Evaluation of minimum factor of safety in slope stability analysis

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This paper explores the feasibility of using optimization methods to search for the minimum factor of safety in slope stability analysis. The routine procedure includes comparing a number of admissible surfaces that are basically selected by random searches. Low efficiency and unreliability are the problems commonly encountered.

Optimization methods allow a mathematically rigorous and reliable search for the minimum factor of safety and its associated "critical slip surfaces." This paper employs simplex, steepest descent, and Davidon−Fletcher−Powell (DFP) methods. The results of a number of test problems, in conjunction with closed-form and grid search solutions, showed that all the above-mentioned methods can provide reasonable results. Case history analyses supported the feasibility of the methods. Modifications to the DFP method were found to be essential for successful implementation of the minimization procedure.

Key words: slope stability, landslide, analysis, limit equilibrium, interslice forces, factor of safety, spline function, optimization methods, case history.

Cet article étudie la faisabilité de l'utilisation des méthodes d'optimisation pour la recherche du coefficient de sécurité minimum dans l'analyse de stabilité des talus. La procédure habituelle compare un nombre de surfaces admissibles qui sont sélectionnées essentiellement par recherches aléatoires. Le faible rendement et le manque de fiabilité sont les problèmes généralement rencontrés.

Les méthodes d'optimisation permettent une recherche mathématiquement rigoureuse et fiable du coefficient de sécurité minimum et des surfaces de glissement critiques correspondantes. Cet article utilise les méthodes simplex, de pente maximale et de Davidson−Fletcher−Powell (DFP). Les résultats d'un certain nombre de problèmes étudiés de concert avec les solutions de recherche le long de mailles et avec cheminement dirigé, ont démontré que toutes les méthodes mentionées ci-haut peuvent fournir des résultats raisonnables. Les analyses de cas ont démontré la faisabilité des méthodés. Les modifications à la méthode DFP se sont avérées essentielles pour utiliser avec succès la procédure de minimisation.

Mots clés : étude de stabilité, glissement, analyse, équilibre limite, forces intertranches, coefficient de sécurité, méthodes d'optimisation, histoire de cas.

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Introduction

The calculation of slope stability safety factors is a routine practice. In general it involves two steps:

First, calculate the factor of safety for a specified slip surface using the method of slices. Extensive studies have been undertaken in this area, and a variety of these methods are available for generalized slip surfaces.

Second, find, among many potential slip surfaces, the "critical" surface that is associated with the minimum factor of safety. One approach to this issue can be elaborated from the calculus of variations (Baker and Garber 1977; Revilla and Castillo 1977). However, its extension to even the simplest geometrically and geologically realistic situation will lead to analytically intractable problems.

With the advent of computers, numerical approaches have become competent in dealing with geotechnical problems that involve heterogeneous and nonlinear physical behaviors as well as complicated boundary conditions. Since the 1950's, the theory of optimization, which acquires extrema by numerical approaches, has been successfully developed and applied in various fields including the minimization of safety factors for slope stability analysis. Boutrup and Lovell (1980) presented a strategy that included random generation of slip surfaces and repeated comparisons of calculated factors of safety. This method is most primitive in the area of optimization. Baker's approach (1980) employed dynamic programming, which, as he stated, is applicable only to "additive functions." To render Spencer's method tractable, Baker had to introduce sophisticated treatments. In each iteration dynamic program-

ming was involved for solving the force and moment equilibrium equations. Coupling the calculation and minimization of safety factors increased the complexity of the algorithm and complicated its extension to other methods of slices, as discussed by Li and White (1987). Celestino and Duncan (1981) adapted an alternative variable method. Their work was extended by Li and White (1987), who introduced a number of techniques, notably the rational approximation method for one-dimensional search and the compound iterative procedure devoted to enhancing the efficiency of the generalized procedure of slices of Fredlund and Krahn (1976). Simplex and complex methods were employed by Nguyen (1985) and Sun (1984) respectively. Sun's successful results seem to contradict Nguyen's argument that "application of the Box method to the search for the critical slip surface may not be relevant."

All these approaches seem to be practical and feasible. The results of practical experience will reveal their merits and promote their application. However, in contrast to the extensive study on the methods of determining safety factors and the methods of optimization, the combination of the two is still being developed. To expedite the implementation of this automatic searching technique as an engineering routine, the authors wish to share the following highlights of their experience: (1) some simple and effective patterns for discretizing slip surfaces; (2) applications of optimization methods that require the evaluation of derivatives, such as the Davidon−Fletcher−Powell method, which are widely acknowledged to be more efficient, especially when the degrees of freedom increase; (3) modifications to Davidon−Fletcher−Powell
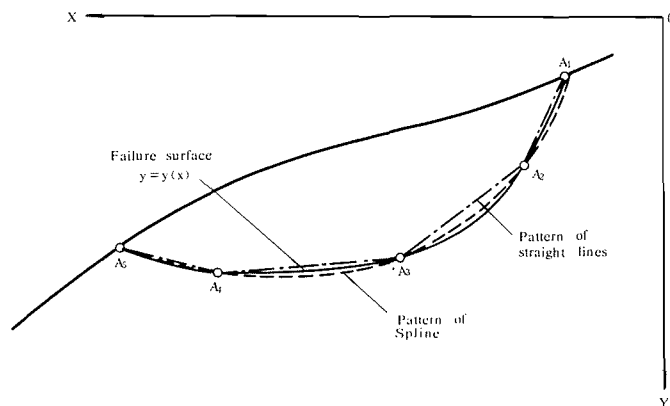
FIG. 1. Discretization patterns.

FIG. 2. Combined pattern of discretization.

methods; (4) validation of the optimized results by closed-form and grid search solutions; (5) the influence of different assumptions of interslice forces upon the optimized results; (6) the analysis of a case history by use of the approaches introduced herein.

The objective function and discretization patterns

Slip surface discretization patterns

The factor of safety F of a specified slip surface, represented by a curve $y = y(x)$ (Fig. 1), can be calculated by conventional methods of slices. Finding the "critical slip surface" related to the minimum safety factor F_m involves a mathematical procedure for minimizing the functional

[1] $F = F(y(x))$

The typical unconstrained optimization technique solves for the minimum of a given objective function

[2] $F = F(Z)$

with respect to variables $z_1, z_2, \ldots, z_n$, which are represented by a vector

[3] $Z = (z_1, z_2, \ldots, z_n)^T$

where n is the degree of freedom.

Therefore, the slip surface $y = y(x)$ should be approximated by the use of certain patterns of discretization:

Pattern 1. The simplest approximation divides a slip surface by a number of nodal points $A_1, A_2, \ldots, A_m$ along the curve $y = y(x)$ and connects each pair of contiguous points by a straight line, as shown in Fig. 1. The variables of Z in [3] are then the coordinates of the individual points $x_1, y_1, x_2, y_2, \ldots, x_m, y_m$, that is,

[4] $Z = (x_1, y_1, x_2, y_2, \ldots, x_m, y_m)^T$

The limitations of this approximation can be eliminated by increasing the number of nodal points.

Pattern 2. Pattern 1 is modified to connect several successive nodal points by a smooth curve, usually constructed by splines as shown in Fig. 1. Fewer nodal points are needed to generate simulations that are just as rational as those generated by the "straight line" pattern. A decrease in degrees of freedom is especially advantageous for numerical convergence of the problem. The present approaches adopt cubic splines with first-kind boundary conditions (Ahiberg et al. 1967).

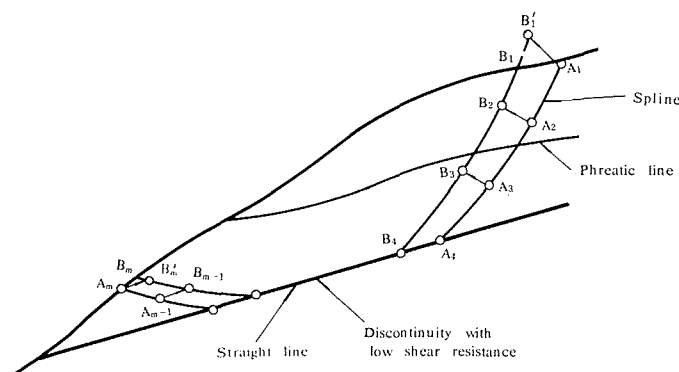It is possible, and in many cases desirable, to combine the two patterns and construct an approximation consisting of both straight and curved lines. This need appears, for instance, in the case of a fairly homogeneous embankment overlying a weak bedding plane, as shown in Fig. 2.

Degree of freedom

The degree of freedom is the number of variables involved in Z. During the minimization of the safety factors, the nodal points move both vertically and horizontally towards their optimal locations. In general, there are $2m$ degrees of freedom for m nodal points. However, it may be reduced to n and the remainder, $2m - n$, may not be included in [4] for the following reasons: (1) In some cases, the left and (or) right ends of the slip surface are well defined. Their coordinates are fixed and will not be variables. (2) A nodal point on a structural discontinuity may sometimes be specified as moving exclusively along that geological bedding plane. Only one of its coordinates is then independent, and the other can be readily determined by the geometry of the discontinuity. The degree of freedom of that point thus falls to 1, with an accompanying specified direction of movement.

During optimization, the points A_1 and A_m, which are originally located at the slope surface, may move inside or outside of the slope to new locations B_1' and B_m', as shown in Fig. 2. Unlike Li and White's approach (1987), which defines these kinds of points as a separate category, the present work finds the upper and lower points of interception B_1 and B_m, and adjusts the slip surface information accordingly. Therefore, these points need no special treatment.

For a problem with n degrees of freedom, the expression of Z in [4] is thus replaced by [3].

Evaluation of the objective function

Various methods of slices can be used to calculate the factor of safety (Bishop 1955; Janbu 1973; Spencer 1967; Morgenstern and Price 1965; Sarma 1973). A comprehensive study was made by Fredlund (1984). Some of the researchers (Baker 1980; Li and White 1987) extended the method of slices they used to meet relatively stringent numerical effectiveness requirements of the optimization algorithm. Extensions to the generalized method of slices (Chen and Morgenstern 1983) were used in this paper to meet the following demands:

(1) The evaluation of safety factors should be reasonably rigorous, capable of providing accurate results consistent with those obtained by closed-form solutions and therefore furnishing convincing evidence of the validity of the optimized results. In the Appendix, a classical problem of the plasticity is solved by the method of slices, using Chen and Morgenstern's extensions with satisfactory results. This problem is subse-
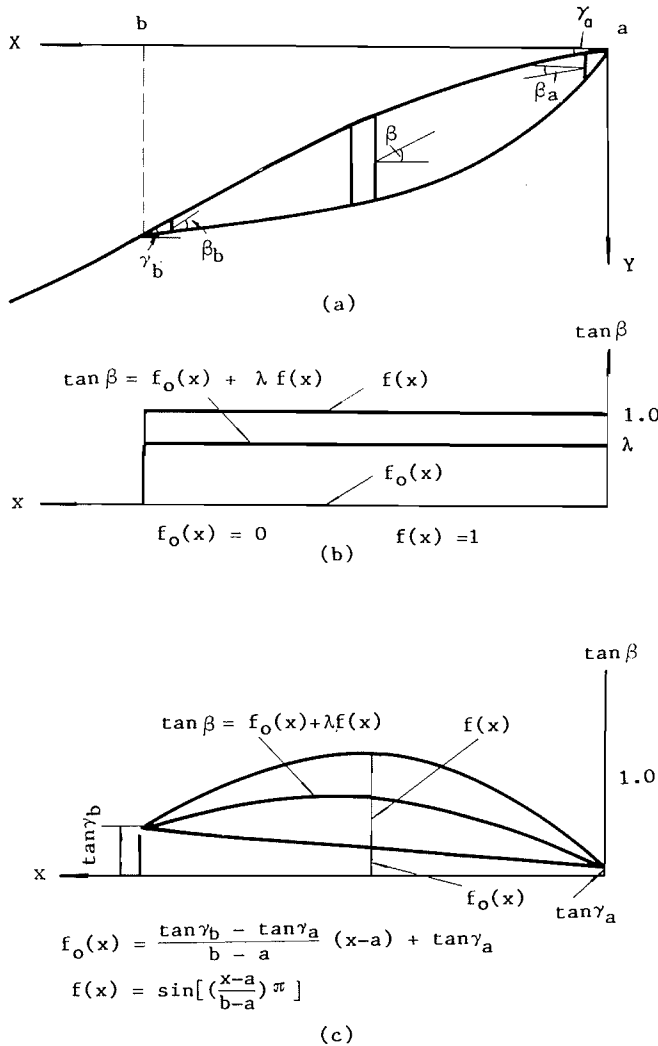
FIG. 3. Assumed functions $f_0(x)$ and $f(x)$ of the interslice force inclination: (a) geometry of the slope; (b) assumption 1, $f_0(x) = 0$, $f(x) = 1$; (c) assumption 2, $f_0(x) = (\tan \gamma_b - \tan \gamma_a)(x - a)/(b - a) + \tan \gamma_a$, $f(x) = \sin [\pi(x - a)/(b - a)]$.

quently employed in the section Test problems and results.

The assumptions concerning the interslice forces affect the results calculated by the method of slices. Chen and Morgenstern suggested that these assumptions be made as follows (Fig. 3):

[5] $\qquad \tan \beta = f_0(x) + \lambda f(x)$

where β is the inclination of the interslice forces with respect to the horizontal; λ is a coefficient to be determined; $f(x)$ is an assumed function that is zero at both ends, and $f_0(x)$ ia another assumed function that has the required values of $\tan \beta$ at both ends when more rigorous analysis is performed.

At the present stage of study, two assumptions, which permit easy access for practitioners, are used and their influence on the optimized results is investigated.

Assumption 1. $f_0(x) = 0$; $f(x) = 1$ (Fig. 3b). This is the simplest and most commonly used assumption, generally referred to as Spencer's method (1967).

Assumption 2. The values of $f_0(x)$ and $f(x)$ are schematically shown in Fig. 3c. This is a more rational assumption, which presents no inconsistency, at the ends of the sliding mass, with

the principle of complementary shear stresses. This argument (Chen and Morgenstern 1983) requires that the interslice forces of the first and last slices be parallel to the slope surfaces at their relevant end points if their widths are sufficiently small, that is (Fig. 3),

[6] $\qquad \beta_a = \gamma_a; \qquad \beta_b = \gamma_b$

where β and γ are the inclinations of the interslice forces and the slope surface, respectively, while the subscripts a and b refer respectively to the right and left ends of the sliding mass.

(2) The iteration process involved in various methods of slices should be effective enough to provide rapid convergence within a minimum allowable error limit of at least 10^{-4} and preferably 10^{-5} or even less. Although engineering practice by no means requires this level of numerical effectiveness and accuracy, it is required in the optimization algorithm because (1) the process involves hundreds of evaluations of F; therefore, the guaranteed convergence of each of these evaluations is essential for smooth execution of the computer program; (2) some of the methods of optimization involve the evaluation of the gradient vector G, which is

[7] $\qquad G = \left(\dfrac{\partial F}{\partial z_1} \dfrac{\partial F}{\partial z_2} \cdots \dfrac{\partial F}{\partial z_n} \right)^{\mathrm{T}}$

and is essentially close to zero when the optimum is being approached. Consequently, sufficient significant figures in the values of F are needed during differencing. Li and White (1987) also pointed out that without adequate accuracy of objective functions, the quadratic fitting technique involved in some optimization methods can be "a risky business."

Table 1 shows iteration details that demonstrate the numerical effectiveness of Chen and Morgenstern's approach. This is an example of its many successful implementations where the converged solution is approached steadily and rapidly.

Methods of optimization employed

The process of minimization starts from an initially estimated slip surface $Z^0 = (z_1^0, z_2^0, \ldots, z_n^0)^{\mathrm{T}}$ with a safety factor F_0 and terminates at a neighboring location $Z^m = (Z^0 + \Delta Z) = (z_1^0 + \Delta z_1, z_2^0 + \Delta z_2, \ldots, z_n^0 + \Delta z_n)$ associated with F_m. Different choices of Z^0 will not affect the solution unless the problem exhibits multiminimum behavior. In that case, several Z^0's from different domains of the slope profile are needed. Each solution is then regarded as a local minimum.

A variety of optimization methods are available. The present approach employs three of them, which are briefly described below. For details the reader should refer to textbooks (see e.g., Walsh 1975).

Nelder and Mead's simplex method (1965)

For a given initial estimate Z^0, a simplex consisting of $n + 1$ vertices R_i ($i = 1, 2, \ldots, n + 1$) is established according to some suggested pattern. By reflections, expansions, and contractions, the simplex is renewed and moves towards the extremum until the convergence criterion is met, that is, until

[8] $\qquad \left\{ \dfrac{1}{n + 1} \sum_{k=1}^{n+1} [F(R_k) - F(R_a)]^2 \right\}^{1/2} < \epsilon$

where

[9] $\qquad R_a = \left(\sum_{k=1}^{n+1} R_k \right) /(n + 1)$

In the present approach $\epsilon = 10^{-5}$.

TABLE 1. Iteration details of the calculation of the factor of safety for slip surface $A_4B_4C_4D_4$ in Fig. 10

Iteration time	Unbalanced force G_n (kN)	Unbalanced moment M_n (kN·m)	F	λ
1	−2511.74	116522.0	1.27451	0.26796
2	547.37	−31017.0	1.08475	0.20779
3	−1.52	−423.9	1.11630	0.22054
4	−0.67	25.4	1.11655	0.22104
5	0.01	−0.4	1.11651	0.22103

NOTE: The unbalanced force G_n and moment M_n are defined in eqs. [40] and [41] found in Chen and Morgenstern (1983). They are both expected to be zero when accurate F and λ are obtained.

The method of steepest descent

For the current step of iteration v, with the slip surface vector Z^v, Z^{v+1} is obtained by a linear search in a certain direction S, that is, Z^{v+1} is a vector that minimizes $Z^v + \alpha S^v$ with respect to α,

$$[10] \quad \frac{\partial F}{\partial \alpha} = \frac{\partial}{\partial \alpha}[F(Z^v + \alpha S^v)] = 0$$

A quadratic fitting technique is used to perform the linear search.

The steepest descent method defines S as the negative gradient vector, that is,

$$[11] \quad S = -G = -\left(\frac{\partial F}{\partial z_1} \quad \frac{\partial F}{\partial z_2} \quad \cdots \quad \frac{\partial F}{\partial z_n}\right)^T$$

The iteration terminates when

$$[12] \quad |F(Z^{v+1}) - F(Z^v)| < \epsilon$$

where ϵ is taken as 1.5×10^{-5}.

Davidon−Fletcher−Powell method and its modification

This method was first suggested by Davidon (1959) and modified by Fletcher and Powell (1963). It is sometimes referred to as the DFP method, and enjoys a reputation for high numerical efficiency.

The computation procedure is similar to that of the method of steepest descent, except that the direction of linear search at step v is taken as

$$[13] \quad S^v = -A^v G^v$$

where A at step $v + 1$ is obtained by

$$[14] \quad A^{v+1} = A^v + C^v - D^v$$

in which $v = 0, 1, 2, \ldots$ and

$$[15] \quad C^v = \frac{\Delta Z_v \Delta Z_v^T}{\Delta Z_v^T Y_v}$$

$$[16] \quad D^v = \frac{A^v Y_v (A^v Y_v)^T}{Y_v^T A^v Y_v}$$

$$[17] \quad Y_v = G^{v+1} - G^v$$

$$[18] \quad \Delta Z_v = Z^{v+1} - Z^v$$

The basic concept of the method involves successively approaching the inverse of the Hessian matrix, H^{-1}, at the extremum, via a series of matrices $A^0, A^1, \ldots, A^n$. The Hessian matrix is defined as

$$[19] \quad H = \begin{bmatrix} \dfrac{\partial^2 F}{\partial z_1^2} & \dfrac{\partial^2 F}{\partial z_1 \partial z_2} & \dfrac{\partial^2 F}{\partial z_1 \partial z_n} \\[2mm] & \dfrac{\partial^2 F}{\partial z_2^2} & \dfrac{\partial^2 F}{\partial z_2 \partial z_n} \\[2mm] & \cdots \cdots \cdots \\[2mm] \text{symmetry} & & \dfrac{\partial^2 F}{\partial z_n^2} \end{bmatrix}$$

A^0 can therefore be regarded as an assumed approximation to H^{-1} and is generally taken as the unit matrix I. $A^1, A^2, \ldots$ are obtained by [14] until the mth iteration, at which A_m is close enough to H^{-1}, and the convergence criterion is met.

However, our experience showed that implementation of the conventional algorithm for minimizing safety factors was not as efficient as anticipated. Investigations revealed that this is attributable to the very large values of the individual elements of H^{-1}. For problems concerned with slope stability, the values of $\partial^2 F/\partial z_i \partial z_j$ $(i, j = 1, \ldots n)$ are generally of the order of $10^{-2} - 10^{-5}$, provided that SI units are employed. It is evident that the first assumed matrix A^0, which is taken as the unity matrix in the conventional algorithm, is too far from H^{-1} to ever reach it. Therefore, the functioning of the algorithm for some problems became less efficient.

A modification was subsequently tested, taking A^0 as

$$[20] \quad A^0 = I \times 10^p$$

where p is a rough estimation of the order of magnitude of the average value of the elements of H^{-1}. In the present work, the estimation takes place after the first linear search is completed and the values of $\partial^2 F/\partial z_i \partial z_j$ $(i, j = 1, \ldots n)$ in the domain concerned can be estimated. The value of p is determined by the inequality

$$[21] \quad 10^p < \frac{\sum\limits_{i=1}^{n} |\partial^2 F/\partial z_1^2|^{-1}}{n} < 10^{p+1}$$

The modification was successful. This argument will be further elucidated in test problem 2 in the next section.

Both the steepest descent and DFP methods require evaluation of the gradient vector G, whose variables are very small in magnitude when the minimum is being approached. The modification recommended by Stewart (1967) was adapted. This approach allows a reasonable estimation of the step size during differencing, based on an approximate balance of the effects of truncation error and cancellation error.
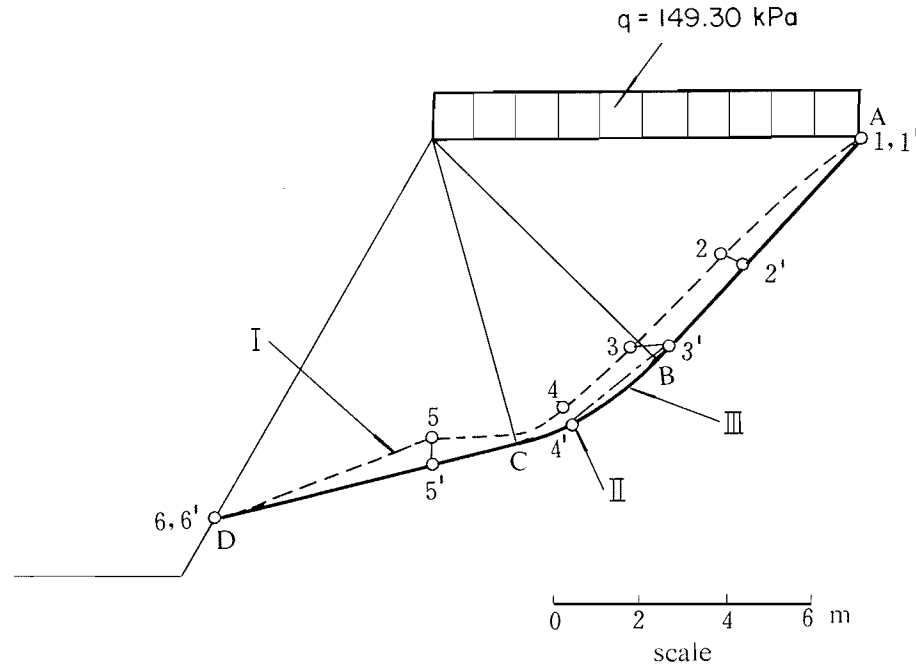
FIG. 4. Slip surfaces of test problem 1 (both ends fixed): I, initially estimated, $F_0 = 1.043$; II, critical $F_m = 0.993$; III, theoretical.

Test problems and results

To verify the feasibility of the approaches, a number of test problems were designed and solved.

Test problem 1

Figure 4 shows a slope that has a closed-form solution. The appendix gives detailed discussions on the location of the theoretical failure surface and its associated factor of safety.

The calculation began with an initial estimate that connected the points $1-6$ with straight lines. Functions $f_0(x)$ and $f(x)$ were set analogous to the theoretical distribution of $\tan \beta$, which is shown in Fig. A3b of the Appendix. Points 1 and 6 were specified as fixed points; the others were allowed to move both vertically and horizontally. The results obtained by the DFP method showed that the optimized locations of points $2'-5'$ fell exactly onto the theoretical slip surface ABCD with a minimum safety factor of 0.993, as shown in Fig. 4. Figure 5 shows another trial, which permitted free movement of the left end along the slope surface. Splines were used. The initial estimate in Fig. 5a was designed to have an irregular shape, while that in Fig. 5b was a straight line connecting five equally spaced points and was purposely designed to be far from the expected theoretical failure surface. Both of the optimized critical slip surfaces agreed closely with the theoretical failure surface.

In the above cases, the assumptions $f_0(x)$ and $f(x)$ were designed to simulate the theoretical distribution of $\tan \beta$, which is generally undeterminable in practical problems. The next investigation compares the results obtained by the two conventional selections of $f_0(x)$ and $f(x)$, previously defined as assumptions 1 and 2. The problem was recalculated using the DFP method and the results in Fig. 6 and Table 2 show that both the locations of critical slip surfaces and the value of F_m are fairly close to the theoretical solution. Assumption 2 yielded more accurate results than assumption 1.

Table 3 compares the numerical effectiveness of different methods. For the same problem, the unmodified DFP method required 15 linear searches, whereas the modified method

TABLE 2. Results calculated using different assumptions of interslice forces for test problem 1 (Fig. 6)

| $f_0(x)$ and $f(x)$ | Slip surface | | | |
| | Initially estimated | | Critical | |
	No.	F_0	No.	F_m
Assumption 1	I	1.265	II	0.966
Assumption 2	I	1.183	III	0.991

needed only 6. In this case, the method of steepest descent was the least efficient.

Test problem 2

Figure 7 shows a simple example with two degrees of freedom, which made it possible to compare the optimized results with those obtained from the grid search. Assumption 1 was used for this problem. The slip surface ABC consisted of a horizontal part of BC along a weak layer and an inclined straight line AB. Point C was fixed, while A and B were specified as moving horizontally. The variables of Z in [3] include the abscissas of points A and B, represented by x_1 and x_2 respectively. The grid search gave the contours of equal F with respect to x_1 and x_2 shown in Figs. 8 and 9, from which F_m can be found to be 1.257 at $Z^m = (92.00,143.00)^T$.

The first, second and third trials using the steepest descent method began at $(84.000,160.000)^T$, $(70.000,145.000)^T$, and $(112.00,150.00)^T$ and converged at virtually the same location, as indicated by the arrows 1, 2, and 3 in Fig. 8. Note that each linear search proceeded normal to the contour of equal F and converged at the "valley" in that transect. This confirmed that the search followed the line of steepest descent, and that the quadratic fitting technique for linear search ([10]) was working.

The fourth calculation, using the simplex method, started from $Z^0 = (63.00,125.00)^T$ and also converged at the same optimized results obtained above, as shown in Fig. 9.
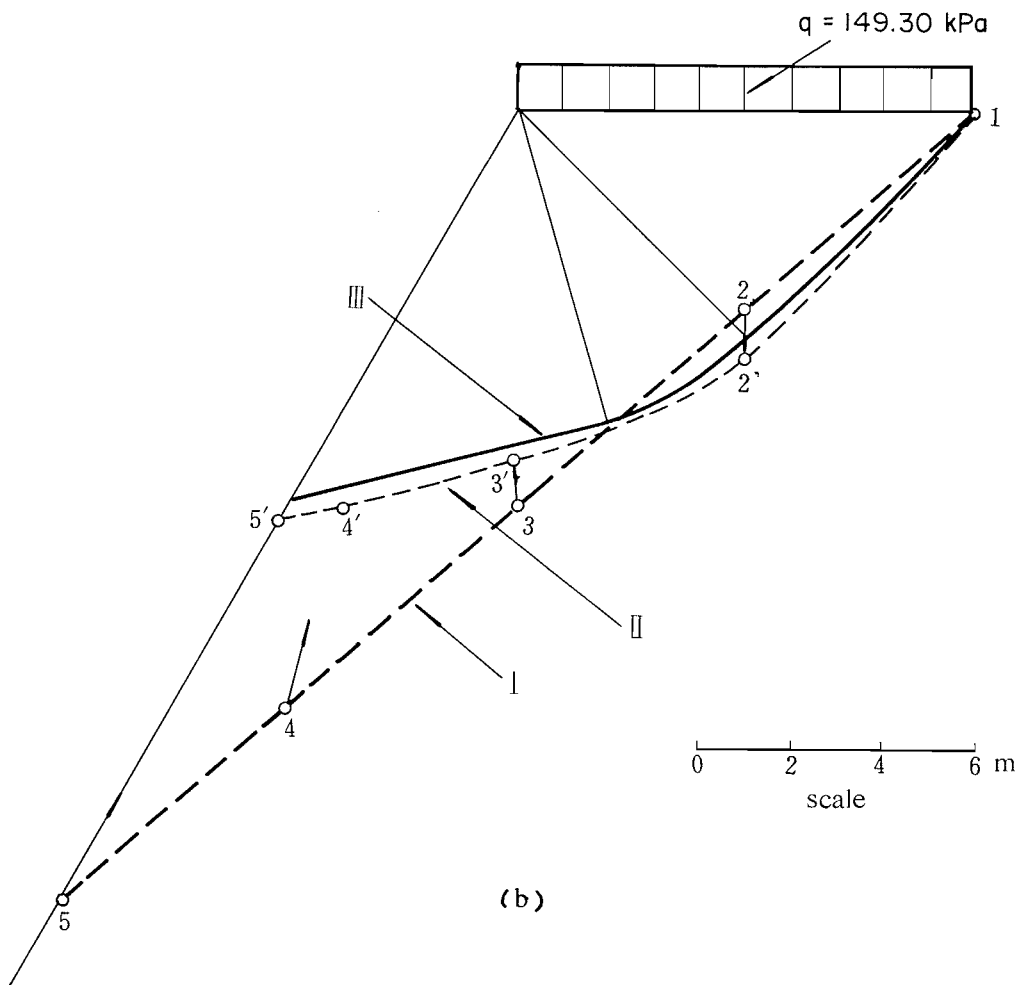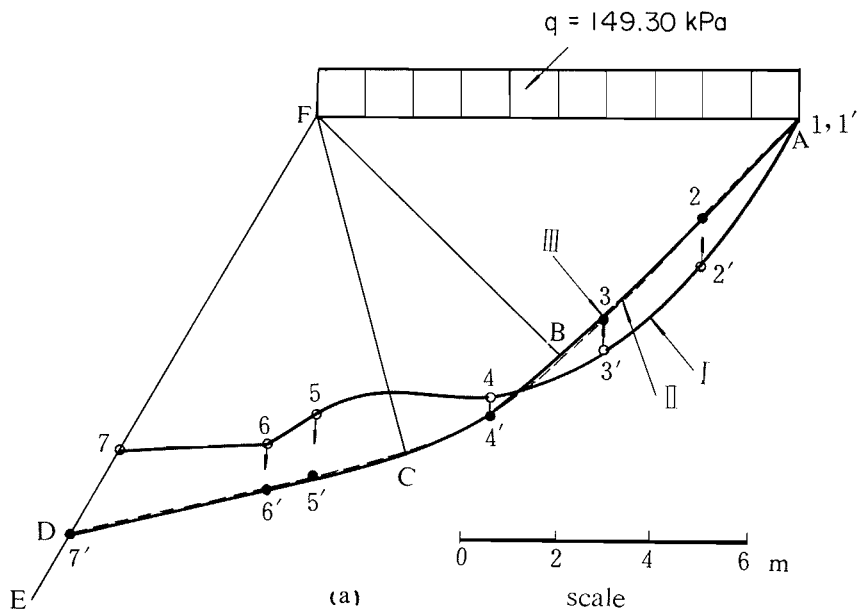
FIG. 5. Slip surfaces of test problem 1 (left end moving along the slope surface, two trials with different initial estimates): (a) I, initially estimated, $F_0 = 1.101$; II, critical, $F_m = 1.009$; method of steepest descent employed; III, theoretical; (b) I, initially estimated, $F_0 = 1.328$; II, critical, $F_m = 1.007$, simplex method employed; III, theoretical.
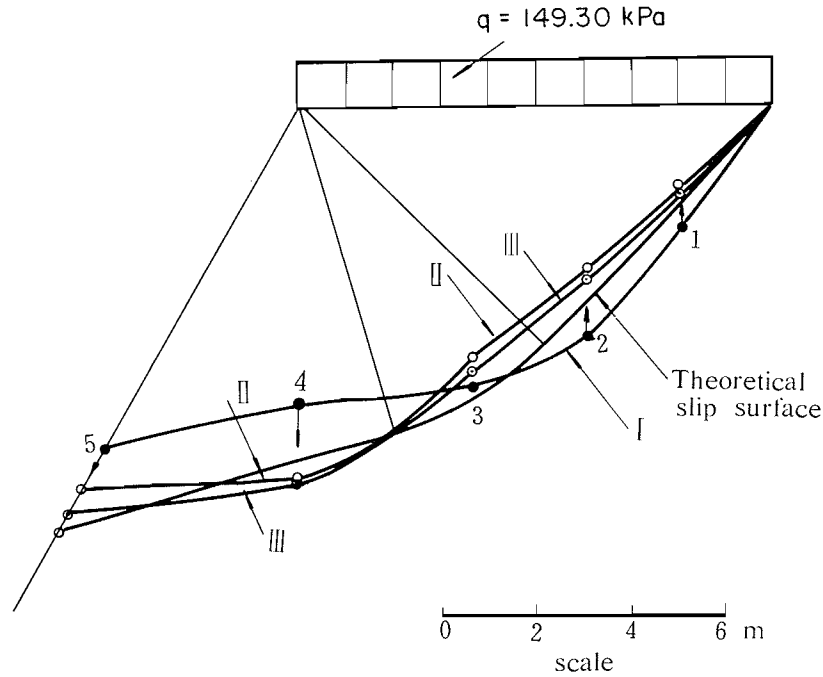
FIG. 6. Test problem 1 analysed using assumptions 1 and 2. For calculated results, see Table 2.
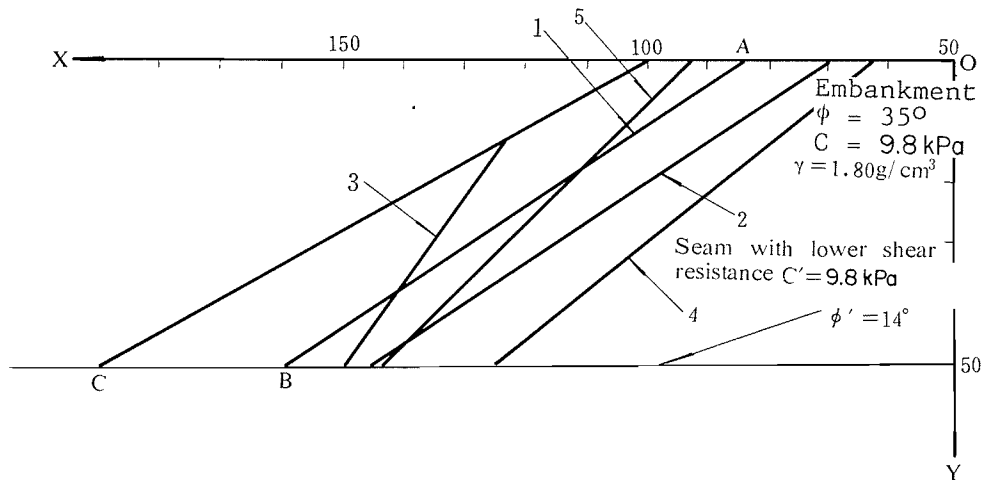

FIG. 7. Slip surfaces of test problem 2: 1, 2, 3, 4, initially estimated; 5, critical. Dimensions in metres.

However, calculations using the DFP method without the modifications for these trials, except for the second one, were unsuccessful. Take the fourth trial as an example. Starting at point A in Fig. 8 where $Z^0 = (63.00, 125.00)^T$, the first linear search proceeded in the direction $S^0 = (0.812, -0.088)^T$, and converged at point B where $Z^1 = (79.259, 123.229)^T$. From [14], it can be found that

$$A^1 = A^0 + C^0 - D^0$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2003.4 & -218.2 \\ -218.2 & 23.7 \end{bmatrix}$$

$$- \begin{bmatrix} 0.5917 & -0.492 \\ -0.492 & 0.408 \end{bmatrix} = \begin{bmatrix} 2003.8 & -217.7 \\ -217.7 & 24.3 \end{bmatrix}$$

It can be seen that C^0 accounted for a great majority of A^1,

TABLE 3. Comparisons of numerical effectiveness using different methods (for the problem with initial estimate I in Fig. 6)

Method	No. of linear searches	CPU time on Hitachi M-160
Steepest descent	38	2 min 46 s
DFP	15	1 min 02 s
Modified DFP	6	0 min 30 s

while the contributions of $(A^0 - D^0)$ to A^1 were negligible.

The direction of the second linear search, determined by [13] as $S^1 = (19.3, -1.7)^T$, is shown as BC_1 in Fig. 8, along which the linear search was fruitless. No further progress in minimization was made after that. Comparing BC_1 with BC_2, the direction of negative gradient at point B, one can find that the

Contours of equal safety factors
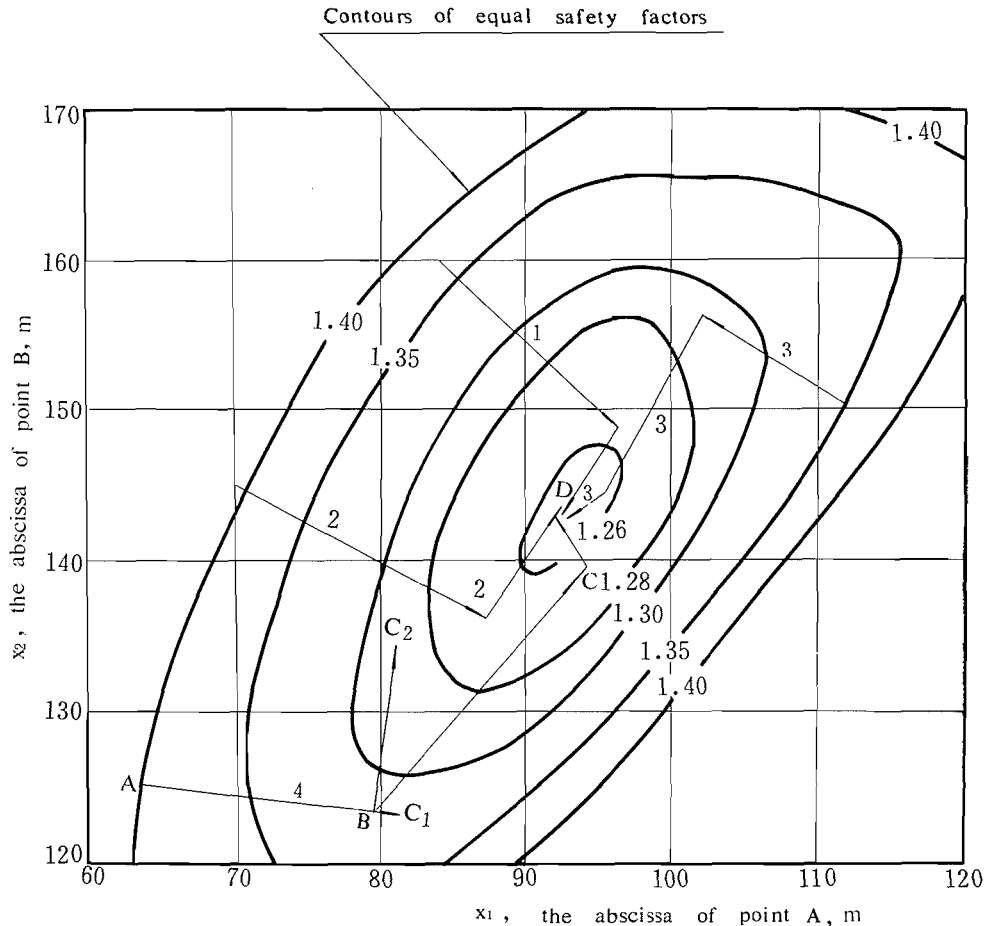

x_1 , the abscissa of point A, m

FIG. 8. Process of optimization of the four trials shown in Fig. 7. The directions of the second linear search are (1) BC_1, the DFP method without modifications; (2) BC, the DFP method with modifications; (3) BC_2, method of steepest descent.

method of steepest descent was even superior to the DFP method in this case.

The modification for the DFP method suggested by the authors was then introduced. The value of p in [20] was estimated by [21] to be 3. A^1 was recalculated as

$$A^1 = A^0 + C^0 - D^0$$

$$= \begin{bmatrix} 1000.0 & 0.0 \\ 0.0 & 1000.0 \end{bmatrix} + \begin{bmatrix} 2003.4 & -218.2 \\ -218.2 & 23.7 \end{bmatrix}$$

$$- \begin{bmatrix} 591.7 & -492.0 \\ -492.0 & 408.0 \end{bmatrix} = \begin{bmatrix} 2411.7 & 273.8 \\ 273.8 & 615.7 \end{bmatrix}$$

It is clear that the contributions of $(A^0 - D^0)$ to A^1 were no longer negligible and the direction of the second linear search was calculated to be $S^1 = (3.07, 3.45)^T$, which is shown as BC in Fig. 8. A great advance was made, eventually giving a final solution of $Z^m = (92.24, 143.26)^T$ and $F_m = 1.257$.

Test problem 3

In this problem, shown in Fig. 10 and Table 4, the four initial estimated slip surfaces were defined by four sets of points A,B,C,D having subscripts 1,2,3,4 respectively. Points A and D moved both vertically and horizontally, while C and B were specified as moving along the weak seam. A and D may be located inside or outside of the slope during optimization. The

TABLE 4. Geotechnical parameters for test problem 3 (Fig. 10)

Soil layer	Density ρ (g/cm^3)	Shear strength parameters	
		ϕ' (deg)	C (kPa)
I	2.00	35	9.8
II	1.90	25	58.8
III	2.15	30	19.8

information about the slip surface was adjusted accordingly. Assumption 1 was used first. The minimized results of the modified DFP method are shown in Fig. 11 and Table 5, from which the differences in the critical slip surface locations and the minimum safety factors derived from the four initial estimates are found to be insignificant. Admittedly, the fourth case yielded F_m of 1.025, which was slightly greater than that of the other three cases. This is because the fourth initial estimate differs appreciably from the other three in both location and shape.

The results gained from the DFP method without modifications are also shown in Table 5. The values of F'_m were generally greater and the critical slip surface location of the fourth case, shown in Fig. 12, indicated even greater deviations. Figure 12 also compares the results obtained by using
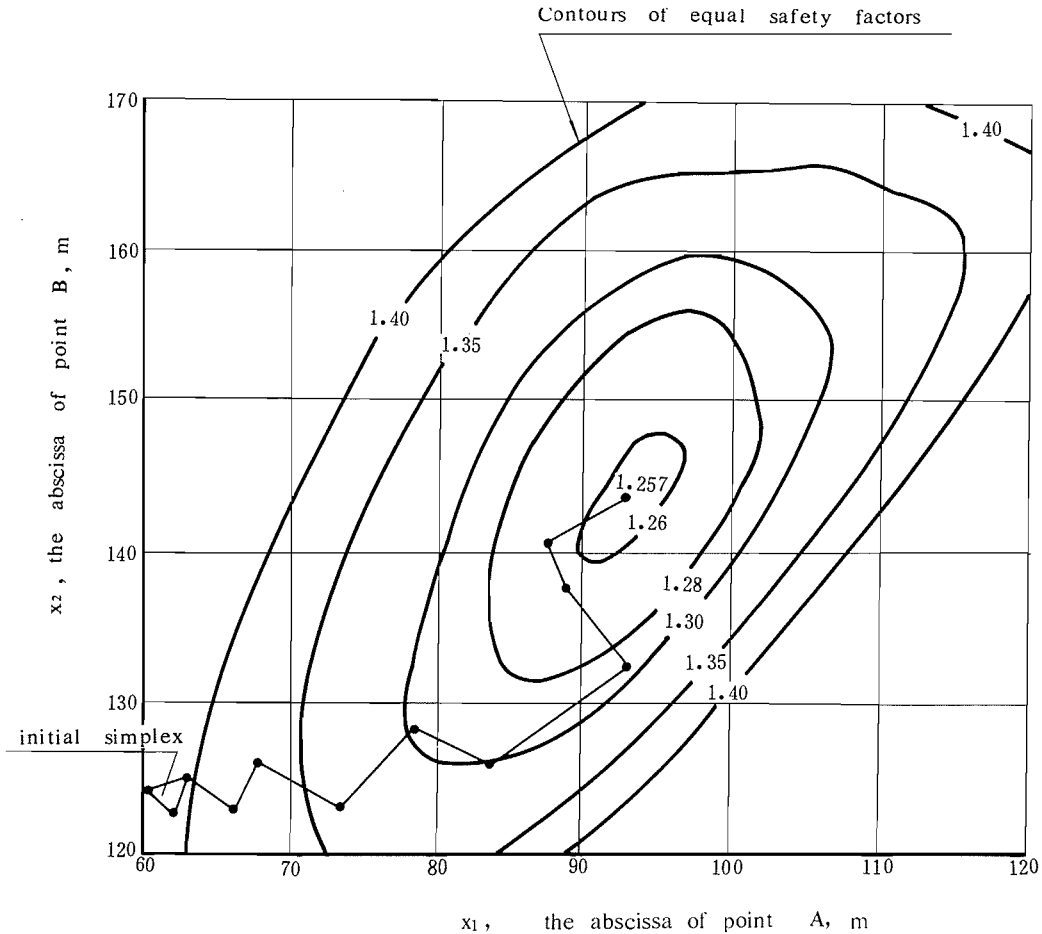
Contours of equal safety factors

FIG. 9. Process of optimization of the fourth trial shown in Fig. 7 using the simplex method.

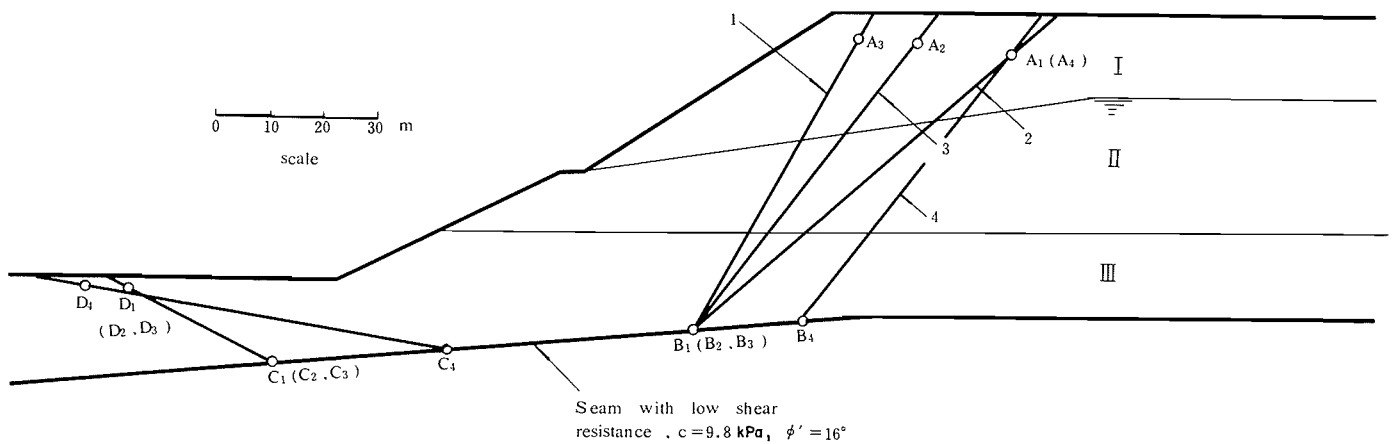Seam with low shear
resistance . c = 9.8 kPa, $\phi' = 16°$

FIG. 10. Four initial estimates of the test problem 3. The respective locations of A_1 and A_4; B_1, B_2, and B_3; C_1, C_2, and C_3; D_1, D_2, and D_3 are identical.

other methods of optimization. Except for the unmodified DFP method, they agree with each other closely.

Figure 13 and Table 6 show the influence of different assumptions of the interslice forces upon the optimized results. The minimum safety factors gained from assumptions 1 and 2 were 1.025 and 1.010 respectively, and the two critical slip surfaces were close together. This implies that different assumptions of interslice forces do not significantly affect the optimized results.

Test problem 4

This problem reevaluates the December 24, 1985, disaster at the Tianshenqiao Hydroelectric Power Project in Guangxi Province, China, in which a landslide killed 48 people.

The slide area is on the right bank of the Nanpanjiang River where the sluice channel of the dam was being constructed. The slope consisted mainly of Quaternary alluvium and talus covered by road fill and underlain by middle Tertiary bedrock, composed of shales and sandstones. There was evidence of
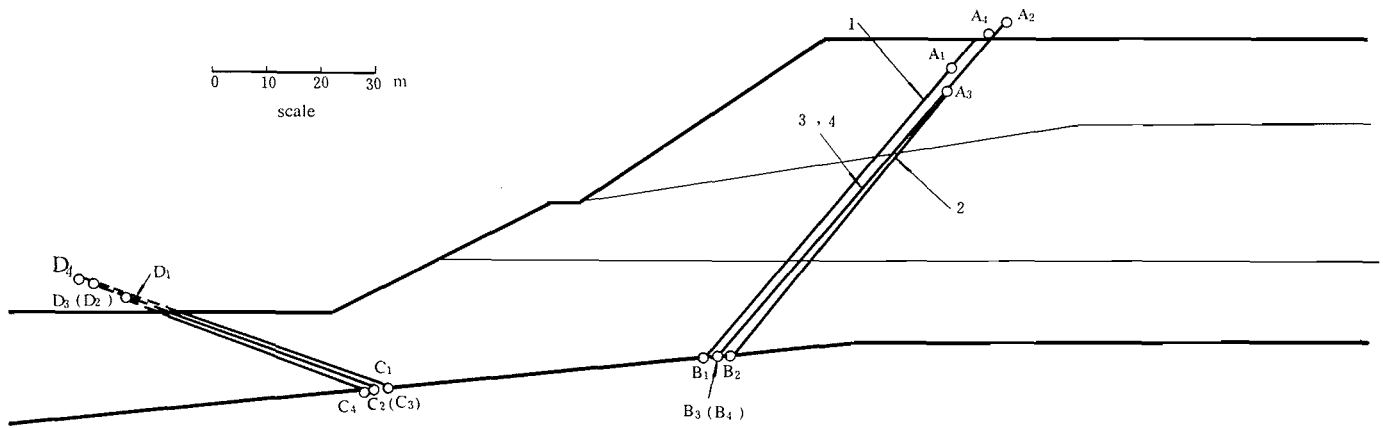
FIG. 11. Critical slip surfaces obtained from the four initial estimates shown in Fig. 10 by the modified DFP method. For calculated results see Table 5. The respective locations of B_3 and B_4; C_2 and C_3; D_2 and D_3 are basically the same.
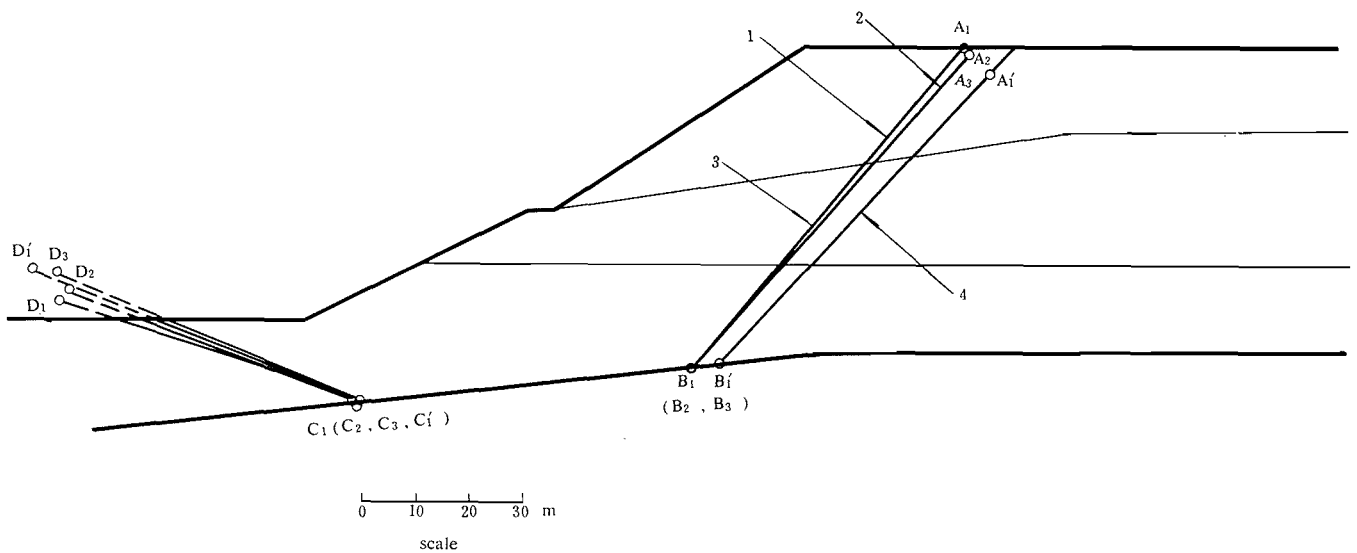
FIG. 12. Critical slip surfaces obtained by different optimization methods based on the initial estimate $A_4, B_4, C_4 D_4$ in Fig. 10: 1, modified DFP, $F_m = 1.025$; 2, steepest descent, $F_m = 1.025$; 3, simplex, $F_m = 1.025$; 4, DFP without modifications, $F'_m = 1.035$. The respective locations of A_2 and A_3; B_1, B_2, and B_3; and C_1, C_2, C_3, and C'_1 are basically the same.
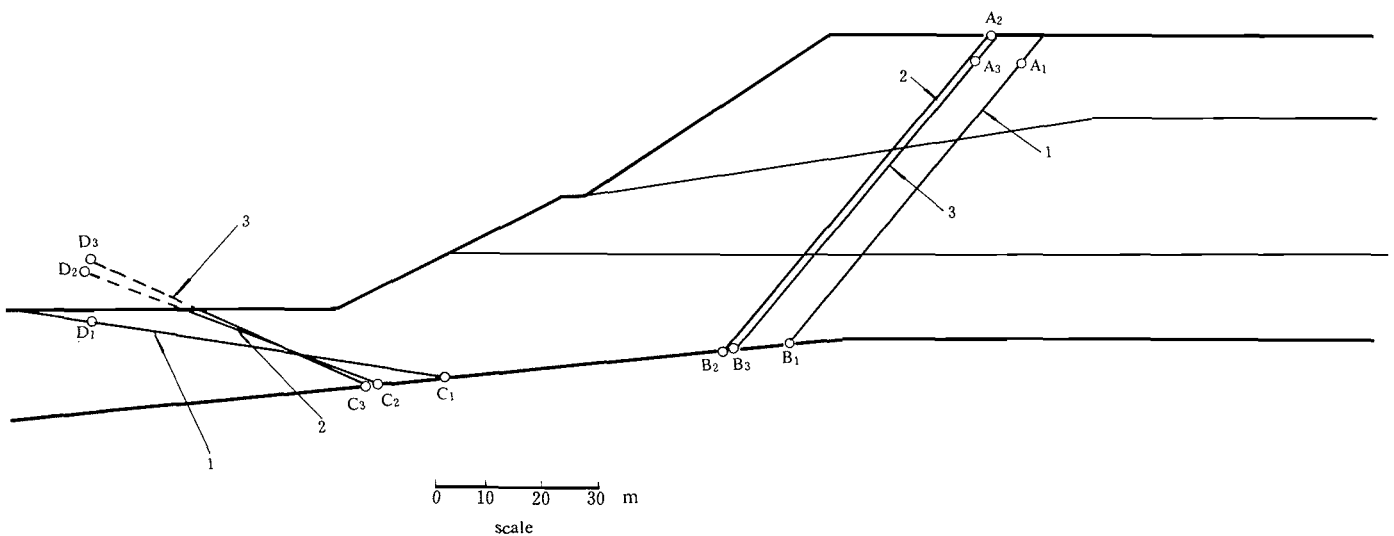
FIG. 13. Test problem 3 analysed using assumptions 1 and 2. For calculated results, see Table 6.
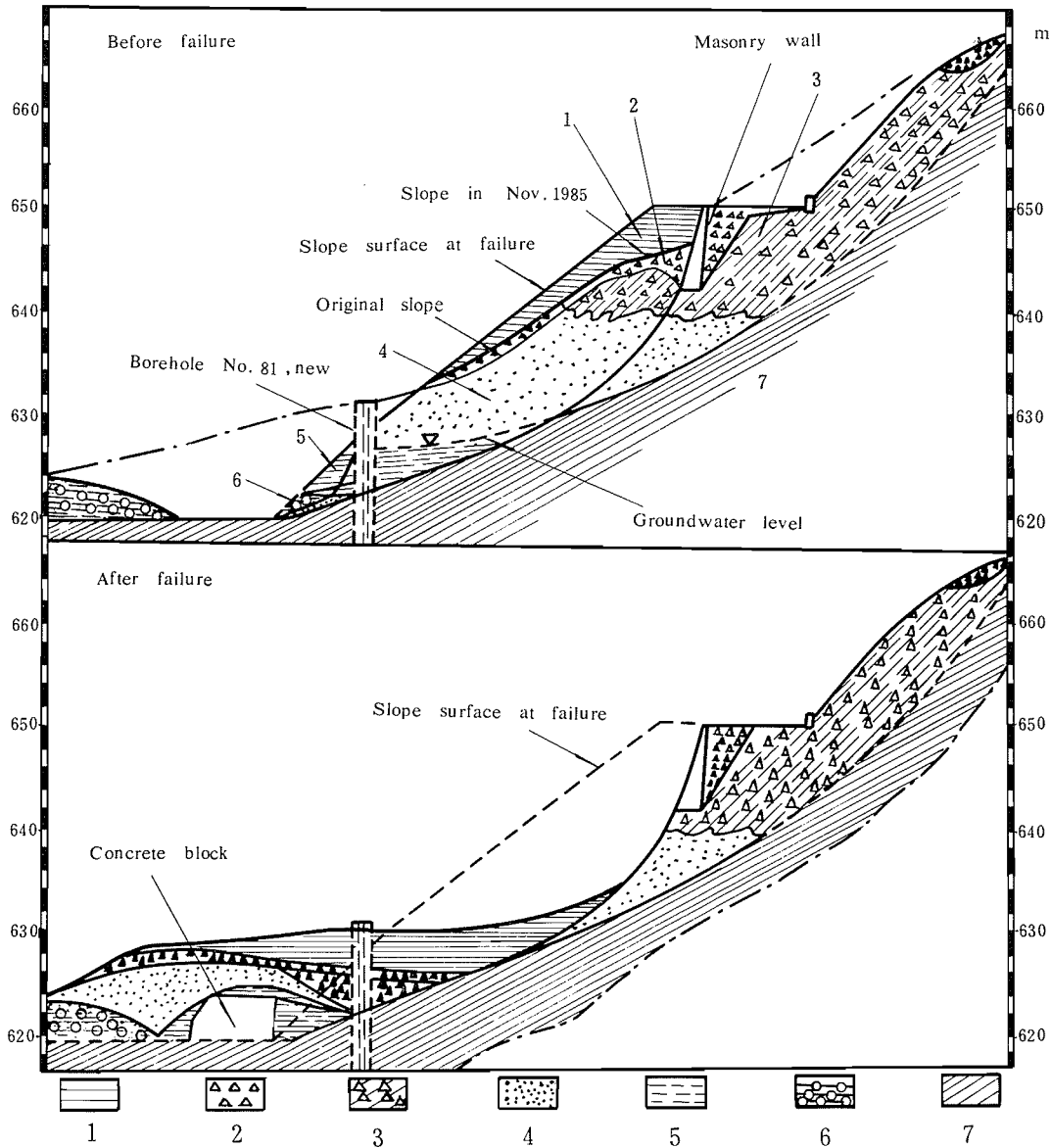
FIG. 14. Test problem 4. Geological profiles before and after failure. Descriptions and geotechnical parameters of layers 1−7 are listed in Table 7.

TABLE 5. Results calculated for test problem 3

			Minimized solution, Fig. 11			
			Modified DFP method		DFP method	
	Initial estimate, Fig. 10					
No.	Slip surface	F_0	Slip surface	F_m	No. of linear searches	F'_m	No. of linear searches
1	$A_1B_1C_1D_1$	1.191	$A_1B_1C_1D_1$	1.011	8	1.014	9
2	$A_2B_2C_2D_2$	1.121	$A_2B_2C_2D_2$	1.009	10	1.011	6
3	$A_3B_3C_3D_3$	1.120	$A_3B_3C_3D_3$	1.009	9	1.012	5
4	$A_4B_4C_4D_4$	1.117	$A_4B_4C_4D_4$	1.025	7	1.035	9

NOTE: F'_m, minimum factor of safety obtained by the DFP method without modifications; F_m, minimum factor of safety obtained by the DFP method with modifications.

groundwater mainly coming from surface runoff of the No. 6 gully and the fissures in the bedrock.

The landslide was triggered by the exposure of the dark grey silty clay layer, caused by the removal of the toe material for the construction of the retaining wall of the sluice channel. The 7115 m³ elliptical sliding mass was 63 m long and 36 m wide. It slumped 20 m down and was 50 m across.

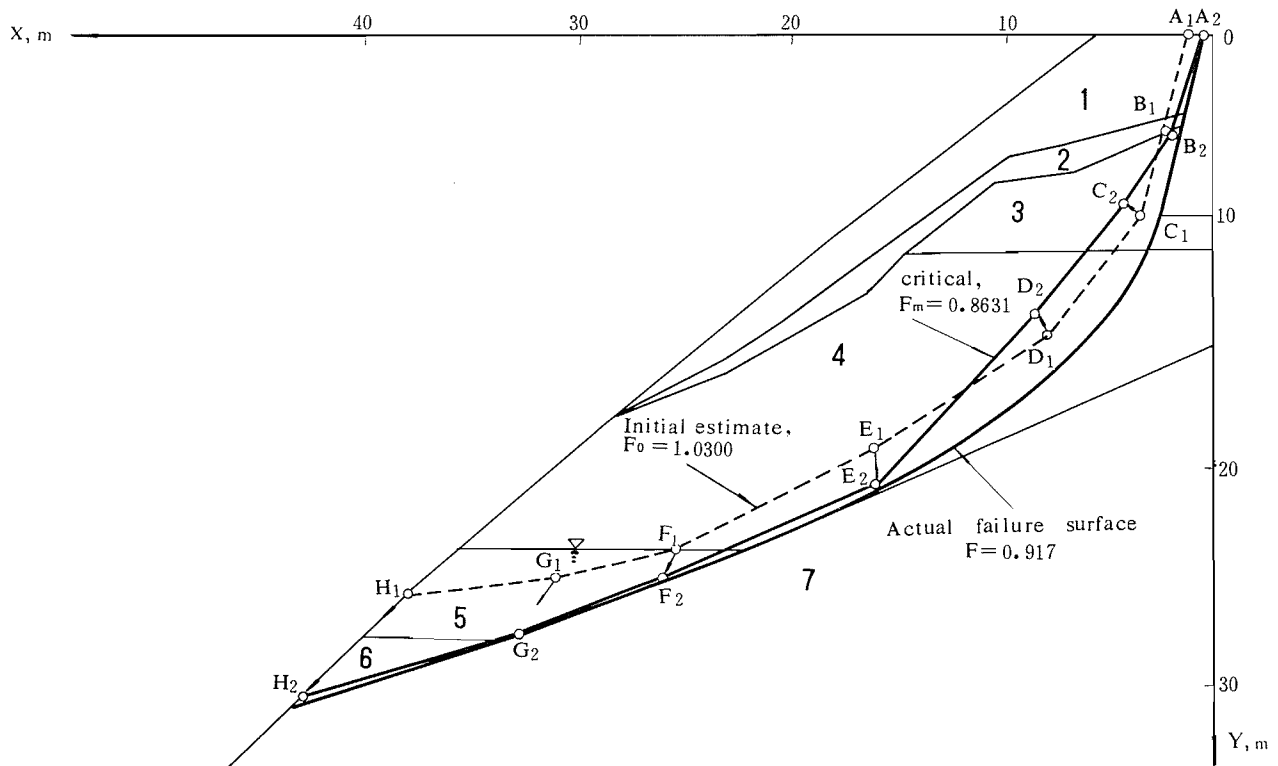The geological profiles shown in Fig. 14 suggest that the

FIG. 15. Test problem 4. Slip surfaces and calculated results.

TABLE 6. Results calculated using different assumptions of interslice forces for test problem 3 (Fig. 13)

	Slip surface			
	Initially estimated		Critical	
$f_0(x)$ and $f(x)$	No.	F_0	No.	F_m
Assumption 1	1	1.117	2	1.025
Assumption 2	1	1.112	3	1.010

failure surface partly followed the interface between soil and the masonry wall near the crown, and partly that between soil and bedrock near the toe.

Stability analysis used the parameters provided by the project designers, as shown in Table 7.

The critical slip surface calculated by the simplex method was fairly close to the measured failure surface as shown in Fig. 15. The optimized location also suggested that the lower part of the critical slip surface followed the soil—bedrock interface. The large values of the shear resistance parameters for the rock layer behaved somewhat like a penalty function of the constrained optimization method, and successfully prevented the slip surface from penetrating that rock layer. The low value of F_m was probably caused by the low shear strength parameters assigned by the designers.

Summary and concluding remarks

The objective of this study is to develop a numerical technique for determining the minimum safety factors in slope stability analysis. The four test problems presented provide convincing evidence that this objective can be met by applying various methods of optimization.

TABLE 7. Geotechnical parameters used in the slope stability analysis of test problem 4 (refer to Fig. 14)

Soil layer	Description	Density ρ (g/cm^3)	Shear strength parameters	
			ϕ (deg)	C (kPa)
1	New fill, clay and debris mixture	1.85	21.8	19.6
2	Old fill, sand, clay, and debris mixture	1.85	21.8	19.6
3	Quaternary talus, clay with rock fragments	1.85	21.8	0.0
4	Quaternary alluvium, fine sand and medium sand	1.85	20.8	29.4
5	Quaternary alluvium, grey and dark silty clay	1.81	10.2	34.3
6	Quaternary alluvium, gravels and sands	1.90	24.2	0.0
7	Tertiary bedrock, shales, sandstones with limestone intercalations	2.40	45.0	39.2

(1) The objective function to be optimized, that is, the factor of safety, is calculated by the conventional method of slices, which involves numerical iterations. The extensions to the generalized method of slices (Chen and Morgenstern 1983) were employed to calculate the objective function because of their ability to obtain guaranteed convergent values of safety factor with adequate accuracy. This is required for the successful implementation of the optimization algorithm.

(2) The validity and feasibility of the optimization methods have been proven by the successful solution of the four test problems.

(3)While various optimization methods are all applicable, the modifications to the Davidon−Fletcher−Powell method suggested by the authors were found to be essential because of the peculiarities involved in slope stability analysis.

(4) The use of different assumptions of the interslice forces has no significant effect upon the optimized results.

Recent work on transferring the computer program to an IBM-PC-XT microprocessor has made it possible to provide engineers access to the present approaches.

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Appendix. The analysis of a classical problem of plasticity by the method of slices

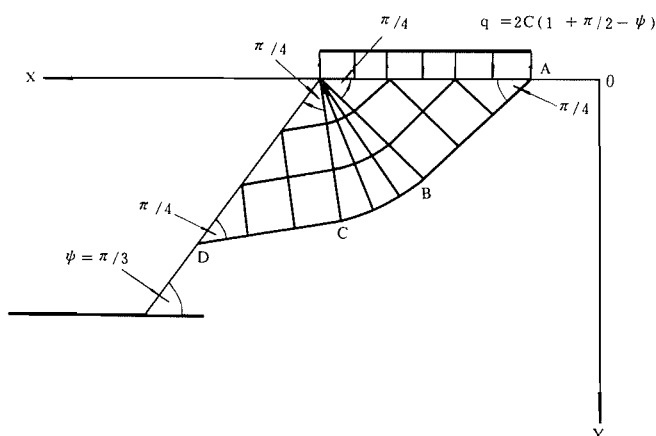Figure A1 shows a weightless, frictionless slope subjected to a vertical load. This is a classical problem in the theory of plasticity first solved by Prandtl (1921). The critical vertical load is determined by

[A1] $q_c = 2C(1 - \psi + \pi/2)$

where C is cohesion and ψ is the inclination of the slope. The derivation of [A1] can be found in Van Iterson (1947).

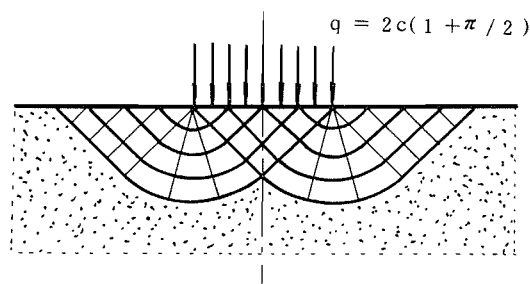FIG. A1. A test problem with closed-form solution.

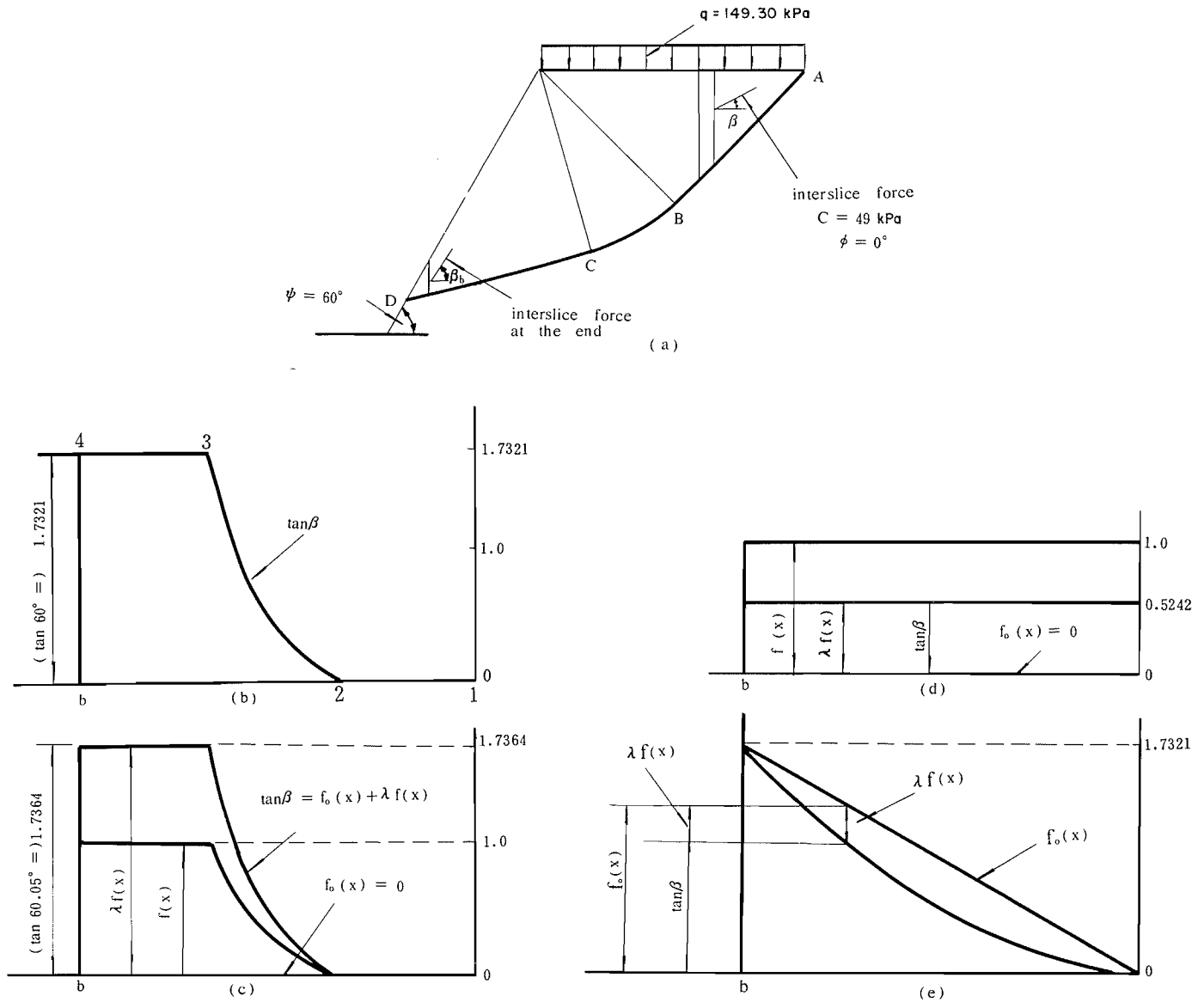FIG. A2. Prantdl solution to bearing capacity of foundations.

FIG. A3. The analysis of the problem shown in Fig. A1: (a) geometry of the slope; (b) theoretical distribution of tan β; (c) $f_0(x), f(x)$ being analogous to the distribution of tan β shown in (b); (d) assumption 1 shown in Fig. 3b; (e) assumption 2 shown in Fig. 3c.

When the inclination of the slope becomes horizontal (see Fig. A2), the problem reduces to the well-known Prandtl solution of bearing capacity (Terzaghi 1943), that is,

[A2] $q_c = 2C(1 + \pi/2)$

Using Hencky's theorem of slip line field, the following points can be made regarding the critical state of the slope:

(1) The slope is divided into two regions. Zone 1 is an area of plastic yielding in which every point is at limit equilibrium. Accordingly, the values of tan β can be identified with the curve defined by 1234 in Fig. A3b. Notice that the value of β at the end point D is 60°, which agrees with the value determined by [6]. In zone II, the elastic state prevails.

(2) The slip line ABCD, which separates the plastic and elastic zones, is composed of straight lines AB and CD and the circular arc BC. They are schematically identified in Fig. A1.

(3) If the method of slices is performed for the slip line ABCD, the lower bound of safety factors should be unity.

The generalized method of slices was then performed, adopting various assumed functions $f_0(x)$ and $f(x)$, which are schematically illustrated in Fig. A3. In this case, $\psi = 60° = \pi/3$ and $C = 49.0$ kPa. Using [A1], the critical vertical load is calculated to be 149.30 kPa. Noting the results shown in Fig. A3:

(1) If the value of $f_0(x)$ was taken to be zero and that of $f(x)$ was purposely made analogous to the theoretical distribution of tan β shown as 1234 in Fig. A3b, the calculated F was 1.002 and β_b was 60.05°, which agreed with the expected theoretical values of 60° determined by [5].

(2) Assumption 1 gave a safety factor of 1.050.

(3) Assumption 2, which is mechanically more reasonable, gave a better result—a safety factor of 1.036.

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