

## $\eta$ -RICCI SOLITONS ON KENMOTSU MANIFOLDS ADMITTING GENERAL CONNECTION

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ABSTRACT. The object of the present paper is to study  $\eta$ -Ricci soliton on Kenmotsu manifold with respect to general connection.

### 1. Introduction

Throughout our paper, we denote Schouten-Van Kampen connection, general connection, Zamkovoy connection, generalized Tanaka-Webster connection, quarter-symmetric connection, Levi-civita connection by the symbols  $\nabla^s$ ,  $\nabla^G$ ,  $\nabla^z$ ,  $\nabla^T$ ,  $\nabla^a$ ,  $\nabla$  respectively.

Recently, Biswas and Baishya ([3], [4]) introduced and studied a new connection, named general connection in the context of Sasakian geometry. The general connection  $\nabla^G$  is defined as

$$(1) \quad \nabla_X^G Y = \nabla_X Y + k_1 [(\nabla_X \eta)(Y) \xi - \eta(Y) \nabla_X \xi] + k_2 \eta(X) \phi Y,$$

for all  $U, V \in \chi(M)$  and the pair  $(\lambda, \mu)$  being real constants. The beauty of such connection  $\nabla^G$  lies in the fact that it has the flavour of

- (i) quarter symmetric metric connection ([11], [5]) for  $(k_1, k_2) \equiv (0, -1)$ ;
- (ii) Schouten-Van Kampen connection [26] for  $(k_1, k_2) \equiv (1, 0)$ ;
- (iii) Tanaka Webster connection [29] for  $(k_1, k_2) \equiv (1, -1)$  and

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(iv) Zamkovoy connection [33] for  $(k_1, k_2) \equiv (1, 1)$ .

In 1982, Hamilton [24] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([19], [20]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$(2) \quad \frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton  $(g, V, \lambda)$  on a Riemannian manifold  $(M, g)$  is a generalization of an Einstein metric such that ([25], [13], [23] [16])

$$(3) \quad (\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

where  $S$  is the Ricci tensor,  $\mathcal{L}_V$  is the Lie derivative operator along the vector field  $V$  on  $M$  and  $\lambda$  is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to  $\lambda$  being negative, zero or positive, respectively. As a generalization of Ricci soliton, the notion of  $\eta$ -Ricci soliton was introduced by Cho and Kimura [13]. They have studied Ricci soliton of real hypersurfaces in a non-flat complex space form and defined  $\eta$ -Ricci soliton, which satisfies the equation

$$(4) \quad (\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0,$$

where  $\lambda$  and  $\mu$  are real numbers. In particular, if  $\mu = 0$ , then the notion of  $\eta$ -Ricci soliton  $(g, V, \lambda, \mu)$  reduces to the notion of Ricci soliton  $(g, V, \lambda)$ . Recently,  $\eta$ -Ricci solitons have been studied by various authors for details we refer ([9], [7], [18], [17], [28] and the reference therein).

This paper is structured as follows: After introduction, a short description of Kenmotsu manifold is given in section 2. In section 3, we have studied some properties of Kenmotsu manifold admitting general connection. Section 4 deals with  $\eta$ -Ricci solitons and Ricci solitons on Kenmotsu manifolds with respect to the general connection admitting some curvature restrictions. Finally in section 6, we have given a non trivial example of  $\eta$ -Ricci solitons and found out the relation between the scalars  $\lambda$  and  $\mu$  on Kenmotsu manifolds with respect to the general connection.

## 2. Preliminaries

Let  $M$  be an  $n(= 2m + 1)$ -dimensional differentiable manifold, it said to be an almost contact Riemannian manifold if either its structural group can be reduced to  $U(n) \times \{I\}$  or there is an almost contact metric structure  $(\phi, \xi, \eta, g)$  consisting of a vector field  $\xi$ ,  $(1, 1)$  tensor field  $\phi$ , 1-form  $\eta$  and Riemannian metric  $g$  satisfying

$$(5) \quad \phi^2 X = -X + \eta(X)\xi,$$

$$(6) \quad \eta(\xi) = 1, \eta(\phi X) = 0, \phi\xi = 0.$$

In Kenmotsu manifolds  $(M^n, g)$  the following relations hold ([17], [28], [14], [30], [1]).

$$(7) \quad g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X), \forall X, Y \in TM$$

$$(8) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), g(QX, Y) = S(X, Y)$$

$$(9) \quad S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y)$$

$$(10) \quad (\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi(X),$$

$$(11) \quad \nabla_X \xi = X - \eta(X)\xi,$$

$$(12) \quad (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y).$$

Further, for Kenmotsu manifold with structure  $(\phi, \xi, \eta, g)$ , following relations holds

$$(13) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(14) \quad S(X, \xi) = -(n - 1)\eta(X),$$

$$(15) \quad R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X,$$

$$(16) \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi$$

$$(17) \quad Q\xi = -(n - 1)\xi,$$

where  $S$  and  $Q$  are Ricci tensor and Ricci operator.

### 3. Kenmotsu manifold admitting general connection

By the help of (5), (11) and (12) the relation (1) reduces to

$$(18) \quad \nabla_X^G Y = \nabla_X Y + k_1 [g(X, Y) \xi - \eta(Y) X] + k_2 \eta(X) \phi Y.$$

Substituting  $Y$  by  $\xi$  in (18) and using (5), (11)

$$(19) \quad \nabla_X^G \xi = (1 - k_1) (X - \eta(X) \xi).$$

Now on an account of (5), (6), (7), (10), (11), (12) and (18), we get the following

$$(20) \quad \nabla_X^G \eta(Y) = \eta(\nabla_X Y) + g(X, Y) - \eta(X) \eta(Y),$$

$$(21) \quad \nabla_X^G (\phi Y) = \nabla_X (\phi Y) + k_1 g(X, \phi Y) \xi - k_2 \eta(X) Y + k_2 \eta(X) \eta(Y) \xi,$$

$$(22) \quad \begin{aligned} \nabla_X^G g(Y, Z) &= g(\nabla_X Y, Z) + k_1 \eta(Z) g(X, Y) - k_1 \eta(Y) g(X, Z) \\ &\quad + k_2 \eta(X) g(\phi Y, Z) + g(Y, \nabla_X Z) + \lambda \eta(Y) g(X, Z) \\ &\quad - k_1 \eta(Z) g(Y, X) + k_2 \eta(X) g(Y, \phi Z). \end{aligned}$$

Now we know that

$$(23) \quad R^G(X, Y)Z = \nabla_X^G \nabla_Y^G Z - \nabla_Y^G \nabla_X^G Z - \nabla_{[X, Y]}^G Z.$$

By using (18), (19), (20), (21) and (22) we obtain the following

$$(24) \quad \begin{aligned} \nabla_{[X, Y]}^G Z &= \nabla_{[X, Y]} Z + k_2 \eta(\nabla_X Y) \phi Z - k_2 \eta(\nabla_Y X) \phi Z \\ &\quad + k_1 [g(\nabla_X Y, Z) \xi - g(\nabla_Y X, Z) \xi - \eta(Z) \nabla_X Y + \eta(Z) \nabla_Y X], \end{aligned}$$

$$\begin{aligned}
 & \nabla_X^G \nabla_Y^G Z \\
 = & \nabla_X (\nabla_Y Z) + k_1 g(X, \nabla_Y Z) \xi - k_1 \eta(\nabla_Y Z) X + k_2 \eta(X) \phi \nabla_Y Z \\
 & + k_1 g(\nabla_X Y, Z) \xi + k_1^2 \eta(Z) g(X, Y) \xi - k_1^2 \eta(Y) g(X, Z) \xi \\
 & + k_1 g(Y, \nabla_X Z) \xi + k_1^2 \eta(Y) g(X, Z) \xi - k_1^2 \eta(Z) g(Y, X) \xi \\
 & + k_1 (1 - k_1) g(Y, Z) X - k_1 (1 - k_1) g(Y, Z) \eta(X) \xi \\
 & + k_1 k_2 \eta(X) g(Y, \phi Z) \xi - k_1 \eta(\nabla_X Z) Y - k_1 g(X, Z) Y \\
 & + k_1 \eta(X) \eta(Z) Y + k_1 k_2 \eta(X) g(\phi Y, Z) \xi - k_1 \eta(Z) \nabla_X Y \\
 & - k_1^2 \eta(Z) g(X, Y) \xi + k_1 \eta(Z) \eta(Y) X - k_1 k_2 \eta(Z) \eta(X) \phi Y \\
 & + k_2 \eta(\nabla_X Y) \phi Z + k_2 g(X, Y) \phi Z - k_2 \eta(X) \eta(Y) \phi Z + k_2 \eta(Y) \nabla_X \phi Z \\
 & + k_1 k_2 \eta(Y) g(X, \phi Z) \xi - k_2^2 \eta(X) \eta(Y) Z + k_2^2 \eta(X) \eta(Y) \eta(Z) \xi.
 \end{aligned}
 \tag{25}$$

Interchanging  $Y$  and  $X$  in (25)

$$\begin{aligned}
 & \nabla_Y^G \nabla_X^G Z \\
 = & \nabla_Y (\nabla_X Z) + k_1 g(Y, \nabla_X Z) \xi - k_1 \eta(\nabla_X Z) Y + k_2 \eta(Y) \phi \nabla_X Z \\
 & + k_1 g(\nabla_Y X, Z) \xi + k_1^2 \eta(Z) g(Y, X) \xi - k_1^2 \eta(X) g(Y, Z) \xi \\
 & + k_1 g(X, \nabla_Y Z) \xi + k_1^2 \eta(X) g(Y, Z) \xi - k_1^2 \eta(Z) g(X, Y) \xi \\
 & + k_1 k_2 \eta(Y) g(X, \phi Z) \xi + k_1 (1 - k_1) g(X, Z) Y \\
 & - k_1 (1 - k_1) g(X, Z) \eta(Y) \xi - k_1 \eta(\nabla_Y Z) X \\
 & - k_1 g(Y, Z) X + k_1 \eta(Y) \eta(Z) X + k_1 k_2 \eta(Y) g(\phi X, Z) \xi \\
 & - k_1 \eta(Z) \nabla_Y X - k_1^2 \eta(Z) g(Y, X) \xi + k_1 \eta(Z) \eta(X) Y - k_1 k_2 \eta(Z) \eta(Y) \phi X \\
 & + k_2 \eta(\nabla_Y X) \phi Z + k_2 g(Y, X) \phi Z - k_2 \eta(Y) \eta(X) \phi Z + k_2 \eta(X) \nabla_Y \phi Z \\
 & + k_1 k_2 \eta(X) g(Y, \phi Z) \xi - k_2^2 \eta(Y) \eta(X) Z + k_2^2 \eta(Y) \eta(X) \eta(Z) \xi.
 \end{aligned}
 \tag{26}$$

Now in reference of (24), (25) and (26) we get from (23)

$$\begin{aligned}
 & R^G(X, Y) Z \\
 = & R(X, Y) Z + (k_1 k_2 - k_2) [\eta(Y) g(X, \phi Z) \xi - \eta(X) g(Y, \phi Z) \xi] \\
 & + (k_1 k_2 - k_2) [\eta(Y) \eta(Z) \phi X - \eta(X) \eta(Z) \phi Y] \\
 & + k_1 (1 - k_1) [g(X, Z) \eta(Y) \xi - g(Y, Z) \eta(X) \xi] \\
 (27) \quad & + k_1 [2 - k_1] g(Y, Z) X - k_1 [2 - k_1] g(X, Z) Y.
 \end{aligned}$$

On contracting (27), we obtain the Ricci tensor  $S^G$  of a Kenmotsu manifold with respect to the general connection  $\nabla^G$  as

$$(28) \quad \begin{aligned} S^G(Y, Z) &= S(Y, Z) + k_2(1 - k_1)g(Y, \varphi Z) \\ &+ k_1(1 - k_1)\eta(Y)\eta(Z) + [2nk_1 - nk_1^2 - 3k_1 + 2k_1^2]g(Y, Z). \end{aligned}$$

This gives

$$(29) \quad \begin{aligned} Q^G Y &= QY - k_2(1 - k_1)\phi Y \\ &+ [2nk_1 - nk_1^2 - 3k_1 + 2k_1^2]Y + k_1(1 - k_1)\eta(Y)\xi. \end{aligned}$$

Again contracting (28) over  $Y$  and  $Z$  we obtain

$$(30) \quad r^G = r + k_1(1 - k_1) + n[2nk_1 - nk_1^2 - 3k_1 + 2k_1^2].$$

Replacing  $Y$  by  $\xi$  in (28) we get

$$(31) \quad S^G(Y, \xi) = (-n + 1)(1 - k_1)^2\eta(Y).$$

By the help of (13), (15), (16) and (27) we obtain the the following

$$(32) \quad \begin{aligned} R^G(\xi, Y)Z &= (1 - k_1)^2\eta(Z)Y - (1 - k_1)g(Y, Z)\xi \\ &- k_2(k_1 - 1)[g(Y, \varphi Z)\xi + \eta(Z)\phi Y] \\ &+ k_1(1 - k_1)\eta(Z)\eta(Y)\xi, \end{aligned}$$

$$(33) \quad \begin{aligned} R^G(Y, Z)\xi &= (1 - k_1)^2\eta(Y)Z - (1 - k_1)^2\eta(Z)Y \\ &+ k_2(k_1 - 1)[\eta(Z)\phi Y - \eta(Y)\phi Z], \end{aligned}$$

$$(34) \quad \begin{aligned} &R^G(Y, \xi)Z \\ &= (1 - k_1)g(Y, Z)\xi - (1 - k_1)^2\eta(Z)Y \\ &+ k_2(k_1 - 1)[g(Y, \phi Z)\xi + \eta(Z)\phi Y] - k_1(1 - k_1)\eta(Z)\eta(Y)\xi. \end{aligned}$$

Thus we can state the following

**THEOREM 3.1.** *Let  $M$  be an  $n$ -dimensional Kenmotsu manifold admitting general connection  $\nabla^G$ . Then (i) the curvature tensor  $R^G$  of  $\nabla^G$  is given by (27), (ii) the Ricci tensor  $S^G$  of  $\nabla^G$  is given by (28) and (iii) the scalar curvature  $r^G$  of  $\nabla^G$  is given by (30).*

**4.  $\eta$ -Ricci solitons on Kenmotsu manifolds admitting general connection**

We consider a Kenmotsu manifold with respect to general connection admitting an  $\eta$ -Ricci soliton  $(g, \xi, \lambda, \mu)$ . Then from (4), it is obvious that

$$(35) \quad (\mathcal{L}_\xi^G g)(X, Y) + 2S^G(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Now, we express the Lie derivative along  $\xi$  on  $M$  with respect to general connection as follows:

$$(36) \quad \begin{aligned} (\mathcal{L}_\xi^G g)(X, Y) &= \mathcal{L}_\xi^G g(X, Y) - g(\mathcal{L}_\xi^G X, Y) - g(X, \mathcal{L}_\xi^G Y) \\ &= \mathcal{L}_\xi^G g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y]). \end{aligned}$$

By the help of (1) and (36), we obtain

$$(37) \quad \begin{aligned} &(\mathcal{L}_\xi^G g)(X, Y) \\ &= \nabla_\xi^G g(X, Y) - g(\nabla_\xi^G X - \nabla_X^G \xi - k_1(X - \eta(X)\xi - k_2\phi X), Y) \\ &- g(X, \nabla_\xi^G Y - \nabla_Y^G \xi - k_1(Y - \eta(Y)\xi - k_2\phi Y)). \end{aligned}$$

Using (18) and (19), the relation (37) reduces to

$$(38) \quad (\mathcal{L}_\xi^G g)(X, Y) = 2g(X, Y) - 2\eta(X)\eta(Y).$$

By virtue of (38), the equation (35) takes the following form

$$(39) \quad S^G(X, Y) = [1 - \mu]\eta(X)\eta(Y) - [1 + \lambda]g(X, Y).$$

Setting  $X = Y = \xi$  in (39), we get

$$(40) \quad \mu + \lambda = (n - 1)(1 - k_1)^2.$$

Thus we can conclude that

**THEOREM 4.1.** *If  $(g, \xi, \lambda, \mu)$  is an  $\eta$ -Ricci soliton on Kenmotsu manifold with respect to quarter symmetric metric connection, then the  $\eta$ -Ricci soliton on  $M$  is expanding, steady or shrinking according as  $(n - 1) \begin{matrix} \geq \\ \leq \end{matrix} \mu$ .*

**THEOREM 4.2.** *If  $(g, \xi, \lambda, \mu)$  is an  $\eta$ -Ricci soliton on Kenmotsu manifold with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection, then the  $\eta$ -Ricci soliton on  $M$  is expanding, steady or shrinking according as  $\mu \begin{matrix} \leq \\ \geq \end{matrix} 0$ .*

DEFINITION 4.3. A Kenmotsu manifold is said to be quasi-conformal like flat with respect to general connection if

$$(41) \quad \omega^G(X, Y)Z = 0,$$

where  $\omega^G$  is the quasi-conformal like curvature tensor with respect to general connection and is given ([2]) by

$$(42) \quad \begin{aligned} \omega^G(X, Y)Z &= R^G(X, Y)Z + a[S^G(Y, Z)X - S^G(X, Z)Y] \\ &\quad - \frac{cr^G}{n} \left( \frac{1}{n-1} + a + b \right) [g(Y, Z)X - g(X, Z)Y] \\ &\quad + b[g(Y, Z)Q^GX - g(X, Z)Q^GY], \end{aligned}$$

for all  $X, Y$  &  $Z \in \chi(M)$ , the set of all vector field of the manifold  $M$ , where scalar triple  $(a, b, c)$  are real constants. The beauty of such *curvature tensor* lies in the fact that it has the flavour of Riemann curvature tensor  $R^G$  if the scalar triple  $(a, b, c) \equiv (0, 0, 0)$ , conformal curvature tensor  $C^G$  ([10]) if  $(a, b, c) \equiv (-\frac{1}{n-2}, -\frac{1}{n-2}, 1)$ , conharmonic curvature tensor  $L^G$  ([12]) if  $(a, b, c) \equiv (-\frac{1}{n-2}, -\frac{1}{n-2}, 0)$ , concircular curvature tensor  $E^G$  ([8], p. 84) if  $(a, b, c) \equiv (0, 0, 1)$ , projective curvature tensor  $P^G$  ([8], p. 84) if  $(a, b, c) \equiv (-\frac{1}{n-1}, 0, 0)$  and  $m$ -projective curvature tensor  $H^G$  [21], if  $(a, b, c) \equiv (-\frac{1}{2n-2}, -\frac{1}{2n-2}, 0)$ , the  $W_1^G$ -curvature tensor [22] if  $(a, b, c) = (\frac{1}{(n-1)}, 0, 0)$ , the  $W_2^G$ -curvature tensor [21], if  $(a, b, c) = (0, -\frac{1}{(n-1)}, 0)$ , the  $W_4^G$ -curvature tensor [22], if  $(a, b, c) = (0, 0, \frac{n}{r})$ .

Contracting  $Y$  over  $Z$  in the above relation, we have

$$(43) \quad \begin{aligned} S^G(X, W) &= -\frac{ar^G}{[1 - a + bn - b]}g(X, W) \\ &\quad + \frac{cr^G}{n} \left( \frac{1}{n-1} + a + b \right) \frac{(n-1)}{[1 - a + bn - b]}g(X, W). \end{aligned}$$



Using (39) in (43), we find

$$\begin{aligned}
 & [1 - \mu] \eta(X) \eta(W) - [1 + \lambda] g(X, W) \\
 &= -\frac{ar^G}{[1 - a + bn - b]} g(X, W) \\
 & \quad + \frac{cr^G}{n} \left( \frac{1}{n-1} + a + b \right) \frac{(n-1)}{[1 - a + bn - b]} g(X, W).
 \end{aligned}
 \tag{44}$$

Putting  $X = W = \xi$  in (44), we get

$$\begin{aligned}
 [\mu + \lambda] &= \frac{a[r + k_1(1 - k_1) + n(2nk_1 - nk_1^2 - 3k_1 + 2k_1^2)]}{[1 - a + bn - b]} \\
 & \quad - \frac{c[r + k_1(1 - k_1) + n(2nk_1 - nk_1^2 - 3k_1 + 2k_1^2)]}{n} \\
 & \quad \left( \frac{1}{n-1} + a + b \right) \frac{(n-1)}{[1 - a + bn - b]}.
 \end{aligned}
 \tag{45}$$

Again, putting  $Y = Z = \xi$ , in (42) and then using (39), we get

$$[\mu + \lambda] = -\frac{1}{b}(1 - k_1)^2 [an - a + 1 - nab + ab - n].
 \tag{46}$$

This leads to the following:

**THEOREM 4.4.** *If  $(g, \xi, \lambda, \mu)$  is an  $\eta$ -Ricci soliton on the quasi-conformal like flat Kenmotsu manifold admitting general connection  $\nabla^G$ , then the scalars  $\lambda$  and  $\mu$  are related by (45).*

**THEOREM 4.5.** *Let  $(g, \xi, \lambda, \mu)$  be a  $\eta$ -Ricci soliton on Kenmotsu manifold with respect to quarter symmetric metric connection. Then the following relation hold*

- (i) *the  $\eta$ -Ricci soliton on  $M$  for each of  $C^G(X, \xi)\xi = 0$  and  $L^G(X, \xi)\xi = 0$  is expanding, steady or shrinking according as  $\left(\frac{n^3 - 4n^2 + 6n - 3}{n-2}\right) \begin{matrix} \leq \\ > \end{matrix} 0$ .*
- (ii) *the  $\eta$ -Ricci soliton on  $M$  having  $H^G(X, \xi)\xi = 0$  is expanding, steady or shrinking according as  $\left(\frac{4n^3 - 10n^2 + 8n - 1}{2n-2}\right) \begin{matrix} \leq \\ > \end{matrix} 0$ .*

**THEOREM 4.6.** *If  $(g, \xi, \lambda, \mu)$  is an  $\eta$ -Ricci soliton admitting  $\omega^G(X, \xi)\xi = 0$  on Kenmotsu manifold, then with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection, the  $\eta$ -Ricci soliton on is expanding, steady or shrinking according as  $\mu \begin{matrix} \leq \\ > \end{matrix} 0$ .*

Now, let  $(g, \xi, \lambda, \mu)$  be an  $\eta$ -Ricci soliton on Kenmotsu manifold admitting general connection such that  $V$  is pointwise collinear with  $\xi$ , that is,  $V = \beta\xi$ , where  $\beta$  is a function. Then obviously (35) holds and we have

$$(47) \quad \begin{aligned} 0 &= (X\beta)\eta(Y) + (Y\beta)\eta(X) + 2\beta g(X, Y) - 2\beta\eta(X)\eta(Y) \\ &\quad + 2S^G(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y). \end{aligned}$$

Putting  $Y = \xi$  in (47) and using (5), (6) and (31) it follows that

$$(48) \quad (X\beta) + (\xi\beta)\eta(X) - 2(n-1)(1-k_1)^2\eta(X) + 2\lambda\eta(X) + 2\mu\eta(X) = 0.$$

Putting  $X = \xi$  in (48) and using (5) and (6) we have

$$(49) \quad (\xi\beta) - (n-1)(1-k_1)^2 + \lambda + \mu = 0.$$

Using (49) in (48), we get

$$(50) \quad (X\beta) - [(n-1)(1-k_1)^2 - \lambda - \mu]\eta(X) = 0.$$

Differentiating (50) covariantly with respect to  $Y$ , we find

$$(51) \quad -[(n-1)(1-k_1)^2 - \lambda - \mu](\nabla_Y\eta)(X) = 0.$$

From (50) and (51), we find

$$(52) \quad [(n-1)(1-k_1)^2 - \lambda - \mu]d\eta = 0.$$

Since  $d\eta \neq 0$ , therefore

$$(53) \quad \lambda + \mu = (n-1)(1-k_1)^2.$$

Substituting (53) in (50), we conclude that  $\beta$  is a constant. Hence it is verified from (47) that

$$(54) \quad S^G(X, Y) = -(\beta + \lambda)g(X, Y) + (\beta - \mu)\eta(X)\eta(Y)$$

In view of (28), the relation (54) takes the form

$$(55) \quad \begin{aligned} &S(X, Y) \\ &= -k_2(1-k_1)g(X, \varphi Y) - [2nk_1 - nk_1^2 - 3k_1 + 2k_1^2 + \beta + \lambda]g(X, Y) \\ &\quad + [\beta - \mu - k_1(1-k_1)]\eta(X)\eta(Y). \end{aligned}$$

Thus we can state

**THEOREM 4.7.** *If  $(g, \xi, \lambda, \mu)$  is an  $\eta$ -Ricci soliton on Kenmotsu manifold with respect to general connection, such that  $V$  is pointwise collinear with  $\xi$ , then  $V$  is a constant multiple of  $\xi$  and the manifold is a generalized  $\eta$ -Einstein manifold with respect to the Levi-Civita connection.*

**COROLLARY 4.8.** *Kenmotsu manifold with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection admitting an  $\eta$ -Ricci soliton  $(g, \xi, \lambda)$  whose potential vector field is pointwise collinear with vector field  $\xi$ , is an  $\eta$ -Einstein manifold with respect to the Levi-Civita connection.*

In particular, for  $\mu = 0$ , (53) yields

$$(56) \quad \lambda = (n - 1)(1 - k_1)^2.$$

Thus we can state

**THEOREM 4.9.** *If Kenmotsu manifold with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection possess a Ricci soliton  $(g, \xi, \lambda)$  whose potential vector field is pointwise collinear with vector field  $\xi$ , then such soliton is always steady .*

**THEOREM 4.10.** *If Kenmotsu manifold with respect to quarter symmetric metric connection possess a Ricci soliton  $(g, \xi, \lambda)$  whose potential vector field is pointwise collinear with vector field  $\xi$ , then such soliton is always expanding.*

### 5. Example

By the help of [3] we introduce an example of 3-dimensional Kenmotsu manifold with respect to Generalised Tanaka-Webster connection. Choosing the linearly independent vector field as

$$(57) \quad e_1 = e^{-z} \frac{\partial}{\partial x}, e_2 = e^{-z} \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$$

at each point of 3-dimensional manifold  $M$ , where  $M = \{(x, y, z) \in R^3 : x \neq 0\}$ . Let  $g$  be the Riemannian metric defined by

$$(58) \quad g(e_i, e_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \text{ for } i, j = 1, 2, 3$$

The 1-form  $\eta$  is defined by  $g(Y, e_3) = \eta(Y)$ , and the (1, 1) tensor field  $\phi$  is defined by  $\phi(e_1) = -e_2$ ,  $\phi(e_2) = e_1$ , and  $\phi(e_3) = 0$ . Let  $\nabla$  be the

Levi-Civita connection with respect to the Riemannian metric  $g$ . Then we have

$$(59) \quad [e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

Considering  $e_3 = \xi$  and using Koszul's formula we get

$$(60) \quad \begin{aligned} \nabla_{e_1} e_3 &= e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = -e_3, \\ \nabla_{e_2} e_3 &= e_2, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \\ \nabla_{e_3} e_1 &= 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

By the help of (59) and (60), we obtain the following

$$(61) \quad \begin{aligned} R(e_1, e_2) e_3 &= 0; R(e_1, e_2) e_2 = -e_1; R(e_1, e_3) e_3 = -e_1; \\ R(e_2, e_2) e_3 &= 0; R(e_2, e_3) e_3 = -e_2; R(e_2, e_1) e_1 = -e_2; \\ R(e_3, e_2) e_2 &= -e_3; R(e_3, e_1) e_2 = 0; R(e_3, e_1) e_1 = -e_3; \\ R(e_3, e_2) e_1 &= R(e_2, e_1) e_3 = R(e_1, e_3) e_2 = 0. \end{aligned}$$

Using (18), (23), (59) and (60), we can easily calculate the following

$$(62) \quad \begin{aligned} \nabla_{e_1}^G e_2 &= 0; \nabla_{e_1}^G e_1 = -e_3 + k_1 e_3; \nabla_{e_1}^G e_3 = e_1 - k_1 e_1. \\ \nabla_{e_2}^G e_3 &= e_2 - k_1 e_2; \nabla_{e_2}^G e_1 = 0; \nabla_{e_2}^G e_2 = -e_3 + k_1 e_3 \\ \nabla_{e_3}^G e_2 &= +k_2 e_1; \nabla_{e_3}^G e_3 = 0; \nabla_{e_3}^G e_1 = -k_2 e_2, \end{aligned}$$

$$(63) \quad \begin{aligned} R^G(e_1, e_2) e_2 &= -e_1 + 2k_1 e_1 + k_1^2 e_1; R^G(e_1, e_2) e_3 = 0; \\ R^G(e_2, e_3) e_3 &= -k_2 e_1 + k_1 k_2 e_1 - e_2 + k_1 e_2; \\ R^G(e_3, e_1) e_1 &= -e_3 + k_1 e_3; R^G(e_3, e_2) e_2 = -e_3 + k_1 e_3; \\ R^G(e_2, e_1) e_1 &= -e_2 + 2k_1 e_2 - k_1^2 e_2; \\ R^G(e_1, e_3) e_3 &= k_2 e_2 - k_1 k_2 e_2 - e_1 + k_1 e_1; \\ R^G(e_1, e_3) e_2 &= -k_2 e_3 + k_1 k_2 e_3; R^G(e_2, e_1) e_3 = 0; \\ R^G(e_1, e_3) e_3 &= k_2 e_2 - k_1 k_2 e_2 - e_1 + k_1 e_1, \end{aligned}$$

$$(64) \quad \begin{aligned} S^G(e_1, e_1) &= k_1^2 + 3k_1 - 2; \\ S^G(e_2, e_2) &= k_1^2 + 3k_1 - 2; \\ S^G(e_3, e_3) &= -2 + 2k_1 \end{aligned}$$

and

$$(65) \quad r^G = -6 + 2k_1^2 + 8k_1.$$

Thus it can be seen that equation (33) is satisfied. Now from (39) and (64) we get

$$(66) \quad \mu + \lambda = 2(1 - k_1).$$

Hence the manifold under consideration satisfies Theorem 2 and Theorem 3.

**THEOREM 5.1.** *There exists a Kenmotsu manifold  $(M^3, g)$  with respect to quarter symmetric metric connection possessing an  $\eta$ -Ricci soliton  $(\xi, \lambda, \mu)$  which is expanding, steady or shrinking according as  $2 \begin{smallmatrix} \geq \\ \leq \end{smallmatrix} \mu$ .*

**THEOREM 5.2.** *There exists a Kenmotsu manifold  $(M^3, g)$  with respect to each of (i) Shouten-Van Kampen connection, (ii) Tanaka Webster connection and (iii) Zamkovoy connection admitting an  $\eta$ -Ricci soliton  $(\xi, \lambda, \mu)$  which is always steady.*

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