ETBR: Extended Truncated Balanced Realization Method for On-Chip Power Grid Network Analysis *

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ABSTRACT

In this paper, we present a novel simulation approach for power grid network analysis. The new approach, called *ETBR* for extended truncated balanced realization, is based on model order reduction techniques to reduce the circuit matrices before the simulation. Different from the (improved) extended Krylov subspace methods EKS/IEKS [15, 2], ETBR performs fast truncated balanced realization on response Grammian to reduce the original system with the similar computation costs of EKS. ETBR also avoids the adverse explicit moment representation of the input signals. Instead, it uses spectrum representation of input signals by fast Fourier transformation. As a result, ETBR is more flexible for different types of input sources and can better capture the high frequency contents than EKS, and this leads to more accurate results especially for fast changing input signals. Experimental results on a number of large networks (up to one million nodes) show that, given the same order of the reduced model, ETBR is indeed more accurate than the EKS method especially for input sources rich in high-frequency components. ETBR also shows similar computation costs of EKS and less memory consumption than EKS.

1. INTRODUCTION

Reliable on-chip power delivery is one of the major challenges for 90nm and below VLSI technology. This situation becomes worse as technology continues to scale owning to the several reasons: First, technology scaling results in decreased interconnect width and increased interconnect resistance in a power supply network. Second, increased device density leads to increased supply current density on a chip. Third, a higher clock frequency gives rise to more significant inductance effect. At the same time, supply voltage continues to decrease, which results in a decreased noise margin for signal transition, and makes transistor more vulnerable to supply voltage degradation. So efficient verification of power integrity becomes critical for final design closure.

Many research works have been done on efficient simulation of on-chip power grid networks. Methods such as multigrid-like [1, 5], hierarchical [16, 2], partition-based [3], fast iterative [13] and random walk based [12, 4] approaches help improve scalability of power gird network analysis. Another approach to fast power grid analysis is based on so-called extended Krylov subspace based methods (EKS) [15, 2]. In EKS methods, both a power grid system and its input signals are used to reduce the original circuits before the simulation. Due to efficiency of Krylov subspace based reduction techniques, EKS/IEKS can deal with very large power grid circuits. But EKS method also suffers several shortcomings. First, the methods need to represent the input signals in the Taylor expansion form or the moment form with respect to complex frequency variable s. This can lead to less accurate results when the input signals are fast changing waveforms with many spike-like shapes. Such high-frequency bearing input waveforms cannot be represented accurately using the moment form owning to the well-known problems in explicit moment matching methods [8, 14]. Second, EKS is based on the Krylov subspace method to reduce the circuit matrices. The Krylov subspace approach leads to localized accuracy due to moment-matching property. Multiple-point moment matching will result in larger reduced systems, which degrades the simulation efficiency.

In this paper, we propose a novel model order reduction based simulation approach. This approach, called *ETBR* for extended truncated balanced realization, is based on the similar idea of the EKS method, where both a system and its input signals are used to reduce the original circuit matrices. But different from the (improved) extended Krylov subspace methods, EKS/IEKS [15, 2], ETBR performs fast truncated balanced realization, which is more accurate than Krylov subspace method used in EKS method, on response Grammian to reduce the original system while with the similar computation costs of EKS. ETBR also avoids the explicit moment representation of the input signals. Instead, it uses spectrum representation of input signals by fast Fourier transformation. As a result, ETBR is much more flexible for different types of input sources and can better capture the high frequency contents than EKS and this leads to more accurate results for fast changing input signals. Experimental results, on a number of large networks up to one million nodes, show that ETBR is indeed more accurate than the EKS/IEKS method especially for current sources rich in high-frequency components. ETBR also shows similar computation costs of EKS and less memory consumption than EKS.

The rest of this paper is organized as follows: Section 2 presents the power grid models used in the paper. Section 3

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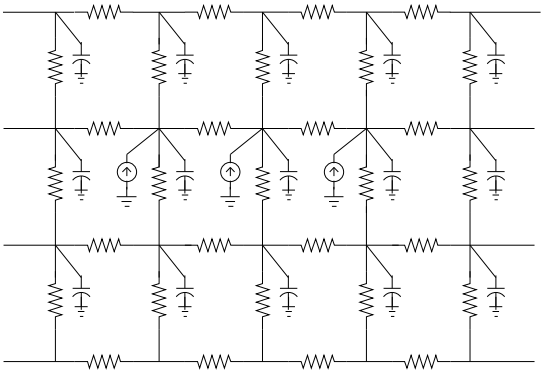


Figure 1: The power grid model used.

reviews the extended Krylov subspace method. Our new ETBR method is presented in Section 4. We also review the standard and fast balanced truncation reduction methods in this section. Section 5 presents the experimental results and Section 6 concludes this paper.

2. POWER GRID NETWORK MODELS

The power grid networks in this paper are modeled as RC networks with known time-variant current sources, which can be obtained by gate level logic simulations of the circuits. Fig. 1 shows the power grid models used in this paper. For a power grid, some nodes having known voltage are modeled as constant voltage sources. For C4 power grids, the known voltage nodes can be internal nodes inside the power grid. Given the current source vector, $u(t)$, the node voltages can be obtained by solving the following differential equations, which is formulated using modified nodal analysis (MNA) approach,

$$Gv(t) + C\frac{dv(t)}{dt} = Bu(t) \tag{1}$$

where $G \in R^{n\times n}$ is the conductance matrix, $C \in R^{n\times n}$ is the matrix resulting from storage elements. $v(t)$ is the vector of time-varying node voltages and branch currents of voltage sources. $u(t)$ is the vector of independent power sources, and B is the input selector matrix.

3. REVIEW OF EXTENDED KRYLOV SUBSPACE METHODS

Krylov subspace based model order reduction methods have been well-accepted in interconnect modeling. But the methods are less efficient for on-chip power supply network analysis, due to the presence of a large number of inputs (supply current excitations) and outputs (potential supply voltage degradation nodes). The main reason is that the computation costs and the projection vectors directly depend on the terminal count and the reduced model may increase very quickly with increasing terminal count.

To mitigate this problem, extended Krylov subspace method was proposed [15, 2]. The idea is to perform the reduction on both model and input in the moment form. As a result, the original multi-input and multi-output reduction problem becomes single-input and multi-output problem. One-side Krylov subspace method like PRIMA [7] can

Algorithm 1: (Improved) Extended Krylov Subspace method (EKS/IEKS)

Input: Circuit of G, C, B, u, reduction order q
Output: Reduced system matrices $\hat{G}$, $\hat{C}$, $\hat{B}$

1. $b_0 = Bu_0$, $m_0 = G^{-1}b_0$, $\hat{v}_0 = \alpha_0 m_0$, $\alpha_0 = norm(m_0)$
2. For $i = 1 : q$
3. $\quad v_i = G^{-1}(\Pi_{j=0}^{i-1}\alpha_j Bu_i - C(\hat{v}_1 + \alpha_{i-1}\sum_{j=0}^{i-1} h_{i-1,j}\hat{v}_j))$
4. $\quad$ For $k = 1 : i-1$
5. $\quad\quad h_{i,k} = \hat{v}_k^T v_i$
5. $\quad \bar{v}_i = v_i - \sum_{j=0}^{i} h_{i,j}\hat{v}_j$
6. $\quad$ If $norm(\bar{v}_i) < \epsilon$ break;
7. $\quad$ Else $\hat{v}_i = \frac{\bar{v}_i}{norm(\bar{v}_i)}$, $\alpha_i = \frac{1}{norm(\bar{v}_i)}$
8. $\hat{G} = V^T GV$,$\hat{C} = V^T CV$, $\hat{B} = V^T B$
9. End

be efficiently used for reducing such systems. The computation now is independent of the number of terminals in the network. IEKS [2] shows that for piece-wise linear (PWL) sources, which is approximated by sums of delayed ramps in Laplace domain, the $1/s$ and $1/s^2$ terms are always zero. So no moment shifting is required as in [15].

Specifically, instead of computing the vectors of the n-th order moments for explicit moment matching, extended Krylov subspace method constructs a modified Krylov subspace by orthonormalizing the moment vectors with current sources. For a RLC network in frequency domain,

$$(G + sC)v(s) = Bu(s) \tag{2}$$

where G is conductance matrix, C is storage element matrix. B is position matrix. If we expand $v(s)$ and $u(s)$ in moment form, we have

$$(G+sC)(m_0+m_1 s+m_2 s^2+...) = B(u_0+u_1 s+u_2 s^2+...) \tag{3}$$

The EKS algorithm essentially performs the orthonormalization on the response moments m_i. The EKS algorithm is shown in *Algorithm* 1.

After we obtain the reduced system, we can perform the transient simulation on the reduced system,

$$\hat{G}\hat{v}(t) + \hat{C}\frac{d\hat{v}(t)}{dt} = \hat{B}u(t) \tag{4}$$

Transient simulation can be carried out on (4), which will be very efficient due to reduced circuit matrices. After this, the original waveforms can be obtained by $v(t) = V\hat{v}(t)$.

4. NEW EXTENDED BALANCED TRUNCATION METHOD

As mentioned before, EKS is less accurate due to its explicit representation of the current sources in moment form and localized moment-matching property. In this paper, we propose an extended truncated balanced realization method, called ETBR, to mitigate the mentioned problems in EKS method.

The new method features two improvements. First, the input signals are represented in its spectrum form in frequency

domain directly by fast Fourier transformation. Second, fast balanced truncation method is used to perform the reduction, which has global accuracy [6, 11].

In the following, we first review the balanced truncation method and then the fast Grammian computation method.

4.1 Review of standard TBR

Given a system in a standard state-space form

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \tag{5}$$

where $A \in R^{n\times n}$, $B \in R^{n\times p}$, $C \in R^{p\times n}$, $y(t)$, $u(t) \in R^p$. The controllable and observable Grammians are the unique symmetric positive definite solutions to the Lyapunov equations.

$$\begin{array}{l} AX + XA^T + BB^T = 0 \\ A^TY + YA + C^TC = 0 \end{array} \tag{6}$$

Since the eigenvalues of the product XY are invariant under similarity transformation, we can perform a similarity transformation ($A_b = T^{-1}AT, B_b = T^{-1}B, C_b = CT$) to diagonalize the product XY such that

$$T^{-1}XYT = \Sigma = diag({\sigma_1}^2, {\sigma_2}^2, \ldots, {\sigma_n}^2) \tag{7}$$

where the Hankel singular values of the system (σ_k), are arranged in a descending order. If we partition the matrices as

$$\left[\begin{array}{c} W_1^T \\ W_2^T \end{array} \right] XY \left[\begin{array}{cc} V_1 & V_2 \end{array} \right] = \left[\begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right] \tag{8}$$

where $\Sigma_1 = diag({\sigma_1}^2, {\sigma_2}^2, \ldots, {\sigma_r}^2)$ are the first r largest eigenvalues of Grammian product XY and W_1 and V_1 are corresponding eigenvectors. A reduced model can be obtained as follows

$$\begin{array}{rcl} \dot{x}(t) & = & A_r x(t) + B_r u(t) \\ y(t) & = & C_r x(t) \end{array} \tag{9}$$

where $A_r = W_1^T A V_1$, $B_r = W_1^T B$, $C_r = CV_1$. The error in the transfer function of the order r approximation is bounded by $2\sum_{i=r+1}^{N} \sigma_k$. In the TBR procedure, the computational cost is dominated by solving Lyapunov equations $O(n^3)$, which makes it too expensive to apply to integrated circuits problems and thus an efficient Grammian approximation technique is highly appreciated.

4.2 Review of fast TBR method: Poor man's TBR

Existing Grammian approximation technique, PMTBR [10], is restricted to a state-space model (5) with $A = A^T$ and $C = B^T$. This is the case for RC and RL circuits. In this symmetrized case, it is easy to see that, both Grammians are equal and are obtained by solving the Lyapunov equation

$$AX + XA^T + BB^T = 0 \tag{10}$$

Since X is symmetric, it is orthogonally diagonalizable, i.e., there exists $T^{-1} = T^T$ such that $T^TXT = \Sigma$. Then, we have

$$T^TXXT = (T^TXT)(T^TXT) = (\Sigma)^2 \tag{11}$$

which means, in this symmetrized case, the eigenspace of Grammian product XX is exactly the eigenspace of each X and we only need to find the dominant invariant subspace of an approximated Grammian $\hat{X}$. In frequency domain, the Grammian X can also be computed from the expression

$$X = \int_{-\infty}^{+\infty} (j\omega I - A)^{-1} BB^T (j\omega I - A)^{-H} d\omega \tag{12}$$

where superscript H denotes Hermitian transpose. Let ω_k be kth sampling point. If we define

$$z_k = (j\omega_k I - A)^{-1} B \tag{13}$$

then X can be approximated as

$$\hat{X} = \sum z_k z_k^H = ZZ^H \tag{14}$$

where $Z = [z_1, z_2, \ldots, z_n]$. Since $\hat{X}$ is symmetric, it is orthogonally diagonalizable.

$$\hat{V}^T \hat{X} \hat{V} = \left[\begin{array}{c} \hat{V}_1^T \\ \hat{V}_2^T \end{array} \right] \hat{X} \left[\begin{array}{cc} \hat{V}_1 & \hat{V}_2 \end{array} \right] = \left[\begin{array}{cc} \hat{\Sigma}_1 & 0 \\ 0 & \hat{\Sigma}_2 \end{array} \right] \tag{15}$$

where $\hat{V}^T\hat{V} = I$. $\hat{V}$ converges to the eigenspaces of X and the dominant eigenvectors $\hat{V}_1$ can be used as the projection matrix in a model reduction approach ($A_r = \hat{V}_1^T A \hat{V}_1, B_r = \hat{V}_1^T B$).

4.3 Response Grammian and fast computation method

Follow the similar strategy of EKS method, we consider the input signals of the system into TBR based reduction framework so that efficient reduction can be done by converting an MIMO system into an SIMO system.

To this end, for linear system in (1), we first define the response Grammian at the frequency domain as:

$$X_r = \int_{-\infty}^{+\infty} (j\omega C + G)^{-1} Bu(j\omega) u^T(j\omega) B^T (j\omega C + G)^{-H} d\omega \tag{16}$$

To fast compute the response Grammian X_r, we can follow the similar strategy in PMTBR method. Specifically, let ω_k be kth sampling point over the frequency range. If we further define

$$z_k^r = (j\omega_k C + G)^{-1} Bu(j\omega_k) \tag{17}$$

then $\hat{X}$ can be computed as

$$\hat{X}_r = \sum z_k^r {z_k^r}^H = Z_r Z_r^H \tag{18}$$

where Z_r is a matrix whose columns are z_k^r. Since the approximate Grammian $\hat{X}_r$ is symmetric, we can obtain the project matrix by singular value decomposition of either $\hat{X}_r$ or Z_r. After this, we can reduce the original matrices into small ones and then perform the transient analysis on the reduced circuit matrices.

Notice that we need frequency response of input signal $u(j\omega_k)$ in (17). This can be obtained by fast Fourier transformation on the input signals in time domain.

4.4 Extended truncated balanced realization method: ETBR

Algorithm 2: Extended Truncated Balanced Realization method (ETBR)

Input: Circuit of G, C, B, $u(t)$, number of samples: q
Output: Reduced system matrices $\hat{G}$, $\hat{C}$, $\hat{B}$

1. Convert all the input signals $u(t)$ into $u(s)$ using FFT.
2. Select q frequency points $s_1, s_2, \ldots, s_q$ over the frequency range
3. Compute $z_k^r = (s_k C + G)^{-1} B u(s_k)$
4. Form the matrix $Z_r = [z_1^r, z_2^r, \ldots, z_q^r]$
5. Perform SVD on Z_r, $Z_r = V_r S_r U_r^T$
6. $\hat{G} = V_r^T G V_r$,$\hat{C} = V_r^T C V_r$, $\hat{B} = V_r^T B$
7. End

In this subsection, we give the algorithm flow of the proposed ETBR method, which is summarized in *Algorithm* 2.

After the algorithm, the reduced system in (4) can be simulated in time domain and the original waveforms can be obtained by $v(t) = V_r \hat{v}(t)$. Note that, like the EKS method, we use congruence transformation for the reduction process with orthogonal columns in the projection matrix (using Arnoldi or Arnoldi-like process), the reduced system must be stable. As far as simulation is concerned, this is good enough. If all the observable ports are also the current source nodes, i.e. $y(t) = B^T v(t)$, where $y(t)$ is the voltage vector at all observable ports, the reduced system is passive.

It was shown in [11] that the fast TBR method has the similar time complexity of the multiple-point Krylov subspace based reduction methods. For the single-point EKS method, ETBR should be slower if the equations (step 3 in both Algorithm 1 and Algorithm 2) are solved the same number of times in theory as EKS requires only one LU decomposition, which should dominate the computation costs. But practically, if we solve for z_k^r in step 3 in Algorithm 2 using non-LU-decomposition based methods, ETBR can take a similar CPU time of EKS for the same number of reduced order. Furthermore, ETBR may even use less memory by avoiding using LU decomposition as there is no need to store L and U matrices. Also it is well known that iterative linear solvers are more memory efficient than direct solvers using LU decomposition. We will have more discussions on CPU time and memory issues in next section of experimental results.

Comparing with the EKS/IKES method, ETBR has the following advantages and features:

1. More accurate over wide band frequency ranges due to the global error bound provided by the TBR based methods.
2. Avoid the explicit moment representation of the input signals, which can lead more accurate results than the EKS method when signals are rich in high frequency components.
3. Can deal with any type of time-domain and frequency-domain input signals. While the EKS method can only deal with input signals in piecewise linear form.
4. Easier to implement than the EKS method and thus more numerical stable than EKS as no explicit input moments are used.
5. Has the similar time complexity of EKS and use less memory than EKS.

Table 1: Test circuits

Test Ckts	#Nodes	#Sources
Ckt1	1,000	100
Ckt2	10,000	100
Ckt3	10,000	1,000
Ckt4	100,000	1,000
Ckt5	100,000	4,000
Ckt6	500,000	5,000
Ckt7	500,000	20,000
Ckt8	1,000,000	50,000

5. EXPERIMENTAL RESULTS

The proposed *ETBR* algorithm has been implemented using Matlab 7.0 and tested on a Intel Xeon 3.0GHz dual CPU workstation with 2GB memory under Linux environment. All the test circuits are randomly generated RC power grid networks up to one million nodes (R on the order of Ω and C on the order of pF), as shown in Table 1. Efficient matrix computations benefit from sparse matrix structure and a parser implemented by Python.

To solve circuits with one million nodes in Matlab, an external linear solver package UMFPACK [17] is used, which is linked with Matlab using Matlab mexFunction. For ETBR, we use a non-LU-decomposition solver in UMFPACK. While for EKS, the LU decomposition solver is used.

In sequel, we will compare our ETBR with IEKS [2], first in accuracy and then in CPU times. In all the test cases, to make a fair comparison, the reduction order q is set to 6 for IEKS and the number of frequency samples used for ETBR is also set to 6.

Fig. 2 shows the simulation results of ETBR and IEKS at the 200th node of Ckt2. The simulation errors compared with SPICE results are shown in Fig. 3. One of the input signal waveforms in both time domain and frequency domain is as shown in Fig. 4. Through Fig. 3, we can see that ETBR is more accurate than IEKS over the entire simulation time.

In the second testing case, we change the input signals so that they can have more fast changing spikes as shown in Fig. 7(a). In other words, current sources are rich in high-frequency components.

We find that ETBR's results are much better than EKS's as shown in Fig. 5. From the simulation errors comparison in Fig. 6, we can see that ETBR is almost 3× more accurate than IEKS (the maximum error: ETBR 0.003 vs IEKS 0.01). This is not a surprise for us if we notice that the input signals shown in Fig. 7(b) have much more high frequency components from 10^7MHz to 10^8MHz than the input signals shown in Fig. 4(b).

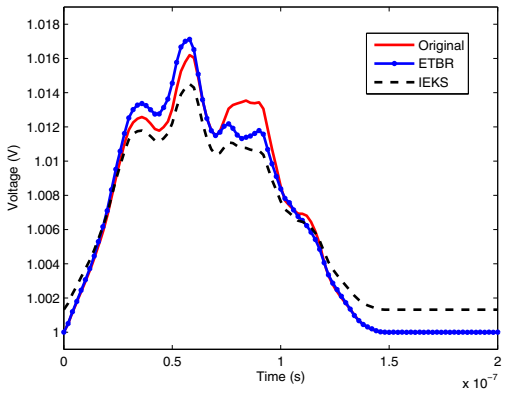


Figure 2: Transient waveform at the 200th node of Ckt2.

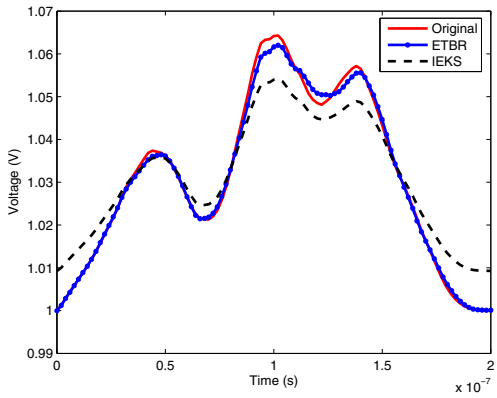


Figure 5: Transient waveform at the 200th node of Ckt2 with fast changing inputs.

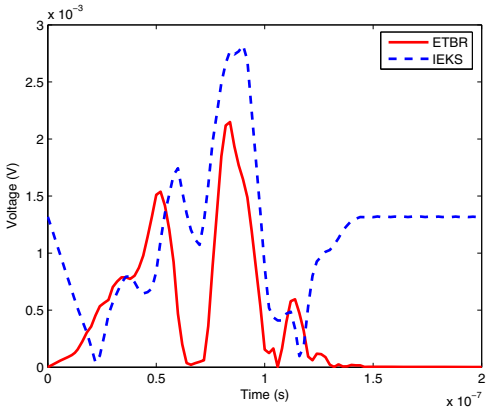


Figure 3: The simulation errors of ETBR and IEKS of Ckt2.

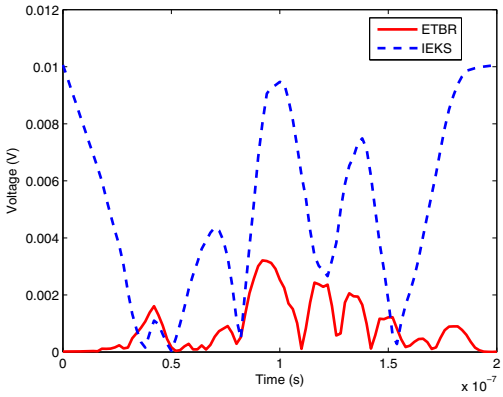
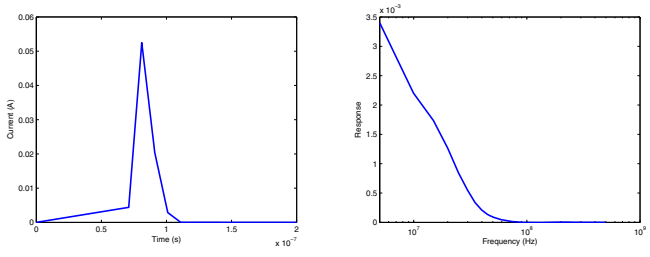
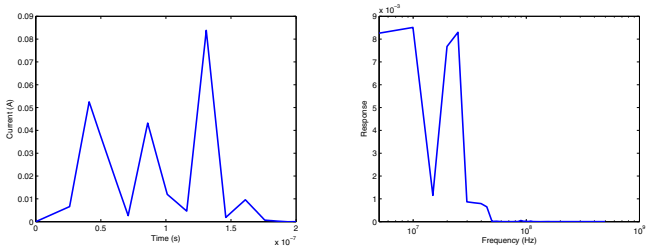


Figure 6: The simulation errors of ETBR and IEKS on Ckt2 with fast changing inputs.



(a) In time domain (b) In frequency domain

Figure 4: Transient waveform at the 5th current source of Ckt2.



(a) In time domain (b) In frequency domain

Figure 7: The transient waveform at the 5th current source of Ckt2.

Finally, we compare the CPU time of the two algorithms on a set of power grid networks up to one million nodes. The capacity of our implementation is mainly limited by the physical memory of our machine (2GB).

Table 2 shows the CPU times of both ETBR (including the cost of FFT) and IEKS on the given set of circuits using the same reduction order $q = 6$. We find that EKS is a bit faster for small circuits. But for Ckt6 and larger circuits, the CPU times are almost the same for both methods. For the largest circuit Ckt8, EKS cannot even finish owning to the memory constraint; while ETBR runs through all the circuits. This clearly shows that ETBR is more memory efficient by using a non-LU decomposition solver than EKS.

Table 2: CPU times (in seconds) comparison of ETBR and IEKS ($q = 6$)

Test Ckts	ETBR (s)	EKS (s)
Ckt1	0.23	0.08
Ckt2	1.28	0.89
Ckt3	1.8	1.4
Ckt4	20.4	18.8
Ckt5	28.6	25.3
Ckt6	152	151
Ckt7	162	160
Ckt8	562	*out of memory*

6. CONCLUSION

In this paper, we have proposed a new power grid analysis approach based on truncated balanced realization reduction techniques. The new simulation method, called *ETBR*, performs the reduction on the system before the transient simulation. But different from the existing extended Krylov subspace methods such as EKS or IEKS, it uses fast truncated balanced realization method on response Grammian to perform the reduction. As a result, ETBR can deliver more accurate results than the EKS method over large frequency range with similar computation costs. Also the new method avoids the explicit moment representation of the input signals. So it can better capture the high frequency contents than the EKS method, which leads to more accurate results for fast changing input signals. Experimental results demonstrated that, given the same order of the reduced model, ETBR is more accurate than EKS especially for input sources rich in high-frequency components. ETBR also shows similar computation costs of EKS and less memory consumption than EKS.

7. REFERENCES

[1] J. Kozhaya, S. R. Nassif and F. Najm, "A Multigrid-like Technique for Power Grid Analysis", in *IEEE Trans. on Computer-Aided Design*, 21(10), pp. 1148-1160, 2002.

[2] Y.-M. Lee, Y. Cao, T.-H. Chen, J. M. Wang, and C. C.-P. Chen, "HiPRIME: Hierarchical and Passivity Preserved Interconnect Macromodeling Engine for RLKC Power Delivery", *IEEE Trans. on CAD of Integrated Circuits and Systems*, Vol. 24, No. 6, 2005, pp. 797-806.

[3] H. Li, Z. Qi, S. X-D. Tan, L. Wu, Y. Cai, and X. Hong, "Partitioning-Based Approach to Fast On-Chip Decap Budgeting and Minimization", in Proc. *Design Automation Conference*, 2005, pp. 170-175.

[4] W. Guo, S. X.-D. Tan, Z. Luo, and X. Hong, "Partial random walk for large linear network analysis", in *Proc. IEEE Int. Symp. on Circuits and Systems (ISCAS)*, page V173-V177, 2004.

[5] S. R. Nassif and J. N. Kozhaya, "Fast Power Grid Simulation", in *Proc. Design Automation Conference*, pp. 156-161, 2000.

[6] B. Moore, "Principle component analysis in linear systems: Controllability, and observability, and model reduction", *IEEE Trans. Automat. Contr.*, vol. 26, no. 1, pp. 17–32, 1981.

[7] A. Odabasioglu, M. Celik, and L. Pileggi, "PRIMA: Passive reduced-order interconnect macromodeling algorithm", *IEEE Trans. on Computer-Aided Design of Integrated Circuits and Systems*, pp. 645–654, 1998.

[8] L. T. Pillage, R. A. Rohrer and C. Visweswariah, *Electronic Circuit and System Simulation Methods*, McGraw-Hill, Inc., 1994.

[9] J. R. Phillips, L. Daniel, and L. M. Silveira, "Guaranteed passive balancing transformations for model order reduction", in *Proc. Design Automation Conf. (DAC)*, pp. 52–57, 2002.

[10] J. Phillips and L.M. Silveira, "Poor man's TBR: a simple model reduction scheme", in *Proc. Design, Automation and Test in Europe Conference*, vol. 2, pp. 938-943, 2004.

[11] ——, "Poor man's TBR: a simple model reduction scheme", *IEEE Trans. on Computer-Aided Design of Integrated Circuits and Systems*, vol. 24, no. 1, pp. 43–55, 2005.

[12] H. Qian, S. R. Nassif and S. S. Sapatnekar, "Random Walks in a Supply Network", in *Proc. Design Automation Conf. (DAC)*, pp. 93-98, 2003.

[13] J. Shi, Y. Cai, J. Fan, S. X.-D. Tan, and X. Hong, "Pattern based method for extreme large power/ground analysis", *IEEE Trans. on Computer-Aided Design of Integrated Circuits and Systems*, 26(4):680–692, Dec. 2007.

[14] S. X.-D. Tan and L. He, *Advanced Model Order Reduction Techniques in VLSI Design*, Cambridge University Press, 2007.

[15] J. M. Wang and T. V. Nguyen, "Extended Krylov Subspace Method for Reduced Order Analysis of Linear Circuit with Multiple Sources," in Proc. DAC, 2000.

[16] M. Zhao, R. V. Panda, S. S. Sapatnekar, T. Edwards, R. Chaudhry and D. Blaauw, "Hierarchical Analysis of Power Distribution Networks," in *Proc. Design Automation Conference*, pp. 150-155, 2000.

[17] UMFPACK, "http://www.cise.ufl.edu/ research/ sparse/ umfpack/"