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EVALUATING DENSITY FORECASTS

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ABSTRACT

We propose methods for evaluating density forecasts. We focus primarily on methods that are applicable regardless of the particular user's loss function. We illustrate the methods with a detailed simulation example, and then we present an application to density forecasting of daily stock market returns. We discuss extensions for improving suboptimal density forecasts, multi-step-ahead density forecast evaluation, multivariate density forecast evaluation, monitoring for structural change and its relationship to density forecasting, and density forecast evaluation with known loss function.

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1. Introduction

Prediction occupies a distinguished position in econometrics; hence, evaluating predictive ability is a fundamental concern. Reviews of the forecast evaluation literature, such as Diebold and Lopez (1996), reveal that most attention has been paid to evaluating *point* forecasts. In fact, the bulk of the literature focuses on point forecasts, while conspicuously smaller sub-literatures treat interval forecasts (e.g., Chatfield, 1993; Christoffersen, 1997) and probability forecasts (e.g., Wallis, 1993; Clemen, Murphy and Winkler, 1995).

Little attention has been given to evaluating *density forecasts*. At least three factors explain this neglect. First, analytic construction of density forecasts has historically required restrictive and sometimes dubious assumptions, such as Gaussian innovations and no parameter estimation uncertainty. Recent work using numerical and simulation techniques to construct density forecasts, however, has reduced our reliance on such assumptions. In fact, improvements in computer technology have rendered the provision of credible density forecasts increasingly straightforward, in both classical and Bayesian frameworks.[1]

Second, until recently there was little demand for density forecasts; historically, point and interval forecasts seemed adequate for most users' needs. Again, however, recent developments have changed the status quo, particularly in quantitative finance. The booming area of financial risk management, for example, is effectively dedicated to providing density forecasts of portfolio values and to tracking certain aspects of the densities. The day will soon arrive in which risk

[1] See, for example, Efron and Tibshirani (1993) and Gelman, Carlin, Stern and Rubin (1995).

management will routinely entail nearly real-time issuance and evaluation of such density forecasts.

Finally, the problem of density forecast evaluation appears difficult. Although it is possible to adapt techniques developed for the evaluation of point, interval and probability forecasts to the evaluation of density forecasts, such approaches lead to incomplete evaluation of density forecasts. For example, using Christoffersen's (1997) method for evaluating interval forecasts, we can evaluate whether the series of 90% prediction intervals corresponding to a series of density forecasts is correctly conditionally calibrated, but that leaves open the question of whether the corresponding prediction intervals at other confidence levels are correctly conditionally calibrated. Correct conditional calibration of density forecasts corresponds to the *simultaneous* correct conditional calibration of all possible interval forecasts, the assessment of which seems a daunting task.

In light of the increasing importance of density forecasts, and lack of attention paid to them in the literature, we propose methods for evaluating density forecasts. Our work is related to the contemporaneous and independent work of Granger and Pesaran (1996), who explore a decision environment with probability forecasts defined over discrete outcomes (an environment different from ours, but closely related), and obtain a result analogous to our Proposition 2. They do not, however, focus on forecast evaluation. Our evaluation methods are based on an integral transform that turns out to have a long history, dating at least to Rosenblatt (1952). Contemporaneous and independent work by Crnkovic and Drachman (1996) is also closely related.

We proceed as follows. In section 2, we present a detailed statement and discussion of the problem, and we provide the theoretical underpinnings of the methods that we introduce subsequently. In section 3 we present methods of density forecast evaluation when the loss function is not known, which is often the relevant case in practice. In section 4, we provide a detailed simulation example of density forecast evaluation in an environment with time-varying volatility. In section 5, we use our tools to evaluate density forecasts of U.S. S&P 500 daily stock returns. In section 6, we discuss extensions for improving suboptimal density forecasts, evaluating multi-step and multivariate density forecasts, monitoring for structural change when density forecasting, and evaluating density forecasts when the loss function is known. We conclude in section 7.

2. Density Forecasts, Loss Functions and Action Choices: Implications for Density Forecast Evaluation

Studying the relationships among density forecasts, loss functions and action choices will help to clarify what can and can not be hoped for when evaluating density forecasts, and will suggest productive directions for density forecast evaluation. We first show that the problem of density forecast evaluation is intrinsically linked to the forecast user's loss function, which would appear to bode poorly for our quest for a universally-applicable approach to density forecast evaluation. We then show that, contrary to first impressions, all is not lost: the analysis suggests an important route to density forecast evaluation, which we pursue in subsequent sections.

<u>The Decision Environment</u>

Let $\{f_t(y_t|\Omega_t)\}_{t=1}^m$ be the sequence of data generating processes governing a series y_t, where $\Omega_t = \{y_{t-1}, y_{t-2}, \dots\}$, and let $\{p_t(y_t|\Omega_t)\}_{t=1}^m$ be a corresponding sequence of 1-step-ahead density forecasts.[2] Finally, let $\{y_t\}_{t=1}^m$ denote the corresponding series of realizations.[3]

Each forecast user has a loss function $L(a, y)$, where 'a' refers to an action choice, and chooses an action to minimize expected loss computed using the density believed to be the data generating process.[4] If she believes that the density forecast $p(y)$ is the correct density, then she chooses an action a^* such that[5]

$$a^*(p(y)) = \underset{a \in A}{\operatorname{argmin}} \int L(a, y) p(y) dy .$$

The action choice defines the loss $L(a^*, y)$ faced for every realization of the process $y \sim f(y)$. This loss is a random variable and possesses a probability distribution, which we call the loss distribution, and which depends only on the action choice.

Expected loss with respect to the true data generating process is

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[2] For notational convenience, we will often not indicate the information set and simply write $f_t(y_t)$ and $p_t(y_t)$, but the dependence on Ω_t should be understood. Moreover, because in this section we consider the relationships among density forecasts, loss functions and actions in a one-period context, we temporarily drop the time subscripts for notational convenience.

[3] We indulge in the standard abuse of notation, which favors convenience over precision, by failing to distinguish between random variables and their realizations. The meaning will be clear from context.

[4] Note the implicit assumption that agents proceed as if p equals f, in spite of the fact that p is only an estimate of f. A richer analysis would account for the estimation uncertainty; see the concluding remarks at the end of this paper.

[5] We assume a unique minimizer. A sufficient condition is that A be compact and that L be strictly convex in 'a'.

$$E[L(a^*, y)] = \int L(a^*, y)\ f(y)dy.$$

The effect of the density forecast on the user's expected loss is easily seen. A density forecast translates into a loss distribution. Two different forecasts will, in general, lead to different action choices and hence different loss distributions. The better is a density forecast, the lower is its expected loss, computed with respect to the true data generating process.

Ranking Two Forecasts

Suppose the user has the option of choosing between two forecasts in a given period, denoted by $p_j(y)$ and $p_k(y)$, where the subscript refers to the forecast. The user will weakly prefer forecast $p_j(y)$ to forecast $p_k(y)$ if

$$\int L(a_j^*, y)\ f(y)dy \le \int L(a_k^*, y)\ f(y)dy,$$

where a_j^* denotes the action that minimizes expected loss when the user bases the action choice on forecast j.

Ideally, we would like to find a ranking of forecasts with which all users would agree, *regardless of their loss function*. Unfortunately, the following proposition shows that such a ranking does not exist.

Proposition 1: Let $f(y)$ be the density of y, let a_j^* be the optimal action based on forecast p_j, and let a_k^* be the optimal action based on forecast p_k. Then there does not exist a ranking r of arbitrary density forecasts p_j and p_k, both distinct from f, such that for all loss functions $L(a, y)$,

$$r_j \geq r_k \Leftrightarrow \int L(a_j^*, y) f(y) dy \geq \int L(a_k^*, y) f(y) dy \ .$$

Proof: In order to establish the result, it is sufficient to find a pair of loss functions L_1 and L_2, a density function f governing y, and a pair of forecasts, p_j and p_k, such that

$$\int L_1(a_k^*, y) f(y) dy < \int L_1(a_j^*, y) f(y) dy,$$

while

$$\int L_2(a_k^*, y) f(y) dy > \int L_2(a_j^*, y) f(y) dy.$$

That is, user 1 does better on average under forecast k, while user 2 does better under forecast j. It is straightforward to construct such an example. Suppose the true density function is N(0,1), and suppose that user 1's loss function is $L_1(a, y) = (y - a)^2$ and user 2's loss function is $L_2(a, y) = (y^2 - a)^2$. The optimal action choices are then $\int y\, p(y)\, dy$ and $\int y^2\, p(y)\, dy$. That is, user 1 bases her action choice on the mean, with higher expected loss occurring with larger errors in the forecast mean, while user 2's actions and expected losses depend on the error in the forecast of the uncentered second moment. In this context, consider two forecasts: forecast j is N(0,2) and forecast k is N(1,1). User 1 prefers forecast j, because it leads to an action choice, and hence a loss distribution, with lower expected loss, but user 2 prefers forecast k for the same reason. □

To repeat: there is no way to rank two incorrect density forecasts such that all users will agree with the ranking.[6] However, it is easy to see that if a forecast coincides with the true data generating process, then it will be preferred by all forecast users, regardless of loss function. Formally, we have the following proposition:[7]

Proposition 2: Suppose that $p_j(y) = f(y)$, so that a_j^* minimizes the expected loss with respect to the true distribution. Then

$$\int L(a_j^*, y)f(y)dy \leq \int L(a_k^*, y)f(y)dy, \ \forall k.$$

Proof: The result follows immediately from the assumption that a_j^* minimizes expected loss over all possible actions, including those which might be chosen under alternative density forecasts. □

The proposition, although simple, is not vacuous. In particular, it suggests a useful direction for evaluating density forecasts. Regardless of loss function, we know that the correct density is weakly superior to all forecasts, which suggests evaluating forecasts by assessing whether the forecast densities are correct, i.e., whether $\{p_t(y_t|\Omega_t)\}_{t=1}^m = \{f_t(y_t|\Omega_t)\}_{t=1}^m$. If not, we know that some users, depending on their loss functions, could potentially be better served by a different density forecast. We now develop that idea in detail.

3. Evaluating Density Forecasts

The task of determining whether $\{p_t(y_t|\Omega_t)\}_{t=1}^m = \{f_t(y_t|\Omega_t)\}_{t=1}^m$ appears difficult, perhaps hopeless, because $\{f_t(y_t|\Omega_t)\}_{t=1}^m$ is never observed, even after the fact. Moreover, and

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[6] The result is analogous to Arrow's celebrated impossibility theorem. The ranking effectively reflects a social welfare function, which does not exist.

[7] Granger and Pesaran (1996) independently arrive at a similar result.

importantly, the true density $f_t(y_t \mid \Omega_t)$ may exhibit structural change, as indicated by its time subscript. As it turns out, the challenges posed by these subtleties are not insurmountable.

The Probability Integral Transform

Our methods are based on the relationship between the data generating process, $f_t(y_t)$, and the sequence of density forecasts, $p_t(y_t)$, as related through the probability integral transform, z_t, of the realization of the process taken with respect to the density forecast. The probability integral transform is defined as

$$\begin{aligned} z_t &= \int_{-\infty}^{y_t} p_t(u)du \\ &= P_t(y_t). \end{aligned}$$

The following lemma describes the distribution, $q_t(z_t)$, of the probability integral transform.

Lemma 1: Let $f_t(y_t)$ be the true density of y_t, let $p_t(y_t)$ be a density forecast of y_t, and let z_t be the probability integral transform of y_t with respect to $p_t(y_t)$. Then assuming that $\frac{\partial P_t^{-1}(z_t)}{\partial z_t}$ is continuous and nonzero over the support of y_t, z_t has support on the unit interval with density

$$\begin{aligned} q_t(z_t) &= \left|\frac{\partial P_t^{-1}(z_t)}{\partial z_t}\right| f_t(P_t^{-1}(z_t)) \\ &= \frac{f_t(P_t^{-1}(z_t))}{p_t(P_t^{-1}(z_t))}. \end{aligned}$$

Proof: Follows from the facts that $p_t(y_t) = \frac{\partial P_t(y_t)}{\partial y_t}$ and $y_t = P_t^{-1}(z_t)$. □

Note, in particular, a key fact: if $p_t(y_t) = f_t(y_t)$, then $q_t(z_t)$ is simply the U(0,1) density. This idea dates at least to Rosenblatt (1952).

Now we go beyond our one-period characterization of the density of z when $p_t(y_t) = f_t(y_t)$ and characterize both the density and dependence structure of the entire z sequence when $p_t(y_t) = f_t(y_t)$.

Proposition 3: Suppose $\{y_t\}_{t=1}^m$ is generated from $\{f_t(y_t|\Omega_t)\}_{t=1}^m$ where $\Omega_t = \{y_{t-1}, y_{t-2}, \ldots\}$. If a sequence of density forecasts $\{p_t(y_t)\}_{t=1}^m$ coincides with $\{f_t(y_t|\Omega_t)\}_{t=1}^m$, then under the usual condition of a non-zero Jacobian with continuous partial derivatives, the sequence of probability integral transforms of $\{y_t\}_{t=1}^m$ with respect to $\{p_t(y_t)\}_{t=1}^m$ is iid U(0,1). That is,

$$\{z_t\}_{t=1}^m \overset{\text{iid}}{\sim} U(0,1).$$

Proof: The joint density of $\{y_t\}_{t=1}^m$ can be decomposed as

$$f(y_m, \ldots, y_1 | \Omega_1) = f_m(y_m|\Omega_m)\, f_{m-1}(y_{m-1}|\Omega_{m-1}) \ldots f_1(y_1|\Omega_1).$$

We therefore compute the joint density of $\{z_t\}_{t=1}^m$ using the change of variables formula

$$q(z_1, z_2, \ldots, z_m) = \begin{vmatrix} \frac{\partial y_1}{\partial z_1} & \cdots & \frac{\partial y_1}{\partial z_m} \\ \vdots & \ddots & \cdots \\ \frac{\partial y_m}{\partial z_1} & \cdots & \frac{\partial y_m}{\partial z_m} \end{vmatrix} f_m(P_m^{-1}(z_m)|\Omega_m)\, f_{m-1}(P_{m-1}^{-1}(z_{m-1})|\Omega_{m-1}) \ldots$$

$$\ldots \times f_1(y_1^{-1}(z_1)|\Omega_1)$$

$$= \frac{\partial y_1}{\partial z_1} \frac{\partial y_2}{\partial z_2} \frac{\partial y_m}{\partial z_m} f_m(P_m^{-1}(z_m)|\Omega_m)\, f_{m-1}(P_{m-1}^{-1}(z_{m-1})|\Omega_{m-1}) \ldots$$

$$\ldots \times f_1(y_1^{-1}(z_1)|\Omega_1),$$

because the Jacobian of the transformation is lower triangular. Thus we have

$$q(z_m,\ldots,z_1|\Omega) = \frac{f_m(P_m^{-1}(z_m)|\Omega_m)}{p_m(P_m^{-1}(z_m))} \cdot \frac{f_{m-1}(P_{m-1}^{-1}(z_{m-1})|\Omega_{m-1})}{p_{m-1}(P_{m-1}^{-1}(z_{m-1}))} \cdots$$

$$\ldots \times \frac{f_1(P_1^{-1}(z_1)|\Omega_1)}{p_1(P_1^{-1}(z_1))}.$$

From Lemma 1, under the assumed conditions, each of the ratios above is a U(0,1) density, the product of which yields an m-variate U(0,1) distribution for $\{z_t\}_{t=1}^m$. Because the joint distribution is the product of the marginals, we have that $\{z_t\}_{t=1}^m$ is distributed iid U(0,1). □

The intuition for the above result may perhaps be better understood from the perspective of Christoffersen (1997), who shows that a correctly conditionally calibrated interval forecast will provide a hit sequence that is distributed iid Bernoulli, with the desired success probability.[8] If a sequence of density forecasts is correctly conditionally calibrated, then *every* interval will be correctly conditionally calibrated and will generate an iid Bernoulli hit sequence. This fact manifests itself in the iid uniformity of the corresponding probability integral transforms.

<u>Practical Application</u>

The theory developed thus far suggests that we evaluate density forecasts by assessing whether the probability integral transform series, $\{z_t\}_{t=1}^m$, is iid U(0,1). Simple tests of iid U(0,1) behavior are readily available, such as runs tests or Kolmogorov-Smirnov tests, all of which are

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[8] The "hit" series is 1 if the realization is contained in the forecast interval, and 0 otherwise.

actually joint tests of uniformity and iid. Such tests, however, are not likely to be of much value in practical applications, because they are not constructive; that is, when rejection occurs, the tests generally provide no guidance as to *why*. If, for example, such a statistic rejects the hypothesis of iid U(0,1) behavior, is it because of violation of unconditional uniformity, violation of iid, or both? Moreover, even if we know that rejection comes from violation of uniformity, we'd like to know more: What, precisely, is the nature of the violation of uniformity, and how important is it? Similarly, even if we know that rejection comes from a violation of iid, what precisely is its nature? Is z heterogeneous but independent, or is z dependent? If z is dependent, is the dependence operative primarily through the conditional mean, or are higher-ordered conditional moments, such as the variance, relevant? Is the dependance strong and important, or is iid an adequate approximation, even if strictly false?

The nonconstructive nature of tests of iid U(0,1) behavior, and the nonconstructive nature of related separate tests of iid and U(0,1), make us eager to adopt more revealing methods of exploratory data analysis. First, as regards evaluating unconditional uniformity, we suggest visual assessment using the obvious graphical tool, a density estimate. Simple histograms are attractive in the present context because they allow straightforward imposition of the constraint that z has support on the unit interval, in contrast to more sophisticated procedures such as kernel density estimates with the standard kernel functions. The estimated density can be visually compared to a U(0,1), and confidence intervals under the null hypothesis of iid U(0,1) are easy to compute.

Second, as regards evaluating whether z is iid, we again suggest visual assessment using the obvious graphical tool, the correlogram, supplemented with the usual Bartlett confidence intervals. The correlogram assists with the detection of particular dependence patterns in z and can provide useful information about the deficiencies of density forecasts. For instance, serial correlation in the z series may indicate that the mean dynamics have been inadequately modeled by the forecaster.[9] Because we're interested in potentially sophisticated nonlinear forms of dependence, in addition to linear dependence, we examine not only the correlogram of $(z-\bar{z})$, but also those of powers of $(z-\bar{z})$. In practice, we have found examination of the correlograms of $(z-\bar{z})$, $(z-\bar{z})^2$, $(z-\bar{z})^3$ and $(z-\bar{z})^4$ to be adequate; it will reveal dependence operative through the conditional mean, conditional variance, conditional skewness, or conditional kurtosis.

4. Application to a Simulated GARCH Process

Before proceeding to apply our density forecast evaluation methods to real data, it is useful to examine their efficacy on simulated data, for which we know the true data-generating process. Hence we examine data simulated from a realistic t-GARCH process designed to mimic high-frequency financial asset return data (Bollerslev, 1987). Specifically, we use a GARCH(1,1) data generating process, the conditional density of which is a standardized Student's-t with six degrees of freedom,

$$y_t = \sqrt{\frac{h_t(v-2)}{v}}\ t(6)$$

$$h_t = \omega + \alpha y_{t-1}^2 + \beta h_{t-1}.$$

We choose the parameters in accordance with those typically obtained when fitting GARCH models to high-frequency financial asset returns: $\omega = 0.01$, $\alpha = 0.13$, and $\beta = 0.86$. We simulate a

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[9] A caveat, however, is that there is in general no one-to-one correspondence between the type of dependence found in z and the dependence in y missed by the forecasts.

series of length 8000, chosen to mimic the sample sizes typical of high-frequency financial data, and we plot it in Figure 1. The persistence in conditional volatility is visually obvious.

We will examine the usefulness of our density forecast evaluation methods in assessing four progressively better density forecasts. Throughout, we split the sample in half and use the "in-sample" observations 1 through 4000 for estimation, and the "out-of-sample" observations 4001 through 8000 for density forecast evaluation.

To establish a benchmark, we first evaluate forecasts that are based on an incorrect and extremely naive assumption that the process is iid N(0,1).[10] That is, in each of the periods 4001-8000, we simply issue the forecast "N(0,1)." In Figure 2a we show two histograms of z, one with 20 bins and one with 40 bins.[11] The histograms have a distinct non-uniform "butterfly" shape -- a hump in the middle and two wings on the sides -- indicating that too many of the realizations fell in middle and in the tails of the forecast densities relative to what we'd expect if the data were really iid normal. This is exactly what we'd hope the histograms to reveal, given that the data-generating process is an unconditionally leptokurtic GARCH(1,1).

In Figure 2b we show the correlograms of $(z - \bar{z})$, $(z - \bar{z})^2$, $(z - \bar{z})^3$ and $(z - \bar{z})^4$.[12] The strong serial correlation in $(z - \bar{z})^2$ and $(z - \bar{z})^4$ makes clear another key deficiency of the N(0,1) forecasts -- they fail to capture the volatility dynamics operative in the process. Again, this

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[10] The process as specified does have mean zero and variance 1, but it is neither iid nor unconditionally Gaussian.

[11] The dashed lines superimposed on the histogram are approximate 95% confidence intervals for the individual bin heights under the null that z is iid U(0,1).

[12] The dashed lines superimposed on the correlograms Bartlett's are approximate 95% confidence intervals under the null that z is iid.

is what we'd hope the correlograms would reveal, given our knowledge of the true data-generating process.

Second, we evaluate forecasts produced under the incorrect assumption that the process is iid but not necessarily Gaussian. We estimate the unconditional distribution from observations 1 through 4000, freeze it, and then issue it as the density forecast in each of the periods 4001 through 8000. Figures 3a and 3b contain the results. The z histogram is now almost perfect (as it must be, apart from estimation error, which is small in a sample of size 4000), but the correlograms correctly continue to indicate neglected volatility dynamics.

Third, we evaluate forecasts that are based on a GARCH(1,1) model estimated under the incorrect assumption that the conditional density is Gaussian. We use observations 1 through 4000 to estimate the model, freeze the estimated model, and then use it to make (time-varying) density forecasts from 4001 through 8000. Figures 4a and 4b contain the z histograms and correlograms. The histograms are closer to uniform and therefore improved, but they still display slight peaks at either end and a hump in the middle. We would hope to see such a reduction, but not elimination, of the butterfly pattern, because allowance for conditionally Gaussian GARCH effects should account for some, but not all, unconditional leptokurtosis.[13] The correlograms now show no evidence of neglected conditional volatility dynamics, again as expected because the conditionally Gaussian GARCH model delivers consistent estimates of the conditional variance parameters, in spite of the fact that the conditional density is misspecified (Bollerslev and Wooldridge, 1992).

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[13] Recall that the data generating process is *conditionally*, as well as unconditionally, fat-tailed.

Finally, we forecast with an estimated correctly-specified t-GARCH(1,1) model. We show the z histogram and correlograms in Figures 5a and 5b. Because we are forecasting with a correctly specified model, estimated using a large sample, we would expect that the histogram and the correlograms would fail to find flaws with the density forecasts, which is the case.

In closing this section, we note that at each step of the above simulation exercise, our density forecast evaluation procedures clearly and correctly revealed the strengths and weaknesses of the various density forecasts. The results, as with all simulation results, are specific to the particular data-generating process examined, but the process and the sample size were chosen to be realistic for the leading applications in high-frequency finance. This gives us confidence that the procedures will perform well on real financial data, to which we now turn, and for which we don't have the luxury of knowing the true data-generating process.

5. Application to Daily S&P 500 Returns

We study density forecasts of daily value-weighted S&P 500 returns, with dividends, from 2/3/62 through 12/29/95; we plot the data in Figure 6. As before, we split the sample into in-sample and out-of-sample periods for model estimation and density forecast evaluation. There are 4133 in-sample observations (7/3/62 - 12/29/78) and 4298 out-of-sample observations (½/79 - 12/29/95). We then assess a series of density forecasts using our evaluation methods.

As in the simulation example, we begin with an examination of N(0,1) density forecasts, in spite of the fact that high-frequency financial data are well-known to be unconditionally leptokurtic and conditionally heteroskedastic.[14] In Figures 7a and 7b we show the histograms and correlograms of z. The histograms have the now-familiar butterfly shape, indicating that the S&P

[14] See, among many others, Bollerslev, Chou and Kroner (1992).

realizations are leptokurtic relative to the N(0,1) density forecasts, and the correlograms of $(z - \bar{z})^2$ and $(z - \bar{z})^4$ indicate that the N(0,1) forecasts are severely deficient, because they neglect strong conditional volatility dynamics.

Next, we generate density forecasts using an apparently much more sophisticated model. Both the Akaike and Schwarz information criteria select an MA(1)-GARCH(1,1) model for the in-sample data, which we estimate, freeze, and use to generate out-of-sample density forecasts. Figures 8a and 8b contain the z histograms and correlograms. The histograms are closer to uniform and therefore improved, although they still display slight peaks at either end and a hump in the middle. The correlograms look even better; all evidence of neglected conditional volatility dynamics has vanished.

Finally, in an effort to remove the last vestiges of non-uniformity from the z histogram, we estimate and then forecast with an MA(1) - t-GARCH(1,1) model. We show the z histogram and correlograms in Figures 9a and 9b. The histogram is improved, albeit slightly, and the correlograms remain good.

6. Extensions

Improving Density Forecasts

We have approached forecast evaluation from an historical perspective, evaluating the ability of a forecaster based on past realizations. The intent, of course, is to gauge the likely future accuracy of the forecaster based on past performance, assuming that the relationship between the correct density and the forecaster's predictive density remains fixed. Given that we observe systematic errors in the historical forecasts, we may wish to simply reject the forecast. It may also turn out that the errors are irrelevant to the user, a case we further examine when we explicitly account for the user's loss function. Nevertheless, it is possible to take the errors into consideration when using the current forecast, just as it is possible to do so in the point forecast case. In the point forecast case, for example, we can regress the y's on the $\hat{y}$'s, the predicted values, and use the estimated relationship to construct an adjusted point forecast.[15]

In the context of density forecasts that produce an iid z sequence, we can construct a similar procedure by rewriting the relationship in Lemma 1. Suppose that the user is in period m and possesses a density forecast of y_{m+1}. From Lemma 1, we have

$$\begin{aligned} f_{m+1}(y_{m+1}) &= p_{m+1}(y_{m+1})\, q_{m+1}(P(y_{m+1})) \\ &= p_{m+1}(y_{m+1})\, q_{m+1}(z_{m+1}). \end{aligned}$$

Thus if we know $q_{m+1}(z_{m+1})$, we would know the actual distribution $f_{m+1}(y_{m+1})$. Because $q_{m+1}(z_{m+1})$ is unknown, an estimate $\hat{q}_{m+1}(z_{m+1})$ can be formed using the historical series of $\{z_t\}_{t=1}^{m}$, and an estimate of the true distribution $\hat{f}_{m+1}(y_{m+1})$ can then be constructed. Standard density estimation techniques can be used to produce the estimate $\hat{q}_{m+1}(z_{m+1})$.[16]

<u>Multi-Step-Ahead Density Forecasts</u>

The evaluation of h-step ahead forecasts can also be evaluated using our methods, except that provisions must be made for autocorrelation in z. This is analogous to expecting MA(h-1) autocorrelation structures for optimal h-step ahead point forecast errors. In this case, it will

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[15] Such a regression is sometimes called a Mincer-Zarnowitz regression, after Mincer and Zarnowitz (1969).

[16] In finite samples, of course, there is no guarantee that the "improved" forecast will actually be superior to the original, because it is based on an estimate of q rather than the true q, and the estimate could be very poor. The same limitation obtains for Mincer-Zarnowitz regressions. The practical efficacy of our improvement methods is an empirical matter, which will have to await future research.

probably be easier to partition the z series into groups for which we expect iid uniformity if the forecasts were indeed correct. For instance, for correct 2-step ahead forecasts, the sub-series $\{z_1, z_3, z_5, \ldots\}$ and $\{z_2, z_4, z_6, \ldots\}$ should each be iid U(0,1), although the full series would not be iid U(0,1).

If a formal test is desired, it may be obtained via Bonferroni bounds, as suggested in a different context by Campbell and Ghysels (1995). Under the assumption that the z series is (h-1)-dependent, each of the following h sub-series will be iid: $\{z_1, z_{1+h}, z_{1+2h}, \ldots\}$, $\{z_2, z_{2+h}, z_{2+2h}, \ldots\}$, ..., $\{z_h, z_{2h}, z_{3h}, \ldots\}$. Thus, a test with size bounded by α can be obtained by performing h tests, each of size α/h, on each of the h sub-series of z, and rejecting the null hypothesis of iid uniformity if the null is rejected for *any* of the h sub-series. With the huge high-frequency datasets now available in finance, such sample splitting, although inefficient, is not likely to cause important power deterioration.

Multivariate Density Forecasts

The principle that governs the univariate techniques in this paper readily extends to the multivariate case, as shown in Diebold, Hickman, Schuermann and Tay (1997). Suppose that the variable of interest y is now an (N x 1) vector, and that we have on hand m multivariate forecasts and their corresponding multivariate realizations. Further suppose that we are able to decompose each period's forecasts into their conditionals, i.e., for each period's forecasts we can write

$$p(y_{1t}, y_{2t}, \ldots, y_{Nt} | \Phi_{t-1}) = p(y_{Nt} | y_{N-1,t}, \ldots, y_{1t}, \Phi_{t-1}) \ldots p(y_{2t} | y_{1t}, \Phi_{t-1}) p(y_{1t} | \Phi_{t-1}),$$

where Φ_{t-1} now refers to the past history of $(y_{1t}, y_{2t}, \ldots, y_{Nt})$. Then for each period we can transform each element of the multivariate observation $(y_{1t}, y_{2t}, \ldots, y_{Nt})$ by its corresponding conditional distribution. This procedure will produce a set of N z-series that will be iid U(0,1) individually, and also when taken as a whole. Note that we will have N! sets of z series, depending on how the joint density forecasts are decomposed, giving us a wealth of information with which to evaluate the forecasts. In addition, the univariate formula for the adjustment of forecasts, discussed above, can be applied to each individual conditional, yielding

$$f(y_{1t}, y_{2t}, \ldots, y_{Nt} | \Phi_{t-1}) = \prod_{i=1}^{N} [p(y_{it} | y_{i-1,t}, \ldots, y_{1t}, \Phi_{t-1}) \, q(P(y_{it} | y_{i-1,t}, \ldots, y_{1t}, \Phi_{t-1}))]$$

$$= p(y_{1t}, y_{2t}, \ldots, y_{Nt} | \Phi_{t-1}) \, q(z_{1t}, z_{2t}, \ldots, z_{Nt} | \Phi_{t-1}) \; .$$

Monitoring for Structural Change When Density Forecasting

Real-time monitoring of adequacy of density forecasts using CUSUM techniques is a simple matter, because under the adequacy hypothesis the z series is iid U(0,1), which is free of nuisance parameters. In particular, if $z_t \overset{iid}{\sim} U(0,1)$, then $z_t \overset{iid}{\sim} \left(\frac{1}{2}, \frac{1}{12}\right)$, so that asymptotically in m,

$$\sum_{t=1}^{m} z_t \sim N\left(\frac{m}{2}, \frac{m}{12}\right),$$

which yields the approximate 95% confidence interval for the CUSUM,

$$\sum_{t=1}^{m} z_t \in \left[\frac{m}{2} \pm 1.96\sqrt{\frac{m}{12}}\right]$$

Similar calculations hold for the CUSUM of squares. Trivial calculations reveal that under the adequacy hypothesis $z_t^2 \overset{iid}{\sim} \left(\frac{1}{3}, \frac{4}{45}\right)$, so that asymptotically in m,

$$\sum_{t=1}^{m} z_t^2 \sim N\left(\frac{m}{3}, \frac{4m}{45}\right),$$

which yields the approximate 95% confidence interval for the CUSUM of squares,

$$\sum_{t=1}^{m} z_t^2 \in \left[\frac{m}{3} \pm 1.96\sqrt{\frac{4m}{45}}\right].$$

Evaluating Density Forecasts Using a Specific Loss Function

If a series of density forecasts has been systematically in error, it may still be the case that for a particular user, depending on her loss function, the systematic errors may be irrelevant. To be precise, the forecast may be such that the action choice induced by the forecast, a_p^*, minimizes the user's actual expected loss.[17] In such cases, which we now consider, the user's loss function can be incorporated into the evaluation process, as is done in other forecasting contexts by Diebold and Mariano (1995) and Christoffersen and Diebold (1996, 1997a, 1997b).

Consider a density forecast series, $\{p_t(y_t)\}_{t=1}^{m}$, and the corresponding action series, $\{a_{p,t}^*\}_{t=1}^{m}$, of a particular user. The series of action choices results in a series of potential losses, $L(a_{p,t}^*, y_t)$. We would like to compare each period's realized loss with that period's expected

[17] Because we have assumed a unique optimal action choice, $a_p^* = a_f^*$.

loss under the optimal action choice $E_{f,t}[L(a^*_{f,t}, y_t)]$. The expected difference will be positive unless $a^*_{p,t} = a^*_{f,t}$.

Unfortunately, we are unable to evaluate $E_{f,t}[L(a^*_{f,t}, y_t)]$. Instead, we will have to use an estimate of $E_{p,t}[L(a^*_{p,t}, y_t)]$ as a proxy for $E_{f,t}[L(a^*_{f,t}, y_t)]$. We can then compute the difference,

$$d_t = L(a^*_{p,t}, y_t) - \frac{1}{m}\sum_{t=1}^{m} L(a^*_{p,t}, y_t).$$

Under the joint null hypothesis that the series of density forecasts is optimal relative to the user's loss function and that the forecaster correctly specifies the expected loss in each period, i.e., $E_{f,t}[L(a^*_{f,t}, y_t)] = E_{p,t}[L(a^*_{p,t} = a^*_{f,t}, y_t)]$, we have $E[d_t]=0$, which can be tested in the same way that Diebold and Mariano (1995) test whether two point forecasts are equally accurate under the relevant loss function.

7. Summary and Concluding Remarks

We have provided a characterization of optimal density forecasts, and we have proposed methods for evaluating whether reported density forecasts coincide with the true sequence of conditional densities. In addition to studying the decision problem associated with density forecasting and showing how to use the series of probability integral transforms to judge the adequacy of a series of density forecasts, we also indicated how to improve a suboptimal density forecast by using information on previously-issued density forecasts and subsequent realizations, how to evaluate multi-step and multivariate density forecasts, and how to monitor for structural change when density forecasting. We did all of this in a framework not requiring specification of the loss function, but when information on the relevant loss function is available, we also showed how to evaluate a density forecast with respect to that loss function.

Notwithstanding their classical feel, our methods are also applicable to Bayesian forecasts issued as predictive probability densities. Superficially, it would seem that strict Bayesians would have little interest in our evaluation methods, on the grounds that conditional on a particular sample path and specification of the prior and likelihood, the predictive density simply is what it is, and there's nothing to evaluate. But such is not the case. A misspecified likelihood, for example, can lead to poor forecasts, whether classical or Bayesian, and density forecast evaluation can help us to flag misspecified likelihoods. It comes as no surprise, therefore, that model checking by comparing predictions to data is emerging as an integral part of modern Bayesian data analysis and forecasting, as highlighted for example in Gelman, Carlin, Stern and Rubin (1995), and our methods are very much in that spirit.

It appears that our methods may also be related to the idea of predictive likelihood, which is based not on the joint density of the sample (the likelihood), but rather the joint density of *future* observations, *conditional* upon the sample (the predictive likelihood).[18] [19] Moreover, in a fascinating development, Clements and Hendry (1993) establish a close link between predictive likelihood and a measure of the accuracy of *point* forecasts that they propose, the generalized forecast error second moment (GFESM). A more detailed investigation of the relationships among our methods, predictive likelihood methods, and the GFESM is beyond the scope of this paper but appears to be a promising direction for future research.

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[18] For a concise introduction to predictive likelihood, see Bjørnstad (1990).

[19] We thank a clever referee for making this observation.

In closing, we wish to focus on the fact that our evaluation tools do not depend on the method used to produce the density forecasts being evaluated; in our framework, the forecasts are the primitives, and in particular, we do not assume that the forecasts are based on a model. This is useful because many density forecasts of interest do not come from models, and even when they do, the forecast evaluator may not have access to the model. Such is the case, for example, with the density forecasts of inflation recorded in the Survey of Professional Forecasters since 1968; for a description of those forecasts and evaluation using our methods, see Diebold, Tay and Wallis (1997).[20] A second and very important example of model-free density forecasts is provided by the recent finance literature, which shows how to use options written at different strike prices to extract a model-free estimate of the market's risk-neutral density forecast of returns on the underlying asset (e.g., Aït-Sahalia and Lo, 1995; Soderlind and Svensson, 1997).

At the same time, we readily acknowledge that many density forecasts *are* based on estimated models, and the sample size sometimes *is* small, in which case it seems clear that it would be useful to extend our methods to account for parameter estimation uncertainty, in a fashion precisely analogous to West's (1996) and West and McCracken's (1997) extensions of Diebold and Mariano (1995).[21] Similarly, the decision-theoretic background that we sketch requires that agents use density forecasts as if they were known to be the true conditional density,

[20] Diebold, Tay and Wallis (1997) also augment the methods proposed here with resampling procedures to approximate better the finite-sample distributions of the test statistics of interest in small macroeconomic, as opposed to financial, samples.

[21] Our simulation results indicate that the effects of parameter estimation uncertainty are inconsequential at least for the comparatively large samples relevant in finance.

in a fashion similar to West, Edison and Cho (1993); it remains to be seen how the decision theory would change if uncertainty were acknowledged.

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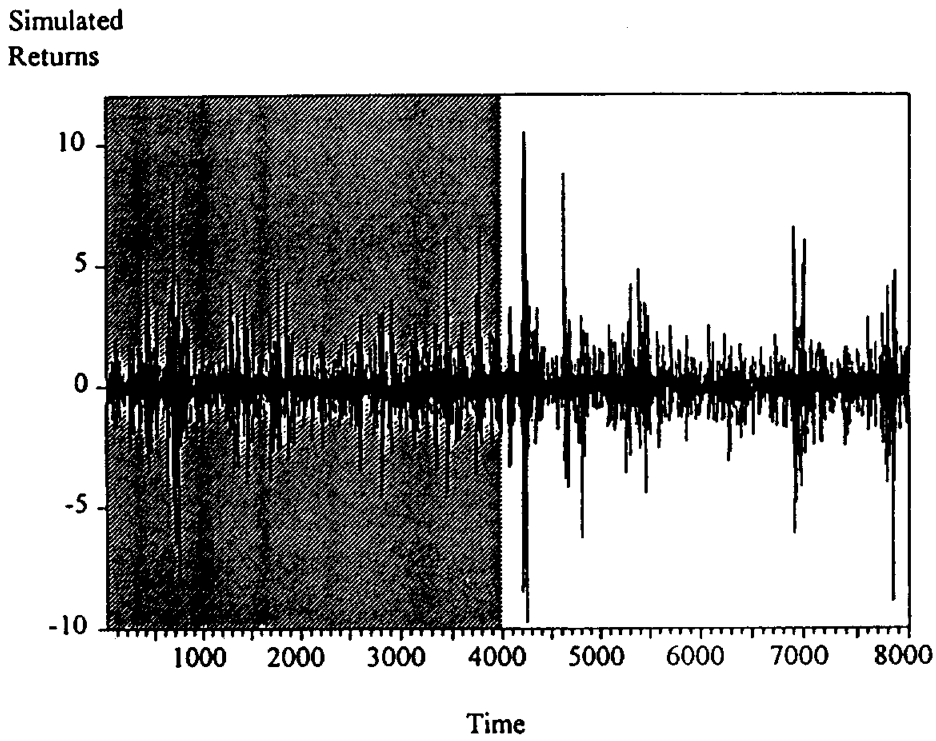
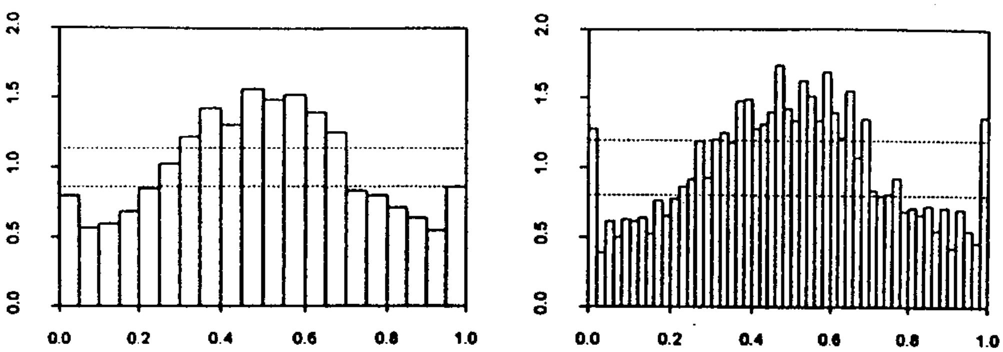
Figure 1
Simulated t-GARCH(1,1) Series (y)



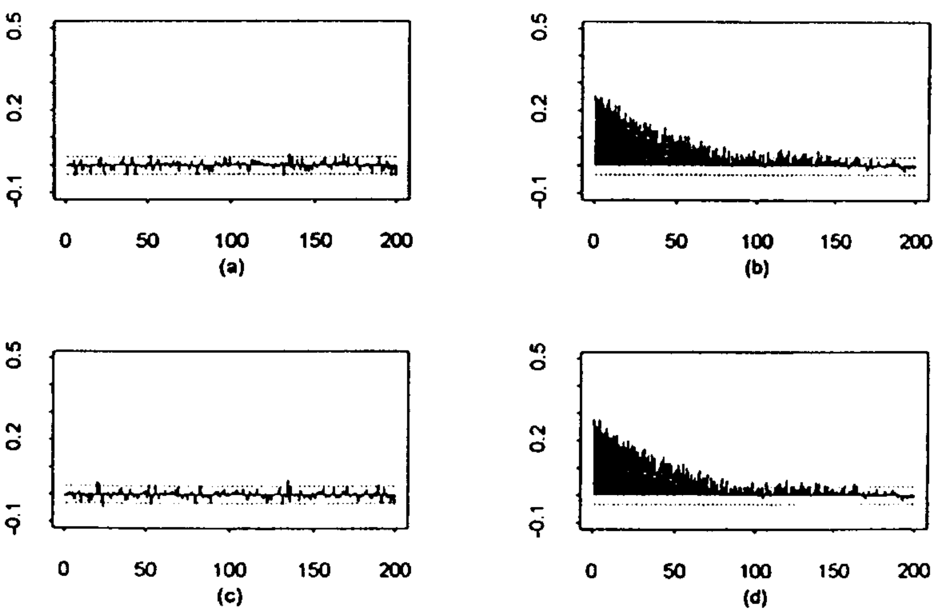
Notes to Figure: The parameters are ω=0.01, α=0.13, and β=0.86. The standardized series is distributed t(6). Data in the shaded region are used for estimation, and data in the unshaded region are used for out-of-sample forecast evaluation.

Figure 2a
Estimates of the Density of z



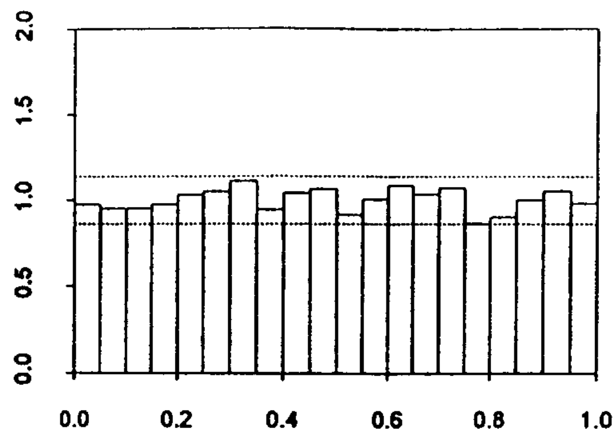
Notes to Figure: z is the probability integral transform of y with respect to density forecasts produced under the incorrect assumption that y is iid N(0,1). See text for details.

Figure 2b
Estimates of the Autocorrelation Functions of Powers of z



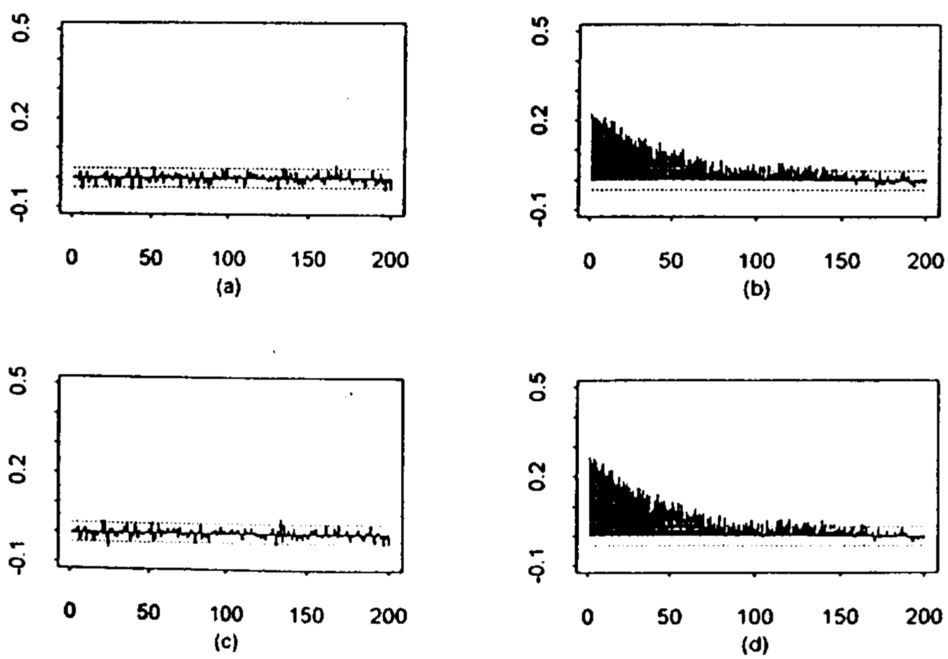
Notes to Figure. Panels (a) to (d) show sample autocorrelations of $(z - \bar{z})$, $(z - \bar{z})^2$, $(z - \bar{z})^3$ and $(z - \bar{z})^4$.

Figure 3a
Estimate of the Density of z



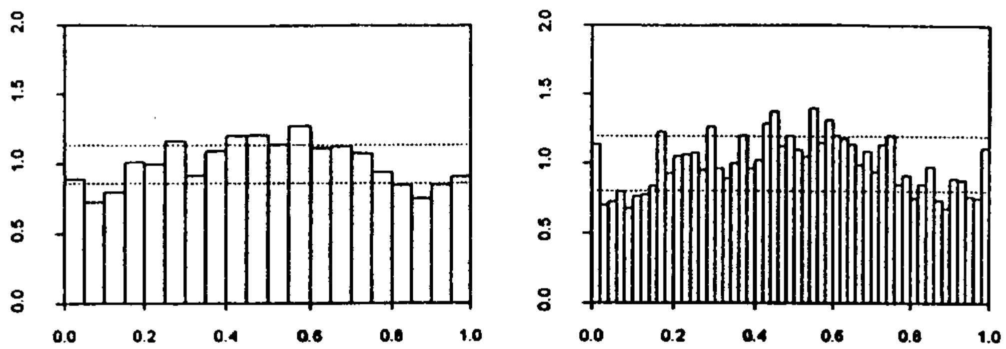
Notes to Figure: z is the probability integral transform of y with respect to density forecasts produced under the incorrect assumption that y is iid with density equal to the unconditional density estimated over periods 1-4000. See text for details.

Figure 3b
Estimates of the Autocorrelation Functions of Powers of z



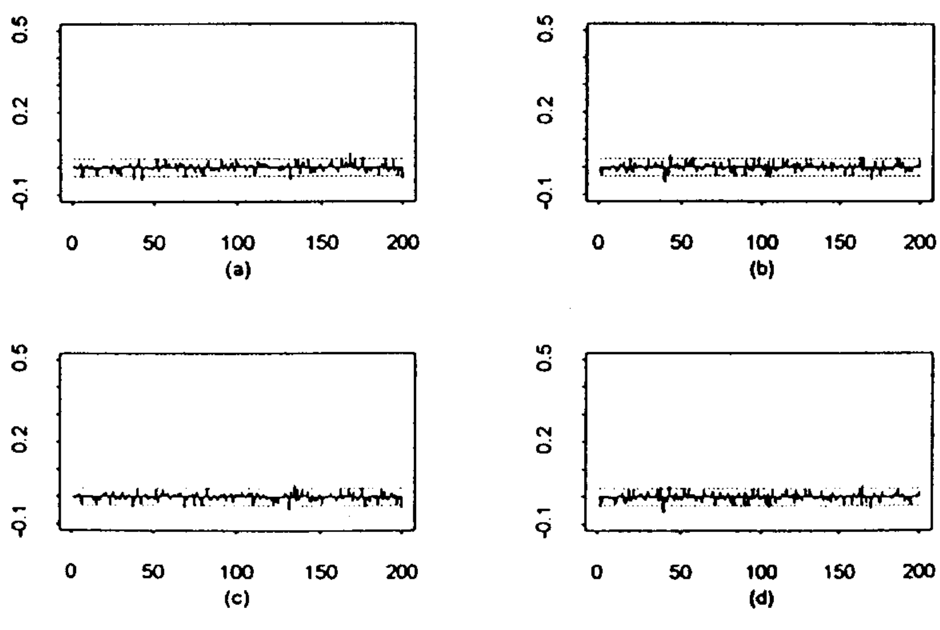
Notes to Figure: Panels (a) to (d) show sample autocorrelations of $(z - \bar{z})$, $(z - \bar{z})^2$, $(z - \bar{z})^3$ and $(z - \bar{z})^4$.

Figure 4a
Estimates of the Density of z



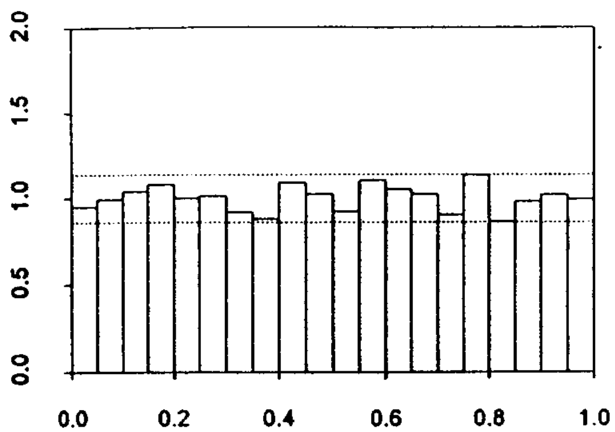
Notes to Figure: z is the probability integral transform of y with respect to density forecasts produced under the incorrect assumption that y is a conditionally Gaussian GARCH(1,1) process with parameters equal to those estimated over periods 1-4000. See text for details.

Figure 4b
Estimates of the Autocorrelation Functions of Powers of z



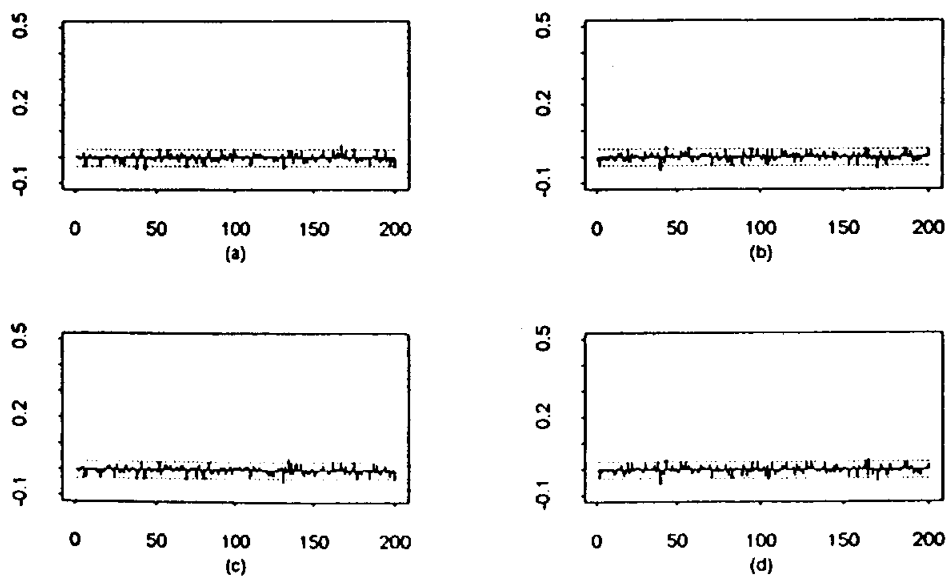
Notes to Figure: Panels (a) to (d) show sample autocorrelations of $(z - \bar{z})$, $(z - \bar{z})^2$, $(z - \bar{z})^3$ and $(z - \bar{z})^4$.

Figure 5a
Estimate of the Density of z



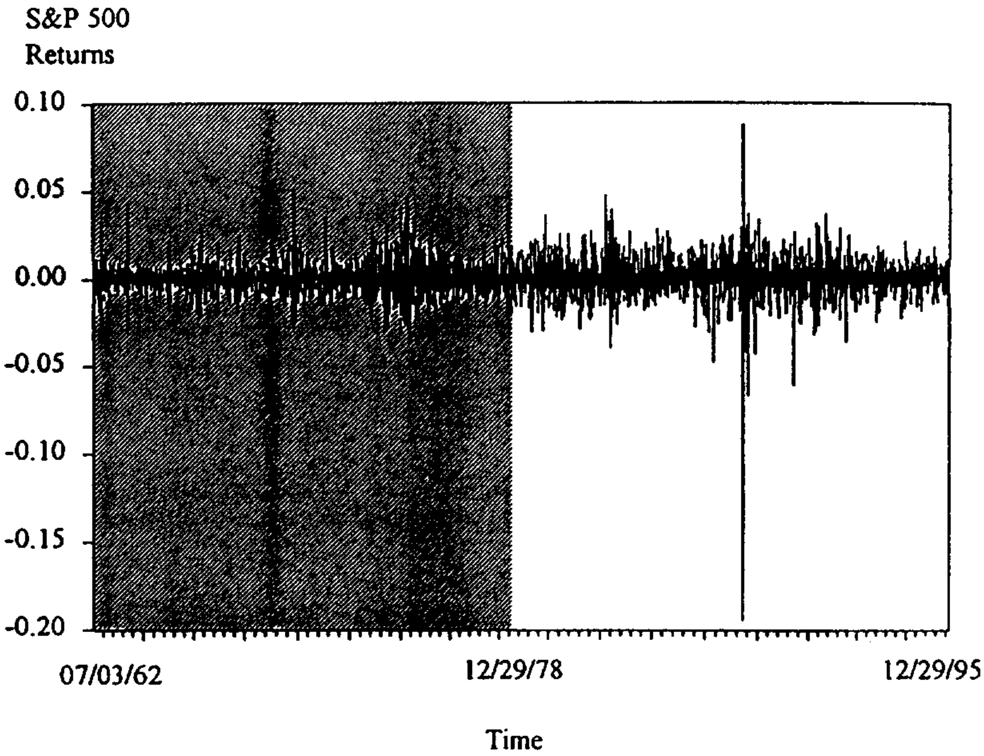
Notes to Figure: Histogram of z series produced from forecasts of simulated t-GARCH(1,1) series based on estimated t-GARCH model. We estimate parameters over 1-4000 and forecast over 4001-8000.

Figure 5b
Estimates of the Autocorrelation Functions of Powers of z



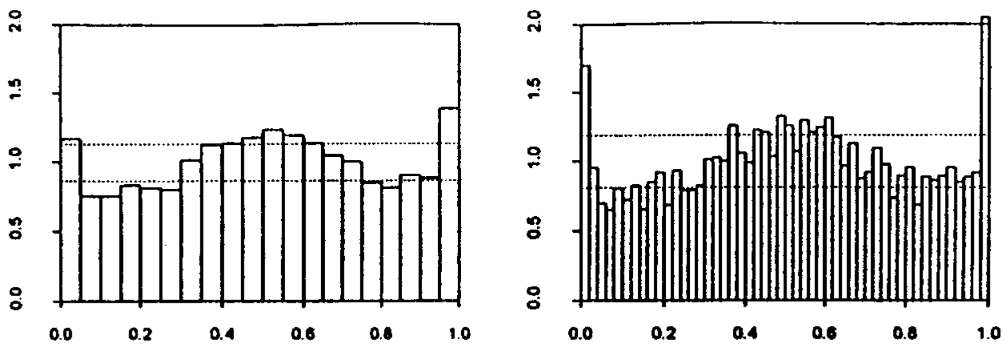
Notes to Figure: Panels (a) to (d) show sample autocorrelations of $(z - \bar{z})$, $(z - \bar{z})^2$, $(z - \bar{z})^3$ and $(z - \bar{z})^4$.

Figure 6
Daily S&P 500 Returns (y)



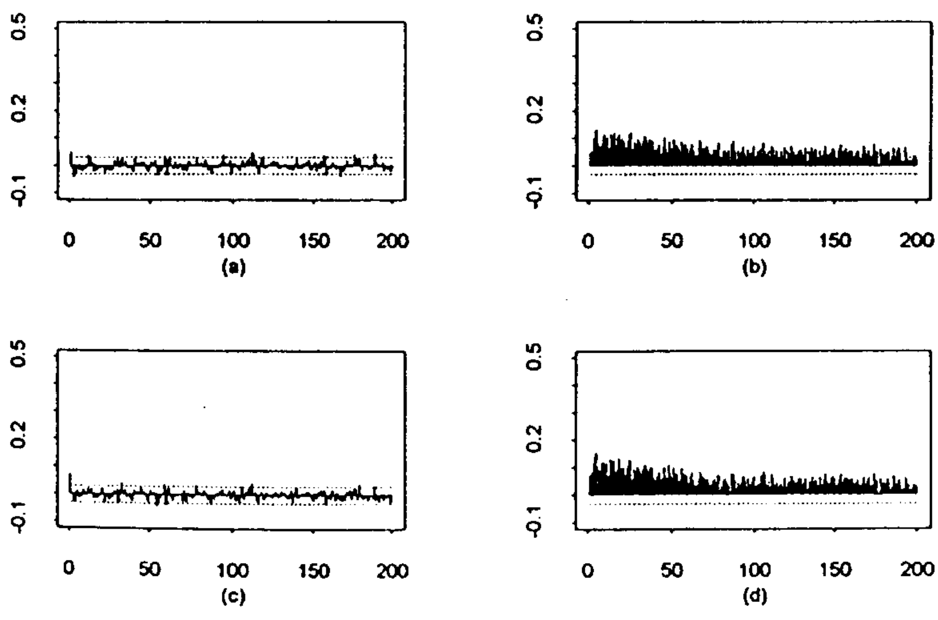
Notes to Figure: We plot value-weighted S&P 500 returns, with dividends, 02/03/62 - 12/29/95. Data in the shaded region are used for estimation, and data in the unshaded region are used for out-of-sample forecast evaluation.

Figure 7a
Estimates of the Density of z



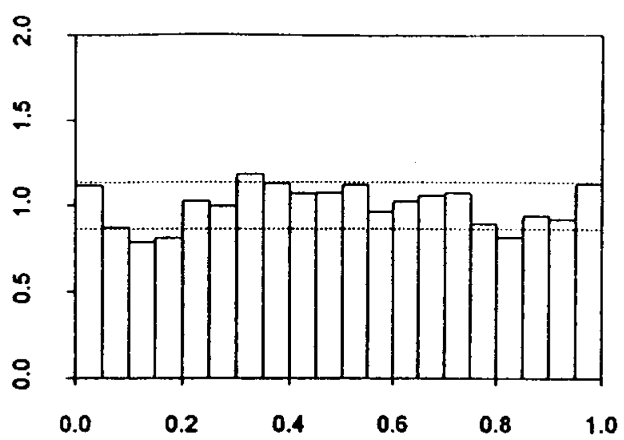
Notes to Figure: z is the probability integral transform of y with respect to density forecasts produced under the assumption that y is iid normal. See text for details.

Figure 7b
Estimates of the Autocorrelation Functions of Powers of z



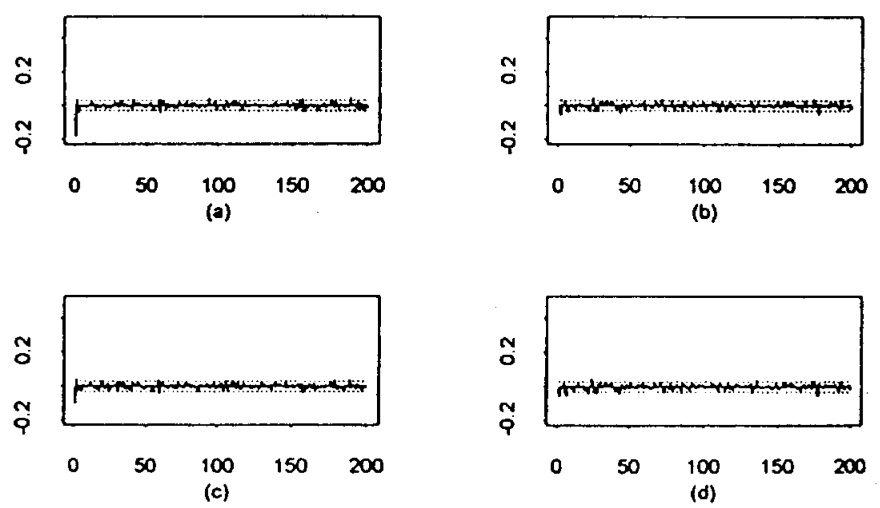
Notes to Figure: Panels (a) to (d) show sample autocorrelations of $(z - \bar{z})$, $(z - \bar{z})^2$, $(z - \bar{z})^3$ and $(z - \bar{z})^4$.

Figure 8a
Estimate of the Density of z



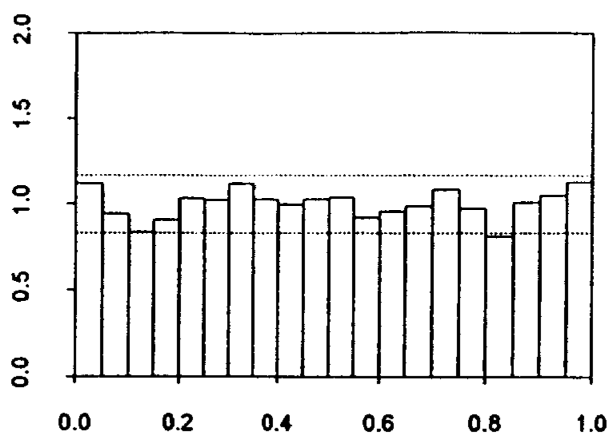
Notes to Figure: z is the probability integral transform of y with respect to density forecasts produced under the assumption that y is a conditionally Gaussian MA(1)-GARCH(1,1) process with parameters equal to those estimated from 7/3/62 to 12/29/78. See text for details.

Figure 8b
Estimates of the Autocorrelation Functions of Powers of z



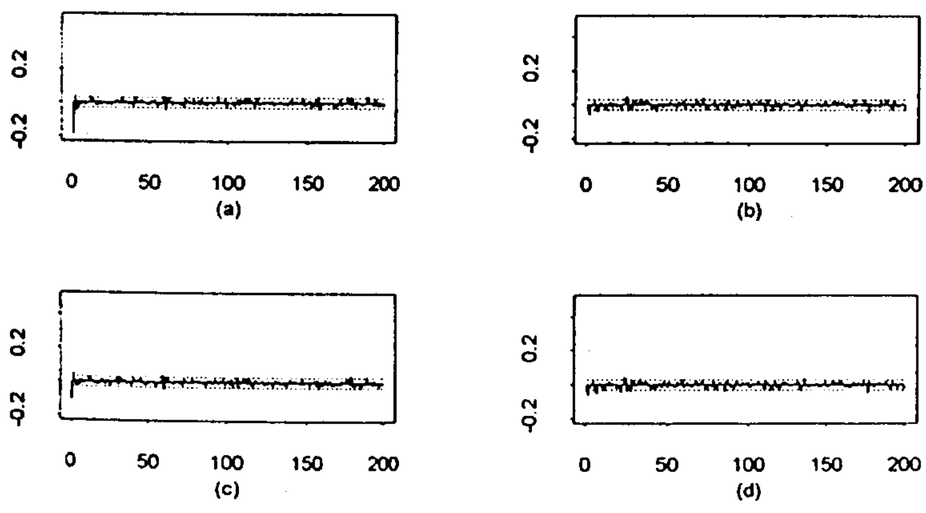
Notes to Figure: Panels (a) to (d) show autocorrelations of $(z - \bar{z})$, $(z - \bar{z})^2$, $(z - \bar{z})^3$ and $(z - \bar{z})^4$.

Figure 9a
Estimate of the Density of z



Notes to Figure: z is the probability integral transform of y with respect to density forecasts produced under the assumption that y is a conditionally Student's t MA(1)-GARCH(1,1) process with parameters equal to those estimated from 7/3/62 to 12/29/78. See text for details.

Figure 9b
Estimates of the Autocorrelation Functions of Powers of z



Notes to Figure: Panels (a) to (d) show sample autocorrelations of $(z - \bar{z})$, $(z - \bar{z})^2$, $(z - \bar{z})^3$ and $(z - \bar{z})^4$.