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ESTIMATION OF VIBRATION FREQUENCIES OF LINEAR ELASTIC MEMBRANES

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Abstract. The free motion of a thin elastic linear membrane is described, in a simplyfied model, by a second order linear homogeneous hyperbolic system of partial differential equations whose spatial part is the Laplace Beltrami operator acting on a Riemannian 2-dimensional manifold with boundary. We adapt the estimates of the spectrum of the Laplacian obtained in the last years by several authors for compact closed Riemannian manifolds. To make so, we use the standard technique of the doubled manifold to transform a Riemannian manifold with nonempty boundary $(M, \partial M, g)$ to a compact Riemannian manifold $(M \sharp M, \widetilde{g})$ without boundary. An easy numerical investigation on a concrete semi-ellipsoidic membrane with clamped boundary tests the sharpness of the method.

Keywords: membrane; Laplacian; estimation of frequencies

MSC 2010: 74K15, 53C20, 53C21, 58C40

1. Introduction

The dynamics of a three dimensional elastic body is described by the well-known Navier-Stokes equations equipped with mixed Dirichlet-Neumann conditions on its boundary; this combination arises from the classical Signorini problem for elastic bodies. It is well known that a general description of the motion is well far from a complete and exhaustive description due to the high complexity of the problem. The construction in total safety of a civil or mechanical structure forces to give a suitable approximation of the problem, solved usually via a Finite Element Program; moreover, the knowledge of the proper frequencies of vibration is necessary to avoid any eventual resonance phenomenon, this is the reason why an easy determination of the same would be desirable in the design phase.

In this paper we furnish some estimate of the vibration frequencies of a linear elastic membrane using a geometric approach viewing the structural element as

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a 2-dimensional Riemannian manifold. In the last years several authors got estimates, from above and from below, of the eigenvalues of the Lapace Beltrami operator acting on a Sobolev space of functions having as the support a compact Riemannian manifold with empty boundary; for the sake of simplicity we shall write *acting on a Riemannian manifold*. In this paper we adapt these results to a vibrating linear elastic membrane. We recall here that the equations of motion of a linear elastic membrane form a system of hyperbolic PDE whose spatial part is strongly elliptic and positive definite; it admits a discrete sequence of nonnegative eigenvalues; this spatial part is the Laplace-Beltrami operator acting on a 2-dimensional Riemannian manifold. The standard technique of separation of variables puts in correspondence the eigenvalues of this operator with the free vibration frequencies. To reach the goal of this paper

1. we reduce the motion of the membrane $\mathcal{M}$ to that of its middle surface depicted as a two dimensional Riemannian manifold with nonempty boundary $(M, \partial M, g)$,
2. we use the standard technique of the "double" of a Riemannian manifold to view the model of the elastic membrane as part of a compact 2-dimensional Riemannian manifold $(M\sharp M, g\sharp g)$ with empty boundary,
3. we arrange the estimates of the eigenvalues of the Laplace Beltrami operator for a closed compact manifold to the vibrating membrane.

2. Dynamics of an elastic membrane

In this part, we summarize the principal and necessary concepts about the dynamics of a thin elastic membrane, necessary for the next developments, referring to the specific literature for a wide and exhaustive treatment (Antman [1]; Carrol and Naghdi [2]; Ericksen and Truesdell [4]; Naghdi [8]; Simo and Fox [10], [11]).

A membrane is a thin three-dimensional body with three peculiar characteristics:

1. it has curved shape in the space,
2. one dimension, in the following named the *thickness*, is smaller than the other two;
3. moreover, the resultant stress at a point m of the membrane is everywhere parallel to the tangent plane at m.

As a three dimensional body, each configuration of the membrane is the set $\mathcal{M} \subset \mathbb{R}^3$ defined as $\mathcal{M} = \mathcal{S} \times \xi\mathbf{n}$, $\mathcal{S}$ being the regular middle surface, $\mathbf{n}$ the unit normal field to $\mathcal{S}$ and $\xi \in [h^-, h^+]$, $h^+ - h^- = h$ the thickness. The motion of a membrane is drawn by that of its middle surface and by the variation of the thickness; the first one is naturally described by its deformation in the ambient space, the description of motion along the thickness can be very complicated but, because of the previous assumptions 1, 2 and 3 and technical reasons allow to neglect it if $h \leqslant \frac{1}{20}D$, D being the minimal dimension between the other two. For the purposes of this paper two further assumptions are necessary:

4. the motion of the membrane is small, i.e., two different configurations of the membrane are "close" to each other, this assumption allows to interchange the two configurations and assume that the balance is achieved in the undeformed one;
5. the relation between stress and strain is linear.

According to the previous assumptions, we refer the motion to the undeformed configuration, taken as *reference configuration*, $\mathcal{S}_0$. Let $\Omega \subset \mathbb{R}^2$ be a a simply connected closed domain of $\mathbb{R}^2$, the parametrization of this configuration is the C^2 map:

$$\varphi_0 \colon \Omega \to \mathbb{R}^3.$$

Setting z^i, $i = 1, 2, 3$, as the coordinates in $\mathbb{R}^3$ and taking x^α, $\alpha = 1, 2$, as the coordinates of a point $p \in \Omega$, the parametrization of the surface in the reference configuration $\mathcal{S}_0$, is given by the functions

$$z^i = \varphi_0^i(x^\alpha).$$

The motion of the membrane is thus given by the map $\mathbf{u} \colon \mathcal{S}_0 \to \mathbb{R}^3$, i.e.

$$u^i = u^i(z^j).$$

The deformation is described by the Lagrange tensor of small deformations:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\sum_{i=1}^{3}\Big(\frac{\varphi_0^i}{\partial x^\alpha}\frac{\partial u^i}{\partial x^\beta} + \frac{\partial\varphi_0^i}{\partial x^\beta}\frac{\partial u^i}{\partial x^\alpha}\Big). \tag{1}$$

R e m a r k 2.1. By definition, the Lagrange strain tensor is a symmetric tensor, it is thus invariant under the swap $\alpha \leftrightarrow \beta$.

We assume that the equilibrium equations are the classical Cauchy equations of a continuum body:

$$\begin{cases} \operatorname{div}\sigma + \mathbf{B} = \varrho\dfrac{\partial^2\mathbf{u}}{\partial t^2}, & \mathbf{P} \in (\mathcal{S}_0 \setminus \partial\mathcal{S}_0), \\ \sigma\mathbf{n} = \mathbf{f}, & \mathbf{P} \in \mathcal{S}_0, \end{cases} \tag{2}$$

where

▷ σ is the Cauchy stress tensor,
▷ $\mathbf{B}$ is the external volume forces,
▷ ϱ is the mass density,
▷ $\mathbf{n}$ is the unit normal point outward the boundary,
▷ $\mathbf{f}$ is the surface contact forces.

In the sequel, we impose on the membrane to be constrained along all its boundary to impede it each kind of motion (Dirichlet conditions for the problem (2)):

$$\mathbf{u}|_{\mathbf{P}\in\partial\mathcal{S}} = \mathbf{0}. \tag{3}$$

We consider linear elastic isotropic bodies, for this kind of materials the relation between the Cauchy stress tensor σ and the Lagrange strain tensor ε is expressed by the *Hook Law*

$$\sigma = [\mathbb{A}(\nu, g)E] \cdot \varepsilon, \tag{4}$$

where $\mathbb{A}$, the *elastic tensor*, depends on the material via the Poisson coefficient ν and on the membrane configuration via the metric tensor g of the undeformed configuration $\mathcal{S}_0$, defined from the isometric immersion $\mathcal{S}_0 \subset \mathbb{R}^3$ as

$$g_{\alpha\beta} = \sum_k \frac{\partial\varphi_0^k}{\partial x^\alpha}\frac{\partial\varphi_0^k}{\partial x^\beta};$$

E is the Young modulus. The components of the elastic tensor are explicitly given by the expression (see [10])

$$A^{\alpha\beta\gamma\delta} = \nu g^{\alpha\beta} g^{\gamma\delta} + \frac{1-\nu}{2}(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}), \tag{5}$$

where $g^{\alpha\beta}$ are the components of the inverse of the metric tensor of the undeformed configuration of the membrane. The assumptions (1) and (4) testify that the material is hyperelastic, i.e. there exists a positive definite stored energy, the Potential Elastic Energy Ψ; as a consequence it follows that the elastic tensor is positive definite.

Remark 2.2. The elastic tensor $\mathbb{A}$ enjoys having three particular symmetries:

1. it is invariant under the interchange of the indices α and β, since the Lagrange strain tensor is symmetric, see Note 2.1;
2. it is invariant under the interchange of the indices γ and δ, since the Cauchy stress tensor is symmetric;
3. it is invariant under the interchange of the couples of indices (α, β) and (γ, δ), since the material is hyperelastic; this property follows from the Schwartz Theorem applied to the Potential Elastic Energy functional Ψ.

We assume at first instance that the external forces are zero. The use of the constitutive law (4) yields the dynamical equations of motion as functions of the displacement $\mathbf{u} = (u_1, u_2, u_3)^{\mathrm{T}}$ only:

$$\begin{cases} \Big[\mathcal{A} - \dfrac{\partial^2}{\partial t^2}\Big](\mathbf{u}) = \mathbf{0}, & \mathbf{P} \in (\mathcal{S}_0 \setminus \partial\mathcal{S}_0), \\ \mathbf{u} = \mathbf{0}, & \mathbf{P} \in \partial\mathcal{S}_0, \end{cases}$$

where $\mathcal{A}\colon H_0^{2,2}(\Omega, \mathbb{R}^3) \to L^2_{\bar{\varrho}}(\Omega, \mathbb{R}^3)$ is defined as

$$(6)\quad \mathcal{A}^i = \frac{1}{\bar{\jmath}\bar{\varrho}} \sum_{\alpha} \frac{\partial}{\partial x^\alpha} \left(\frac{\bar{\jmath}Eh}{2(1-\nu^2)} \sum_{\beta,\gamma,\delta} A^{\alpha\beta\gamma\delta} \frac{1}{2} \frac{\partial \varphi_0^i}{\partial x^\beta} \left(\sum_k \frac{\partial \varphi_0^k}{\partial x^\gamma} \frac{\partial u^k}{\partial x^\delta} + \frac{\partial \varphi_0^k}{\partial x^\delta} \frac{\partial u^k}{\partial x^\gamma} \right) \right),$$

where $H_0^{2,2}(\Omega, \mathbb{R}^3)$ is the Sobolev space of the functions satisfying the boundary conditions (3) equipped with the usual norm and $L^2_{\bar{\varrho}}(\Omega, \mathbb{R}^3)$ is the Lebesgue space equipped with the scalar product (*weighed mass scalar product*)

$$\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle_{L^2_{\bar{\varrho}}} = \int_\Omega \bar{\varrho} \langle \mathbf{u}(x), \mathbf{v}(x) \rangle \, \mathrm{d}\mu(\Omega).$$

In (6) we have

▷ $\bar{\jmath} = \|\partial_{x^1}\varphi_0 \wedge \partial_{x^2}\varphi_0\|$ the local element of area,

▷ $\bar{\varrho} = \bar{\jmath}^{-1} \int_{h^-}^{h^+} \varrho \bar{\jmath} \, \mathrm{d}\xi$ the surface mass density.

The assumption of hyperelasticity is equivalent to the symmetry of the operator $\mathcal{A}$ in $L^2_{\bar{\varrho}}(\Omega, \mathbb{R}^3)$, see e.g. the definition of the elastic tensor expressed by relation (5) and in Remark 2.2. The main existence theorem for linear elastodynamics is as follows:

Theorem 2.3. *Let the material be hyperelastic and let $\mathbb{A}$ be its elasticity tensor. Let the symmetric operator be $\mathcal{A}\colon H_0^{2,2}(\Omega) \to L^2_{\bar{\varrho}}(\Omega)$. Then $\mathcal{A}$ is strongly elliptic.*

Proof. See [7], p. 346. □

The elasticity of the material implies that there are not dissipative phenomena inside the body, hence the work done by the external forces and internal stress in a consistent field of displacement and the corresponding strain resultant is always positive, i.e. $\langle\langle \mathbf{u}, \mathcal{A}(\mathbf{v}) \rangle\rangle_{L^2_{\bar{\varrho}}} \geqslant 0$. This inequality expresses that the linear operator $\mathcal{A}$ is positive definite. We conclude with the following

Proposition 2.4. *The linear operator $\mathcal{A}\colon H_0^{2,2}(\Omega, \mathbb{R}^3) \to L^2_{\bar{\varrho}}(\omega)$ is symmetric, positive definite and strongly elliptic. It admits a nondecreasing discrete set of positive eigenvalues λ_N.*

Moreover, we recall here that the linear operator $\mathcal{A}$ acts as div grad on the vector field $\mathbf{u}\colon \mathcal{S}_0 \to \mathbb{R}^3$, i.e., it is the well-known Laplacian-Beltrami operator Δ (or, more briefly, the *Laplacian*).

If the membrane has constant thickness h and if the material is homogeneous the elastic constants are independent of x and t, and they can be shifted out from the differential operator. Using the standard techniques of the variables separation: $\mathbf{u}(x,t) = X(x)T(t)$, we get the eigenvalues-eigenfunctions problem

$$\Delta X(x) + \lambda X(x) = 0, \tag{7a}$$

$$\frac{2\varrho(1-\nu^2)}{Eh}\frac{\partial^2 T(t)}{\partial t^2} + \lambda T(t) = 0. \tag{7b}$$

Equation (7a), accompanied with the boundary condition (3), expresses the usual problem of the spectrum of a manifold $\mathcal{S}_0$ with Dirichlet data on the boundary; the equation (7b) is the classical one-dimensional oscillator equation. Searching the solutions of (7b) in the form $T(t) = T_0 \mathrm{e}^{\mathrm{i}\omega t}$ (T_0 is the initial amplitude, ω the oscillation frequency and i the imaginary unit) the relation between the Nth frequency ω_N and the Nth eigenvalue λ_N of the operator Δ is given by

$$\omega_N = \sqrt{\frac{Eh}{2\varrho(1-\nu^2)}\lambda_N}.$$

3. Estimation of free frequencies

In the last years several authors got some estimates, from above and from below, of the spectrum of the Laplace Beltrami operator acting on a compact n dimensional Riemannian manifold without boundary. The proposed model of vibrating membrane is that of a 2-dimensional compact Riemannian manifold *with nonempty boundary.* Gluing two isometric copies of one Riemannian manifold with boundary $(M, \partial M, g)$ along their common boundary we get a *doubled* Riemannian manifold $(M\sharp M, \widetilde{g})$ which is compact and with empty boundary. It is easy to prove (see Appendix and [9]) that the spectrum of the doubled manifold splits in two components, the first one relative to the Dirichlet spectrum of each copy on $(M, \partial M, g)$, the other one relative to the Neumann spectrum. If any assumption on the curvature of the manifolds is done, the estimates depending on the topology, the diameter or the volume of the manifold $(M, \partial M, g)$ are available, since the new metric in the doubled manifold $(M\sharp M, \widetilde{g})$ is the C^0-limit of C^∞-metrics $(M\sharp M, g\sharp g)$, and the diameter and the volume go to the limit (see Lemma 5.3 in the Appendix at the end of this paper). On the contrary, for other estimates it is necessary to fix a lower bound of the Ricci curvature of the boundaryless manifold, but in the doubled manifold the curvature can reach negative values and can still be high in a neighborhood of the "*equator*" (the line of junction of the two copies of $(M, \partial M, g)$). The following example explains the problem:

E x a m p l e 3.1. The equator is in general a discontinuity line for the curvature of the doubled manifold. To get a suitable regularity it is necessary to smooth the metric in a suitable neighborhood of this line. If we paste in the common boundary two copies of a membrane having the shape of a "trumpet bell", in a neighborhood of the equator we get a regularized surface of junction with positive Gaussian curvature. On the contrary, pasting two isometric copies of membrane shaped as "Russian dome", in the neighborhood of the equator the Gaussian curvature of the surface is negative and its value depends on the attack angle between the two copies, which is the reason why the Gaussian curvature can assume negative values still high.

Definition 3.2. A Riemannian manifold $(M, \partial M, g)$ with nonempty boundary is said to have *convex boundary* if the second fundamental form of the boundary $II_{\partial M}$ is negative definite with respect to the inward normal N.

Definition 3.3. Let $m \in M$ be a point of the manifold and let $D_m(r)$ be the geodesic disk of radius r centered at m. If there exists a value i_m such that for $r \geqslant i_m$ the disk has auto-intersections, then i_m is the *injectivity radius of M at m*; running m over all M, the $\min\limits_{m \in M} i_m$ is the *injectivity radius of M*.

R e m a r k 3.4. We emphasize here that the boundary of a compact 2-dimensional manifold is a 1-dimensional manifold, i.e. a *closed curve*, thus the *injectivity radius* is the *diameter* and it makes no sense to talk about the *Second Fundamental Form*, since it reduces to the *curvature function* $k\colon \partial M \to \mathbb{R}^+ \cup \{0\}$. Let $P \in \partial M$ be any point of the membrane belonging to the boundary and let $\mathbf{N}$ and $\mathbf{n}$, respectively, be the normal to the surface and the normal to the boundary (seen as a line isometrically embedded in $\mathbb{R}^3$). We say that the boundary is convex if $\langle \mathbf{N}, \mathbf{n} \rangle < 0$. The inverse of the radius of the osculating circle at this point is the curvature of the boundary at this point. The convexity of the boundary is reflected in the fact that the osculating circle and the normal $\mathbf{N}$ lie on the opposite sides. The assumption "*sectional curvature of the boundary* $\geqslant -k^2$" becomes simply fulfilled at every point of the osculating circle; we take the minimum of this value as reference value k. For the same reason in this case the bound η of the second fundamental form coincides with the same value k.

R e m a r k 3.5. We recall here that for manifolds of dimension 2, the Gaussian curvature K yields the whole Riemann Curvature Tensor R, reduced to the single term $R_{1212} = \mathrm{scal} = 2K$, where scal is the scalar curvature, and the Ricci curvature tensor becomes $\mathrm{Ric} = 2K \cdot g$.

The estimates for which it is necessary to fix a lower bound of curvature of the boundaryless manifold are not available, in the first instance, without any adaptation of the metric. We have to get a limit metric obeying the following conditions:

1. if the boundary is convex, it is enough to have uniform control from below of sectional curvature of each copy of $(M, \partial M, g)$; the assumption

$$\mathrm{Ric} \geqslant -(n-1)\delta^2 g$$

remains indeed still true in the whole doubled manifold $M\sharp M$ thanks to Theorem 5.7 (see Appendix at the end to this paper and [9]);
2. if the boundary is not convex, we have to get a metric $\widetilde{g}$ which is at the same time
   (a) close enough to the metric $g\sharp g$ in such a way that both spectra are of the same order and
   (b) far enough from it in such a way that the sectional curvature is bounded.

In Theorem 5.8 a lower and an upper bound of this new metric are shown.

In what follows, we denote by $\mathcal{M}(h, E, \nu, \varrho)$ an elastic membrane of thickness h, Young modulus E, Poisson coefficient ν, and density ϱ. Moreover, in the estimates we will use all the corrections due to the regularization of the metric.

Estimates from below are available only for the first eigenvalue assuming that the n-dimensional compact Riemannian manifold (M, g) without boundary has diameter D and Ricci curvature limited from below: $\mathrm{Ric} \geqslant (n-1)kg$ and $k \geqslant 0$. These estimates are given by the Lichnerowicz formula (see [5], p. 210)

$$\lambda_1(M, g) \geqslant nk$$

and the Li-Yau formula (see [6], p. 189)

$$\lambda_1(M, g) \geqslant \frac{\pi^2}{2D^2}.$$

The direct use of the Lichnerowicz formula leads to

Lemma 3.6. *Let $\mathcal{M}(h, E, \nu, \varrho)$ be an elastic membrane such that its Gaussian curvature K is greater than zero. Then*

$$\omega_1(\mathcal{M}) \geqslant \sqrt{\frac{EhK}{2\varrho(1-\nu^2)}}. \tag{8}$$

In a similar way, using the Li-Yau formula and the regularized metric of the doubled manifold (see formula (15)) we get

Lemma 3.7. *Let $\mathcal{M}(h, E, \nu, \varrho)$ be an elastic membrane of diameter D and Gaussian curvature K greater than zero, whose boundary has diameter a and upper bound of the curvature k. Then*

$$\omega_1(\mathcal{M}) \geqslant \frac{\pi}{2\overline{D}'}\sqrt{\frac{Eh}{2\varrho(1-\nu^2)}}, \tag{9}$$

where $\overline{D}' = \overline{D}$ if the boundary is convex and $\overline{D}' = \left(\cosh \frac{1}{4}ka + \sinh \frac{1}{4}ka\right)\overline{D}$ if the boundary is not convex.

Estimates from above depend in general on the topology via the genus γ of the surface (roughly speaking each surface can be deformed continuously and with no lacerations to a sphere with γ handles, this number is the genus γ of the surface), the volume, the diameter, the curvature of the undeformed configuration of the membrane, moreover, some estimate depends also on the curvature and the injectivity radius of the boundary, because of the regularization of the metric.

Lemma 3.8. *The upper bound of the first vibration frequency of a linear elastic membrane $\mathcal{M}(h, E, \nu, \varrho)$ with nonempty and connected boundary, of genus γ and area A is given by*

$$\omega_1(\mathcal{M}) = \min[\omega_1^D(\mathcal{M}), \omega_1^N(\mathcal{M})] \leqslant \sqrt{\frac{Eh\pi(2\gamma+1)}{4\varrho(1-\nu^2)A}}. \tag{10}$$

P r o o f. If (M, g) is a 2-dimensional compact Riemannian manifold of genus γ and with empty boundary, the Yang-Yau Theorem ensures that $\lambda_1(M, g) \leqslant 8\pi(\gamma+1)/A$ (see [12]). We proved in [9] that, if $(M, g, \partial M)$ is a 2-dimensional compact Riemannian manifold of genus γ and with nonempty boundary ∂M, then $\lambda_1^N(M, g) \cdot \mathrm{Vol}_g(M, g) \leqslant 4\pi(2\gamma+1)$ or $\lambda_1^D(M, g) \cdot \mathrm{Vol}_g(M, g) \leqslant 4\pi(2\gamma+1)$. Moreover, setting $\mathrm{Vol}_g(M, g) = A(M)$, $A(M) = 2A(\mathcal{M})$, we get the estimate (10). □

Lemma 3.9. *The upper bound of the Nth vibration frequency of an elastic membrane $\mathcal{M}(h, E, \nu, \varrho)$ of diameter D is given by*

$$\omega_N(\mathcal{M}) \leqslant \frac{N}{D}\sqrt{\frac{3Eh}{\varrho(1-\nu^2)}}. \tag{11}$$

P r o o f. Let $(M, \partial M, g)$ be a compact Riemannian manifold with nonempty boundary and $(M\sharp M, g_{M\sharp M})$ its double, it is clear that $\mathrm{diam}_{M\sharp M} \geqslant \mathrm{diam}_M$; Corollary 2.2 of Cheng, [3], ensures that, if D is the diameter of a compact n-dimensional boundaryless Riemannian manifold, its Nth eigenvalue is raised by $N^2 n(n+1)/D^2$, in the case of the elastic membrane we have that $\mathrm{diam}_{M\sharp M} \geqslant D$, $n = 2$, which yields (11). □

Lemma 3.10. *The upper bound of the Nth vibration frequency of an elastic membrane $\mathcal{M}(h, E, \nu, \varrho)$ of diameter D and the Gaussian curvature K whose boundary has diameter a and upper bound of the curvature k, is given by*

$$\omega_N(\mathcal{M}) \leqslant \sqrt{\frac{Eh}{2\varrho(1-\nu^2)}\Big(\frac{K}{2} + \frac{16N^2\pi^2}{D'^2}\Big)}, \tag{12}$$

where $D' = D$ if the boundary is convex or $D' = \big(\sinh \frac{1}{4}ka / \sinh \frac{1}{2}ka\big)D$ if the boundary is not convex.

P r o o f. *Convex boundary*: As in the previous estimation, we consider the boundaryless doubled manifold $M\sharp M$ with sectional curvature equal to $\frac{1}{2}K$ and diameter $d_{M\sharp M}$ such that $d_{M\sharp M} \geqslant d_M$ where M is the middle surface of the shell $\mathcal{M}$ embedded isometrically in the ambient space $\mathbb{R}^3$. Corollary 2.3 in Cheng [3] gives, when $n = 2(m+1)$, $m = 0, 1, 2, \ldots$

$$\lambda_N(M) \leqslant \frac{(2m+1)^2}{4}\frac{K}{2} + \frac{4N^2(1+2^m)^2\pi^2}{d^2},$$

and when $n = 2m+3$, $m = 1, 2, \ldots$

$$\lambda_N(M) \leqslant \frac{(2m+2)^2}{4}\frac{K}{2} + \frac{4N^2(1+\pi^2)(1+2m^2)^2}{d^2}.$$

Keeping $m = 0$, we get the estimation for the Nth eigenvalue

$$\lambda_N(M\sharp M) \leqslant \frac{K}{2} + \frac{16N^2\pi^2}{D^2}$$

and hence directly (12).

Nonconvex boundary: The proof is still valid but the diameter has to be corrected by the factor depending on the curvature and the injectivity radius of the boundary, conformably with relation (15) of Theorem 5.8 in Appendix. □

The next estimate is usable for compact Riemannian manifolds with convex boundary and such that the diameter of the doubled manifold is the same as the each copy from which the manifold is obtained. It is impossible to bound in a easy way the functions of D bounding the frequencies.

Lemma 3.11. *Let $\mathcal{M}(h, E, \nu, \varrho)$ be an elastic membrane with convex boundary, whose diameter, the Gaussian curvature and volume are respectively D, K and A and such that its Ricci curvature is greater than $\frac{1}{2}Kg$. There exists a critical number $N_0 = 2\pi D/A$ such that*

1. *if $N \leqslant N_0$ then*

$$\omega_N(\mathcal{M}) \leqslant \frac{4N}{D} \cosh \frac{D\sqrt{K/2}}{4N} \cdot \sqrt{\frac{Eh}{2\varrho(1-\nu^2)}}, \tag{13}$$

2. *if $N > N_0$ then*

$$\omega_N(\mathcal{M}) \leqslant \frac{8N}{D} \sqrt{\frac{(N+1)\pi}{KA}} \sinh \frac{D\sqrt{K/2}}{N} \cosh \frac{D\sqrt{K/2}}{4N} \cdot \sqrt{\frac{Eh}{2\varrho(1-\nu^2)}}. \tag{14}$$

P r o o f. The previous estimates follow directly from the theorem of Chen, see [5], p. 209, remembering that the area of the unit 2-ball is π. □

4. A numerical investigation

To test the accuracy of the above estimate we proceeded to a numerical simulation via a Finite Element Program on a half ellipsoid of rotation membrane, whose geometric and mechanical characteristics are reported in the following Table 1 and Table 2.

Vertical semi-axes	5.00 m
Horizontal semi-axes	6.82 m
Diameter	21.42 m
Thickness	0.15 m
Minimal Gaussian curvature	0.0012 m^{-2}
Maximal Gaussian curvature	0.0040 m^{-2}
Area	241.90 m^2

Table 1. Geometrical characteristics of the membrane.

ϱ	2.35	$\mathrm{KNs^2m^{-4}}$
E	3 102 600	$\mathrm{KNm^{-2}}$
ν	0.15	

Table 2. Mechanical characteristics of the membrane.

In Table 3 we ordered the estimated and calculated frequencies. On the first sight we emphasize that all estimates depending on the Gaussian curvature are less accurate (both from above and from below) than the other ones; this depends on the particular shape of Chen estimates and not on the model proposed. For the first nonzero eigenvalue of the canonical unit sphere $(\mathbb{R}^2, \mathrm{can})$ the direct calculation gets $\lambda_1 = 2$, on the contrary the estimations (12) and (13) give back values at least ten times larger than the explicitly calculated value. The Cheng's estimates have, for the moment, a strong theoretical importance only. We conclude that with a good approximation the proposed method is accurate enough.

Estimate	(8)	(9)	Calculated via F.E.P.	(11)	(13)	(10)	(12)
Frequency (Hz)	20.14	23.34	36.20	37.59	59.94	102.57	186.88

Table 3. First free frequency.

5. Appendix

We sketch here the main ideas and results of the method of the doubled manifold, referring to [9] for all technical details, in particular how to gain a C^∞-metric with a uniform control from below of the curvature.

5.1. Definition of the doubled manifold $(M\sharp M, g\sharp g)$ **and general properties of the spectrum.** Let $(M, \partial M, g)$ be a Riemannian manifold with compact and differentiable boundary ∂M. From the disjoint union $M_1 \amalg M_2$ of two copies of the manifold M and the canonical maps ψ_1 and ψ_2 of M on M_1 and M_2 we get the *double* $M\sharp M$ of $(M, \partial M)$ as the quotient manifold of $M_1 \amalg M_2$ via the following equivalence relation: $\psi_1(x) \sim \psi_2(x)$ if and only if $x \in \partial M$. In other words, we define the doubled manifold as $(M \times \{1, -1\})/\sim$, where the equivalence relation $\sim$ is defined as

$$(x, i) \sim (y, j) \ \text{ if and only if } \ (x = y \text{ and } i = j) \text{ or } (x = y \in \partial M \text{ and any } i, j).$$

The two boundaries that in this way are identified yield an $(n-1)$-hyper-surface named "the equator" of $M\sharp M$. The manifold $M\sharp M$ can be equipped with a structure of a C^∞ manifold in the following way:

Let $p\colon (M \times \{1,-1\}) \to M\sharp M$ be the canonical surjection,

$$U \subset p(\partial M \times \{-1\}) = p(\partial M \times \{1\})$$

an open neighborhood in $M\sharp M$ and N the g-unitary inward normal field of ∂M, the local chart Φ is defined as

$$\Phi(t,x) = \begin{cases} p(\exp_x[t \cdot N(x)], 1) & \text{if } t \geqslant 0, \\ p(\exp_x[-t \cdot N(x)], -1) & \text{if } t < 0. \end{cases}$$

If $\varepsilon \leqslant \mathrm{inj}_M$ (inj_M is the injectivity radius of M), the exponential normal map is a diffoeomorphism of $]0,\varepsilon[\times \partial M$ on its image in M and the changes of charts are C^∞-maps.

Let $j\colon M \to M \times \{1\}$ be the isometric immersion of M in $M \times \{1\}$ and let $\Sigma\colon M\sharp M \to M\sharp M$ be the symmetry with respect to the equator swapping the two copies of M in $M\sharp M$: $\Sigma(M \times \{1\}) = (M \times \{-1\})$. The map j induces on $M \times \{1\}$ and on $M \times \{-1\}$ the metrics $g_1 = j^*(g)$ and the metric $g_{-1} = \Sigma^*(g_1)$, respectively. The passage to the quotient with respect to the equivalence relation $\sim$ induces the metric $g\sharp g$ on $M\sharp M$.

Fact 5.1. The metric $g\sharp g$ as defined above on $M\sharp M$ is C^0 but not C^1. Moreover it is a C^0-limit of C^∞-metrics g_k defined on $M\sharp M$.

Let (M,g) be a closed C^∞ Riemannian manifold of dimension n. We write the metric g and the Laplace operator in a local system of coordinates $(x^1, x^2, \ldots, x^n)$, respectively as $g = \sum_{i,j} g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$ and $\Delta = \sqrt{\det g^{-1}} \cdot \frac{\partial}{\partial x^i}(\sqrt{\det g} \cdot g^{ij} \cdot \frac{\partial}{\partial x^j})$; it is well known that the Laplacian is a self-adjoint elliptic operator having a discrete sequence of positive eigenvalues going to infinity:

$$0 \leqslant \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_i \leqslant \ldots$$

Moreover, each eigenspace $E(\lambda_i)$ has finite dimension, the direct sum of them is dense in $C^\infty(M)$ and the Hilbert space $L^2(M, \mathrm{d}v_g)$ ($\mathrm{d}v_g$ is the Riemannian measure on M) has a Hilbertian base of eigenfunctions.

Lemma 5.2. *For C^0-metrics on $(M\sharp M, g\sharp g)$ the spectrum of the Laplacian coincides with the critical values of the functional*

$$u \mapsto R(u) = \frac{\int_{M\sharp M} |\mathrm{d}u|^2_{(g\sharp g)}\, \mathrm{d}v_{(g\sharp g)}}{\int_{(M\sharp M)} u^2\, \mathrm{d}v_{(g\sharp g)}}$$

defined on $\mathcal{H}_0 = H_1(M,g)\backslash\{0\}$*. The critical points are calculated using the* min-max *principle or the* max-min *principle, i.e.*

$$\begin{aligned}\lambda_i(M\sharp M, g\sharp g) &= \inf_{\mathcal{E}_{i+1}} \max_{u\in\mathcal{E}\backslash\{0\}} \frac{\int_{M\sharp M} |\mathrm{d}u|^2_{g\sharp g}\,\mathrm{d}v_{g\sharp g}}{\int_{(M\sharp M)} u^2\,\mathrm{d}v_{g\sharp g}}\\ &= \sup_{\mathcal{E}_i} \inf_{u\in\mathcal{E}_i^{\perp}\backslash\{0\}} \frac{\int_{M\sharp M} |\mathrm{d}u|^2_{g\sharp g}\,\mathrm{d}v_{g\sharp g}}{\int_{(M\sharp M)} u^2\,\mathrm{d}v_{(g\sharp g)}},\end{aligned}$$

where $\mathcal{E}_i \subset \mathcal{H}_0$ *is any vectorial subspace of dimension* i *in* $\mathcal{H}_0$*.*

Proof. See [5], par 2. □

Lemma 5.3. *Let* $\{g_k\}_{k\in\mathbb{N}}$ *be a sequence of* C^∞*-metrics converging in the* C^0*-topology to a* C^0*-limit metric on* $M\sharp M$*, then:*

(i) $\operatorname{diam}(M\sharp M, g\sharp g) = \lim_{k\to\infty} \operatorname{diam}(M\sharp M, g_k)$,

(ii) $\operatorname{Vol}(M\sharp M, g\sharp g) = \lim_{k\to\infty} \operatorname{Vol}(M\sharp M, g_k)$,

(iii) $\lambda_i(M\sharp M, g\sharp g) = \lim_{k\to\infty} \lambda_i(M\sharp M, g_k)$.

Proof. See [9]. □

Definition 5.4. A function $u \in C^\infty(M, \partial M, g)$ solves the *Dirichlet problem* when

$$\begin{cases}\Delta u = 0,\\ u|_{\partial M} = 0,\end{cases}$$

and the *Neumann problem* when

$$\begin{cases}\Delta u = 0,\\ \dfrac{\partial u}{\partial N}\Big|_{\partial M} = 0,\end{cases}$$

where N is the inward unit normal to the boundary ∂M.

Lemma 5.5. *Let* $(M, \partial M, g)$ *be a Riemannian manifold with nonempty boundary and* $(M\sharp M, g\sharp g)$ *its double. Then*

(i) *the spectrum of* $(M\sharp M, g\sharp g)$ *is the union of the Dirichlet and the Neumann spectra of* $(M, \partial M)$*:*

$$\{\lambda_i(M\sharp M, g\sharp g);\ i\in\mathbb{N}\} = \{\lambda_i^D(M)_i;\ i\in\mathbb{N}\setminus\{0\}\} \cup \{\lambda_i^N(M)_i;\ i\in\mathbb{N}\},$$

where each eigenvalue has to be counted with its own multiplicity,

(ii) *there exists a Hilbertian base of eigenfunctions such that the restriction to each copy of M is an eigenfunction of the Dirichlet or Neumann problem.*

Example (classic) 5.6. The sphere $(\mathbb{S}^1, \mathrm{can})$ is the double of the Riemannian manifold with boundary $[0, \pi]$.

▷ The Dirichlet spectrum of $[0, \pi]$ is $\Sigma^D = \{k^2;\ k \in \mathbb{N} \setminus \{0\}\}$, the multiplicity of each eigenvalue is 1 and the eigenspace associated to k^2 is given by

$$E_{k^2}^{[0,\pi]} = \mathrm{Span}(\sin kt),$$

▷ the Neumann spectrum of $[0, \pi]$ is $\Sigma^N = \{k^2;\ k \in \mathbb{N}\}$, the multiplicity of each eigenvalue is 1 and the eigenspace associated to k^2 is given by

$$E_{k^2}^{[0,\pi]} = \mathrm{Span}(\cos kt),$$

▷ the spectrum of $(\mathbb{S}^1, \mathrm{can})$ is thus $\Sigma^{\mathbb{S}^1} = \{k^2;\ k \in \mathbb{N}\}$, the zero-eigenvalue has multiplicity 1 and each strictly positive eigenvalue has multiplicity equal to 2; counting each eigenvalue with its multiplicity, for $k \geqslant 1$ we get

$$E_{k^2}^{\mathbb{S}^1} = \mathrm{Span}(\sin kt, \cos kt),$$

or equally

$$\Sigma^{\mathbb{S}^1} = \Sigma^D \cup \Sigma^N.$$

5.2. Regularization of the metric. The following theorems give the relevant conditions on the limit metric in the case of a manifold with convex boundary (Theorem 5.7) or nonconvex boundary (Theorem 5.8). The deduction of them is complex and lies beyond the goal of this paper, we refer to [9] to the complete argumentation.

Theorem 5.7. *Let $(M, \partial M, g)$ be a Riemannian manifold with nonempty convex boundary. Then there exist metrics $g_k \in C^\infty(M\sharp M)$ converging to g in the C^0-topology such that, taking the minimum of the sectional curvature* Sec *on all the 2-dimensional tangent planes to M, we have* $\min(\mathrm{Sec}_{g_k}) \geqslant \min(\mathrm{Sec})$.

If the boundary is not convex it is necessary to "sweeten" the metric in a suitable neighborhood of the equator. To get a C^∞-metric $\widetilde{g}$ with a control from below of the sectional curvature it is necessary to regularize, in a suitable neighborhood of the equator, the C^0-metric $g\sharp g$ obtained by gluing the two copies of M with the strong condition that the new metric is isometric to the metric of each part of the doubled manifold. To make so we “twist” the metric $g\sharp g$ in the direction of the normal N to the equator, pointing inward to each copy of M, in such a way as to maintain the lengths and control from below the sectional curvature. Roughly speaking, we interpose between the two copies of M a thin cylinder with a “smoothed and isometric” junction between each of the parts. What we have just said is summarized in the following theorem whose proof is given in [9]:

Theorem 5.8. *Let $(M, \partial M, g)$ be an n-dimensional Riemannian manifold with nonempty boundary and let k, a and η be three real numbers limiting respectively the sectional curvature of $(M, \partial M, g)$, the injectivity radius of the boundary ∂M, and its second fundamental form h, i.e.,*

$$\mathrm{Sec}_g \geqslant -k^2, \quad \mathrm{inj}_{\partial M} \geqslant a, \quad h_{\partial M} \leqslant \eta.$$

Then there exists on $(M, \partial M)$ a metric $\widetilde{g}$ such that $\widetilde{g}\sharp\widetilde{g}$ is C^∞ on $M\sharp M$ and such that

$$\left(\frac{\sinh(\frac{1}{4}ka)}{\sinh(\frac{1}{2}ka)}\right)^2 \cdot g \leqslant \widetilde{g} \leqslant \left[\cosh\Big(\frac{ka}{4}\Big) + \sup\Big(1, \frac{\eta}{k}\Big) \cdot \sinh\Big(\frac{ka}{4}\Big)\right]^{2n-2} \cdot \left(\frac{\sinh(\frac{1}{2}ka)}{\sinh(\frac{1}{4}ka)}\right)^{2n-4} \cdot g \tag{15}$$

and

$$\mathrm{Sec}_{\widetilde{g}} \geqslant -\frac{4k^2}{\tanh^2(\frac{1}{4}ka)}.$$

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