348532	.003794	.015863	.140204	April 1974	1.014496	.026828	.015775	.206406
Nov. 1967	.261725	.002853	.015855	.136774	May 1974	.988371	.017411	.015775	.182171
Dec. 1967	.366155	.002502	.015849	.135467	June 1974	.711190	.016189	.015775	.178783
Jan. 1968	.412265	.002835	.015842	.136665	July 1974	.869816	.012150	.015775	.167106
Feb. 1968	.424413	.002479	.015837	.135339	Aug. 1974	.796295	.015466	.015775	.176751
March 1968	.333501	.002371	.015832	.134919	Sept. 1974	.954646	.010882	.015775	.163271
April 1968	.462387	.002476	.015828	.135294	Oct. 1974	.800807	.014681	.015775	.174517
May 1968	.417502	.002297	.015824	.134613	Nov. 1974	.691141	.015687	.015775	.177375
June 1968	.427862	.003090	.015820	.137513	Dec. 1974	.956826	.012617	.015775	.168498
July 1968	.393096	.002663	.015816	.135941	Jan. 1975	.371898	.016067	.015775	.178443
Aug. 1968	.406695	.002956	.015814	.137002	Feb. 1975	.528631	.011186	.015775	.164197
Sept. 1968	.390811	.003034	.015811	.137274	March 1975	.344435	.010840	.105775	.163141
Oct. 1968	.505491	.002687	.015808	.135995	April 1975	.456692	.011197	.105775	.164229
Nov. 1968	.373272	.002279	.015806	.134480	May 1975	.261674	.008629	.015775	.156218
Dec. 1968	.455372	.002627	.015803	.135757	June 1975	.485385	.008703	.015775	.156454
Jan. 1969	.405788	.002746	.015801	.136188	July 1975	.631167	.026754	.015775	.206224
Feb. 1969	.372005	.004271	.015799	.141669	Aug. 1975	.691944	.011649	.015775	.165602
March 1969	.336123	.004378	.015798	.142040	Sept. 1975	.419884	.015026	.015775	.175502
April 1969	.555862	.003935	.015796	.140468	Oct. 1975	.568815	.008529	.015775	.155897
May 1969	.391042	.002893	.015795	.136702	Nov. 1975	.350230	.008723	.015775	.156516
June 1969	.558338	.003060	.015793	.137308	Dec. 1975	.556091	.006723	.015775	.149991
July 1969	.480662	.002108	.015792	.133791	Jan. 1976	.429548	.005613	.015775	.146244
Aug. 1969	.447825	.003148	.015791	.137617	Feb. 1976	.157377	.023609	.015775	.198454
Sept. 1969	.502701	.003750	.015790	.139785	March 1976	.171241	.008231	.015775	.154938
Oct. 1969	.470417	.004343	.015789	.141888	April 1976	.186770	.011615	.015775	.165496
Nov. 1969	.433698	.006022	.015788	.147683	May 1976	.210665	.011210	.015774	.164268
Dec. 1969	.654710	.003904	.015787	.140323	June 1976	.356741	.006427	.015774	.149002
Jan. 1970	.487577	.002749	.015786	.136144	July 1976	.307489	.005152	.015774	.144660
Feb. 1970	.503435	.002897	.015785	.136684	Aug. 1976	.342123	.003775	.015774	.139820
March 1970	.414428	.004010	.015785	.140692	Sept. 1976	.372251	.003294	.015774	.138089
April 1970	.377556	.003358	.015784	.138356	Oct. 1976	.406500	.002671	.015774	.135814
May 1970	.459002	.003155	.015783	.137619	Nov. 1976	.317764	.002756	.015774	.136127
June 1970	.330371	.006192	.015783	.148238	Dec. 1976	.357940	.003526	.015774	.138928
July 1970	.296244	.003083	.015782	.137353	Jan. 1977	.299603	.002945	.015774	.136821
Aug. 1970	.277078	.003901	.015782	.140297	Feb. 1977	.467422	.004549	.015774	.142560
Sept. 1970	.363102	.002959	.015781	.136895	March 1977	.428932	.004097	.015774	.140967
Oct. 1970	.309036	.002964	.015781	.136912	April 1977	.427606	.003755	.015774	.139748
Nov. 1970	.349346	.002757	.015781	.136154	May 1977	.398111	.003313	.015774	.138156
Dec. 1970	.310305	.002097	.015780	.133707	June 1977	.468643	.002634	.015774	.135679

Table 3 (continued)

Month	$\sum_\theta \hat{\pi}^e(t)$ / 200	Parameter uncertainty	Filter uncertainty	Standard deviation of the econometric uncertainty	Month	$\sum_\theta \hat{\pi}^e(t)$ / 200	Parameter uncertainty	Filter uncertainty	Standard deviation of the econometric uncertainty
July 1977	.432242	.002812	.015774	.136333	Jan. 1981	.782313	.024367	.015774	.200354
Aug. 1977	.442687	.003437	.015774	.138604	Feb. 1981	.646986	.022546	.015774	.195757
Sept. 1977	.377332	.003854	.015774	.140101	March 1981	1.130567	.025621	.015774	.203457
Oct. 1977	.360030	.007978	.015774	.154117	April 1981	.420064	.025759	.015774	.203798
Nov. 1977	.470847	.008754	.015774	.156616	May 1981	.681751	.020248	.015774	.189795
Dec. 1977	.431026	.006051	.015774	.147735	June 1981	.440380	.019218	.015774	.187062
Jan. 1978	.528069	.004783	.015774	.143377	July 1981	.511838	.019017	.015774	.186523
Feb. 1978	.473220	.002900	.015774	.136655	Aug. 1981	.407620	.023193	.015774	.197403
March 1978	.523630	.002721	.015774	.135997	Sept. 1981	.645894	.020885	.015774	.191465
April 1978	.654615	.009584	.015774	.159244	Oct. 1981	.309660	.019470	.015774	.187734
May 1978	.698524	.008117	.015774	.154567	Nov. 1981	.429655	.020797	.015774	.191237
June 1978	.774900	.005243	.015774	.144974	Dec. 1981	.139263	.021625	.015774	.193389
July 1978	.657250	.005147	.015774	.144643	Jan. 1982	.460011	.016248	.015774	.178947
Aug. 1978	.576595	.004228	.015774	.141430	Feb. 1982	.160507	.021514	.015774	.193101
Sept. 1978	.564464	.003588	.015774	.139150	March 1982	.089540	.027168	.015774	.207225
Oct. 1978	.658797	.004527	.015774	.142484	April 1982	−.071015	.024849	.015774	.201552
Nov. 1978	.658264	.003863	.015774	.140133	May 1982	.119616	.018390	.015774	.184836
Dec. 1978	.668566	.005195	.015774	.144809	June 1982	.235169	.059178	.015774	.273774
Jan. 1979	.783193	.013509	.015774	.171123	July 1982	.524152	.018616	.015774	.185446
Feb. 1979	.889885	.010337	.015774	.161591	Aug. 1982	.067334	.021337	.015774	.192642
March 1979	.934237	.007282	.015774	.151843	Sept. 1982	.076071	.016365	.015774	.179274
April 1979	.885449	.007578	.015774	.152814	Oct. 1982	.194601	.024952	.015774	.201808
May 1979	.869079	.005695	.015774	.146523	Nov. 1982	.077405	.018542	.015774	.185248
June 1979	.923886	.005090	.015774	.144443	Dec. 1982	−.143560	.015948	.015774	.178107
July 1979	.822160	.005174	.015774	.144734	Jan. 1983	−.005608	.009571	.015774	.159202
Aug. 1979	.842537	.004149	.015774	.141151	Feb. 1983	−.126137	.012138	.015774	.167069
Sept. 1979	.814703	.004711	.015774	.143128	March 1983	.063916	.006654	.015774	.149759
Oct. 1979	.798863	.003766	.015774	.139785	April 1983	.192518	.020751	.015774	.191117
Nov. 1979	.782230	.005444	.015774	.145667	May 1983	.266590	.007191	.015774	.15[illegible]543
Dec. 1979	.766099	.005811	.015774	.146921	June 1983	.170954	.012638	.015774	.168561
Jan. 1980	.731874	.010159	.015774	.161039	July 1983	.287377	.007963	.015774	.154069
Feb. 1980	.907036	.005506	.015774	.145876	Aug. 1983	.253636	.013691	.015774	.171655
March 1980	1.019419	.014157	.015774	.173006	Sept. 1983	.245478	.008167	.015774	.154730
April 1980	.846046	.014801	.015774	.174858	Oct. 1983	.219928	.008529	.015774	.155895
May 1980	.532216	.024310	.015774	.200211	Nov. 1983	.102955	.012004	.015774	.166670
June 1980	.719212	.017863	.015774	.183404	Dec. 1983	.171813	.010257	.015774	.161341
July 1980	.665867	.019208	.015774	.187035	Jan. 1984	.409875	.013266	.015774	.170413
Aug. 1980	.612224	.020003	.015774	.189149	Feb. 1984	.223426	.013910	.015774	.172291
Sept. 1980	.671003	.016044	.015774	.178378	March 1984	.274642	.009861	.015774	.160111
Oct. 1980	.575305	.018028	.015774	.183854	April 1984	.185243	.009202	.015774	.158038
Nov. 1980	.643496	.017944	.015774	.183625	May 1984	.053552	.014439	.015774	.173821
Dec. 1980	.752406	.017718	.015774	.183009					

NOTE: Parameter uncertainty = $\text{var}_\theta \hat{\pi}^e(t)$. Filter uncertainty = $[\Sigma_\theta \text{ var } \hat{\pi}^e(t)]/200 = [\Sigma_\theta P_{11}(t)]/200$. Standard deviation of the econometric uncertainty = the square root of the sums of columns 3 and 4.

series is to determine whether the forecast error $\hat{e}(t) = \pi(t) - \hat{\pi}^e(t)$ can be predicted either by lagged values of itself or by other economic variables known at time t. To examine these questions, we ran ordinary least squares of the form

$$\hat{e}(t) = \beta_0 + \beta_1 x(t-1) + \beta_2 x(t-2) + \beta_3 x(t-3) + \beta_4 x(t-4) + \varepsilon(t) \quad (14)$$

for alternative choices of the variable x. The critical .05 value of $F(4, \infty)$ is 2.37. Thus for every case reported in Table 7, we cannot reject the null hypothesis that $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$; we have not found any economic variables that will help forecast unexpected inflation.

We also performed an alternative test to verify that $\hat{e}(t)$ is mean zero: an ordinary least squares regression on a constant yielded a value of −.00032 with a t statistic of −.0179 and a Durbin–Watson (DW) statistic of 2.01.

Finally, we ran an ordinary least squares regression of actual $\pi(t)$ on a constant, $\pi(t-1), \ldots, \pi(t-4)$, and $i(t-1), \ldots, i(t-4)$ for the same sample period used to estimate $\hat{\pi}(t)$. If agents use information in our model efficiently, the standard error for this regression should be greater than our estimated value of $\hat{\sigma}_e$. In fact, the standard error of the latter regression is .355, compared with our value of $\hat{\sigma}_e = .304$, a 65% reduction.

Hence we see that our estimated $\hat{\pi}^e(t)$ series does not contradict the assertion that inflationary expectations are unbiased, rational, and efficient.

8. AN APPLICATION OF THE ESTIMATED EXPECTED INFLATION SERIES TO ASSET PRICING

The Ross (1976) APT predicts that

$$\rho_t = E\rho_t + bf_t + \varepsilon_t, \quad (15)$$

where ρ_t = nominal return on an asset; $E\rho_t$ = expected return on an asset; $f_t = (f_{1t}, \ldots, f_{nt})'$ = actual values of systematic factors influencing the return on the asset; $b = (b_1, \ldots, b_n)$ = the asset's sensitivity to a change in the systematic factors; and ε_t = the realization of the unsystematic, idiosyncratic factors.

We employed four macroeconomic factors, defined as

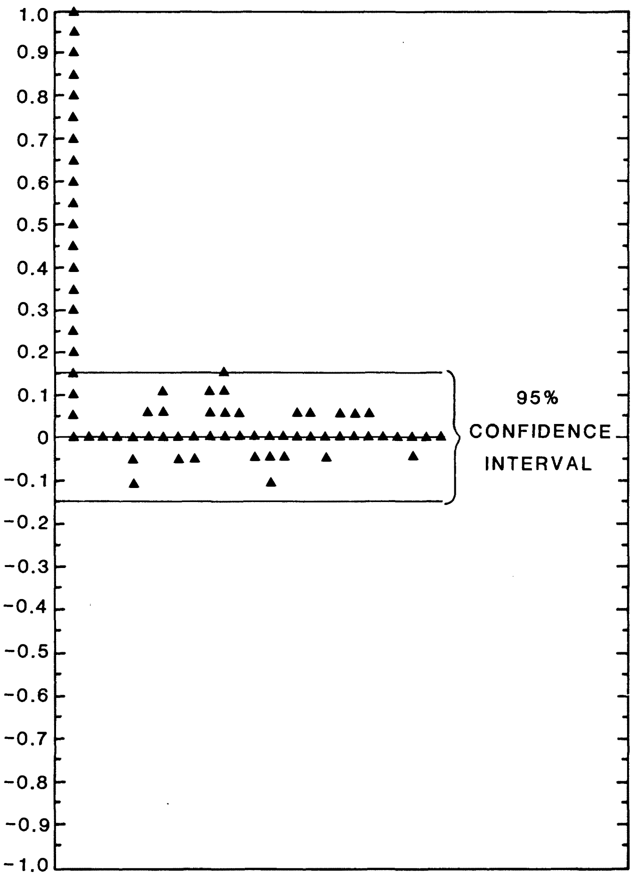

Figure 1. Correlogram for Estimated Expected Inflation Forecast Error.

follows [all data are from Ibbotson Associates (1984) and Ibbotson and Sinquefield (1982)]: f_{1t} = $UPRIS_t$ = unanticipated change in risk premium measured as the return on corporate bonds in month t minus return on government bonds in month t; f_{2t} = UTS_t = unanticipated change in the term structure measured as the return on government bonds in month t minus return on Treasury bills in month $t - 1$; f_{3t} = UI_t = unanticipated inflation measured as $\pi(t) - \hat{\pi}^e(t)$ (from Table 2); and f_{4t} = DEI_t = the difference in expected inflation measured as $\hat{\pi}^e(t + 1) - \hat{\pi}^e(t)$ (from Table 2). An economic justification for these factor measures is provided in Chen, Roll, and Ross (1983), Roll and Ross (1984), and Burmeister and Wall (1986).

To demonstrate that our inflationary expectations work has a useful application, we computed an ordinary least squares regression for the equation

$$\text{RSPI}_t = \beta_0 + \beta_1 UPRIS_t + \beta_2 UTS_t + \beta_3 UI_t + \beta_4 DEI_t + \varepsilon_t \quad (16)$$

over a sample from July 1964 to May 1984, where $RSPI_t$ denotes the return on the Standard & Poor's (S&P) 500 stock index in month t. [The use of our estimated expected inflation series in variables on the right side of Equation (16) entails a partial equilibrium methodology and should be viewed with the usual caution. Of course, unobserved expectations and observed realizations are best estimated jointly in a general equilibrium framework. When one does not know the correct general equilibrium model, however—which is certainly the case for stock market returns—the only feasible alternative is a partial equilibrium specification.] We suppose that $RSPI_t = \rho_t$ is the return on a portfolio, and the Ross APT implies that there exist $(\lambda_0, \lambda_1, \ldots, \lambda_4)$ such that

$$E\rho_t \doteq \lambda_0 + \sum_{i=1}^{4} b_i \lambda_i,$$

where b_i = sensitivity of the S&P index to factor i and λ_i is the risk premium for factor i. Substituting the latter into

$$\rho_t = E\rho_t + \sum_{i=1}^{4} b_i f_{it} + \varepsilon_t$$

gives

$$\rho_t = \lambda_0 + \sum_{i=1}^{4} b_i \lambda_i + \sum_{i=1}^{4} b_i f_{it} + \varepsilon_t,$$

which is exactly the form of the preceding regression equation.

We obtained the following results (for which the t statistics are reported in parentheses):

$$\hat{\beta}_0 = .0079 \ (3.28) \quad \hat{\beta}_1 = 1.05 \ (5.44) \quad \hat{\beta}_2 = -.412 \ (-2.18)$$

$$\hat{\beta}_3 = -1.99 \ (-2.22) \quad \hat{\beta}_4 = 4.37 \ (2.37)$$

$$R^2 = .22 \quad F(4, 234) = 18.0 \quad DW = 1.93.$$

Both $\hat{\beta}_3$ and $\hat{\beta}_4$ are significant, a strong indication that our estimated expected inflation series has useful economic content (assuming that the APT is correct). Moreover, we conclude that over the July 1964–May 1984 monthly sample period, unanticipated inflation has a negative impact on stock market returns. On the other hand, the difference in expected

Table 4. Autocorrelations and Q Statistics for Inflation Forecast Errors

Lags	Autocorrelations	Standard errors	Q statistics
1	−.528E-02	.647E-01	.672E-02
2	.894E-02	.647E-01	.261E-01
3	−.207E-01	.647E-01	.130
4	−.863E-01	.647E-01	1.94
5	.334E-01	.652E-01	2.21
6	.918E-01	.653E-01	4.28
7	−.715E-01	.658E-01	5.54
8	−.355E-01	.661E-01	5.86
9	.121	.662E-01	9.44
10	.138	.671E-01	14.2
11	.427E-01	.683E-01	14.7
12	−.685E-01	.684E-01	15.9
13	−.114	.687E-01	19.1
14	−.254E-01	.695E-01	19.3
15	.274E-01	.695E-01	19.6
16	.472E-01	.696E-01	20.2
17	−.568E-01	.697E-01	21.0
18	.332E-01	.699E-01	21.3
19	.413E-01	.700E-01	21.8
20	.636E-01	.701E-01	22.9

Table 5. Characteristic Roots of the Polynomial (11) for the Estimated Parameter Values $\hat{\theta}$

Root	Absolute value
$\lambda_1 = 1.038 + .03545i$	1.0386
$\lambda_2 = 1.038 - .03545i$	1.0386
$\lambda_3 = -1.155$	1.155
$\lambda_4 = 1.195 + 1.156i$	1.6626
$\lambda_5 = 1.195 - 1.156i$	1.6626
$\lambda_6 = -.5530 + 1.935i$	2.012
$\lambda_7 = -.5530 - 1.935i$	2.012
$\lambda_8 = -4.590$	4.590

inflation, which in this model can be interpreted as an unanticipated expectation that the level of inflation will rise, has a positive impact on nominal stock market returns. A more complete examination of the APT, using the expected inflation series derived here, is made in Burmeister and Wall (1986), and a new method for jointly estimating the b's and the λ's is presented in McElroy, Burmeister, and Wall (1985).

9. CONCLUSION

We have only scratched the surface of the research opportunities opened by these new techniques. In particular, in future work we intend to estimate other expected inflation series by using two alternative measures of the price level: the consumer price index (CPI) and a constructed monthly implicit deflator for the gross national product. It will be interesting to learn how these expected inflation measures compare with the one given here and, most important, to see whether conclusions such as those in Hamilton (1985) about forecast errors and business cycles are robust with respect to these different measures of the price level.

A second important task for future work is to examine the stability of our parameter estimates over time. There is some evidence that financial markets underwent a structural change around October 1979, and we plan to test whether the parameter values agents implicitly use to forecast inflation also shifted around this date.

Finally, there are a host of interesting economic questions that can be investigated with our expected inflation series: What is the relationship between the level of inflation and the variance of forecast errors? Does the latter relationship have anything to do with business cycles? Why have recent real interest rates been so high? How do inflationary expectations influence long-term interest rates, and does the variance of forecast errors play any role in determining the term structure of interest rates? We hope to address some of these issues, as well as others, in our future research.

Table 6. Characteristic Roots of the Polynomial (14) for the Estimated Parameter Values $\hat{\theta}$

Root	Absolute value
$\lambda_1 = 1.048$	1.048
$\lambda_2 = -1.118$	1.118
$\lambda_3 = 2.082 - 2.094i$	2.95
$\lambda_4 = 2.082 + 2.094i$	2.95

Table 7. Tests on the Estimated Inflation Forecast Error

Definition of the variable x(t)	$\bar{R}^2$	F(4, 234)	DW
$\hat{e}(t)$	−.0086	.49	1.99
Return on government bonds in month t minus return on Treasury bills in month $t-1$	.0017	1.10	2.02
Return on government bonds in month t minus return on corporate bonds in month t	.0014	1.87	2.00
Rate on 4–6-month commercial paper in month t	−.014	.176	2.00
Return on Treasury bills in month t	−.015	.103	2.00
$\pi(t)$	−.015	.14	2.02
Inflation in month t as measured by the CPI	.0023	.135	1.96
Return on the Standard & Poor's 500 stock index in month t	.014	1.87	1.98
Return on corporate bonds in month t	.014	1.85	2.00
Return on government bonds in month t	.0017	1.10	2.02
Trade-weighted value of the U.S. dollar in month t	−.0035	.887*	1.97

*This value is $F(4, 124)$ because of data limitations.

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APPENDIX A: IDENTIFICATION OF THE STATE-SPACE MODEL

Our state-space model, (6), possesses a structure that is a special case of that considered in Wall (1984), and its identification can be established by the theory developed there. Specifically, we make use of Theorem 2 of that paper, with minor modifications.

Proposition 1. Let $\{F, G, Q, H, D, R\}$ be as specified in (6). If $\psi_4 - \phi_4$ and $\beta_4 - \alpha_4$ are nonzero and if at least one element of the first row of G is nonzero, then the state-space model is minimal.

Proof. Consider first the controllability condition. The state-space model is completely controllable iff the rank of $[G\ FG\ F^2G \cdots F^4G]$ is 5. The companion form of F and the special structure of G ensure that one will always be able to find embedded in the controllability matrix a 5×5 upper triangular submatrix. This can be seen by straightforward but tedious calculation. For example, suppose that g_{11} is nonzero; then we could form an upper triangular submatrix from the first columns from each of G, FG, F^2G, F^3G, and F^4G. This submatrix would have g_{11} down the diagonal and hence be of rank 5. If $g_{11} = 0$, then we could repeat the exercise by using g_{12}. We are guaranteed that we can form at least one such full-rank triangular submatrix because of the assumption in the proposition.

Next consider the observability condition. Here we desire the rank of observability matrix $[H' \; F'H' \; (F^2)'H' \cdots (F^4)'H']$ to be 5. Once again the companion form of F, the special form of H, and the assumptions of the proposition yield observability. This follows from the fact that F and the second row of H form an observable pair for a fourth-order state-space model when we drop from consideration the last column of F and the last element of the second row of H. Thus working with just the second row of H, HF, HF^2, HF^3, and HF^4, we can always form a submatrix of rank 4. Indeed, it will again be triangular with 1 in the first position and $\beta_4 - \alpha_4$ down the diagonal. Combining this submatrix with the first row of H and the assumption in the proposition concerning $\psi_4 - \phi_4$ guarantees that we always can find five linearly independent columns in the observability matrix.

Proposition 2. Let the conditions of Proposition 1 hold. Then the state-space model (6) is identified.

Proof. For minimal state-space models, the form of the equivalence relation that leads to lack of identification is known to be represented by a nonsingular coordinate transformation in the state space (see Wall 1984 and the references cited therein). Specifically, if $\{F_1, G_1, Q_1, H_1, D_1, R_1\}$ and $\{F_2, G_2, Q_2, H_2, D_2, R_2\}$ are any two members of an equivalence class, then they are related by

$$F_2 = T^{-1}F_1T, \quad G_2 = T^{-1}G_1, \quad Q_2 = Q_1,$$

$$H_2 = H_1T, \quad D_2 = D_1, \quad R_2 = R_1.$$

Identification obtains whenever there are enough restrictions in the specification that the only matrix T consistent with these and satisfying the above equations is $T = I$. In this way we will be assured of selecting one and only one representative from each equivalence class. We now show that there are enough restrictions in (6) to force $T = I$.

Consider first the relationship between F_1 and F_2. We have T as the solution to the matrix equation $TF_2 - F_1T = 0$. It can be shown (see Lukes, in press or Wall 1984) that all solutions to the foregoing equation, with F as specified in (6), must be of the form

$$T = [c, \; -p_1(F)c, \; -p_2(F_1)c, \; -p_3(F_1)c, \; -p_4(F_1)c],$$

where $c \in N$ with N the null space of $p_5(F_1)$ and $p_n(F)$ the characteristic polynomial of the respective $n \times n$ submatrix of F (where $n = 1$ corresponds to the upper left corner 1×1 submatrix defined by $\beta_1 - \alpha_1$).

Now consider the relationship between G_1 and G_2. T must satisfy the matrix equation $TG_2 = G_1$. From the assumptions of the proposition, at least one element of the first row of G is nonzero, so without loss of generality let us assume that it is α_1. The first columns on each side of the matrix equation must satisfy

$$\begin{bmatrix} \alpha_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [c, \; -p_1(F_1)c, \; -p_2(F_1)c, \; -p_3(F_1)c, \; -p_4(F_1)c] \begin{bmatrix} \alpha_1' \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where α_1' denotes the g_{11} element of G_2 and α_1 denotes the g_{11} element of G_1. It is easy to see that comparison of the first elements on both sides of this equation requires c_1 to be nonzero, whereas all of the subsequent comparisons require $c_2 = c_3 = c_4 = c_5 = 0$. Thus the vector c has all elements zero except the first.

Finally, let us consider the relationship between H_1 and H_2, stipulated by the matrix equation $H_2 = H_1T$. From the second row of this equation we find that

$$\begin{aligned}[1 \;\; 0 \;\; 0 \;\; 0 \;\; 0] &= [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]T \\ &= [c_1 \;\; * \;\; * \;\; * \;\; *],\end{aligned}$$

where the asterisks denote generally nonzero entries. Hence $c_1 = 1$, and we conclude that the first column of T is the first column of the identity matrix. (We also conclude that the first row of T is the first row of an identity matrix, but this is not needed for the proof.) Insertion of this result into each of the remaining four columns of T reveals that each of these is, in turn, the corresponding column of a 5×5 identity matrix. Thus $T = I$ and the model specification is identified.

APPENDIX B: DESCRIPTION OF THE KALMAN-FILTERING ALGORITHM AND UNOBSERVED VARIABLE ESTIMATION

For expositional convenience first consider the problem of estimating $x(t)$ given the parameters of F, G, Γ, H, and D, together with the first two moments of $y(t)$ and $z(t)$. If $\hat{x}(t, \tau)$ denotes the minimum mean squared error estimate of $x(t)$ given the model and all observed data up through time τ,

$$Y^\tau = \{y(1), y(2), \ldots, y(\tau)\},$$

$$Z^\tau = \{z(1), z(2), \ldots, z(\tau)\},$$

then $\hat{x}(t, t)$ is produced by the following recursive computation:

$$\hat{x}(t + 1, t) = F\hat{x}(t, t) + Gz(t), \tag{B.1}$$

$$P(t + 1, t) = FP(t, t)F' + \Gamma Q \Gamma', \tag{B.2}$$

$$B(t + 1, t) = HP(t + 1, t)H' + R, \tag{B.3}$$

$$\hat{\varepsilon}(t + 1, t) = y(t + 1) - H\hat{x}(t + 1, t) - Dz(t + 1), \tag{B.4}$$

$$K(t + 1) = P(t + 1, t)H'B^{-1}(t + 1, t), \tag{B.5}$$

$$\hat{x}(t + 1, t + 1) = \hat{x}(t + 1, t) + K(t + 1)\hat{\varepsilon}(t + 1, t), \tag{B.6}$$

$$P(t + 1, t + 1) = [I - K(t + 1)H]P(t + 1, t), \tag{B.7}$$

for $t_0 \leq t \leq T$. $P(t + 1, t)$ is the var–cov matrix of the estimation matrix error in $\hat{x}(t + 1, t)$; that is,

$$P(t + 1, t) = E\{[x(t + 1) - \hat{x}(t + 1, t)] \times [x(t + 1) - \hat{x}(t + 1, t)]'\}.$$

$B(t+1, t)$ is the var–cov matrix of the innovation; that is,

$$B(t+1, t) = E\{\hat{\varepsilon}(t+1, t)\hat{\varepsilon}(t+1, t)'\}.$$

The initial values for $\hat{x}(t, t)$ and $P(t, t)$ are assumed to be known and are given by

$$\hat{x}(t_0, t_0) = \hat{x}(0) = E\{x(t_0) \text{ given all information at time } t_0\},$$

$$P(t_0, t_0) = P(0) = E\{[x(t_0) - \hat{x}(0)][x(t_0) - \hat{x}(0)]'\}.$$

Thus $P(t, \tau)$ is the var–cov matrix of the error in estimating $x(t)$ given all observations up through time $\tau \leq t$. The vector $\hat{\varepsilon}(t+1, t)$ represents the innovations process and is analogous to the model residuals used in econometric estimation. Equations (B.1)–(B.7) constitute the Kalman filter.

More efficient estimates of the states can be obtained by using all of the sample information available, that is, $\hat{x}(t, T)$. This is referred to as the smoothed estimate. It is derived from the filtered estimate, $\hat{x}(t, t)$, by means of a reverse "sweep" over the data from T back to $t + 1$. Broadly speaking, computation is as follows: the recursive Kalman filter is employed in reverse time "beginning" at time T using a diffuse "prior" for $\hat{x}(T, t+1)$, that is, $P(T, T+1) = \infty$. For any time t in the closed interval $[0, T]$, this reverse time filter produces an estimate, $\hat{x}(t, t+1)$, along with its corresponding var–cov matrix, $P(t, t+1)$. This represents our best estimate of $x(t)$, using data only over the interval $[t+1, T]$. Combining this with our forward time estimate, $\hat{x}(t, t)$, using only data over the interval $[0, t]$, gives us the desired result $\hat{x}(t, T)$. The method of combination follows from a classical result in probability and statistics; namely, the optimal combination of two independent estimates $\hat{x}(t, t)$ [with precision matrix $P^{-1}(t, t)$] and $\hat{x}(t, t+1)$ [with precision matrix $P^{-1}(t, t+1)$] is

$$\hat{x}(t, T) = P(t, T)[P^{-1}(t, t)\hat{x}(t, t) + P^{-1}(t, t+1)\hat{x}(t, t+1)],$$

with corresponding precision matrix

$$P^{-1}(t, T) = P^{-1}(t, t) + P^{-1}(t, t+1).$$

Details of the smoothing algorithms are given in Cooley, Rosenberg, and Wall (1977). Thus once filtered estimates are obtained, they can be revised by the smoothing algorithm to produce the most efficient estimates of $x(t)$. [It can be shown that $P(t, t) \geq P(t, T)$ for $t_0 \leq t \leq T$. This should be intuitively clear, since by definition $\hat{x}(t, T)$ uses more information than $x(t, t)$. See Jazwinski (1970, chap. 7) or Bryson and Ho (1969, chap. 13).]

Using the Kalman filter to generate model residuals enables the formation of a loss function that can be used in parameter estimation. The parameters to be estimated may include not only the unknown elements of H, D, and R (the parameters of the behavioral equations) but, more important, those of F, G, Γ, and Q. The algorithms for estimation of the unknowns in this manner are called *prediction error* methods and, like the Kalman filter, are thoroughly treated in the control literature (see Ljung and Sorderstrom 1983).

The algorithm employed in the present study is outlined by the following steps:

1. Collect the unknown parameters into a vector θ of dimension $N \times 1$. Denote an initial guess at its true value by θ^0, and insert this into the Kalman filter Equations (B.1)–(B.7). Set $i = 0$.
2. Using the Kalman filter Equations (B.1)–(B.7), compute the model innovations sequence $\{\hat{\varepsilon}(t+1, t); t_0 \leq t \leq T-1\}$, where $\hat{\varepsilon}(t+1, t) = \hat{\varepsilon}(t+1, t, \theta^i)$ is an implicit function of θ^i.
3. Form the loss function $J(\theta^i)$, where

$$J(\theta^i) = \frac{1}{2}\sum_{t_0}^{T-1} [\hat{\varepsilon}(t+1, t)'\Lambda_{t+1|t}^{-1}\hat{\varepsilon}(t+1, t) + \ln(\det \lambda_{t+1|t})] \quad \text{(B.8)}$$

and $\Lambda_{t+1|t}$ is some positive definite weighting matrix.
4. Compute an improved estimate of θ, denoted by θ^{i+1}, such that $J(\theta^{i+1}) \leq J(\theta^i)$. Use

$$\theta^{i+1} = \theta^i - \rho^i M^{-1}\partial J('\theta^i)/\partial\theta,$$

where ρ^i is a (scalar) step size parameter and M_i^{-1} is a positive definite $N \times N$ matrix such that in the limit (as $i \to \infty$) it tends to the inverse Hessian of J. (See discussion after step 5.)
5. Check to see whether $\|\theta^{i+1} - \theta^i\| < \delta_1$ and/or $\|\partial J(\theta^{i+1})/\partial\theta\| \leq \delta_2$. If so, stop; θ^{i+1} is accepted as the "best" estimate of θ. Otherwise, set θ^i to θ^{i+1}, $i = i + 1$, and return to step 2. If it is assumed that $\varepsilon(t)$ is normally distributed for each t and $\Lambda_{t+1|t}$ is set equal to $B(t+1, t)$, then approximate maximum likelihood estimates are obtained. The approximation involved relates to the prior on the unknown coefficient vector θ. The difference between our likelihood function and the exact likelihood function is of the order $1/T$. Thus asymptotically ($T \to \infty$) there is no difference. (See Pagan 1980 for a discussion of this fact.)

The iterative algorithm given earlier requires an initial estimate, θ^0; a convergence criterion, δ_1 and/or δ_2; and expressions for the components of the gradient vector $\partial J(\theta)/\partial\theta$. The gradients may be computed numerically, using simple finite first differences of $J(\theta)$, or analytically, using a straightforward application of differential calculus to (B.8). The method by which ρ^i and M_i are computed depends on the particular function minimization algorithm employed. A Davidson–Fletcher–Powell variable metric algorithm is used here, since M_i^{-1} is then computed automatically, with $\partial J(\theta)/\partial\theta$ being the only user-supplied information. On convergence M_i^{-1} is the inverse Hessian of $J(\theta)$ (i.e., the information matrix) and yields valuable information concerning estimated parameter standard errors, correlations (covariances), and identifiability. In particular, once the algorithm converges, a simple scaling of M_i^{-1} produces an estimate of the parameter var–cov matrix. This estimate is useful not only in hypothesis tests on elements of θ but also in examining identifiability. Since local identifiability and asymptotic nonsingularity of the Hessian matrix are equivalent, a nearly singular M_i^{-1} indicates identification problems. In practice, this is most easily tested by converting the parameter var–cov matrix to a correlation matrix and examining the off-diagonal elements. Interparameter correlations near unity, say ±.996, lead to a singular condition suggesting an overparameterized specification and lack of complete identification. Moreover, a singular Hessian for $J(\theta)$ results in nonconvergence of the numerical optimization algorithm, so lack of convergence and lack of identification are highly related.

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