Euler's Constant to 1271 Places

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Abstract. The value of Euler's or Mascheroni's constant

$$\gamma = \lim_{n\to\infty} (1 + \tfrac{1}{2} + \cdots + (1/n) - \ln n)$$

has now been determined to 1271 decimal places, thus extending the previously known value of 328 places. A calculation of partial quotients and best rational approximations to γ was also made.

1. Historical Background. Euler's constant was, naturally enough, first evaluated by Leonhard Euler, and he obtained the value 0.577218 in 1735 [1]. By 1781 he had calculated it more accurately as 0.5772156649015325 [2]. The calculations were carried out more precisely by several later mathematicians, among them Gauss, who obtained

$$\gamma = 0.57721566490153286060653.$$

Various British mathematicians continued the effort [3], [4]; an excellent account of the work done on evaluation of γ before 1870 is given by Glaisher [5]. Finally, the famous mathematician-astronomer J. C. Adams [6] laboriously determined γ to 263 places. Adams thereby extended the work of Shanks, who had obtained 110 places (101 of which were correct).

Adams' result stood until 1952, when Wrench [9] calculated 328 decimal places. Although much work has been done trying to decide whether γ is rational, the evaluation has not been carried out any more precisely. With the use of high-speed computers, the constants π and e have been evaluated to many thousands of decimal places [11], [12]. A complete bibliography for π appears in [11]. The evaluation of γ to many places is considerably more difficult.

2. Evaluation of γ. The technique used here to calculate γ is essentially that used by Adams and earlier mathematicians. A complete derivation of the method is given by Knopp [7]. We use Euler's summation formula in the form

$$\sum_{i=1}^{n} f(i) = \int_1^n f(x)\,dx + \tfrac{1}{2}(f(n) + f(1)) + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!}[f^{(2j-1)}(n) - f^{(2j-1)}(1)] + R_k \tag{1}$$

where B_m are the Bernoulli numbers defined symbolically by

$$e^{Bx} = \frac{x}{e^x - 1}. \tag{2}$$

With this notation, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, etc. Here the remainder

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R_k is given by

$$R_k = \frac{1}{(2k+1)!}\int_1^n P_{2k+1}(x) f^{(2k+1)}(x)\,dx; \tag{3}$$

and $P_{2k+1}(x)$ is a periodic Bernoulli polynomial, symbolically

$$P_{2k+1}(x) = (\{x\} + B)^{2k+1} = (-1)^{k-1}(2k+1)!\sum_{r=1}^{\infty}\frac{2\sin 2r\pi x}{(2r\pi)^{2k+1}} \tag{4}$$

where $\{x\}$ is the fractional part of x.

Now we put $f(x) = 1/x$, obtaining from (1)

$$\begin{aligned} 1 + \tfrac{1}{2} + \cdots + \frac{1}{n} &= \ln n + \frac{1}{2} + \frac{1}{2n} \\ &+ \frac{B_2}{2}\left(1 - \frac{1}{n^2}\right) + \cdots + \frac{B_{2k}}{2k}\left(1 - \frac{1}{n^{2k}}\right) - \int_1^n \frac{P_{2k+1}(x)}{x^{2k+2}}\,dx. \end{aligned} \tag{5}$$

Taking the limit in (5) as $n \to \infty$, we find

$$\gamma = \frac{1}{2} + \frac{B_2}{2} + \cdots + \frac{B_{2k}}{2k} - \int_1^\infty \frac{P_{2k+1}(x)}{x^{2k+2}}\,dx. \tag{6}$$

Subtracting (5) from (6) gives

$$\begin{aligned} \gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \frac{1}{2n} + \frac{B_2}{2n^2} \\ + \cdots + \frac{B_{2k}}{(2k)n^{2k}} - \int_n^\infty \frac{P_{2k+1}(x)\,dx}{x^{2k+2}}. \end{aligned} \tag{7}$$

If the remainder is discarded and we consider (7) as an infinite series in k, it diverges as $k \to \infty$. It still yields a good method for calculating γ, however, since

$$|P_{2k+1}(x)| \leqq \frac{2(2k+1)!}{(2\pi)^{2k+1}}\sum_{r=1}^{\infty}\frac{1}{r^{2k+1}}, \tag{8}$$

and by applying Stirling's formula to (8) we obtain

$$\left|\int_n^\infty \frac{P_{2k+1}(x)\,dx}{x^{2k+2}}\right| \leqq \frac{4}{n}\sqrt{\frac{k}{\pi}}\left(\frac{k}{n\pi e}\right)^{2k}. \tag{9}$$

Put $k = 250$ and $n = 10000$ to obtain a remainder

$$\left|\int_{10000}^\infty \frac{P_{501}(x)\,dx}{x^{502}}\right| < 10^{-1269}, \tag{10}$$

so these values may be used in (7) to determine γ to at least 1269 places. This particular choice of k and n was made for convenience on a decimal computer, in an attempt to obtain the greatest precision in a reasonable time.

3. Details of the Computation. The sum $1 + \frac{1}{2} + \cdots + \frac{1}{10000}$ was evaluated as

$$S_{10000} = \frac{3}{2} + \frac{7}{12} + \cdots + \frac{19999}{99990000} = 9.787606036\cdots. \tag{11}$$

Combining terms in this way reduced the number of necessary divisions. The natural logarithm of 10000 was then determined by

(12) $\ln 10000 = -252 \ln (1 - .028) + 200 \ln (1 + .0125) + 92 \ln (1 - .004672).$

Such an expansion was designed for fast convergence and for convenience on a decimal computer. It is a simple matter to obtain such an expansion by hand calculation; we seek integers (x, y, z) such that $2^x 3^y 5^z \approx 1$ and $y \geqq 0$. If three linearly independent solutions are obtained, one can calculate ln 2, ln 3, and ln 5, and, in particular, ln 10. If $2^{x_1}3^{y_1}5^{z_1} > 1$ and $2^{x_2}3^{y_2}5^{z_2} < 1$, suitable positive integral combinations of (x_1, y_1, z_1) and (x_2, y_2, z_2) will give closer approximations. The method is to find small values of (x, y, z) so that $x + y \log_2 3 + z \log_2 5 \approx 0$, then combine these to get better and better approximations. The expansion (12) corresponds to the solutions $(-1, 5, -3)$, $(-4, 4, -1)$, and $(6, 5, -6)$. For a binary computer the extra requirement $z \geqq 0$ makes it more difficult, but solutions can be used such as

(12a) $\ln 10000 = 160 \ln 2^{-32}3^7 5^9 - 864 \ln 2^{-11}3^4 5^2 + 292 \ln 2^{-15}3^8 5.$

Finally, Bernoulli numbers $B'_{2k} = 10^{-8k} B_{2k}$ were evaluated using the recursion relation

$$(13) \quad \binom{2k+1}{2k} B'_{2k} + 10^{-8} \binom{2k+1}{2k-2} B'_{2k-2} + \cdots + 10^{8-8k} \binom{2k+1}{2} B_2' = (2k-1)/2 \cdot 10^{8k}.$$

From the fact that

$$(14) \quad \left| \frac{B_{2k}}{B_{2k-2}} \right| \approx \frac{2k(2k-1)}{4\pi^2}$$

it can be seen that the recursion (13) does not cause truncation errors to propagate. Furthermore, 1300 decimal places were used in all calculations.

When using (13) to calculate B'_{2k}, first all the positive terms were added together, then all the negative terms added together and finally the two were combined. This gave extra speed to the calculations. Care was also taken to avoid multiplying by zero. The evaluation of B'_{2k} becomes more difficult as k increases, because of the number of terms and the size of the binomial coefficients. Since the B_n alternate in sign, the actual error in the calculation of γ is less than $\frac{B'_{502}}{502} \approx +0.25 \times 10^{-1271}$, so the value obtained here should be correct to 1271 decimals. The fact that the final answer agreed with Adams' value and that numerous checks were made on all the arithmetical routines provides a good basis for guaranteeing the stated accuracy of the results. Dr. Wrench has independently verified the approximations to 1039 decimal places.

The present calculations were performed on a Burroughs 220 computer. The evaluation of S_{10000} required approximately one hour, and each of the logarithms required about six minutes. Evaluation of the 250 Bernoulli numbers was the most troublesome part of the calculations, and the total time for their calculation was approximately eight hours. A table of the Bernoulli numbers B' to 1270D has been sent to the Unpublished Mathematical Tables file of the journal, *Mathematics of Computation*.

4. Determination of Partial Quotients. To find best rational approximations to γ, we represent it as a continued fraction

$$\gamma = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}\,. \tag{15}$$

Put $P_1 = Q_0 = 1$, $Q_1 = P_0 = 0$, and for $i \geqq 1$

$$\begin{aligned} P_{i+1} &= a_i P_i + P_{i-1}\,. \\ Q_{i+1} &= a_i Q_i + Q_{i-1}\,. \end{aligned} \tag{16}$$

In matrix notation,

$$\begin{pmatrix} P_{i+1} & P_i \\ Q_{i+1} & Q_i \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

Then $\lim P_i/Q_i = \gamma$. The fractions P_i/Q_i represent the best approximations to γ in the sense that

$$| Q_i\gamma - P_i | < | q\gamma - p |, \quad \text{if} \quad q < Q_i\,, \quad i \geqq 3. \tag{18}$$

We have then $a_i \neq 0$ $(i > 1)$, if and only if γ is irrational, and the sequence of partial quotients a_i will be periodic if and only if γ is quadratic, that is,

$$\gamma = r + \sqrt{s}, \quad r \quad \text{and} \quad s \quad \text{rational.}$$

For proofs of these well-known results see Cassels [8].

The algorithm used to determine the partial quotients a_i, using limited decimal precision, is as follows:

Set $\gamma_1 = \gamma$, and

$$a_i = [\gamma_i], \quad \gamma_{i+1} = \{\gamma_i\}^{-1}, \quad i \geqq 1. \tag{19}$$

We have decimal numbers r_1 and s_1 such that

$$r_1 \leqq \gamma_1 \leqq s_1\,.$$

We successively find numbers r_i, s_i such that

$$r_i \leqq \gamma_i \leqq s_i\,. \tag{20}$$

If $[r_i] \neq [s_i]$, then the algorithm terminates. If $[r_i] = [s_i]$, then $[r_i] = a_i$ and

$$\{r_i\} \leqq \{\gamma_i\} \leqq \{s_i\}.$$

Hence

$$\{s_i\}^{-1} \leqq \gamma_{i+1} \leqq \{r_i\}^{-1}.$$

Choose decimal numbers r_{i+1} and s_{i+1} so that $r_{i+1} \leqq \{s_i\}^{-1}$ by truncation, $s_{i+1} \geqq \{r_i\}^{-1}$ by rounding up. Then the algorithm continues, until $[r_i] \neq [s_i]$.

The method used for calculating $\{s_i\}^{-1}$ when $\{s_i\}$ has several hundred decimal places was adapted from that of Pope and Stein [10]. Approximately six seconds was required to obtain each quotient. If t partial quotients are desired, the total time is proportional to t^3.

Table 1 gives the value of γ to 1271 decimal places. Table 2 gives the first 372 partial quotients of γ. Only 372 are given, although the value in Table 1 would have probably yielded over 1000 partial quotients. Table 3 gives for the reader's convenience the first few "best rational approximations" to γ. Here the ratio 228/395 gives a remarkably good value, correct to six decimal places.

From Table 2 one can compute

$$Q_{373} \approx 1.135 \times 10^{193}, \tag{21}$$

and we can conclude that if γ is rational its denominator must be larger than Q_{373}. Another consequence is that only about 385 decimal places of Table 1 were needed to obtain the 372 partial quotients. The referee has pointed out that Lehman [14] had already calculated the first 315 partial quotients for γ on the basis of Wrench's 328-place value [9]. These are in perfect agreement with the values obtained here.

The partial quotients of γ, as calculated in Table 2, appear to be "random" in some sense. Almost all real numbers have partial quotients satisfying

$$\lim \sqrt[n]{a_2 a_3 \cdots a_{n+1}} = K \tag{22}$$

where $K \approx 2.685$ is Khintchine's constant [13]. In this case,

$$\sqrt[371]{a_2 a_3 \cdots a_{372}} \approx 2.692,$$

a reasonable approximation to K.

Table 1
Value of Euler's Constant

.57721	56649	01532	86060	65120	90082	40243	10421	59335	93992
35988	05767	23488	48677	26777	66467	09369	47063	29174	67495
14631	44724	98070	82480	96050	40144	86542	83622	41739	97644
92353	62535	00333	74293	73377	37673	94279	25952	58247	09491
60087	35203	94816	56708	53233	15177	66115	28621	19950	15079
84793	74508	57057	40029	92135	47861	46694	02960	43254	21519
05877	55352	67331	39925	40129	67420	51375	41395	49111	68510
28079	84234	87758	72050	38431	09399	73613	72553	06088	93312
67600	17247	95378	36759	27135	15772	26102	73492	91394	07984
30103	41777	17780	88154	95706	61075	01016	19166	33401	52278
93586	79654	97252	03621	28792	26555	95366	96281	76388	79272
68013	24310	10476	50596	37039	47394	95763	89065	72967	92960
10090	15125	19595	09222	43501	40934	98712	28247	94974	71956
46976	31850	66761	29063	81105	18241	97444	86783	63808	61749
45516	98927	92301	87739	10729	45781	55431	60050	02182	84409
60537	72434	20328	54783	67015	17739	43987	00302	37033	95183
28690	00155	81939	88042	70741	15422	27819	71652	30110	73565
83396	73487	17650	49194	18123	00040	65469	31429	99297	77956
93031	00503	08630	34185	69803	23108	36916	40025	89297	08909
85486	82577	73642	88253	95492	58736	29596	13329	85747	39302
37343	88470	70370	28441	29201	66417	85024	87333	79080	56275
49984	34590	76164	31671	03146	71072	23700	21810	74504	44186
64759	13480	36690	25532	45862	54422	25345	18138	79124	34573
50136	12977	82278	28814	89459	09863	84600	62931	69471	88714
95875	25492	36649	35204	73243	64109	72682	76160	87759	50880
95126	20840	45444	77992	3(0)					

TABLE 2
Partial Quotients

	000	001	001	002	001	002	001	004	003	013	005	001	001	008
001	002	004	001	001	040	001	011	003	007	001	007	001	001	005
001	049	004	001	065	001	004	007	011	001	399	002	001	003	002
001	002	001	005	003	002	001	010	001	001	001	001	002	001	001
003	001	004	001	001	002	005	001	003	006	002	001	002	001	001
001	002	001	003	016	008	001	001	002	016	006	001	002	002	001
007	002	001	001	001	003	001	002	001	002	013	005	001	001	001
006	001	002	001	001	011	002	005	006	001	001	001	006	001	002
002	001	005	006	002	001	001	007	013	004	001	002	004	001	004
001	001	023	001	009	005	002	001	001	001	008	003	002	004	002
033	005	001	002	001	003	002	004	002	001	005	012	001	017	006
002	032	005	003	001	006	001	003	001	002	001	018	001	002	017
001	006	001	021	001	006	001	071	018	001	006	058	002	001	013
055	001	103	001	014	001	005	008	001	002	010	002	001	001	003
003	002	001	182	001	004	003	002	004	001	002	001	001	001	006
001	001	001	006	001	003	002	069	002	001	006	002	002	012	001
001	001	008	001	002	003	002	001	052	001	025	004	002	018	001
040	001	018	001	002	014	001	002	002	010	001	001	002	006	071
007	001	010	002	001	001	001	002	001	003	002	004	001	006	003
001	001	029	001	029	001	001	003	004	007	001	001	010	002	002
030	001	021	003	012	001	039	008	007	001	002	001	002	002	001
001	002	003	001	013	001	002	003	001	001	001	001	008	007	001
001	001	004	002	005	012	001	015	005	001	007	001	005	001	001
001	006	005	001	041	001	005	001	009	013	001	001	005	021	025
008	005	001	014	001	001	001	006	003	001	100	001	265		

TABLE 3
Best Rational Approximations

1/2	.50
3/5	.60
4/7	.571
11/19	.579
15/26	.5769
71/123	.57724
228/395	.5772152
3035/5258	.57721567
15403/26685	.5772156642
18438/31943	.5772156654
33841/58628	.57721566487

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