Estimators for Orientation and Anisotropy in Digitized Images

Lucas J. van Vliet and Piet W. Verbeek

Pattern Recognition Group of the Faculty of Applied Physics
Delft University of Technology
Lorentzweg 1, 2628 CJ Delft, The Netherlands

e-mail: lucas@ph.tn.tudelft.nl

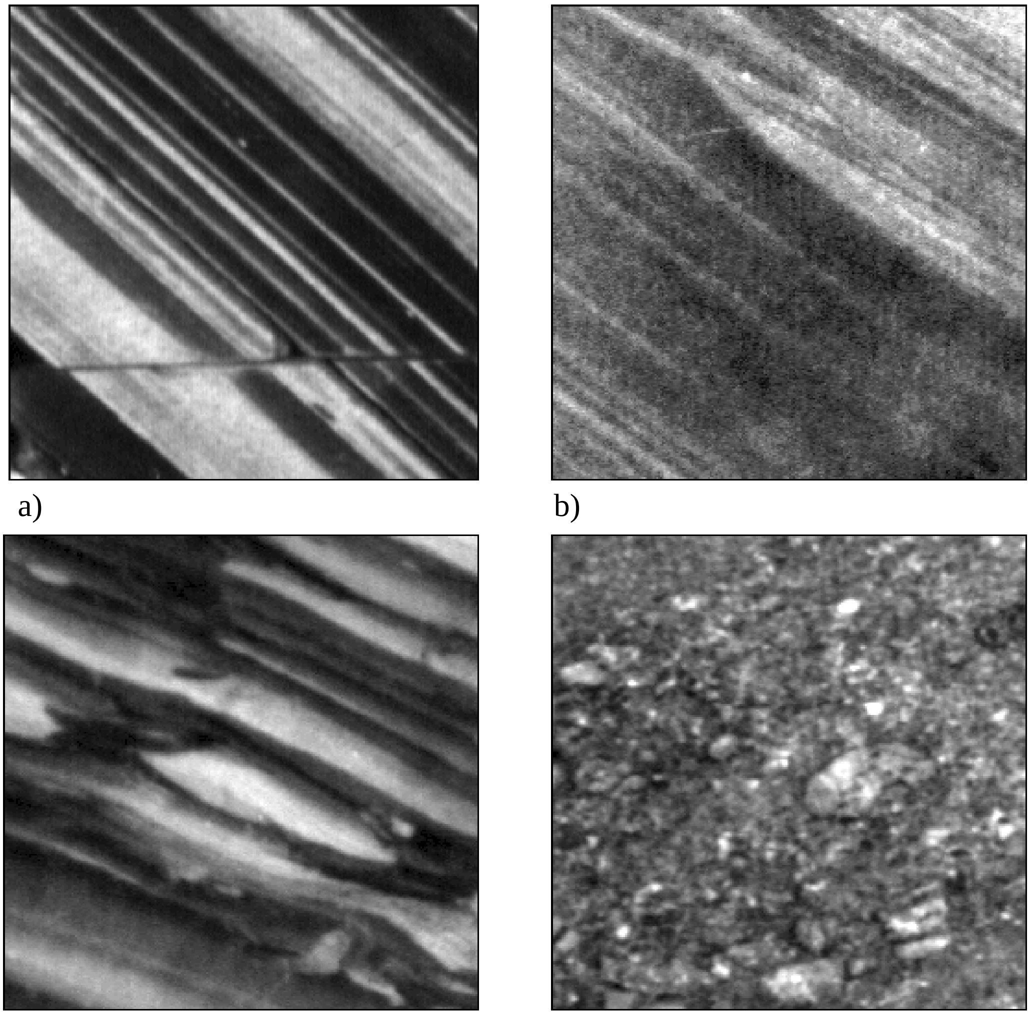
Abstract
This paper describes a technique for characterization and segmentation of anisotropic patterns that exhibit a single local orientation. Using Gaussian derivatives we construct a gradient-square tensor at a selected scale. Smoothing of this tensor allows us to combine information in a local neighborhood without canceling vectors pointing in opposite directions. Whereas opposite vectors would cancel, their tensors reinforce. Consequently, the tensor characterizes orientation rather than direction. Usually this local neighborhood is at least a few times larger than the scale parameter of the gradient operators. The eigenvalues yield a measure for anisotropy whereas the eigenvectors indicate the local orientation. In addition to these measures we can detect anomalies in textured patterns.

1. Introduction

Information from subsurface structures may help geologists in their search for hydrocarbons (oil and gas). In addition to seismic measurements which are performed at the earth's surface important information can be extracted from a borehole. This can be done either by downhole imaging of the borehole wall or by analyzing the removed borehole material “the core”. Core imaging requires careful drilling with a hollow drillbit. The cores are transported to the surface for further analysis. Apart from physical measurements geologists are interested in the spatial organization of the acquired rock formations. We show that this can be done with the help of quantitative image analysis. The cylindrical cores can be cut longitudinally (slabbed) and digitization of the flat surface yields a 2D slabbed core image.

Quantitative information about the layer structure in a borehole may help the geologist to improve their interpretation. The approach to be followed is guided by a simple layer model of the earth's subsurface. These layers can be described by a number of parameters which may all vary as a function of depth. Some of these parameters have a direct geometric meaning (dip and azimuth) whereas others are much more difficult to express quantitatively in a unique way. In this paper we will focus on orientation and anisotropy measurements applied to slabbed core images.

ASCI'95, Proceedings of the first Conference of the Advanced School for Computing and Imaging, Heijen (The Netherlands), May 16-18, 1995, pp. 442-450.



a) b) c) d)

Figure 1: Four samples of slabbed core images. **a)** planar region of high SNR; **b)** planar region of low SNR; **c)** discontinuous layers of high SNR; **d)** isotropic region of low SNR.

Deposition of various sediments (sand, shale) over time allows us to use layers for modeling the earth's subsurface. The core images under study exhibit a single dominant local 3D orientation caused by cutting through a large number of mainly parallel layers. The orientation of these layers can be described by two angles called dip and azimuth. For slabbed core images cut along the azimuth we can find the local dip from the 2D cross section. Due to the nature of the deposition in combination with other environmental conditions the layers do not display themselves as high contrast, planar, continuous boundaries (cf. figure 1). This prohibits the use of edge or line detectors followed by high level edge tracking or edge linking procedures. To estimate orientation and anisotropy as well as to detect anomalies we have extended the orientation estimators of [1,2].

2. Gradient-square tensor

Important features of core images are the amount and orientation of local anisotropy (cf. figures 1) which can be used to characterize laminar beds. To all pixels of the image we assign a vector, the gradient vector. The magnitude of the vector can be seen as the image contrast in the direction of the vector. In a window around an edge all these vectors point roughly in the same direction. A weighted sum would result in an average direction. Thus an edge can be uniquely characterized by a vector. The vector not only describes the edge orientation, but also which side has the higher grey level. In a window around a line all gradient vectors on one side point roughly in the same direction. However, gradient vectors from opposite sides of the line have the same orientation, but have opposite signs (direction). In a weighted sum, vectors from opposite sides of a line cancel out. To avoid cancellation of vectors pointing in opposite direction we need a quantity that depends on sin(2φ) and cos(2φ), i.e. a double angle. This way vectors pointing in opposite direction do not cancel, but reinforce each other. The first derivative or gradient consists of terms cos(φ) and sin(φ). The second derivative depends on the double angle terms sin(2φ) and cos(2φ). In two-dimensions the second derivative function space is defined by a 2x2 Hessian matrix. In order to use the first derivative we must use the dyadic product of the gradient vector with itself $\mathbf{g} \cdot \mathbf{g}^t$: the gradient-square tensor **G**.

$$\mathbf{G} = \mathbf{g} \bullet \mathbf{g}^t = \begin{pmatrix} g_x^2 & g_x g_y \\ g_x g_y & g_y^2 \end{pmatrix} \tag{1}$$

An alternative interpretation is as follows. Again we start with a set (window) of gradient vectors. We transfer all vectors to a common origin. For line-like objects we have seen that vectors from opposite sides of the line cancel out. To avoid this we replace the vectors by their endpoints. For windows containing lines, this result yields a cloud of points centered around the origin. To analyze the properties of such a cloud we calculate the sample covariance matrix. If $g_x = g_y = 0$ then

$$\operatorname{cov}(g_x, g_y) = \begin{pmatrix} \overline{g_x g_x} & \overline{g_x g_y} \\ \overline{g_x g_y} & \overline{g_y g_y} \end{pmatrix} \equiv \overline{\mathbf{G}} = \overline{\mathbf{g} \bullet \mathbf{g}^t} \tag{2}$$

with $\bar{\ }$ the average of the set (window). Thus for all point clouds with zero mean ($g_x = g_y = 0$) the covariance matrix equals the smoothed gradient-square tensor $\overline{\mathbf{G}}$. The eigenvalues of the covariance matrix correspond to the variances along the axes of inertia of the cloud. The axis with the lowest variance corresponds to the orientation of anisotropic features in the input image. For windows containing strong edges, the mean gradient (the cloud of endpoints) lies far away from the origin ($\bar{g} >> 0$). Note that in general the smoothed gradient-square tensor $\overline{\mathbf{G}}$ is different from the covariance matrix.

3. Orientation estimation from a Gradient-square tensor

The gradient-square tensor **G** at each image location is per definition one-dimensional, i.e. it contains a single vector. A weighted sum of such gradient-square tensors yields a smoothed tensor $\overline{\mathbf{G}}$ that represents a window with potentially more than one orientation. An arbitrary smoothing filter can be used to produce a tensor $\overline{\mathbf{G}}$ with orientation information from an arbitrary weighted neighborhood. The eigenvalues and corresponding eigenvectors yield specific information about the local neighborhood.

λ_1 largest eigenvalue: tensor energy in the direction of the first eigenvector $\mathbf{v}_1$;

λ_2	smallest eigenvalue: tensor energy in the direction of the second eigenvector $\mathbf{v}_2$;
$1-\lambda_2/\lambda_1$	anisotropy (consistency of local orientation);
φ	local orientation;
$\lambda_1 + \lambda_2$	smoothed gradient-square magnitude.

A smoothed gradient-square tensor can be written as

$$\overline{\mathbf{G}} = \overline{\mathbf{g} \bullet \mathbf{g}^t} = \begin{pmatrix} \overline{g_x g_x} & \overline{g_x g_y} \\ \overline{g_x g_y} & \overline{g_y g_y} \end{pmatrix} \tag{3}$$

The eigenvalues can be found by solving $|\mathbf{G} - \lambda \mathbf{I}| = 0$ resulting in

$$\lambda_{1,2} = \tfrac{1}{2}\left(\overline{g_x^2} + \overline{g_y^2}\right) \pm \tfrac{1}{2}\sqrt{\left(\overline{g_x^2} - \overline{g_y^2}\right)^2 + 4\left(\overline{g_x g_y}\right)^2} \tag{4}$$

The normalized eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that correspond to the eigenvalues λ_1 and λ_2 are of course orthogonal. The eigenvalues satisfy the relations $\mathbf{v}_1^{\mathbf{t}}\overline{\mathbf{G}}\mathbf{v}_1 = \lambda_1$ and $\mathbf{v}_2^{\mathbf{t}}\overline{\mathbf{G}}\mathbf{v}_2 = \lambda_2$. The orientations of both eigenvectors are

$$\varphi_1 = \arctan\left(\frac{\lambda_1 - \overline{g_x^2}}{\overline{g_x g_y}}\right) \qquad \varphi_2 = \arctan\left(\frac{\overline{g_x g_y}}{\lambda_2 - \overline{g_y^2}}\right) \tag{5}$$

Note that φ_1 is the local gradient direction. This solution uses a simple rule borrowed from linear algebra which solves the double angle problem implicitly. The relation between the gradient direction and the double angle phenomenon is presented in Appendix A. The local orientation φ corresponds to the direction of the smallest eigenvalue.

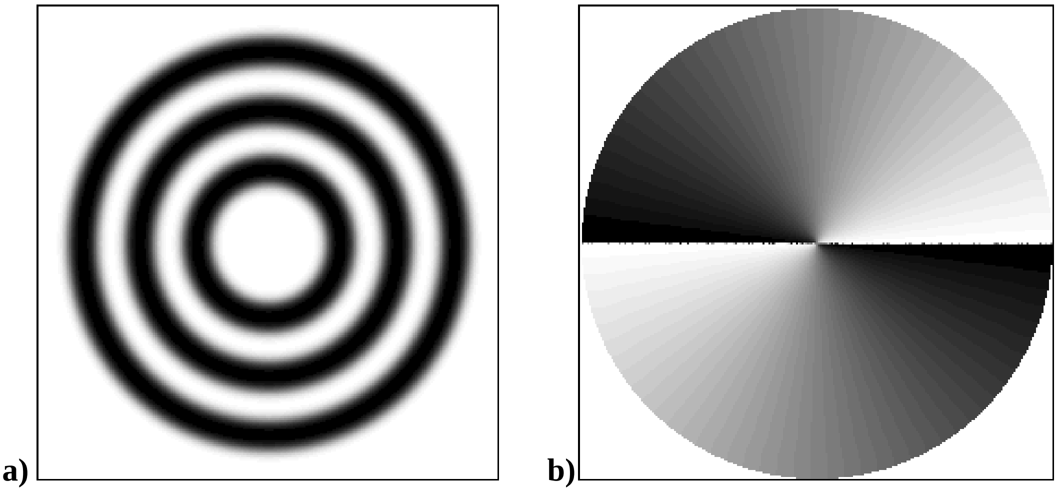
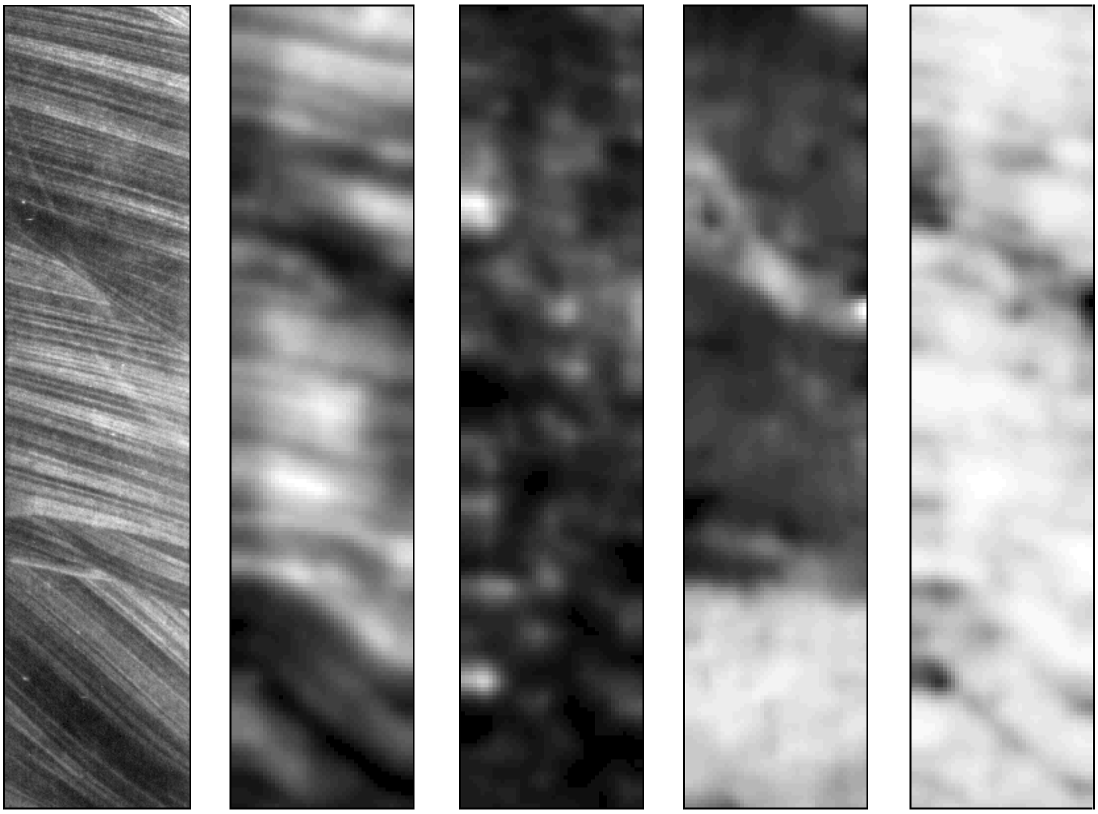


Figure 2: a) concentric circles; **b)** local orientation $\left\lfloor -\tfrac{1}{2}\pi, \tfrac{1}{2}\pi \right\rfloor$ obtained by smoothed gradient-square tensor.

The local orientation is automatically mapped into interval $\left\lfloor -\tfrac{1}{2}\pi, \tfrac{1}{2}\pi \right\rfloor$. The double-angles (introduced by the dyadic product) give rise to two discontinuities in the orientation field per revolution of the input pattern. Figure 2 shows the relation between the direction of anisotropy in the input image and the corresponding local orientation φ. The discontinuities occur for vertically oriented patterns which have a very low frequency of occurrence in our core images. Note that in all our images we have mapped monotonically increasing pixel values to monotonically increasing brightness levels. Where it can be of help to interpret the images we have indicated the interval that corresponds to the brightness values ranging from black to white.

The tensor **G** is constructed from first derivatives which are implemented as convolutions with a derivative-of-Gaussian. The Gaussian is applied as regularization function, to suppress noise and allows scale selection. The scale parameter should be large enough to suppress noise and small enough to detect thin linear structures that present themselves at a small scale (very thin layers of sand and shale). The smoothing of the tensor allows us to combine information in a local neighborhood. This smoothing describes the orientation inside that local neighborhood at the selected scale. Usually this local neighborhood is at least a few times larger than the scale parameter embedded in the gradient operators. The tensor smoothing can be accomplished by any low-pass filter. Since the smoothing size will be large we restrict ourselves to low-pass filters that have processing times independent of the filter size. We have two such filters at our disposal: the square-shaped uniform filter and a recursive implementation of the Gaussian filter [3]. The uniform filter takes approximately 70 ms and the recursive Gaussian filter takes approximately 100 ms when applied to an image of size 256x256. Figure 3 illustrates the information embedded in the smoothed gradient-square tensor. It shows that the method is suitable for measuring the local orientation. All sub-images use the full intensity scale (contrast stretching). The input image can be segmented into regions containing a single dominant orientation.

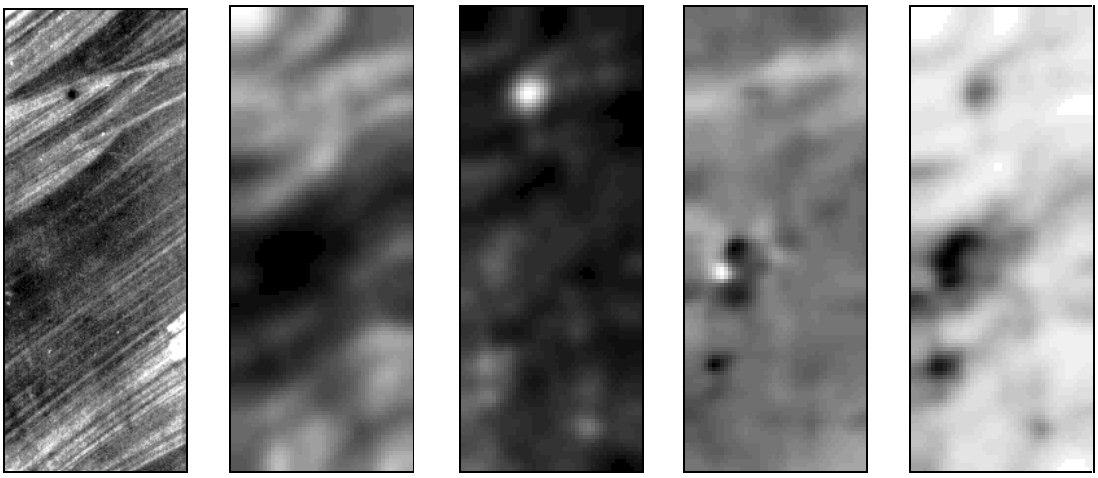


a) b) [0, 1.5] c) [0, 0.3] d) [0.2, 0.9] e) [0.4, 1]

Figure 3: A smoothed gradient-square tensor yields the following information. **a)** input image; **b)** largest eigenvalue λ_1; **c)** smallest eigenvalue λ_2; **d)** local orientation φ; **e)** anisotropy ($1-\lambda_2/\lambda_1$).

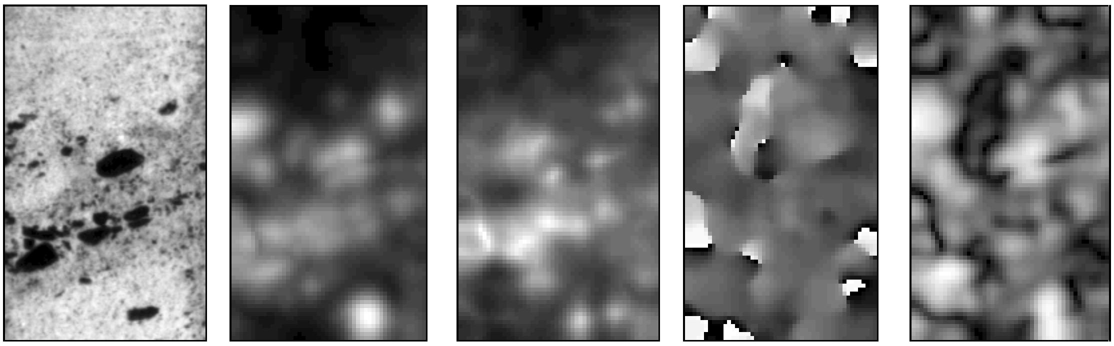
4. Anisotropy estimation from a Gradient-square tensor

The ratio λ_2/λ_1 lies in the interval [0,1] and therefore $1-\lambda_2/\lambda_1$ lies in the interval [0,1] as well. The quantity $1-\lambda_2/\lambda_1$ is a measure of anisotropy. A value equal to zero indicates perfect isotropy whereas a value equal to one indicates perfect anisotropy. In practice perfect anisotropy does not occur since the embedded noise is independent from pixel to pixel. The noise power always contributes a certain amount of isotropy. Note that for images with a high SNR the quantity $1-\lambda_2/\lambda_1$ is dominated by the signal, whereas for images with a very low SNR this quantity is dominated by the noise (c.f. figure 4). Figure 4 shows that a laminar region with low SNR cannot be detected from the eigenvalues but is easily identified from the ratios of these eigenvalues ($1-\lambda_2/\lambda_1$) (cf. the dark area in figure 4e). However, areas where ($1-\lambda_2/\lambda_1$) is low do not always correspond to laminar regions of low SNR. Figure 5 shows that low values of ($1-\lambda_2/\lambda_1$) also arise from isotropic regions of any SNR. Isotropic regions of high SNR can be distinguished from laminar regions of low SNR after examining the images with eigenvalues. Figures 4 and 5 show that both images contain regions with very low values for anisotropy (cf. dark areas in figures 4e and 5e). In figure 4 this region contains very little contrast and is dominated by noise. The isotropic regions have very low values of λ_1 and λ_2. In figure 5 very low anisotropy values occur also in regions with high contrast (high SNR). Here we notice that isotropy occurs due to the isotropic characterization of the signal (high values of λ_1 and λ_2).



a) b) [0, 1.2] c) [0, 0.4] d) [–0.9, –0.1] e) [0, 0.85]

Figure 4: Gradient-square tensor method applied to a region of low SNR. In the middle-left part of the image the noise dominates the gradient-square tensor producing an unreliable estimate of local orientation. **a)** input image; **b)** largest eigenvalue λ_1; **c)** smallest eigenvalue λ_2; **d)** local orientation φ; **e)** anisotropy $(1-\lambda_2/\lambda_1)$.

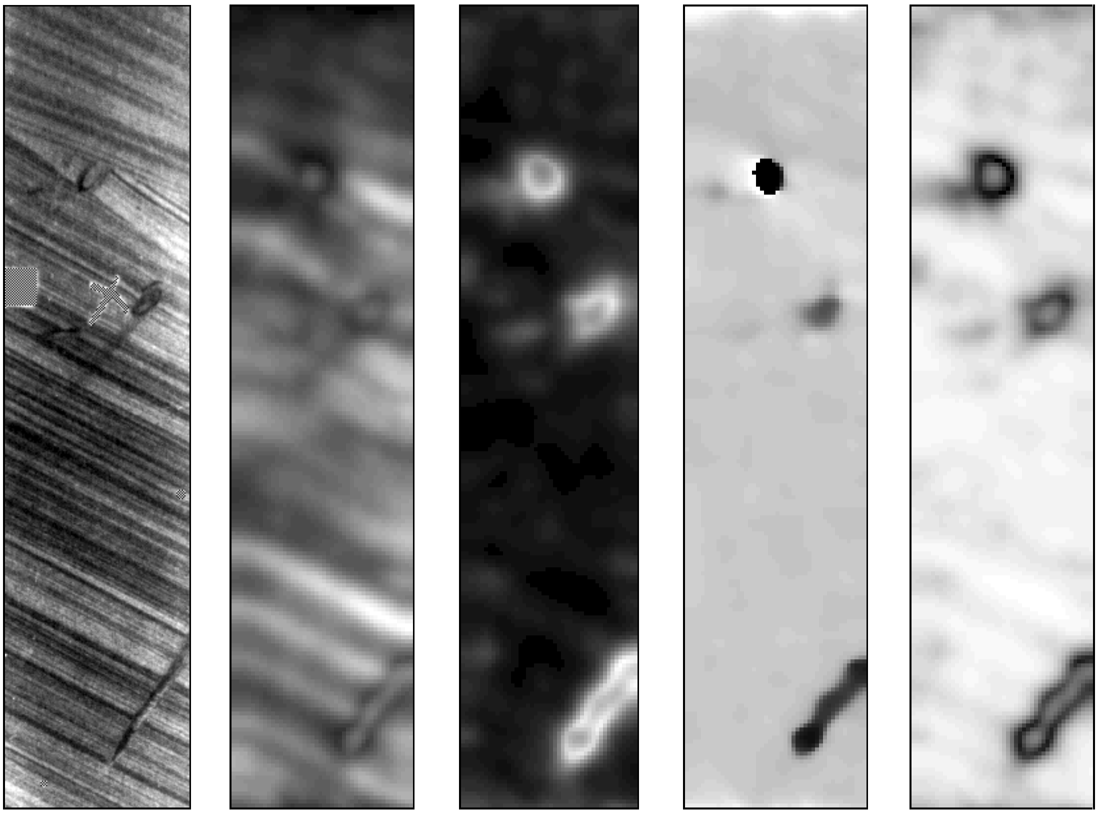


a) b) [0, 7.8] c) [0, 4.1] d) $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$ e) [0, 0.8]

Figure 5: Gradient-square tensor method applied to a region without anisotropy of high SNR. **a)** input image; **b)** largest eigenvalue λ_1; **c)** smallest eigenvalue λ_2; **d)** local orientation φ; **e)** anisotropy $(1-\lambda_2/\lambda_1)$.

5. Detection of anomalies in laminar beds

An important application of this technique is the detection of anomalies in laminar beds. At the position of such anomalies the local one-dimensional pattern is disturbed. Here λ_2 is higher. This is clearly demonstrated in figure 6. The anomalies show up as well defined regions in the smallest eigenvalue λ_2 as well as the anisotropy image $(1-\lambda_2/\lambda_1)$.



a) b) [0, 1.6] c) [0, 0.6] d) [–1.0, 1.0] e) [0, 1]

Figure 6: Gradient-square tensor method applied to a laminar region with a few anomalies. **a)** input image; **b)** eigenvalue λ_1; **c)** eigenvalue λ_2; **d)** local orientation φ; **e)** anisotropy $(1-\lambda_2/\lambda_1)$.

6. Conclusions

In this paper we showed that the smoothed gradient-square tensor is a powerful tool for analyzing oriented patterns. The size of the smoothing filter allows us to choose between a very localized measure with low signal-to-noise ratio and a measure over a larger region with a higher SNR. Applied to core images we obtained a robust estimate of local orientation when the diameter of the smoothing filter is equal to the diameter of the core cylinder (thus equal to the width of the image). This estimates yields exactly one orientation for every depth. Statistics as a function of depth transform the 2D images $(\varphi, (1-\lambda_2/\lambda_1))$ into a series of 1D logs. From anisotropy we derive a second measure called consistency. Consistency is derived from range of anisotropy values found. A small difference between the upper and lower values of anisotropy indicates a consistent orientation over the entire width of the core diameter (at that position). A large difference indicates that an anisotropic pattern is locally disturbed by anomalies such as fractures, nodules, lenses or other artifacts.

Acknowledgment

We thank Dr. W.J.M. Epping and Dr. ir. M.A. Kraaijveld for the stimulating discussions. This work was partially supported by the ASCI PostDoc program, the Rolling Grants program of the Netherlands Organization for Fundamental Research of

Matter (FOM) and Shell Research as part of the research program “3D Computer Vision”, sponsored by the Netherlands Foundation for Computer Vision (SCVR).

References

[1] Kass and Witkin, Analyzing oriented patterns, Computer Vision Graphics and Image Processing, vol. 37, 1987, pp. 362-385.

[2] L. Haglund, Adaptive multi-dimensional filtering, Ph.D. thesis, Linköping, 1992.

[3] I.T. Young and L.J. van Vliet, Recursive implementation of the Gaussian filter, accepted by Signal Processing, 1995.

Appendix A: Double angle of the gradient square tensor

The gradient vector **g** has orientation ϕ ($\phi = \text{atan}(g_y/g_x)$). The dyadic product of the gradient vector with itself yields the gradient square tensor $\mathbf{g} \bullet \mathbf{g}^t = \mathbf{G}$. The gradient square in a direction φ equals

$$
\begin{aligned}
\left(g^2\right)_\varphi \equiv \left(\mathbf{g} \bullet \mathbf{g}^t\right)_\varphi &= (\cos\varphi \quad \sin\varphi)\,\mathbf{G}\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix} \\
&= (\cos\varphi \quad \sin\varphi)\begin{pmatrix} g_x^2 & g_x g_y \\ g_x g_y & g_y^2 \end{pmatrix}\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix} \\
&= g_x^2 \cos^2\varphi + g_y^2 \sin^2\varphi + 2 g_x g_y \cos\varphi \sin\varphi \\
&= \tfrac{1}{2}\left(g_x^2 + g_y^2\right) + \tfrac{1}{2}\left(g_x^2 - g_y^2\right)\left(\cos^2\varphi - \sin^2\varphi\right) + \tfrac{1}{2} 2 g_x g_y 2\cos\varphi\sin\varphi \\
&= \tfrac{1}{2}\left(g_x^2 + g_y^2\right) + \tfrac{1}{2}\left(g_x^2 - g_y^2\right)\cos 2\varphi + \tfrac{1}{2} 2 g_x g_y \sin 2\varphi \\
&= \tfrac{1}{2}\left(g_x^2 + g_y^2\right)\left(1 + \frac{g_x^2 - g_y^2}{g_x^2 + g_y^2}\cos 2\varphi + \frac{2 g_x g_y}{g_x^2 + g_y^2}\sin 2\varphi\right)
\end{aligned}
\tag{A1}
$$

this expression can be simplified by introducing a double-angle 2χ

$$
\begin{aligned}
\left(g^2\right)_\varphi &= \tfrac{1}{2}\left(g_x^2 + g_y^2\right)\left(1 + \cos 2\chi \cos 2\varphi + \sin 2\chi \sin 2\varphi\right) \\
&= \tfrac{1}{2}\left(g_x^2 + g_y^2\right)\left(1 + \cos\left(2\varphi - 2\chi\right)\right)
\end{aligned}
\tag{A2}
$$

with

$$
\begin{aligned}
\tan 2\chi &= \frac{2 g_x g_y}{g_x^2 - g_y^2} \\
\sin 2\chi &= \frac{2 g_x g_y}{g_x^2 + g_y^2} \\
\cos 2\chi &= \frac{g_x^2 - g_y^2}{g_x^2 + g_y^2}
\end{aligned}
\tag{A3}
$$

The gradient square in a direction φ depends on angle between the chosen direction φ and the gradient direction ϕ. It is maximal for $\varphi = \chi$, thus $\phi \equiv \chi$.

Smoothing of the tensor yields the average orientation χ'.

$$\tan 2\chi' = \frac{2\overline{g_x g_y}}{\overline{g_x^2} - \overline{g_y^2}} \tag{A4}$$

Note that the average orientation of the gradient-square tensor is expressed in double angles. The orientation is estimated without using the eigenvalues of **G**. Below we show that the orientation of first eigenvector φ_1 (the first eigenvector $\mathbf{v}_1$ corresponds to the largest eigenvalue λ_1) equals the angle χ'.

Using two simple matrix properties $|\mathbf{G} - \lambda \mathbf{I}| = 0$ and $(\mathbf{G} - \lambda_i \mathbf{I})\mathbf{v_i} = 0$ yields

$$\lambda_1 = \tfrac{1}{2}\left(\overline{g_x^2} + \overline{g_y^2}\right) + \tfrac{1}{2}\sqrt{\left(\overline{g_x^2} - \overline{g_y^2}\right)^2 + 4\left(\overline{g_x g_y}\right)^2} \tag{A5}$$

$$\tan \varphi_1 = \frac{\lambda_1 - \overline{g_x^2}}{\overline{g_x g_y}} \tag{A6}$$

Substitution of λ_1 into $\tan\varphi_1$ yields

$$\begin{aligned}\tan \varphi_1 &= \frac{\lambda_1 - \overline{g_x^2}}{\overline{g_x g_y}} \\ &= \frac{\overline{g_y^2} - \overline{g_x^2}}{2\overline{g_x g_y}} + \sqrt{\frac{\left(\overline{g_y^2} - \overline{g_x^2}\right)^2}{4\left(\overline{g_x g_y}\right)^2} + 1} \\ &= \frac{-\cos 2\chi'}{\sin 2\chi'} + \sqrt{\frac{\cos^2 2\chi'}{\sin^2 2\chi'} + 1} \\ &= \frac{-\cos 2\chi'}{\sin 2\chi'} + \frac{1}{\sin 2\chi'} = \frac{1 - \cos 2\chi'}{\sin 2\chi'}\end{aligned} \tag{A7}$$

using simple trigonometry the double angle can be expressed in terms of χ'

$$\tan \varphi_1 = \frac{2\sin^2 \chi'}{2\sin \chi' \cos \chi'} = \tan \chi' \tag{A8}$$

From this we may conclude that the direction of the first eigenvector φ_1 equals the angle χ' and thus is identical to the average gradient orientation ϕ'.

$$\varphi_1 \equiv \chi \equiv \phi \tag{A9}$$