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EULER CHARACTERISTICS FOR GAUSSIAN FIELDS ON MANIFOLDS

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We are interested in the geometric properties of real-valued Gaussian random fields defined on manifolds. Our manifolds, M, are of class C^3 and the random fields f are smooth. Our interest in these fields focuses on their excursion sets, $f^{-1}[u, +\infty)$, and their geometric properties. Specifically, we derive the expected Euler characteristic $\mathrm{E}[\chi(f^{-1}[u, +\infty))]$ of an excursion set of a smooth Gaussian random field. Part of the motivation for this comes from the fact that $\mathrm{E}[\chi(f^{-1}[u, +\infty))]$ relates global properties of M to a geometry related to the covariance structure of f. Of further interest is the relation between the expected Euler characteristic of an excursion set above a level u and $\mathrm{P}[\sup_{p\in M} f(p) \geq u]$. Our proofs rely on results from random fields on $\mathbb{R}^n$ as well as differential and Riemannian geometry.

1. Introduction. In reviewing the literature on smooth random fields of the past few decades, it is clear that the study of smooth random fields has profited from classical ideas in integral geometry. Details of this approach can be found, for example, in [18, 22] and the recent review [3].

The study of the Euler characteristic, χ, of the excursion sets $f^{-1}[u, +\infty)$, of a smooth random field f on $\mathbb{R}^n$, began in the 1970s, both as a multiparameter extension of the concept of the number of upcrossings of a one parameter process and as an object of intrinsic interest in describing the properties of random fields. In recent years, the study of the Euler characteristic of the excursions of random fields has undergone a revival of sorts due to its applications in the statistics of medical imaging and astrophysics (see, e.g., [5, 6, 21, 22]). The basic results used in all of these applications are explicit formulae for the expectation $\mathrm{E}[\chi(f^{-1}[u, +\infty))]$, which has been computed for a large number of random fields defined on $\mathbb{R}^n$, combined with the following approximation for the distribution of the maximum of a smooth, isotropic Gaussian random field or a so-called finite Karhunen–Loève expansion constant variance Gaussian field [3, 18]

$$\mathrm{P}\Big[\sup_{t\in D} f_t \geq u\Big] \simeq \mathrm{E}\big[\chi\big(D \cap f^{-1}[u, +\infty)\big)\big].$$

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Our main contribution will be the extension of these formulae for

$$\mathrm{E}\big[\chi\big(f^{-1}[u,+\infty)\big)\big]$$

to the case in which f is a Gaussian field indexed by an abstract manifold M, with or without boundary. Our motivation comes from this same class of applications, in particular, the interest in extending results for isotropic fields on $\mathbb{R}^n$ to arbitrary nonstationary fields on $\mathbb{R}^n$ and the emerging interest in fields defined on or restricted to submanifolds of Euclidean space in applications such as medical imaging. We also feel that the subject has significant mathematical interest in its own right.

For isotropic Gaussian random fields and those with a finite Karhunen–Loève expansion, $\mathrm{E}[\chi(D \cap f^{-1}[u,+\infty))]$ has a simple expression in terms of the geometric properties of the parameter space D, which, as we already noted, leads to an accurate approximation of $\mathrm{P}[\sup_{t\in D} f(t) \geq u]$. More generally, if f is an isotropic random field (satisfying certain regularity conditions) on D, a compact set in $\mathbb{R}^n$ whose boundary ∂D is a C^2 hypersurface, then

$$\mathrm{E}\big[\chi\big(D \cap f^{-1}[u,+\infty)\big)\big] = \sum_{j=0}^{n} a_{j,n} \mathcal{M}_{n-j}(D) \lambda^{j/2} \rho_{f,j}(u) \tag{1.1}$$

where $\mathcal{M}_j$ are integral invariants (under the group of isometries of $\mathbb{R}^n$) referred to as the Minkowski functionals of D, the $a_{j,n}$ are constants independent of f and D, λ is a spectral parameter of the process and the functions $\rho_{j,f}(u)$ are referred to as Euler characteristic, or EC densities. For details see [22]. In the Gaussian (unit variance) case, it was proven in [1] that for $j \geq 1$,

$$\rho_{j,f}(u) = (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-u^2/2},$$

where H_j is the jth Hermite polynomial. For other examples of $\rho_{j,f}$ for various Gaussian related fields, see [1, 5, 6, 21].

Steiner's formula [11, 15] (which can be used to define the Minkowski functionals) states that the Minkowski functionals $(\mathcal{M}_j(D))_{0\leq j\leq n}$ satisfy

$$\mathcal{H}_n\big(T(D,\rho)\big) = \sum_{j=0}^{n} \frac{\rho^j}{j!} \mathcal{M}_j(D)$$

where

$$T(D,\rho) = \{x \in \mathbb{R}^n : d(x,D) \leq \rho\}$$

is the ρ-tube around D in $\mathbb{R}^n$, $d(\cdot, D)$ is the standard distance function of D and $\mathcal{H}_n$ is n-dimensional Hausdorff measure. With this definition of the Minkowski functionals, they can be extended to functionals of so-called "sets of positive reach" [9]. Note, however, that they depend on the embedding of D

in $\mathbb{R}^n$. To see this, consider $m > n$, and note that the polynomials $\mathcal{H}_n(T(D,\rho))$ and $\mathcal{H}_m(T(i_{n,m}(D),\rho))$ with $i_{n,m}$ the standard inclusion of $\mathbb{R}^n$ into $\mathbb{R}^m$ will be different. For example, take a curve C in $\mathbb{R}^2$. Then $\mathcal{M}_1(C)$ will be proportional to the arc length of C and $\mathcal{M}_2(C) = 0$, while $\mathcal{M}_2(i_{2,3}(C))$ is nonzero and is also proportional to arc length.

Weyl [10, 20] studied the volume of tubes around embedded submanifolds M of Euclidean space, endowed with the Riemannian metric g induced by $\mathbb{R}^n$. He showed that the Minkowski functionals have a normalization independent of the embedding dimension. Specifically, he showed that if we define the Lipschitz–Killing curvatures of (M, g) as

$$\mathcal{L}_j(M) = \frac{1}{(n-j)!\omega_{n-j}} \mathcal{M}_{n-j}(M),$$

where $r^n\omega_n$ is the volume of $B_{\mathbb{R}^n}(0,r)$, the r-ball in $\mathbb{R}^n$, then the $\mathcal{L}_j(M)$ are intrinsic to (M, g), that is, they do not depend on the embedding of M into $\mathbb{R}^n$, only on the Riemannian metric g. With this normalization [7, 11] Weyl's Tube formula has the form

$$\mathcal{H}_n\big(T(M,\rho)\big) = \sum_{j=0}^{n} \mathcal{L}_j(M)\omega_{n-j}\rho^{n-j}.$$

Weyl also gave explicit formulas for the $\mathcal{L}_j(M)$ in the case when M has no boundary

$$(1.2)\quad \mathcal{L}_j(M) = \begin{cases} (2\pi)^{-(n-j)/2} \displaystyle\int_M \left(\left(\frac{n-j}{2}\right)!\right)^{-1} \operatorname{Tr}^M(-R)^{(n-j)/2}\,\mathrm{Vol}_g, & \\ \qquad\qquad\qquad\qquad n-j \ge 0 \text{ is even}, & \\ 0, \qquad\qquad\qquad\quad\; n-j \text{ is odd}, & \end{cases}$$

where Vol_g is the volume form of (M, g), R is the curvature tensor of (M, g) and the trace Tr^M on the algebra of double forms is given pointwise by $\operatorname{Tr}^M(\alpha) = \operatorname{Tr}^{T_pM}(\alpha_p)$ (see Section 2.1). For the expression when M has boundary, see [7] and Section 5.

Note that a real-valued Gaussian random field can always be viewed as a map Ψ from the parameter space, in this case a manifold M, into a Hilbert space, that is, the RKHS of the field or $L^2(\Omega, \mathcal{F}, \mathrm{P})$. If this map is smooth and nondegenerate, that is if it is an immersion, then the random field induces a Riemannian metric on M given by the pull-back $\Psi^*(\langle\cdot,\cdot\rangle_{L^2(\Omega,\mathcal{F},\mathrm{P})})$ of the standard structure on $L^2(\Omega, \mathcal{F}, \mathrm{P})$. Specifically, the Riemannian metric is given by

$$g_p(X_p, Y_p) = \mathrm{E}[X_p f Y_p f]$$

for tangent vectors X_p and Y_p. The main results of the paper, Theorems 4.1 and 5.1, show that (1.1) holds for smooth Gaussian fields on a manifold M, with the $a_{j,n}\mathcal{M}_{n-j}(D)\lambda^{j/2}$ replaced by $\mathcal{L}_j(M)$, calculated with respect to this pulled-back Riemannian metric.

The outline of the paper is as follows: in Section 2 we recall some of the geometric material and linear algebra needed for the subsequent calculations. For a more complete discussion of differential geometry, the reader is referred, for example, to [4, 8, 12]. At the end of Section 2 we give a brief list of the notation needed for what follows. In Section 3 we describe the regularity conditions we will require our Gaussian random fields f to satisfy and give sufficient conditions for them to hold. In Section 4 we derive a formula for $\mathrm{E}[\chi(f^{-1}[u,+\infty))]$ when f is defined on a manifold without boundary M and in Section 5, we treat the case when M is a manifold with boundary. Finally, in Section 6, we conclude with some examples, including nonstationary Gaussian random fields on $\mathbb{R}^n$, stationary fields on Lie groups, and isotropic fields on the sphere and spherical caps.

2. Geometric material. In this section we recall the geometric material needed in further sections. We begin by looking at random (specifically Gaussian) *double forms* on a vector space V, some of which can, when V is equipped with an inner product, be identified with random matrices. Following this, we state a version of Morse's theorem, which we later use to calculate the Euler characteristic of excursions of Gaussian fields. We conclude this section with a brief review of the relevant Riemannian geometry used in the sequel. In particular, we describe the Riemannian geometry induced by certain nondegenerate random fields.

2.1. *Gaussian double forms.* Given V an n-dimensional vector space, we denote by $(\bigwedge^*(V),\wedge)$ the Grassmann algebra of V equipped with the wedge product $\wedge$, that is, $\bigwedge^*(V)=\bigoplus_{j=0}^n \Lambda^j(V)$, where

$$\Lambda^j(V)=\left\{\alpha\in L\left(\bigoplus_{j=1}^r V;\mathbb{R}\right): \alpha(v_{\sigma(1)},\dots,v_{\sigma(j)})=\varepsilon_\sigma\alpha(v_1,\dots,v_j)\right.$$
$$\left.\forall v_1,\dots,v_j\in V,\sigma\in S(j)\right\},$$

with $L(E;F)$ the set of linear maps between the vector spaces E and F, and $S(j)$ the symmetric group on j letters and ε_σ is the sign of the permutation σ. Recall that the wedge product $\wedge:\Lambda^r(V)\times\Lambda^s(V)\to\Lambda^{r+s}(V)$ for all r and s is defined on $\alpha\in\Lambda^r(V)$, $\beta\in\Lambda^s(V)$ by

$$\alpha\wedge\beta=\frac{(r+s)!}{r!s!}\mathcal{A}(\alpha\otimes\beta)$$

where $\mathcal{A}$ is the alternating projection.

If $B_{V^*} = \{\theta_1, \dots, \theta_n\}$ is a basis for V^*, the dual of V, then a basis $B_{\bigwedge^*(V)}$ for $\Lambda^*(V)$ is given by

$$B_{\bigwedge^*(V)} = \bigcup_{j=1}^{n} \{\theta_{i_1} \wedge \cdots \wedge \theta_{i_j} : i_1 < i_2 < \cdots < i_j\}.$$

Any inner product $\langle \cdot, \cdot \rangle$ on V extends naturally to an inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^*(V)$ as follows. Given any orthonormal basis $B_V = (v_1, \dots, v_n)$ for V, there is a uniquely defined dual basis $B_{V^*} = (\theta_1, \dots, \theta_n)$ of V^*. Carrying out the construction of the basis for $\bigwedge^*(V)$ as above and declaring this to be an orthonormal basis defines the desired inner product.

We set $\Lambda^{r,s}(V) = \Lambda^r(V) \otimes \Lambda^s(V)$, the linear span of the image of $\Lambda^r(V) \times \Lambda^s(V)$ under the map $\otimes$ and let $\bigwedge^*(V) \otimes \bigwedge^*(V) = \bigoplus_{r,s=0}^{\infty} \Lambda^{r,s}(V)$. We define the product $\cdot$, which we refer to as the *double wedge* product, on $\bigwedge^*(V) \otimes \bigwedge^*(V)$, the linear span of $\bigoplus_{r,s=0}^{\infty} \Lambda^r(V) \otimes \Lambda^s(V)$, as follows:

$$(\alpha \otimes \beta) \cdot (\gamma \otimes \theta) = (\alpha \wedge \gamma) \otimes (\beta \wedge \theta).$$

We shall be most interested in the restriction of the double wedge product $\cdot$ to $\bigwedge^{*,*}(V) = \bigoplus_{j=0}^{\infty} \Lambda^j(V) \otimes \Lambda^j(V)$, which makes the pair $(\bigwedge^{*,*}(V), \cdot)$ into a commutative algebra. For $\gamma \in \bigwedge^{*,*}(V)$, we define the polynomial γ^j as the product of γ with itself j times, and $\gamma^0 = 1$. We now fix an orthonormal basis $B_V = (v_1, \dots, v_n)$ with dual basis $B_{V^*} = (\theta_1, \dots, \theta_n)$ (as identified by $\langle \cdot, \cdot \rangle$). Since any inner product $\langle \cdot, \cdot \rangle$ on V induces an inner product on $\bigwedge^*(V)$, $\langle \cdot, \cdot \rangle$ induces a real-valued linear map on $\bigwedge^{*,*}(V)$, the trace, denoted by Tr, which acts on an element $\gamma = \alpha \otimes \beta$

$$\mathrm{Tr}(\gamma) = \langle \alpha, \beta \rangle,$$

and which we extend linearly. A quick calculation shows that for $\gamma \in \Lambda^{k,k}(V)$, we have

$$\mathrm{Tr}(\gamma) = \frac{1}{k!} \sum_{a_1, \dots, a_k = 1}^{n} \gamma\big((v_{a_1}, \dots, v_{a_k}), (v_{a_1}, \dots, v_{a_k})\big).$$

If there is more than one vector space under consideration, we will use the notation Tr^{V_1} and Tr^{V_2} where necessary. For instance, if α is a section of $\bigwedge^{*,*}(M)$, that is, for each p, $\alpha_p \in \bigwedge^{*,*}(T_pM)$ where T_pM is the tangent space to M at p, we write

$$\mathrm{Tr}^{T_pM}(\alpha_p)$$

and use the notation $\mathrm{Tr}^M(\alpha)$ to denote the real valued function on M given by

$$\mathrm{Tr}^M(\alpha)_p \triangleq \mathrm{Tr}^{T_pM}(\alpha_p).$$

If $\gamma \in \Lambda^{0,0}(V)$, then $\gamma \in \mathbb{R}$ and we define $\mathrm{Tr}(\gamma) = \gamma$. Note also that $\gamma \in \Lambda^{k,k}(V)$ can be identified with a linear map $T_\gamma : \Lambda^k(V) \to \Lambda^k(V)$ by defining

$$T_\gamma(\theta_{i_1} \wedge \cdots \wedge \theta_{i_k})(w_1, \dots, w_k) = \gamma(v_{i_1}, \dots, v_{i_k}, w_1, \dots, w_k).$$

We can then extend T_γ by linearity. Clearly the identity can be represented as

$$I = \sum_{i=1}^{n} \theta_i \otimes \theta_i.$$

We shall need one useful formula [9] later when we calculate the expected Euler characteristic of a Gaussian random field on a manifold M. Choose $A \in \Lambda^{k,k}(V)$ and choose $0 \le j \le n-k$. Then

$$\operatorname{Tr}(AI^j) = \frac{(n-k)!}{(n-k-j)!} \operatorname{Tr}(A). \tag{2.1}$$

For a more complete description of the properties of Tr, the reader is referred to Section 2 of [9].

In later sections, we shall need to be able to evaluate the expectation of determinants of symmetric matrices whose elements are Gaussian random variables. We shall now derive such a formula, ignoring the symmetry requirement. If we view an $n \times n$ matrix (A_{ij}) as representing a linear mapping T_A from $\mathbb{R}^n$ to $\mathbb{R}^n$, with $A_{ij} = \langle e_i, T_A e_j \rangle$, then A can also be represented by $\gamma_A \in \Lambda^{1,1}(\mathbb{R}^n)$, and from the discussion above,

$$\det(A) = \frac{1}{n!} \operatorname{Tr}^{\mathbb{R}^n}\big((\gamma_A)^n\big). \tag{2.2}$$

This makes the calculations we have to do later much simpler as we exploit the fact that $(\bigwedge^{*,*}(V), \cdot)$ is a commutative algebra.

We now come to the main computational tool in what follows, which simplifies the formula for computing the expectation of the determinant of an $n \times n$ Gaussian matrix to a formula no more complicated than the formula for $E[Z^n]$ where $Z \sim N(\mu, \sigma^2)$.

LEMMA 2.1. *Suppose that W is a Gaussian double form, that is, there exists a probability space $(\Omega, \mathcal{F}, P)$ along with an $\mathcal{F}/\mathcal{B}(\Lambda^{1,1}(V))$ measurable map W such that, for any basis $B_V = \{v_1, \dots, v_n\}$ the matrix with entries $W(v_i, v_j)$ is Gaussian. Then,*

$$E[W^k] = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{(k-2j)!\,j!\,2^j} \mu^{k-2j} C^j$$

in the sense that, for all choices $v_1, \dots, v_k, v'_1, \dots, v'_k \in V$,

$$\begin{aligned} &E\big[W^k\big((v_1, \dots, v_k), (v'_1, \dots, v'_k)\big)\big] \\ &\quad = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!}{(k-2j)!\,j!\,2^j} \mu^{k-2j} C^j\big((v_1, \dots, v_k), (v'_1, \dots, v'_k)\big) \end{aligned} \tag{2.3}$$

where $\mu = \mathrm{E}[W]$ *and* $C \in \Lambda^{2,2}(V)$ *is defined as*

$$\begin{aligned}
&C\big((v_1, v_2) \otimes (v_1', v_2')\big)\\
&\qquad = \mathrm{E}\big[\big(W - \mathrm{E}[W]\big)^2\big((v_1, v_2) \otimes (v_1', v_2')\big)\big]\\
&\qquad = 2\Big(\mathrm{E}\big[\big(W(v_1, v_1') - \mathrm{E}[W(v_1, v_1')]\big)\big(W(v_2, v_2') - \mathrm{E}[W(v_2, v_2')]\big)\\
&\qquad\qquad - \big(W(v_1, v_2') - \mathrm{E}[W(v_1, v_2')]\big)\big(W(v_2, v_1') - \mathrm{E}[W(v_2, v_1')]\big)\big]\Big).
\end{aligned}$$

PROOF. We first prove that, in the case that $\mu = 0$,

$$\mathrm{E}[W^k] = \begin{cases} 0, & \text{if } k \text{ is odd},\\ \dfrac{(2j)!}{j!2^j} C^j, & \text{if } k = 2j, \end{cases} \tag{2.4}$$

from which the case $\mu \neq 0$ easily follows from the binomial theorem.

It is easy to show by induction that

$$W^{2j}\big((v_1, \dots, v_{2j}), (v_1', \dots, v_{2j}')\big) = \sum_{\pi,\sigma \in S(2j)} \varepsilon_\pi \varepsilon_\sigma \prod_{k=1}^{2j} W(v_{\pi(k)}, v_{\sigma(k)}').$$

We now evaluate its expectation, abbreviating the left-hand side by W^{2j}:

$$\begin{aligned}
\mathrm{E}[W^{2j}] &= \sum_{\pi,\sigma \in S(2j)} \varepsilon_\pi \varepsilon_\sigma \mathrm{E}\Bigg[\prod_{k=1}^{2j} W(v_{\pi(k)}, v_{\sigma(k)}')\Bigg]\\
&= K_j \sum_{\pi,\sigma \in S(2j)} \varepsilon_\pi \varepsilon_\sigma \prod_{k=1}^{j} \mathrm{E}\big[W(v_{\pi(2k-1)}, v_{\sigma(2k-1)}') W(v_{\pi(2k)}, v_{\sigma(2k)}')\big]\\
&= \frac{K_j}{2^j} \sum_{\pi,\sigma \in S(2j)} \varepsilon_\pi \varepsilon_\sigma \prod_{k=1}^{j} \mathrm{E}\big[W(v_{\pi(2k-1)}, v_{\sigma(2k-1)}') W(v_{\pi(2k)}, v_{\sigma(2k)}')\\
&\qquad\qquad - W(v_{\pi(2k)}, v_{\sigma(2k-1)}') W(v_{\pi(2k-1)}, v_{\sigma(2k)}')\big]\\
&= \frac{K_j}{2^{2j}} \sum_{\pi,\sigma \in S(2j)} \varepsilon_\pi \varepsilon_\sigma \prod_{k=1}^{j} C\big((v_{\pi(2k-1)}, v_{\pi(2k)}), (v_{\sigma(2k-1)}', v_{\sigma(2k)}')\big)\\
&= \frac{K_j}{2^j} C^j\big((v_1, \dots, v_{2j}), (v_1', \dots, v_{2j}')\big)
\end{aligned}$$

where K_j is a combinatorial constant, depending only on j. The only step that needs justification is the step from the first to the second line, which is justified by using the Wick formula for the expectation of a product of zero mean Gaussian random variables. The Wick formula states that the expectation of a product of zero mean Gaussian random variables is the sum of the products of the expectations of pairs, summed over all possible groupings into pairs. In the above expression, however, we note that in the second line, we already have each possible grouping into pairs for each of the summands in the first line.

It remains to calculate the constant K_j, and for this we use Lemma 5.3.2 of [1], which we restate here, for completeness:

LEMMA 2.2. *Let Z^{2n} be a $2n \times 2n$ symmetric matrix of Gaussian random variables, with covariances satisfying*

$$\mathrm{E}\big[Z_{ij}^{2n} Z_{kl}^{2n}\big] - \mathrm{E}\big[Z_{kj}^{2n} Z_{il}^{2n}\big] = \delta_{kj}\delta_{il} - \delta_{ij}\delta_{kl}.$$

Then

$$\mathrm{E}\big[\det(Z^{2n})\big] = \frac{(-1)^n(2n)!}{n!2^n}. \tag{2.5}$$

Setting $V = \mathbb{R}^{2n}$ with its standard inner product, and defining $W(e_i, e_j) = Z_{ij}^{2n}$, where $(e_i)_{1\le i\le 2n}$ is the standard basis for $\mathbb{R}^{2n}$, we see that we are indeed in the same situation as in the lemma and $C = -I^2$. Next, we note that

$$\begin{aligned}
&\mathrm{E}\big[W^{2n}((e_1,\dots,e_n),(e_1,\dots,e_n))\big]\\
&\quad = \mathrm{E}\left[\sum_{\pi,\sigma\in S(2n)} \varepsilon_\sigma\varepsilon_\pi \prod_{j=1}^{2n} W(e_{\sigma(j)}, e_{\pi(j)})\right]\\
&\quad = \mathrm{E}\big[(2n)!\det(Z^{2n})\big]\\
&\quad = \frac{K_n}{2^n}(-1)^n I^{2n}\big((e_1,\dots,e_{2n}),(e_1,\dots,e_{2n})\big)\\
&\quad = \frac{K_n}{2^n}(-1)^n(2n)!,
\end{aligned}$$

where we have used (2.5). Equating the second and the fourth lines, combined with (2.5), we see

$$K_n = \frac{(2n)!}{n!}.$$

This completes the proof. □

2.2. *Morse's theorem.* In this section we state, without proof, a version of Morse's theorem for C^2 Riemannian manifolds with boundary (N, h) isometrically embedded in some orientable ambient manifold (M, g). Morse's theorem relates the Euler characteristic, a topological invariant, to the critical points of *nondegenerate* functions, to be defined below. The interested reader is referred to [13, 14, 19].

First, we begin with some definitions and notation. For a manifold M, a critical point of $f \in C^1(M)$ is a point p such that the differential of f, $df_p \equiv 0$, where df is the one-form defined by $df_p(X_p) = X_p f$. The points $\{t \in \mathbb{R} : f_p = t$ and $df_p \equiv 0\}$ are called the *critical values* of f and the points $\{t \in \mathbb{R} : f_p = t$ and $df_p \neq 0\}$ the *regular values*. For $f \in C^1(M)$, where (M, g) is a Riemannian manifold, we recall that the *gradient* of f, ∇f, is the unique continuous vector field such that

$$\langle \nabla f, X \rangle = Xf$$

for every vector field X. Next, we recall that the Hessian $\nabla^2 f$ of a function $f \in C^2(M)$ on a Riemannian manifold (M, g) is the bilinear symmetric map from $C^1(T(M)) \times C^1(T(M)) \to C^0(M)$, given by

$$\nabla^2 f(X, Y) = g(\nabla_X \nabla f, Y) = XYf - \nabla_X Yf,$$

where ∇ is the Levi–Civita connection of (M, g). Note that, $\nabla^2 f \in C^0(\Lambda^{1,1}(M))$ and, at a critical point, the Hessian is independent of the metric g. A critical point p is called nondegenerate if the bilinear mapping $\nabla^2 f(\cdot, \cdot)|_{T_pM}$ is nondegenerate. A function $f \in C^2(M)$ is said to be nondegenerate if all its critical points are nondegenerate. The index of a nondegenerate critical point p is the dimension of the largest subspace L of T_pM, such that $\nabla^2 f(\cdot, \cdot)|_L$ is negative definite.

If O is an open set of M such that ∂O is a C^2 embedded submanifold of M, then we call the C^2 submanifold with boundary $\overline{O}$, a C^2 *domain*. In what follows, we denote counting measure on a set S by $\#_S$, and for $S \subset M$, the restriction of f to S by $f_{|S}$. The proof of the following, which is an extension of Morse's original theorem to cases for which $f^{-1}[u, +\infty) \cap \partial M \neq \varnothing$, can be found in [19], or derived as a corollary to Proposition A3 of [18].

THEOREM 2.3 (Morse's theorem). (i) *Suppose f is a nondegenerate function on M and u is a regular value of f. Then,*

$$\chi\big(f^{-1}[u, +\infty)\big) = \sum_{k=0}^{n} (-1)^k \#_M\big\{f > u, df = 0, \operatorname{index}(-\nabla^2 f) = k)\big\}.$$

(ii) *Suppose N is a compact C^2 domain of M, with outward pointing unit normal vector $\nu_\perp$, and f is a nondegenerate function on M such that $f_{|\partial N}$ is also nondegenerate. Suppose further that u is a regular value of f and $f_{|\partial N}$. Then,*

$$\begin{aligned}
&\chi\big(f^{-1}[u, +\infty) \cap N\big) \\
&\quad = \sum_{k=0}^{n} (-1)^k \#_N\big\{f > u, df = 0, \operatorname{index}(-\nabla^2 f) = k\big\} \\
&\qquad + \sum_{k=0}^{n-1} (-1)^k \#_{\partial N}\big\{f > u, df = 0, \operatorname{index}(-\nabla^2 f) = k, \langle \nabla f, \nu_\perp \rangle > 0\big\}.
\end{aligned}$$

2.3. *Densities of point processes on manifolds.* In this section we state two lemmas which deal with point processes on a manifold M, derived from smooth random fields on M.

We call a map $\theta : \Omega \to \bigwedge^*(M)(\bigwedge^{*,*}(M))$ a random (double) differential form if the coefficients of θ with respect to the natural basis for $\bigwedge^*(M)(\bigwedge^{*,*}(M))$ in any chart (U, φ) are $\mathcal{F} \otimes \mathcal{B}(M)$-measurable. We can unambiguously talk of the expectation $\mathrm{E}[\theta]$, as well as its conditional expectation $\mathrm{E}[\theta|\mathcal{G}]$ for $\mathcal{G} \subset \mathcal{F}$, by taking expectations of its coefficients. If, furthermore, a random differential form θ is P-almost surely integrable, then Fubini's theorem allows us to make sense out of expressions like

$$\mathrm{E}\left[\int_M \theta\right] = \int_M \mathrm{E}[\theta].$$

We shall use random differential forms to count the critical points and their indices in order to calculate $\mathrm{E}[\chi(M \cap f^{-1}[u, +\infty)]$ for both Gaussian and other smooth random fields. To start, we state a lemma which generalizes Theorem 5.1.1 of [1] to manifold-valued random fields defined on a manifold M (i.e., $H : M \times \Omega \to N$, where N is some other manifold).

LEMMA 2.4. *Let M and N be two oriented n-dimensional manifolds. Given some probability space $(\Omega, \mathcal{F}, P)$, suppose H is an almost surely C^1 N-valued random field on M, and G an almost surely continuous E-valued random field on M where E is some topological space. We are interested in the number of points in M such that $H(p) = q_0$ and $G(p) \in A$, where q_0 is some arbitrary point in N and A is some open set in E. Suppose that, with probability* 1:

(i) *There are no points $p \in M$ satisfying both $H(p) = q_0$ and either $G(p) \in \partial A$ or* $\mathrm{Rank}(H_*)_p < n$. *Here* $\mathrm{Rank}(H_*)_p$ *is the rank of the linear mapping $(H_*)_p : T_pM \to T_{H(p)}N$.*

(ii) *There are only a finite number of points $p \in M$ satisfying $H(p) = q_0$.*

Suppose further that we have a family $(\alpha_\varepsilon)_{\varepsilon>0}$ of n-forms on N and a chart (U, φ) with coordinates $(y_1, \dots, y_n)$ such that

$$\alpha_\varepsilon(\varphi(p)) = \beta_\varepsilon(\varphi(p))\left(\bigwedge_{j=1}^n dy_i\right)$$

$$\triangleq \varepsilon^{-n}\omega_n^{-1}\mathbb{1}_{\{\varphi(p)\in B(0,\varepsilon)\}}\left(\bigwedge_{j=1}^n dy_i\right).$$

Then we have, with probability 1,

$$\#\{p \in M : H(p) = q_0, G(p) \in A\} = \lim_{\varepsilon\to 0}\int_M \mathbb{1}_{\{G\in A\}}|H^*\alpha_\varepsilon|,$$

where, for an n-form α on M,

$$|\alpha| = \left|d\alpha/d\Theta\right| \cdot \Theta$$

for some volume form Θ that determines the orientation of M and

$$d\alpha/d\Theta \stackrel{\Delta}{=} \alpha(X_1, \dots, X_n)/\Theta(X_1, \dots, X_n),$$

for any frame $(X_1, \dots, X_n)$ of linearly independent vector fields on M.

PROOF. The proof of this is a straightforward modification of the proof of Theorem 5.1.1 in [1]. □

The reason we use this rather than the original of [1] is that, when we later take expectations, we can unambiguously talk about the expectation of the integral $\int_M \mathbb{1}_{\{G\in A\}}|H^*\alpha_\varepsilon|$ by taking the expectation of the coefficients of the differential without worrying about triangulating the manifold and calculating the integrals locally.

Lemma 2.4 gives an almost sure representation for certain point processes on M, while we are interested in the density of these point processes

$$\mathrm{E}\left[\lim_{\varepsilon\to 0}\int_M \mathbb{1}_{\{G\in A\}}|H^*\alpha_\varepsilon|\right].$$

Under certain conditions, we can interchange the limit and expectation above to get

$$\mathrm{E}\left[\lim_{\varepsilon\to 0}\int_M \mathbb{1}_{\{G\in A\}}|H^*\alpha_\varepsilon|\right] = \int_M \lim_{\varepsilon\to 0}\mathrm{E}\left[\mathbb{1}_{\{G\in A\}}|H^*\alpha_\varepsilon|\right].$$

The following lemma gives sufficient conditions for this, in the case that H is an $\mathbb{R}^n$-valued random field.

LEMMA 2.5. *Suppose (M, g) is an oriented Riemannian manifold and $H = (H_1, \dots, H_n)$ is an $\mathbb{R}^n$-valued random field, satisfying the conditions of Lemma* 2.4. *Further, suppose for any $\varepsilon > 0$,*

$$\mathrm{P}\left[\max_{1\le i\le n}\sup_{p\in M}\sup_{q\in B_\tau(p,h)}|H_i(p) - H_i(q)| > \varepsilon\right] = o(h^n)$$

and

$$\mathrm{P}\left[\max_{1\le i\le n}\sup_{p\in M}\sup_{q\in B_\tau(p,h)}|XH_i(p) - XH_i(q)| > \varepsilon\right] = o(h^n)$$

for any unit vector field X, where $B_\tau(p, h)$ is the ball in the metric induced by g. Further, suppose that

$$\frac{d\mathrm{E}[\mathbb{1}_{\{G\in A\}}|H^*\alpha_\varepsilon|]}{d\,\mathrm{Vol}_g}$$

is bounded, for $\varepsilon < \varepsilon_0$ by $d\Theta/d\,\mathrm{Vol}_g$ for some integrable form Θ. Then,

$$\mathrm{E}\left[\lim_{\varepsilon\to 0}\int_M \mathbb{1}_{\{G\in A\}}|H^*\alpha_\varepsilon|\right] = \int_M \lim_{\varepsilon\to 0}\mathrm{E}\big[\mathbb{1}_{\{G\in A\}}|H^*\alpha_\varepsilon|\big].$$

PROOF. This is also a straightforward modification of the proof of Lemmas 5.2.1 and 5.2.2 in [1]. □

2.4. *The Riemannian structure induced by a smooth Gaussian field.* In this section we describe the Riemannian structure g induced on M by the field f in terms of the covariance function of the field.

As mentioned in Section 1, the Riemannian structure induced by f is the pull-back of the standard structure on $L^2(\Omega, \mathcal{F}, \mathrm{P})$ and is given by

$$g(X, Y) = \mathrm{E}[XfYf].$$

We now describe the Levi–Civita connection ∇ and the curvature tensor R of (M, g) in terms of the covariance structure of f. The following relation, known as Koszul's formula (cf., e.g., [12]) gives a coordinate free formula that determines ∇ for C^1 vector fields X, Y, Z we have

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y)\\ &\quad + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z]).\end{aligned}$$

A simple calculation then shows that

$$g(\nabla_X Y, Z) = \mathrm{E}[(\nabla_X Yf)(Zf)] = \mathrm{E}[(XYf)(Zf)]. \tag{2.6}$$

Next, we show how the curvature tensor of (M, g) is related to the covariance structure of f. If g is C^2, for C^2 vector fields X, Y, Z, W, we have

$$\begin{aligned}&\mathrm{E}\big[(\nabla^2 f)^2\big((X, Y), (Z, W)\big)\big]\\
&\quad = 2\mathrm{E}\big[\nabla^2 f(X, Z)\nabla^2 f(Y, W) - \nabla^2 f(X, W)\nabla^2 f(Y, Z)\big]\\
&\quad = 2\mathrm{E}\big[(XZf - \nabla_X Zf)(YWf - \nabla_Y Wf)\\
&\qquad\qquad - (XWf - \nabla_X Wf)(YZf - \nabla_Y Zf)\big]\\
&\quad = 2\big(\mathrm{E}[XZfYWf] - g(\nabla_X Z, \nabla_Y W)\big)\\
&\qquad - 2\big(\mathrm{E}[XWfYZf] - g(\nabla_X W, \nabla_Y Z)\big)\\
&\quad = 2\big(X\mathrm{E}[ZfYWf] - \mathrm{E}[ZfXYWf] - g(\nabla_X Z, \nabla_Y W)\big)\\
&\qquad - 2\big(Y\mathrm{E}[XWfZf] - \mathrm{E}[ZfYXWf] - g(\nabla_X W, \nabla_Y Z)\big)\\
&\quad = 2\big(Xg(Z, \nabla_Y W) - g(\nabla_X Z, \nabla_Y W) - g(Z, \nabla_{[X,Y]} W)\big)\\
&\qquad - 2\big(Yg(\nabla_X W, Z) - g(\nabla_X W, \nabla_Y Z)\big)\end{aligned}$$

$$
\begin{aligned}
&= 2\big(g(\nabla_X Z, \nabla_Y W) + g(Z, \nabla_X \nabla_Y W) - g(\nabla_X Z, \nabla_Y W) - g(Z, \nabla_{[X,Y]} W)\big) \\
&\quad - 2\big(g(\nabla_Y \nabla_X W, Z) + g(\nabla_Y Z, \nabla_X W) - g(\nabla_X W, \nabla_Y Z)\big) \\
&= 2\big(g(Z, \nabla_X \nabla_Y W) - g(\nabla_Y \nabla_X W, Z) - g(Z, \nabla_{[X,Y]} W)\big) \\
&= 2R\big((X, Y), (W, Z)\big) \\
&= -2R\big((X, Y), (Z, W)\big),
\end{aligned}
$$

where R is the curvature tensor of (M, g). Equivalently, as double differential forms,

$$
\mathrm{E}\big[(\nabla^2 f)^2\big] = -2R. \tag{2.7}
$$

From our assumptions, it is not clear that the terms $XYWf$ and $YXWf$ make proper sense, though their difference, $[X, Y]Wf$, which is what appears above, is well defined.

2.5. *Notation used in following sections.* Here is the promised list of notation, requested by a referee to ease the pain of some of the following technicalities:

M	abstract manifold with or without boundary,
$S(H)$	unit sphere in a Hilbert space H,
$S_\rho(H)$	ρ-sphere in a Hilbert space H,
$(\bigwedge^*(V), \wedge)$	Grassman algebra of a vector space V,
$(\bigwedge^{*,*}(V), \cdot)$	algebra of double forms of a vector space V,
Tr^H	trace on $(\bigwedge^{*,*}(H), \cdot)$ of a Hilbert space H,
I	identity mapping in $\Lambda^{1,1}(H)$ for a Hilbert space H,
(M, g)	abstract Riemannian manifold (M, g),
∇	Lévi–Civita connection of Riemannian manifold (M, g),
∇f	gradient of a function f on a Riemannian manifold (M, g),
$\nabla^2 f$	Hessian of a function f on a Riemannian manifold (M, g),
R	curvature tensor of Riemannian manifold (M, g),
S	second fundamental form of ∂M in M of a Riemannian manifold with boundary (M, g),
$\mathcal{H}_j$	j-dimensional Hausdorff measure on a metric space (T, τ),
$\mathcal{M}_j(D)$	jth Minkowski functional of $D \subset \mathbb{R}^n$,
$\mathcal{L}_j(M)$	jth Lipschitz–Killing curvature of Riemannian manifold (M, g),
ω_n	volume of the unit ball in $\mathbb{R}^n$,
C_f	covariance function of a random field f.

3. Suitable regularity. In this section we discuss the regularity conditions we require our Gaussian random fields to satisfy. For a parameter space, T, if we are given a nonnegative definite, symmetric function $C : T \times T \to \mathbb{R}$, it is well known that we can construct a Gaussian random field f on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with parameter space T and covariance function C.

Suppose that T is an open subset of $\mathbb{R}^n$. We say that f has jth order derivatives in the L^2 sense if

$$\lim_{\|t'\|, \|s'\| \to 0} \mathrm{E}[F(t, t')F(s, s')]$$

exists for all $s, t \in T$ and sequences $s', t' \in \bigoplus^j \mathbb{R}^n$, where $F(t, t')$ is the symmetrized difference

$$F(t, t') = \frac{1}{\prod_{i=1}^j \|t'_i\|_{\mathbb{R}^n}} \sum_{s \in \{0,1\}^j} (-1)^{j - \sum_{i=1}^j s_i} f\left(t + \sum_{i=1}^j s_i t'_i\right).$$

In this case, for $(t, t') \in T \times \bigoplus^j \mathbb{R}^n$, we denote the jth order derivative in the direction t' by $D^j_{L^2} f(t, t')$, that is, $D^j_{L^2} f(t, t')$ is defined by the following L^2 limit:

$$D^j_{L^2} f(t, t') = \lim_{h \to 0} F(t, ht').$$

It is clear that $D^j_{L^2} f$ is also a Gaussian field on $T \times \bigoplus^j \mathbb{R}^n$, since, for each (t, t'), it is an L^2 limit of Gaussian random variables. We endow the space $\mathbb{R}^n \times \bigoplus^j \mathbb{R}^n$ with the product norm

$$\|(s, s')\|_{n,j} = \|s\|_{\mathbb{R}^n} + \|s'\|_{\bigoplus^j \mathbb{R}^n} = \|s\|_{\mathbb{R}^n} + \left(\sum_{i=1}^j \|s'_i\|^2_{\mathbb{R}^n}\right)^{1/2},$$

and we write $B_{n,j}(y, h)$ for the h-ball centered at $y = (t, t')$ in the metric induced by $\| \cdot \|_{n,j}$. Further, we write,

$$T_{j,\rho} = T \times \left\{t' : \|t'\|_{\bigoplus^j \mathbb{R}^n} \in (1 - \rho, 1 + \rho)\right\}$$

for the product of T with the ρ-tube around the unit sphere in $\bigoplus^j \mathbb{R}^n$.

If the covariance function C is smooth enough, then the existence of jth order derivatives in the L^2 sense implies the existence of j almost surely continuous derivatives. The following lemma gives sufficient conditions for the almost sure jth order differentiability of f.

LEMMA 3.1. *Suppose f is a centered Gaussian random field on T, a bounded open set in $\mathbb{R}^n$, such that f has a jth order derivative in the L^2 sense.*

Suppose furthermore that there exists $0 < K < \infty$, *and* $\rho, \delta, h_0 > 0$ *such that for* $0 < \eta_1, \eta_2, h < h_0$,

$$\begin{aligned} &\mathrm{E}\big[\big(F(t, \eta_1 t') - F(s, \eta_2 s')\big)^2\big] \\ &\qquad < \frac{K}{(-\log(\|(t,t') - (s,s')\|_{n,j} + |\eta_1 - \eta_2|))^{1+\delta}}, \end{aligned} \tag{3.1}$$

for all $((t,t'),(s,s')) \in S(\rho, h)$ *where*

$$S(\rho, h) = \big\{\big((t,t'),(s,s')\big) \in T_{j,\rho} \times T_{j,\rho} : (s,s') \in B_{n,j}\big((t,t'),h\big)\big\}.$$

Then, there exists a continuous modification, $\hat{f}$ *of* f, *such that* $\hat{f} \in C^j(T)$, *with probability* 1. *Denoting the derivatives by* $D^j \hat{f}$, *we have*, *for any* $\varepsilon, M > 0$,

$$\mathrm{P}\Bigg[\sup_{(t,t') \in T \times S(\bigoplus^j \mathbb{R}^n)} \sup_{s \in B_{\mathbb{R}^n}(t,h)} \big|D^j \hat{f}(t;t') - D^j \hat{f}(s;t')\big| > \varepsilon\Bigg] = o(h^M).$$

PROOF. If we define the Gaussian field

$$\tilde{F}(t, t', \eta) = \begin{cases} F(t, \eta t'), & \eta \neq 0, \\ D^j_{L^2} f(t, t'), & \eta = 0, \end{cases}$$

on $T_{j,\rho} \times (-h, h)$, an open subset of the finite dimensional vector space $\mathbb{R}^n \times \bigoplus^j \mathbb{R}^n \times \mathbb{R}$ with norm

$$\|(t, t', \eta)\|_{n,j,1} = \|(t,t')\|_{n,j} + |\eta|,$$

the continuity of the jth derivative follows from well-known results for the continuity of Gaussian processes on general spaces (cf. Chapter 4 in [2]). The second conclusion follows from an application of the Borell–Cirelson inequality [2]. □

Next, we define suitably regular Gaussian fields on manifolds.

DEFINITION 3.2. A Gaussian random field on a C^k ($k \geq 3$) manifold M is *suitably regular* if it satisfies the following conditions:

(i) $f(\cdot, \omega) \in C^2(M)$ almost surely.

(ii) The symmetric two-tensor field g induced by f is C^2 and nondegenerate, where g is defined by,

$$g_p(X_p, Y_p) = \mathrm{E}[X_p f Y_p f].$$

In other words f induces a C^2 Riemannian metric on M, given by the pull-back of the standard structure on $L^2(\Omega, \mathcal{F}, P)$ under the map $p \mapsto f_p$.

(iii) For every $\varepsilon > 0$, $X, Y \in C^1(T(M))$,

$$\mathrm{P}\left[\sup_{p\in M}\sup_{q\in B_\tau(p,h)}|XYf_p - XYf_q| > \varepsilon\right] = o(h^n)$$

where τ is the metric on M induced by the Riemannian metric g.

The third condition replaces the analogous condition in Theorem 5.2.2 in [1].

The following lemma gives sufficient conditions for suitable regularity on a compact C^k ($k \geq 3$) manifold.

LEMMA 3.3. *Suppose f is a Gaussian random field on a C^k ($k \geq 3$) compact manifold M that induces a C^2 Riemannian metric on M as described above. Furthermore, suppose that M has a countable atlas $\mathcal{A} = (U_i, \varphi_i)_{i\in I}$ such that for every i the Gaussian field $f_i = f \circ \varphi_i^{-1}$ on $\varphi_i(U_i) \subset \mathbb{R}^n$ satisfies* (3.1) *with $j = 2$, $T = \varphi_i(U_i)$ and $f = f_i$ and some $K_i, \delta_i, h_{0,i} > 0$. Then f is suitably regular.*

PROOF. It follows from Lemma 3.1 that for any i, $f_i \in C^2(\varphi_i(U_i))$ almost surely. Since $\mathcal{A}$ is countable it follows by definition that $f \in C^2(M)$ almost surely. Furthermore since M is compact, there exists a finite atlas $\mathcal{A}'$ with index set $I(\mathcal{A}')$, such that for every chart in $\mathcal{A}'$, (3.1) holds with a K and δ independent of the chart. Without loss of generality, we can choose $\mathcal{A}'$ such that $\varphi_i(U_i)$ is a bounded open set in $\mathbb{R}^n$.

Next we note that, for h small enough, denoting by τ the Riemannian metric induced by f,

$$\sup_{p\in M}\sup_{q\in B_\tau(p,h)}|XYf_p - XYf_q| = \max_{i\in I(\mathcal{A}')}\sup_{p\in U_i}\sup_{q\in B_\tau(p,h)\cap U_i}|XYf_p - XYf_q|.$$

Therefore, to show

$$\mathrm{P}\left[\sup_{p\in M}\sup_{q\in B_\tau(p,h)}|XYf_p - XYf_q| > \varepsilon\right] = o(h^n),$$

it is sufficient to show that it holds in every U_i. Denoting by τ_i the metric on $\varphi_i(U_i)$ induced by τ, and by d_i the standard metric on $\mathbb{R}^n$ restricted to $\varphi_i(U_i)$, we have for h sufficiently small, there exists $C_{1i}, C_{2i} > 0$, such that,

$$B_{d_i}(x, C_{1i}h) \subset B_{\tau_i}(x,h) \subset B_{d_i}(x, C_{2i}h).$$

So we just have to prove

$$\mathrm{P}\left[\sup_{x\in\varphi_i(U_i)}\sup_{y\in B_{d_i}(x,h)\cap\varphi_i(U_i)}\left|X^iY^if_i(x) - X^iY^if_i(y)\right| > \varepsilon\right] = o(h^n), \tag{3.2}$$

where $X^i = \varphi_{i*}X$, $Y^i = \varphi_{i*}Y$. We can write $X^iY^if_i(x)$ as follows:

$$X^iY^if_i(x) = \sum_{j=1}^n a_j(x)\frac{\partial f_i}{\partial x_j}(x) + \sum_{k,l=1}^n b_{kl}(x)\frac{\partial^2 f_i}{\partial x_k\,\partial x_l}(x), \tag{3.3}$$

where the a_j, b_{kl} are bounded continuous functions on $\varphi_i(U_i)$ (or, if necessary we can enlarge the atlas $\mathcal{A}'$ so that they are bounded). It follows from our assumptions, that for all h,

$$\sup_{x\in\varphi_i(U_i)} \sup_{y\in B_{d_i}(x,h)\cap\varphi_i(U_i)} \sup_{c\in S(\mathbb{R}^n)} \mathrm{E}\left[\left(\sum_{k=1}^n c_k\left(\frac{\partial f_i}{\partial x_k}(x) - \frac{\partial f_i}{\partial x_k}(y)\right)\right)^2\right] < h^2 K_i',$$

for some $K_i > 0$, expressible in terms of the variance of the second order L^2 derivatives of f_i. Combining this with the fact that the functions a_j and b_{kl} are bounded, and Lemma 3.1 completes the proof. □

4. Expected Euler characteristics for smooth Gaussian random fields on manifolds without boundary. In this section we derive a formulae for $\mathrm{E}[\chi(f^{-1}[u,+\infty))] = \mathrm{E}[\chi(M \cap f^{-1}[u,+\infty))]$ when f is a centered, unit variance Gaussian random field defined on a compact, oriented, manifold M, with or without boundary. We begin with the case when M has no boundary.

Part (i) of Theorem 2.3 enables us to derive a point set representation for the Euler characteristic of a C^2 manifold with boundary N. In particular,

$$\chi\big(M \cap f^{-1}[u,+\infty)\big) = \sum_{k=0}^n (-1)^k \#_M\{f > u, df = 0,\ \mathrm{index}(-\nabla^2 f) = k\},$$

so that $\chi\big(M \cap f^{-1}[u,+\infty)\big)$ can be represented as the total number of points in $n+1$ different point processes. Our assumptions on the Gaussian field will allow us to use Lemmas 2.4 and 2.5, and we have the following:

THEOREM 4.1. *Let f be a suitably regular, centered, unit variance Gaussian field on a C^3 compact manifold M (cf. Definition* 3.2). *Then*

$$\mathrm{E}\big[\chi\big(M \cap f^{-1}[u,+\infty)\big)\big] = \sum_{j=0}^n \mathcal{L}_j(M)\rho_j(u)$$

where $\mathcal{L}_j(M)$ are the Lipschitz–Killing curvatures (1.2) *of M, calculated with respect to the metric induced by f and ρ_j is given by*

$$\begin{aligned}\rho_j(u) &= \frac{1}{(2\pi)^{(j+1)/2}}\int_u^\infty H_j(t)e^{-t^2/2}\,dt \\ &= \begin{cases} \dfrac{1}{(2\pi)^{(j+1)/2}}H_{j-1}(u)e^{-u^2/2}, & j \ge 1, \\ 1-\Phi(u), & j = 0,\end{cases}\end{aligned}$$

where

$$H_j(x) = \sum_{l=0}^{\lfloor j/2\rfloor} \frac{(-1)^l j!}{(j-2l)!\,l!\,2^l}x^{j-2l}$$

is the jth Hermite polynomial and Φ is the cumulative distribution of a standard Gaussian random variable.

PROOF. As mentioned above, we shall use Lemma 2.4 to count the critical points of f in M above the level u, with index k and Lemma 2.3 to calculate $\chi(M \cap f^{-1}[u, +\infty))$. In the notation of Lemma 2.4, $N = \mathbb{R}^n$; $E = \mathbb{R} \times \mathrm{Sym}_{n\times n}$ (= the space of symmetric $n \times n$ matrices), $M = M \cap f^{-1}[u, +\infty)$ and $q_0 = 0$. The maps G and H are defined as follows: $G = (f, -\nabla^2 f_E)$ and $H = \nabla f_E$, where $\nabla^2 f_E$ and ∇f_E take values in $\mathrm{Sym}_{n\times n}$ and $\mathbb{R}^n$ and are the coefficients of the tensors $\nabla^2 f$ and ∇f read off in some fixed orthonormal frame field $E = (E_1, \dots, E_n)$. Specifically, they are defined by

$$\nabla^2 f_{E,ij} = \nabla^2 f(E_i, E_j)$$

and

$$\nabla f_{E,i} = E_i f.$$

Finally, we set $A_k = \{(x, T) : x > u, \mathrm{index}(T) = k\} \subset \mathbb{R} \times \mathrm{Sym}_{n\times n}$. It follows from the assumptions on f that it is almost surely nondegenerate (cf. Chapter 3 in [1]). This implies that f has, almost surely, a finite number of critical points, which are all nondegenerate. These are the regularity conditions needed for Lemma 2.4, so, after applying Lemma 2.4 $n + 1$ times, we see that, with probability 1,

$$\chi\big(M \cap f^{-1}[u, +\infty)\big) = \lim_{\varepsilon\to 0} \int_M \theta_f^\varepsilon(u).$$

Here $\theta_f^\varepsilon(u)$ is a random n-form on M, given in any chart (U, φ) with coordinates $(x_1, \dots, x_n)$ by

$$\begin{aligned}
\theta_f^\varepsilon(u)|_U &= \sum_{k=0}^n (-1)^k |\nabla f_E^*(\alpha_\varepsilon)| \mathbb{1}_{A_k}\big(f, -\nabla^2 f(E_i, E_j)\big) \\
&= \sum_{k=0}^n (-1)^k \left|\det\left(\frac{\partial E_i f}{\partial x_j}\right)\right| \mathbb{1}_{A_k}\big(f, -\nabla^2 f(E_i, E_j)\big) \\
&\qquad \times \beta_\varepsilon(\nabla f_E)\left(\bigwedge_{i=1}^n dx_i\right) \qquad (4.1) \\
&= \sum_{k=0}^n (-1)^k \sqrt{\det(g_{ij})} |\det(-E_i E_j f)| \mathbb{1}_{A_k}\big(f, -\nabla^2 f(E_i, E_j)\big) \\
&\qquad \times \beta_\varepsilon(\nabla f_E)\left(\bigwedge_{i=1}^n dx_i\right) \\
&= \det(-E_i E_j f) \beta_\varepsilon(\nabla f_E) \mathbb{1}_{[u,+\infty)}(f) \mathrm{Vol}_g
\end{aligned}$$

where $\alpha_\varepsilon(y) = \beta_\varepsilon(y)(\bigwedge_{i=1}^n dy_i)$ is a family of forms on $\mathbb{R}^n$, satisfying the requirements of Lemma 2.4.

What remains to be done to calculate $\mathrm{E}[\chi(M \cap f^{-1}[u, +\infty))]$ is to interchange the order of integration, (which will be justified by suitable regularity and Lemma 2.5), calculate $\lim_{\varepsilon\to 0} \mathrm{E}[\theta_f^\varepsilon(u)]$ and integrate the resulting expression over M.

In order to apply Lemma 2.5, we must show that, for ε sufficiently small and for every k,

$$\frac{d\mathrm{E}[\mathbb{1}_{\{(f,\nabla^2 f_E)\in A_k\}}|\nabla f_E^*\alpha_\varepsilon|]}{d\mathrm{Vol}_g} \le \frac{d\Theta}{d\mathrm{Vol}_g},$$

for some integrable form Θ. It suffices, then, to prove that for ε sufficiently small

$$\frac{d\mathrm{E}[|\nabla f_E^*\alpha_\varepsilon|]}{d\mathrm{Vol}_g} = \mathrm{E}\big[|\det(E_iE_jf)|\beta_\varepsilon(\nabla f_E)\big] \le \frac{d\Theta}{d\mathrm{Vol}_g}.$$

Noting that

$$E_iE_jf = \nabla^2 f(E_i, E_j) + \nabla_{E_i}E_jf$$

the determinant in the above expression can be bounded by a polynomial in the absolute value of the terms of ∇f_E with coefficients depending only on $\nabla^2 f_E$. Since the density of ∇f_E is bounded and $\nabla^2 f_E$ is independent of ∇f_E, we can indeed find such a Θ.

Continuing with the calculation,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathrm{E}\big[\det(-E_iE_jf)\beta_\varepsilon(\nabla f_E)\mathbb{1}_{f>u}\big]\\
&\quad= \int_u^\infty \mathrm{E}\big[\det(-E_iE_jf)\big|\nabla f_E = 0, f = t\big]\varphi_{f,\nabla f_E(t,0)}\,dt\\
&\quad= \int_u^\infty \mathrm{E}\big[\det\big(-\nabla^2 f(E_i,E_j) - \nabla_{E_i}E_jf\big)\big|\nabla f_E = 0, f = t\big]\varphi_{f,\nabla f_E}(t,0)\,dt\\
&\quad= \int_u^\infty \mathrm{E}\big[\det\big(-\nabla^2 f(E_i,E_j)\big)\big|\nabla f_E = 0, f = t\big]\varphi_{f,\nabla f_E}(t,0)\,dt\\
&\quad= \int_u^\infty \frac{1}{n!}\mathrm{E}\big[\mathrm{Tr}\big((-\nabla^2 f)^n\big)\big|\nabla f_E = 0, f = t\big]\varphi_{f,\nabla f_E}(t,0)\,dt,
\end{aligned}$$

where $\varphi_{f,\nabla f_E}$ is the joint density of f and ∇f_E and in the last line we have used (2.2). Since $\nabla^2 f$ is a Gaussian double form, we can use Lemma 2.1 to calculate the above expectation. We first calculate the conditional expectation of $\nabla^2 f$, as well as that of $(\nabla^2 f)^2$. We have

$$\mathrm{E}\big[(\nabla^2 f)\big|\nabla f_E = 0, f = t\big] = -tI,$$

$$\mathrm{E}\big[\big(\nabla^2 f - \mathrm{E}[(\nabla^2 f)|\nabla f_E = 0, f = t]\big)^2\big|\nabla f_E = 0, f = t\big] = -(2R + I^2)$$

where I is the identity double form, determined by g, described in Section 2.1. The proof of the above is straightforward, following the ideas Section 2.4 and making specific use of (2.6) and (2.7), combined with standard results about multivariate Gaussian conditional distributions. The expectation can thus be written as

$$\begin{aligned}
&\frac{(-1)^n}{n!}\mathrm{E}\big[\mathrm{Tr}^M(\nabla^2 f)^n\big|df=0, f=t\big] \\
&\quad= \sum_{j=0}^{\lfloor n/2\rfloor}\frac{(-1)^j}{(n-2j)!j!2^j}\,\mathrm{Tr}^M\big((tI)^{n-2j}(I^2+2R)^j\big) \\
&\quad= \sum_{j=0}^{\lfloor n/2\rfloor}\sum_{l=0}^{j}\frac{(-1)^j}{j!2^j(n-2j)!}t^{n-2j}\,\mathrm{Tr}^M\left(I^{n-2l}\binom{j}{l}(2R)^l\right) \\
&\quad= \sum_{l=0}^{\lfloor n/2\rfloor}\frac{(-1)^l}{l!}\,\mathrm{Tr}^M\left(R^l\sum_{k=0}^{\lfloor (n-2l)/2\rfloor}\frac{(-1)^k}{2^k(n-2k-2l)!k!}t^{n-2k-2l}I^{n-2l}\right) \\
&\quad= \sum_{l=0}^{\lfloor n/2\rfloor}\frac{(-1)^l}{l!}\,\mathrm{Tr}^M(R^l)H_{n-2l}(t)
\end{aligned}$$

where, in the last line, we have used (2.1). We conclude that

$$\begin{aligned}
\theta_f(u) &\stackrel{\Delta}{=} \lim_{\varepsilon\to 0}\mathrm{E}[\theta_f^\varepsilon(u)] \\
&= \sum_{l=0}^{\lfloor n/2\rfloor}\left[\int_u^\infty \frac{1}{(2\pi)^{(n+1)/2}}H_{n-2l}(t)e^{-t^2/2}\,dt\right]\frac{(-1)^l}{l!}\,\mathrm{Tr}^M(R^l)\mathrm{Vol}_g \\
&= \sum_{l=0}^{\lfloor n/2\rfloor}\frac{\rho_{n-2l}(u)}{(2\pi)^l}\frac{(-1)^l}{l!}\,\mathrm{Tr}^M(R^l)\mathrm{Vol}_g.
\end{aligned}$$

The conclusion of the theorem follows by integrating $\theta_f(u)$ over M. □

5. The manifold with boundary case.

In what follows now we shall consider the case when M is a manifold with boundary. In this situation, to use Morse's theorem we need to assume that M is an embedded submanifold with boundary of some ambient manifold N and our fields f are suitably regular when viewed as random fields on N. We do not have to consider $M \subset \mathbb{R}^n$, M could be a compact C^2 domain of any ambient manifold N.

THEOREM 5.1. *Let f be a suitably regular zero mean, unit variance Gaussian field on a C^3 manifold N (cf. Definition 3.2). Suppose M is an embedded*

C^2 submanifold with boundary of N, the boundary being denoted by ∂M, with outward pointing unit normal vector field ν. Then

$$\mathrm{E}\big[\chi\big(M \cap f^{-1}[u,+\infty)\big)\big] = \sum_{j=0}^{n} \mathcal{L}_j(M)\rho_j(u), \tag{5.1}$$

where $\mathcal{L}_j(M)$ are the Lipschitz–Killing curvatures of M, calculated with respect to the metric induced by f and are defined by

$$\mathcal{L}_j(M) = \mathcal{L}_j(\mathring{M}) + \int_{\partial M} Q_j \operatorname{Vol}_{\partial M,g},$$

with

$$Q_j = \sum_{k=0}^{\lfloor (n-1-j)/2 \rfloor} \frac{1}{(2\pi)^k s_{n-j-2k}} \frac{(-1)^k}{k!(n-1-j-2k)!} \operatorname{Tr}^{\partial M}(S^{n-1-j-2k} R^k),$$

where $\mathring{M}$ is the interior of M, S is the second fundamental form of ∂M in M, defined by

$$S(X,Y) = -g(\nabla_X \nu, Y)$$

and

$$s_j = \frac{2\pi^{j/2}}{\Gamma(\frac{j}{2})}$$

is the surface area of the unit sphere in $\mathbb{R}^j$.

PROOF. Following the arguments in Theorem 4.1, we apply Lemmas 2.4 and 2.5 along with (ii) of Lemma 2.3. From these it follows that

$$\mathrm{E}\big[\chi\big(M \cap f^{-1}[u,+\infty)\big)\big] = \int_{\mathring{M}} \theta_f(u) + \int_{\partial M} \theta_f^{\partial M}(u)$$

where $\theta_f^{\partial M}(u)$ is an $(n-1)$-form on ∂M. Assuming that the orthonormal frame field E is chosen so that $E_n = \nu$, the unit outward pointing normal vector field on ∂M, we see that in any chart (U,φ) such that $U \cap \partial M \neq \varnothing$, we have

$$\begin{aligned}\theta_f^{\partial M}(u) = \frac{1}{(2\pi)^{(n-1)/2}} \int_u^\infty \int_0^\infty \mathrm{E}\big[&\det(-E_i E_j f)_{1\le i,j\le n-1} \big| f = t,\\ &\nabla f_E = (0,\dots,0,y)\big]\\ &\times \varphi_{f,\nabla f_E}\big(t,(0,\dots,0,y)\big)\, dy\, dt \operatorname{Vol}_{\partial M,g}.\end{aligned}$$

The conditional covariances remain the same as in the previous section, as we are still conditioning on the vector $(f, \nabla f_E)$. Specifically, the restriction of $\nabla^2 f$ to ∂M satisfies

$$\mathrm{E}\big[(\nabla^2 f_{|\partial M})^2 \big| f = t, \nabla f_E = (0,\dots,0,y)\big] = -(2R + I^2).$$

As for conditional means,

$$
\begin{aligned}
\mathrm{E}&\big[XYf\,\big|\,f=t,\nabla f_E=(0,\dots,0,y)\big]\\
&=\mathrm{E}\big[\nabla^2 f_{|\partial M}(X,Y)\,\big|\,f=t,\nabla f_E=(0,\dots,0,y)\big]\\
&=-g(X,Y)t-S(X,Y)y
\end{aligned}
$$

for any $X, Y \in C^2(T(\partial M))$. Equivalently,

$$
\mathrm{E}\big[\nabla^2 f_{|\partial M}\,\big|\,f=t,\nabla f_E=(0,\dots,0,y)\big]=-tI-yS.
$$

The expectation in the expression for $\theta_f^{\partial M}(u)$, can then be written as follows:

$$
\begin{aligned}
&\frac{(-1)^{n-1}}{(n-1)!}\mathrm{E}\big[\mathrm{Tr}^{\partial M}(\nabla^2 f_{|\partial M})^{n-1}\,\big|\,f=t,\nabla f_E=(0,\dots,0,y)\big]\\
&\quad=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\frac{(-1)^j}{(n-1-2j)!\,j!\,2^j}\,\mathrm{Tr}^{\partial M}\big((tI+yS)^{n-1-2j}(I^2+2R)^j\big)\\
&\quad=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\sum_{l=0}^{j}\frac{(-1)^j}{(n-1-2j)!}\frac{1}{2^{j-l}l!(j-l)!}\,\mathrm{Tr}^{\partial M}\big((tI+yS)^{n-1-2j}I^{2j-2l}R^l\big)\\
&\quad=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\sum_{k=0}^{n-1-2j}\sum_{l=0}^{j}\frac{(-1)^j(n-1-2l-k)!}{(n-1-2j-k)!\,k!}\frac{1}{2^{j-l}l!(j-l)!}\\
&\qquad\qquad\times y^k t^{n-1-2j-k}\,\mathrm{Tr}^{\partial M}(S^kR^l)\\
&\quad=\sum_{k=0}^{n-1}\sum_{j=0}^{\lfloor (n-1-k)/2\rfloor}\sum_{l=0}^{j}\frac{(-1)^j(n-1-2l-k)!}{(n-1-2j-k)!\,k!}\frac{1}{2^{j-l}l!(j-l)!}\\
&\qquad\qquad\times y^k t^{n-1-2j-k}\,\mathrm{Tr}^{\partial M}(S^kR^l)\\
&\quad=\sum_{k=0}^{n-1}\sum_{l=0}^{\lfloor (n-1-k)/2\rfloor}\sum_{j=l}^{\lfloor (n-1-k)/2\rfloor}\frac{(-1)^j(n-1-2l-k)!}{(n-1-2j-k)!\,k!}\frac{1}{2^{j-l}l!(j-l)!}\\
&\qquad\qquad\times y^k t^{n-1-2j-k}\,\mathrm{Tr}^{\partial M}(S^kR^l)\\
&\quad=\sum_{k=0}^{n-1}\sum_{l=0}^{\lfloor (n-1-k)/2\rfloor}\frac{(-1)^l y^k\,\mathrm{Tr}^{\partial M}(S^kR^l)}{k!\,l!}\\
&\qquad\qquad\times\sum_{i=0}^{\lfloor (n-1-2l-k)/2\rfloor}\frac{(-1)^i(n-1-2l-k)!}{(n-1-2l-k-2i)!}\frac{1}{2^i i!}t^{n-1-2l-k-2i}
\end{aligned}
$$

$$= \sum_{k=0}^{n-1} \sum_{l=0}^{\lfloor (n-1-k)/2 \rfloor} \frac{(-1)^l y^k}{l!k!} \mathrm{Tr}^{\partial M}(S^k R^l) H_{n-1-2l-k}(t)$$

$$= \sum_{m=0}^{n-1} \sum_{l=0}^{\lfloor m/2 \rfloor} \frac{(-1)^l y^{m-2l}}{l!(m-2l)!} \mathrm{Tr}^{\partial M}(S^{m-2l} R^l) H_{n-1-m}(t)$$

$$= \sum_{j=0}^{n-1} \sum_{l=0}^{\lfloor (n-1-j)/2 \rfloor} \frac{(-1)^l y^{n-1-j-2l}}{l!(n-1-j-2l)!} \mathrm{Tr}^{\partial M}(S^{n-1-j-2l} R^l) H_j(t).$$

After integrating over $(u, +\infty) \times (0, +\infty)$, we conclude that

$$\theta_f^{\partial M}(u) = \sum_{j=0}^{n-1} Q_j \rho_j(u) \mathrm{Vol}_{\partial M, g}.$$

The conclusion of the theorem now follows from integration over ∂M. □

6. Examples. In this section we derive the expected Euler characteristic for some specific manifolds and random fields defined on them.

6.1. *Nonstationary fields on* $\mathbb{R}^n$. Since $\mathbb{R}^n$ itself is a manifold, it is natural to consider suitably regular Gaussian random fields f on $\mathbb{R}^n$. We first fix T, a compact C^2 domain in $\mathbb{R}^n$. Furthermore, we can relax the C^3 assumption present in Theorem 5.1 because ∂T is a hypersurface in $\mathbb{R}^n$. We now proceed to give an outline of how one would calculate $\mathrm{E}[\chi(T \cap f^{-1}[u, +\infty))]$ by deriving a formula for R in terms of derivatives of C_f the covariance function of f. In general, though, it is obvious that there is no closed form for the expression in Theorem 5.1, except as a universal functional of the derivatives of C_f. However, the integrals can be calculated numerically for a specific example.

In order to carry out the numeric integration, we must evaluate the function $\mathrm{Tr}^{\mathbb{R}^n}(R^k)$ for $k = 1, \dots, \lfloor n/2 \rfloor$. If $(E_i)_{1 \le i \le n}$ are the coordinate vector fields on $\mathbb{R}^n$ [i.e., the natural basis in the global chart $(\mathbb{R}^n, i)$ where i is the inclusion map], then it is an exercise to show that, for any C^2 Riemannian metric g on $\mathbb{R}^n$, with R its curvature tensor,

$$\begin{aligned} R^E_{ijkl} &\triangleq R\big((E_i, E_j), (E_k, E_l)\big) \\ &= \sum_{s=1}^{n} \big[g_{sl}\big(E_i(\Gamma^s_{jk}) - E_j(\Gamma^s_{ik})\big) + \Gamma_{isl}\Gamma^s_{jk} - \Gamma_{jsl}\Gamma^s_{ik} \big] \\ &= E_i \Gamma_{jkl} - E_j \Gamma_{ikl} + \sum_{s,t=1}^{n} (\Gamma_{iks} g^{st} \Gamma_{jlt} - \Gamma_{jks} g^{st} \Gamma_{ilt}) \end{aligned}$$

where $g_{ij} = g(E_i, E_j)$, g^{ij}, Γ_{ijk} and Γ_{ij}^k are defined by the following relations:

$$\Gamma_{ijk} = g(\nabla_{E_i} E_j, E_k) = \tfrac{1}{2}[E_j g_{ik} - E_k g_{ij} + E_i g_{jk}],$$

$$\Gamma_{ij}^k = \sum_{s=1}^n g^{ks}\Gamma_{ijs},$$

$$\sum_{k=1}^n g_{ik} g^{kj} = \delta_{ij}.$$

We saw in (2.6) that if g is the metric induced by a random field f,

$$g_p(\nabla_X Y_p, Z_p) = \mathrm{E}[XYf_p Zf_p],$$

from which

$$\Gamma_{ijk}(x) = \frac{\partial^3 C_f(t,s)}{\partial t_i \partial t_j \partial s_k}\bigg|_{(x,x)}.$$

Denoting by $(de_i)_{1\le i\le n}$ the dual basis of $(E_i)_{1\le i\le n}$, we can express the curvature tensor as

$$R = \tfrac{1}{4}\sum_{i,j,k,l=1}^n R_{ijkl}^E (de_i \wedge de_j) \otimes (de_k \wedge de_l).$$

For any measurable section $(X_i)_{1\le i\le n}$ of $\mathcal{O}(T)$ with dual frames $(\theta_i)_{1\le i\le n}$ such that

$$\theta_i = \sum_{i'=1}^n g_{ii'}^{1/2} de_{i'},$$

where $g^{1/2}$, given by

$$(g^{1/2})_{ij} = g(E_i, X_j),$$

is a (measurable) square root of g. It follows that

$$R = \tfrac{1}{4}\sum_{i,j,k,l=1}^n R_{ijkl}^X (\theta_i \wedge \theta_j) \otimes (\theta_k \wedge \theta_l),$$

where

$$\begin{aligned} R_{ijkl}^X &= \sum_{i',j',k',l'=1}^n R_{i'j'k'l'}^E g_{ii'}^{-1/2} g_{jj'}^{-1/2} g_{kk'}^{-1/2} g_{ll'}^{-1/2} \\ &= R((X_i, X_j), (X_k, X_l)). \end{aligned}$$

We next define the curvature forms for the section X by

$$R = \tfrac{1}{2}\Theta_{ij} \otimes \theta_i \wedge \theta_j.$$

From the definition of the double wedge product "·", R^k can therefore be expressed as

$$R^k = \frac{1}{2^k} \sum_{i_1,\dots,i_{2k}=1}^{n} \left(\bigwedge_{l=1}^{k} \Theta_{i_{2l-1}i_{2l}} \right) \otimes \left(\bigwedge_{l=1}^{k} (\theta_{i_{2l-1}} \wedge \theta_{i_{2l}}) \right).$$

It follows that

$$R^k\big((X_{a_1},\dots,X_{a_{2k}}),(X_{a_1},\dots,X_{a_{2k}})\big) = \frac{1}{2^{2k}} \sum_{i_1,\dots,i_{2k}=1}^{n} \delta^{(a_1,\dots,a_{2k})}_{(i_1,\dots,i_{2k})} \left(\sum_{\sigma\in S(2k)} \varepsilon_\sigma \prod_{l=1}^{k} R^X_{i_{2l-1}i_{2l}a_{\sigma(2l-1)}a_{\sigma(2l)}} \right)$$

where, for all m,

$$\delta^{(c_1,\dots,c_m)}_{(b_1,\dots,b_m)} = \begin{cases} \varepsilon_\sigma, & \text{if } c=\sigma(b), \text{ for some } \sigma\in S(m), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} \mathrm{Tr}^T(R^k) &= \frac{1}{(2k)!} \sum_{a_1,\dots,a_{2k}=1}^{n} R^k\big((X_{a_1},\dots,X_{a_{2k}}),(X_{a_1},\dots,X_{a_{2k}})\big) \\ &= \frac{1}{2^{2k}} \sum_{a_1,\dots,a_{2k}=1}^{n} \left(\sum_{\sigma\in S(2k)} \varepsilon_\sigma \prod_{l=1}^{k} R^X_{a_{2l-1}a_{2l}a_{\sigma(2l-1)}a_{\sigma(2l)}} \right). \end{aligned} \tag{6.1}$$

The same method can be used to derive a formula for $\mathrm{Tr}^{\partial T}(R^k S^j)$ ($j \le n-2k$). We also note that, the above calculation gives a formula that can be used to calculate $\mathrm{Tr}^M(R^k)$ in any given chart for any Riemannian manifold (M,g).

6.2. *Lie groups and stationary Gaussian fields.* A Lie group G is a group that is also a C^∞ manifold, such that the map taking g to g^{-1} is C^∞ and the map taking (g_1,g_2) to g_1g_2 is also C^∞. We denote the identity element of G by e and by L_g and R_g the left and right multiplication maps and by $I_g = L_g \circ R_g^{-1}$ the inner automorphism of G induced by g.

We recall that a vector field X on G is said to be *left invariant* if for all $g,g'\in G$, $(L_g)_* X_{g'} = X_{gg'}$. Similarly, a covariant tensor field Φ is said to be left invariant (resp. right invariant) if, for every g_0, g in G, $L^*_{g_0}\Phi_{g_0 g} = \Phi_g$ (resp. $R^*_{g_0}\Phi_{g g_0} = \Phi_g$), Φ is said to be *bi-invariant* if it is both left and right invariant. If h is a (left, right, bi-) invariant Riemannian metric on G, then it is clear that, for every g, the map (L_g, R_g, I_g) is an isometry of (G,h). In particular, the curvature tensor R of h, is (left, right, bi-) invariant. This means that for Gaussian random fields that induce such Riemannian metrics, the integrals needed to evaluate $\mathrm{E}[\chi(M\cap f^{-1}[u,+\infty))]$ are significantly easier to calculate.

LEMMA 6.1. *Suppose f is a suitably regular, centered, unit variance Gaussian random field on a compact n-dimensional Lie group G, such that the Riemannian metric h induced by f is (left, right, bi-) invariant. Then,*

$$\mathrm{E}[\chi(M \cap f^{-1}[u, +\infty))] = \sum_{l=0}^{\lfloor n/2 \rfloor} \frac{\rho_{n-2l}(u)}{(2\pi)^l} \frac{(-1)^l}{l!} \mathrm{Tr}^{T_e G}(R_e^l)\mu_h(G),$$

where μ_h is the Riemannian measure of (G, h).

PROOF. We have to prove the following:

$$\int_G \mathrm{Tr}^G(R^l)_g \, d\mu_h(g) = \mathrm{Tr}^{T_e G}(R_e^l)\mu_h(G).$$

Suppose X, Y, Z, W are left-invariant vector fields. Since $g' \mapsto L_g g'$ is an isometry for every g, we have

$$\begin{aligned} R_g\big((X_g, Y_g), (Z_g, W_g)\big) &= R_e\big((L_{g^{-1}*}X_g, L_{g^{-1}*}Y_g), (L_{g^{-1}*}Z_g, L_{g^{-1}*}W_g)\big) \\ &= R_e\big((L_{g*})^{-1}X_g, (L_{g*})^{-1}Y_g, (L_{g*})^{-1}Z_g, (L_{g*})^{-1}W_g\big) \\ &= R_e(X_e, Y_e, Z_e, W_e). \end{aligned}$$

Therefore, if $(X_i)_{1 \le i \le n}$ is an orthonormal set of left-invariant vector fields,

$$\begin{aligned} &(R_g)^l\big((X_{i_1 g}, \dots, X_{i_l g}), (X_{j_1 g}, \dots, X_{j_l g})\big) \\ &\quad = (R_e)^l\big((X_{i_1 e}, \dots, X_{i_l e}), (X_{j_1 e}, \dots, X_{j_l e})\big), \end{aligned}$$

from which it follows that

$$\mathrm{Tr}^{T_g G}\big((R_g)^l\big) = \mathrm{Tr}^{T_e G}\big((R_e)^l\big),$$

which completes the proof. □

A natural class of examples which induce invariant Riemannian metrics on a Lie group G are the stationary random fields on G. Since G is not necessarily Abelian, Bochner's spectral representation theorem does not hold and we must distinguish between left and right stationary. For a discussion on representations of stationary random fields on general groups, not just Lie groups, see [23]. We say that a random field f on G is *left stationary* if for all n, and all $(g_1, \dots, g_n)$, and any g_0, the following holds:

$$\big(f(g_1), \dots, f(g_n)\big) \stackrel{\mathcal{D}}{=} \big(f \circ L_{g_0}(g_1), \dots, f \circ L_{g_0}(g_n)\big)$$

where $\stackrel{\mathcal{D}}{=}$ means equality in distribution. We say that f is *right stationary* if $f'(g) \stackrel{\Delta}{=} f(g^{-1})$, is left stationary, and that f is *bi-stationary* if it is both left and right stationary. As in the case where $G = \mathbb{R}^n$, for a Gaussian random field f on G, there is a simple condition on the covariance function of f that implies stationarity. Specifically, it is easy to check that if C_f satisfies

$$C_f(g_1, g_2) = C'_f(g_1^{-1} g_2),$$

for some function $C' : G \to \mathbb{R}$, then f is left stationary. Similarly, if C_f satisfies

$$C_f(g_1, g_2) = C''_f(g_1 g_2^{-1}),$$

for some function C'', then f is right stationary.

Intuitively, we expect a left-stationary random field to induce a left-invariant Riemannian metric, which as the following lemma shows is indeed the case.

LEMMA 6.2. *Let f be a* (*left*, *right*, *bi*)*-stationary random field on a Lie group G with a derivative in the L^2 sense. If f induces a Riemannian metric h, then h is* (*left*, *right*, *bi-*) *invariant.*

PROOF. We prove the result when f is left-stationary, the proofs are identical for the right- or bi-stationary case. For a fixed $g_0 \in G$, we define the random field f^{g_0} by $f^{g_0} = f \circ L_{g_0}$. By left stationarity, for every g_0, $f^{g_0} \stackrel{\mathcal{D}}{=} f$ as a random field. In particular, the metric induced by f^{g_0} is the same as that induced by f, in other words, denoting by h the metric on G induced by f,

$$\begin{aligned} h_g(X_g, Y_g) &= \mathrm{E}[X_g f Y_g f] \\ &= \mathrm{E}[X_g f^{g_0} Y_g f^{g_0}] \\ &= \mathrm{E}[(L_{g_0 *} X_g) f (L_{g_0 *} Y_g) f] \\ &= (L_{g_0}^* h_{g_0 g})(X_g, Y_g). \end{aligned}$$

This completes the proof. □

Next, we give an example of how to construct a (left, right)-stationary Gaussian random field on a G. Note that the above lemma shows that we can only construct smooth bi-stationary Gaussian fields on G if it is unimodular, that is, if any left Haar measure on G is also right invariant. We give the construction for a left-stationary field. The construction also immediately gives a construction for a right-stationary Gaussian field as follows: if f is left stationary, then it is straightforward to check that the process $\tilde{f}(g) \stackrel{\Delta}{=} f(g^{-1})$ is right stationary.

We say G has a smooth (C^∞) (left) action on a smooth (C^∞) n-manifold M, if there exists a map $\theta : G \times M \to M$ satisfying, for all $x \in M$, and $g_1, g_2 \in G$,

$$\theta(e, x) = x,$$

$$\theta(g_2, \theta(g_1, x)) = \theta(g_2 g_1, x).$$

We write $\theta_g : M \to M$ for the partial mapping $\theta_g(x) = \theta(g, x)$. Suppose μ is a measure on M, such that, for all $g \in G$,

$$\theta_{g*}(\mu) \ll \mu,$$
$$\frac{d\theta_{g*}(\mu)}{d\mu}(x) = D(g),$$

where $\theta_{g*}(\nu)$ is the push-forward of the measure ν under the map θ_g and $d\nu/d\mu$ is the Radon–Nikodym derivative of ν with respect to μ. We call such a measure μ (*left*) *relatively invariant* under G. It is easy to see that $D(g)$ is a C^∞ homomorphism from G into the multiplicative group of positive real numbers, that is, $D(g_1 g_2) = D(g_1)D(g_2)$, so that μ is left invariant with respect to G if, and only if, it is left relatively-invariant and $D \equiv 1$, that is, D is the trivial homomorphism.

LEMMA 6.3. *Suppose G acts smoothly on a smooth manifold M, and μ is relatively invariant under G. Let D denote the related homomorphism from G to $\mathbb{R}$ and let W be a Gaussian white noise on $L^2(M, \mathcal{B}(M), \mu)$. Then, for any $F \in L^2(M, \mathcal{B}(M), \mu)$,*

$$f(g) = \frac{1}{\sqrt{D(g)}} W(F \circ \theta_{g^{-1}}),$$

is a left-stationary Gaussian random field on G.

PROOF. We must prove that

$$\mathrm{E}[f(g_1)f(g_2)] = C(g_1^{-1}g_2)$$

for some $C : G \to \mathbb{R}$. From the definition of W we have

$$\begin{aligned}
\mathrm{E}[f(g_1)f(g_2)] &= \frac{1}{\sqrt{D(g_1)D(g_2)}} \int_M F\big(\theta_{g_1^{-1}}(x)\big) F\big(\theta_{g_2^{-1}}(x)\big)\mu(dx) \\
&= \frac{1}{\sqrt{D(g_1)D(g_2)}} \int_M F\big(\theta_{g_1^{-1}}\big(\theta_{g_2}(x)\big)\big) F(x)\theta_{g_2*}(\mu)(dx) \\
&= \frac{D(g_2)}{\sqrt{D(g_1)D(g_2)}} \int_M F\big(\theta_{g_1^{-1}g_2}(x)\big) F(x)\mu(dx) \\
&= \sqrt{D(g_1^{-1}g_2)} \int_M F\big(\theta_{g_1^{-1}g_2}(x)\big) F(x)\mu(dx) \\
&= C(g_1^{-1}g_2).
\end{aligned}$$

This completes the proof. □

It is clear that the regularity of F determines the regularity of its related random field f. For instance, it is easy to show that if $F \in C_c^0(M)$ (c for compact support), and

$$\sup_{x\in M}\sup_{g\in B_\tau(e,h)}|F(x)-F(gx)|<\frac{K}{(-\log h)^{1+\delta}}$$

where τ is the metric associated to a left-invariant Riemannian metric on G, then the stationary field related to F has a continuous modification. It is also easy to show that if $F\in C_c^3(M)$ and the left-invariant two-tensor h on G induced by F is nondegenerate, then f, the Gaussian random field related to F is suitably regular.

The most natural example of a Lie group acting on a manifold is its action on itself. In particular, any right Haar measure is left relatively-invariant. Another natural example of a group G acting on M is given by $G=GL(n,\mathbb{R})\times\mathbb{R}^n$ acting on $M=\mathbb{R}^n$. For $g=(A,t)$ and $x\in\mathbb{R}^n$ we have

$$\theta(g,x)=Ax+t.$$

In this example, it is easy to see that Lebesgue measure dx is relatively-invariant with respect to G with $D(g)=\det(A)$. Furthermore, any Lie subgroup $H\subset G$ acts on $\mathbb{R}^n$ and Lebesgue measure is clearly relatively-invariant with respect to H. Consider the following examples:

(i) $H=\mathbb{R}^n$. In this situation the field constructed is just a stationary random field on $\mathbb{R}^n$ in the usual sense.

(ii) $H=\mathbb{R}^+I\times\mathbb{R}^n$, that is,

$$\theta(g,x)=\theta\big((s,t),x\big)=(sI)x+t.$$

This example was used in [17], though only up to $n=3$ and with the assumption that $F(x)=F(-x)$. The calculations were done directly and did not use the fact that the random field was in fact stationary.

(iii) $H=G$. This situation was studied in [16], but only up to $n=2$, due to the significant algebra involved. Even in the case $n=2$, the symbolic computation software MAPLE was used to perform the calculations. No use was made of the fact that the random field was stationary in this situation either.

6.3. *Spheres and spherical caps.* Throughout this section, Z_n will denote a suitably regular isotropic Gaussian random field on $\mathbb{R}^n$, such that both it and its first order partial derivatives have unit variance.

For our first example, we restrict Z_n to $S_a(\mathbb{R}^n)$, the sphere of radius a in $\mathbb{R}^n$. Since $S_a(\mathbb{R}^n)$ is a space of constant curvature a^{-2}, we have

$$R=-\frac{1}{2a^2}I^2$$

and hence

$$\mathcal{L}_{n-2l}(S_a(\mathbb{R}^n))=\frac{2a^{n-2l-1}\pi^{(n-2l)/2}}{\Gamma\left(\frac{n}{2}\right)}\frac{(n-1)!}{2^l l!(n-1-2l)!},$$

$$\mathcal{L}_{n-2l-1}=0,$$

so that, by Theorem 4.1,

$$\mathrm{E}\big[\chi\big(S_a(\mathbb{R}^n)\cap Z_n^{-1}[u,+\infty)\big)\big] = \sum_{l=0}^{\lfloor (n-1)/2\rfloor} \frac{2a^{n-2l-1}\pi^{(n-2l)/2}}{\Gamma(\frac{n}{2})}\frac{(n-1)!}{2^l l!(n-1-2l)!}\rho_{n-2l}(u).$$

As a second example, consider a spherical cap in $S_a(\mathbb{R}^n)$, that is, a geodesic ball in $S_a(\mathbb{R}^n)$, which, without loss of generality, we center around the point $x=(0,\dots,0,a)$. Since the spherical distance between two points $x,y\in S_a(\mathbb{R}^n)$ is given by

$$d(x,y)=a\cos^{-1}\big(a^{-2}\langle x,y\rangle\big),$$

a geodesic ball of radius r is therefore

$$B_{S_a(\mathbb{R}^n)}(x,r)=\big\{y\in S_a(\mathbb{R}^n):\langle x,y\rangle\ge a^2\cos(r/a)\big\}.$$

The interior terms of $\mathcal{L}_{n-2l}(B_{S_a(\mathbb{R}^n)}(x,r))$ required by Theorem 5.1 will clearly be

$$\mathcal{H}_{n-1}\big(B_{S_a(\mathbb{R}^n)}(x,r)\big)\frac{(n-1)!}{(n-1-2l)!(2\pi)^l l!}\frac{1}{2^l a^{2l}}.$$

As for the boundary terms, note that the boundary lies in the hyperplane $x_n=\cos(r/a)$, which implies that the shape operator of $\partial B_{S_a(\mathbb{R}^n)}(x,r)$ in $S_a(\mathbb{R}^n)$ is equal to $\cos(r/a)$ times the shape operator of $\partial B_{S_a(\mathbb{R}^n)}(x,r)$ in the hyperplane $x_n=\cos(r/a)$, which is just $(a\sin(r/a))^{-1}I$. Using these facts we see that

$$\int_{\partial B_{S_a(\mathbb{R}^n)}(x,r)} Q_j \,\mathrm{Vol}_{\partial B_{S_a(\mathbb{R}^n)}(x,r)} = \sum_{k=0}^{\lfloor (n-2-j)/2\rfloor}\frac{a^j}{(4\pi)^k}\frac{s_{n-1}}{s_{n-1-j-2k}}\frac{(n-2)!}{(n-2-j-2k)!}\frac{\cot(r/a)^{n-2-j-2k}}{j!k!}.$$

Substituting the interior and boundary terms into Theorem 5.1 gives us the required formula for

$$\mathrm{E}\big[\chi\big(B_{S_a(\mathbb{R}^n)}(x,r)\cap Z_n^{-1}[u,+\infty)\big)\big].$$

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