

EVALUATION OF TWO CONVOLUTION SUMS  
INVOLVING THE SUM OF DIVISORS FUNCTION

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The convolution sums  $\sum_{m < n/5} \sigma(m)\sigma(n - 5m)$  and  $\sum_{m < n/7} \sigma(m)\sigma(n - 7m)$  are evaluated for all  $n \in \mathbb{N}$ .

1. INTRODUCTION.

Let  $\mathbb{N}$  denote the set of natural numbers. We set

$$\sigma_k(n) = \sum_{d|n} d^k, \quad n, k \in \mathbb{N},$$

where  $d$  runs through the positive integers dividing  $n$ . If  $n \notin \mathbb{N}$  we set  $\sigma_k(n) = 0$ . We write  $\sigma(n)$  for  $\sigma_1(n)$ . We define the convolution sum  $W_k(n)$  by

$$(1.1) \quad W_k(n) := \sum_{m < n/k} \sigma(m)\sigma(n - km),$$

where  $m$  runs through the positive integers  $< n/k$ . The sum  $W_k(n)$  has been evaluated for  $k = 1, 2, 3, 4, 6, 8, 9$  and  $16$  for all  $n \in \mathbb{N}$ , see [4, 7] ( $k = 1$ ), [7, 8, 9] ( $k = 2, 3, 4$ ), [2] ( $k = 6$ ), [11] ( $k = 8$ ), [8, 9, 10, 12] ( $k = 9$ ), [1] ( $k = 16$ ). For  $k = 5$  Melfi [8, 9] has shown that for  $n \equiv 8 \pmod{16}, n \not\equiv 0 \pmod{5}$

$$(1.2) \quad W_5(n) = \frac{5}{312}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n).$$

In this paper we make use of some recent results of Berndt, Chan, Sohn and Son [3] to evaluate  $W_5(n)$  and  $W_7(n)$  for all  $n \in \mathbb{N}$ . We prove

**THEOREM 1.** For all  $n \in \mathbb{N}$

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{5}\right) - \frac{1}{130}c_5(n),$$

where the  $c_5(n)$  ( $n \in \mathbb{N}$ ) are integers defined by

$$(1.3) \quad q \prod_{n=1}^{\infty} (1 - q^n)^4 (1 - q^{5n})^4 = \sum_{n=1}^{\infty} c_5(n)q^n.$$

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The first twenty values of  $c_5(n)$  are  $c_5(1) = 1, c_5(2) = -4, c_5(3) = 2, c_5(4) = 8, c_5(5) = -5, c_5(6) = -8, c_5(7) = 6, c_5(8) = 0, c_5(9) = -23, c_5(10) = 20, c_5(11) = 32, c_5(12) = 16, c_5(13) = -38, c_5(14) = -24, c_5(15) = -10, c_5(16) = -64, c_5(17) = 26, c_5(18) = 92, c_5(19) = 100, c_5(20) = -40.$

**THEOREM 2.** For all  $n \in \mathbb{N}$

$$W_7(n) = \frac{1}{120}\sigma_3(n) + \frac{49}{120}\sigma_3\left(\frac{n}{7}\right) + \left(\frac{1}{24} - \frac{1}{28}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{7}\right) - \frac{1}{70}c_7(n),$$

where the  $c_7(n)$  ( $n \in \mathbb{N}$ ) are integers defined by

$$(1.4) \quad q \left( \prod_{n=1}^{\infty} (1 - q^n)^{16} (1 - q^{7n})^8 + 13q \prod_{n=1}^{\infty} (1 - q^n)^{12} (1 - q^{7n})^{12} + 49q^2 \prod_{n=1}^{\infty} (1 - q^n)^8 (1 - q^{7n})^{16} \right)^{1/3} = \sum_{n=1}^{\infty} c_7(n)q^n.$$

The first twenty values of  $c_7(n)$  are  $c_7(1) = 1, c_7(2) = -1, c_7(3) = -2, c_7(4) = -7, c_7(5) = 16, c_7(6) = 2, c_7(7) = -7, c_7(8) = 15, c_7(9) = -23, c_7(10) = -16, c_7(11) = -8, c_7(12) = 14, c_7(13) = 28, c_7(14) = 7, c_7(15) = -32, c_7(16) = 41, c_7(17) = 54, c_7(18) = 23, c_7(19) = -110, c_7(20) = -112.$

2. NOTATION.

Let  $z = x + iy \in \mathbb{C}$  be such that  $y > 0$ . Set  $q = e^{2\pi iz}$  so that  $|q| = e^{-2\pi y} < 1$ . The Dedekind eta function  $\eta(z)$  is defined by

$$(2.1) \quad \eta(z) := e^{2\pi iz/24} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).$$

Following Ramanujan (see [3, p. 82]) we set

$$(2.2) \quad f(-q) := e^{-2\pi iz/24} \eta(z) = \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) = \prod_{n=1}^{\infty} (1 - q^n).$$

It is convenient to set

$$(2.3) \quad A := f(-q), \quad B := f(-q^5), \quad C := f(-q^7).$$

We define  $c_5(n)$  and  $c_7(n)$  by (1.3) and (1.4) respectively. It is clear that  $c_5(n) \in \mathbb{Z}$ . We shall show that  $c_7(n) \in \mathbb{Z}$  in the proof of Theorem 2. From (1.3), (1.4), (2.2) and (2.3), we obtain

$$(2.4) \quad qA^4B^4 = \sum_{n=1}^{\infty} c_5(n)q^n.$$

$$(2.5) \quad qA^3B^3 \left( \frac{A^7}{C} + 13qA^3C^3 + 49q^2 \frac{C^7}{A} \right)^{1/3} = \sum_{n=1}^{\infty} c_7(n)q^n.$$

The Eisenstein series  $P(q)$  and  $Q(q)$  (see [3, p. 81]) are defined by

$$(2.6) \quad P(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n$$

and

$$(2.7) \quad Q(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n.$$

### 3. PROOF OF THEOREM 1.

We begin with a classical result.

**LEMMA 3.1.**

$$P(q)^2 = 1 + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n))q^n.$$

PROOF: See for example Glaisher [5, 6]. □

**LEMMA 3.2.**

$$P(q^5)^2 = 1 + \sum_{n=1}^{\infty} \left( 240\sigma_3\left(\frac{n}{5}\right) - \frac{288}{5}n\sigma\left(\frac{n}{5}\right) \right) q^n.$$

PROOF: Letting  $q \rightarrow q^5$  in Lemma 3.1 we obtain the result. □

**LEMMA 3.3.**

$$(P(q) - 5P(q^5))^2 = 16\frac{A^{10}}{B^2} + 352qA^4B^4 + 2000q^2\frac{B^{10}}{A^2}.$$

PROOF: This follows from [3, (3.21), (3.34)] and (2.3). □

**LEMMA 3.4.**

$$(a) \quad Q(q) = \frac{A^{10}}{B^2} + 250qA^4B^4 + 3125q^2\frac{B^{10}}{A^2}.$$

$$(b) \quad Q(q^5) = \frac{A^{10}}{B^2} + 10qA^4B^4 + 5q^2\frac{B^{10}}{A^2}.$$

PROOF: This follows from [3, Theorem 3.1, p. 85] and (2.3). □

**LEMMA 3.5.**

$$(a) \quad \frac{A^{10}}{B^2} = 1 + \sum_{n=1}^{\infty} \left( -\frac{5}{13}\sigma_3(n) + \frac{3125}{13}\sigma_3\left(\frac{n}{5}\right) - \frac{125}{13}c_5(n) \right) q^n.$$

$$(b) \quad q^2\frac{B^{10}}{A^2} = \sum_{n=1}^{\infty} \left( \frac{1}{13}\sigma_3(n) - \frac{1}{13}\sigma_3\left(\frac{n}{5}\right) - \frac{1}{13}c_5(n) \right) q^n.$$

PROOF: From Lemma 3.4 we obtain

$$(3.1) \quad \frac{A^{10}}{B^2} = -\frac{1}{624}Q(q) + \frac{625}{624}Q(q^5) - \frac{125}{13}qA^4B^4$$

and

$$(3.2) \quad q^2 \frac{B^{10}}{A^2} = \frac{1}{3120}Q(q) - \frac{1}{3120}Q(q^5) - \frac{1}{13}qA^4B^4.$$

Letting  $q \rightarrow q^5$  in (2.7), we obtain

$$(3.3) \quad Q(q^5) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3\left(\frac{n}{5}\right)q^n.$$

Using (2.4), (2.7) and (3.3) in (3.1) and (3.2), we obtain the asserted results. □

**LEMMA 3.6.**

$$(P(q) - 5P(q^5))^2 = 16 + \sum_{n=1}^{\infty} \left( \frac{1920}{13}\sigma_3(n) + \frac{4800}{13}\sigma_3\left(\frac{n}{5}\right) + \frac{576}{13}c_5(n) \right)q^n.$$

PROOF: This follows from Lemma 3.3, Lemma 3.5 and (2.4). □

**LEMMA 3.7.**

$$P(q)P(q^5) = 1 + \sum_{n=1}^{\infty} \left( \frac{120}{13}\sigma_3(n) + \frac{3000}{13}\sigma_3\left(\frac{n}{5}\right) - \frac{144}{5}n\sigma(n) - 144n\sigma\left(\frac{n}{5}\right) - \frac{288}{65}c_5(n) \right)q^n.$$

PROOF: This follows from Lemmas 3.1, 3.2, 3.6 and the identity

$$P(q)P(q^5) = \frac{1}{10}P(q)^2 + \frac{5}{2}P(q^5)^2 - \frac{1}{10}(P(q) - 5P(q^5))^2.$$
□

PROOF OF THEOREM 1. Letting  $q \rightarrow q^5$  in (2.6), we obtain

$$(3.4) \quad P(q^5) = 1 - 24 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{5}\right)q^n.$$

Next

$$\begin{aligned} \sum_{n=1}^{\infty} W_5(n)q^n &= \sum_{n=1}^{\infty} \left( \sum_{m < n/5} \sigma(m)\sigma(n - 5m) \right)q^n \\ &= \left( \sum_{l=1}^{\infty} \sigma(l)q^l \right) \left( \sum_{m=1}^{\infty} \sigma(m)q^{5m} \right) \\ &= \left( \frac{1 - P(q)}{24} \right) \left( \frac{1 - P(q^5)}{24} \right) \\ &= \frac{1}{576} - \frac{1}{576}P(q) - \frac{1}{576}P(q^5) + \frac{1}{576}P(q)P(q^5). \end{aligned}$$

Appealing to (2.6), (3.1) and Lemma 3.7, and equating coefficients of  $q^n$  ( $n \in \mathbb{N}$ ), we obtain the required result.  $\square$

Since  $c_5(mn) = c_5(m)c_5(n)$  for all  $m, n \in \mathbb{N}$  with  $(m, n) = 1$  and  $c_5(8) = 0$  (see [9]), we deduce that  $c_5(n) = 0$  for all  $n \in \mathbb{N}$  with  $n \equiv 8 \pmod{16}$ . Hence for  $n \equiv 8 \pmod{16}$  Theorem 1 gives

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{5}\right).$$

Melfi's result (1.2) is the special case of this result when  $n \not\equiv 0 \pmod{5}$ .

We show that the evaluation

$$(3.5) \quad c_5(5^k) = (-1)^k 5^k, \quad k \in \mathbb{N} \cup \{0\},$$

is a consequence of Theorem 1. Let  $n \in \mathbb{N}$ . The elementary identities

$$(3.6) \quad \sigma(5n) = 6\sigma(n) - 5\sigma(n/5)$$

and

$$(3.7) \quad \sigma_3(5n) = 126\sigma_3(n) - 125\sigma_3(n/5)$$

are easily proved. Thus

$$\begin{aligned} W_5(5n) &= \sum_{m < n} \sigma(m)\sigma(5n - 5m) \\ &= 6 \sum_{m < n} \sigma(m)\sigma(n - m) - 5 \sum_{m < n} \sigma(m)\sigma((n - m)/5). \end{aligned}$$

It is a classical result that

$$\sum_{m < n} \sigma(m)\sigma(n - m) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{n}{2}\right)\sigma(n),$$

see for example [7]. Also

$$\sum_{m < n} \sigma(m)\sigma((n - m)/5) = \sum_{t < n/5} \sigma(n - 5t)\sigma(t) = W_5(n).$$

Hence

$$W_5(5n) + 5W_5(n) = \frac{5}{2}\sigma_3(n) + \left(\frac{1}{2} - 3n\right)\sigma(n).$$

Appealing to Theorem 1, (3.6) and (3.7), we obtain

$$c_5(5n) = -5c_5(n), \quad n \in \mathbb{N},$$

from which (3.5) follows.

4. PROOF OF THEOREM 2.

It is convenient to set

$$(4.1) \quad H := \left( \frac{A^7}{C} + 13qA^3C^3 + 49q^2\frac{C^7}{A} \right)^{1/3},$$

where  $A$  and  $C$  are defined in (2.3).

**LEMMA 4.1.**

$$P(q^7)^2 = 1 + \sum_{n=1}^{\infty} \left( 240\sigma_3\left(\frac{n}{7}\right) - \frac{288}{7}n\sigma\left(\frac{n}{7}\right) \right) q^n.$$

**PROOF:** Letting  $q \rightarrow q^7$  in Lemma 3.1, we obtain the result. □

**LEMMA 4.2.**

$$(P(q) - 7P(q^7))^2 = 36H^4.$$

**PROOF:** From [3, Lemma 6.2, p. 109] we obtain

$$P(q) - 7P(q^7) = -6\left(\frac{A^7}{C}\right)^{2/3} \left( 1 + 13q\frac{C^4}{A^4} + 49q^2\frac{C^8}{A^8} \right)^{2/3}.$$

Squaring both sides we obtain

$$\begin{aligned} (P(q) - 7P(q^7))^2 &= 36\left(\frac{A^7}{C}\right)^{4/3} \left( 1 + 13q\frac{C^4}{A^4} + 49q^2\frac{C^8}{A^8} \right)^{4/3} \\ &= 36\left(\frac{A^7}{C} + 13qA^3C^3 + 49q^2\frac{C^7}{A}\right)^{4/3} \\ &= 36H^4. \end{aligned}$$
□

**LEMMA 4.3.**

$$(a) \quad Q(q) = \left( \frac{A^7}{C} + 245qA^3B^3 + 2401q^2\frac{C^7}{A} \right) H.$$

$$(b) \quad Q(q^7) = \left( \frac{A^7}{C} + 5qA^3B^3 + q^2\frac{C^7}{A} \right) H.$$

**PROOF:** This follows from [3, Theorem 5.1, p. 100]. □

**LEMMA 4.4.**

$$qA^3B^3H = \sum_{n=1}^{\infty} c_7(n)q^n.$$

**PROOF:** This follows from (2.5) and (4.1). □

**LEMMA 4.5.**

$$(a) \quad \frac{A^7}{C}H = 1 + \sum_{n=1}^{\infty} \left( -\frac{1}{10}\sigma_3(n) + \frac{2401}{10}\sigma_3\left(\frac{n}{7}\right) - \frac{49}{10}c_7(n) \right) q^n.$$

$$(b) \quad q^2\frac{C^7}{A}H = \sum_{n=1}^{\infty} \left( \frac{1}{10}\sigma_3(n) - \frac{1}{10}\sigma_3\left(\frac{n}{7}\right) - \frac{1}{10}c_7(n) \right) q^n.$$

PROOF: From Lemma 4.3 we obtain

$$(4.2) \quad \frac{A^7}{C} = -\frac{1}{2400}Q(q) + \frac{2401}{2400}Q(q^7) - \frac{49}{10}qA^3C^3H$$

and

$$(4.3) \quad q^2\frac{C^7}{A}H = \frac{1}{2400}Q(q) - \frac{1}{2400}Q(q^7) - \frac{1}{10}qA^3C^3H.$$

Letting  $q \rightarrow q^7$  in (2.7) we obtain

$$(4.4) \quad Q(q^7) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3\left(\frac{n}{7}\right)q^n.$$

Using (2.7), (4.4) and Lemma 4.3 in (4.2) and (4.3), we obtain the asserted results.  $\square$

LEMMA 4.6.

$$(P(q) - 7P(q^7))^2 = 36 + \sum_{n=1}^{\infty} \left( \frac{864}{5}\sigma_3(n) + \frac{42336}{5}\sigma_3\left(\frac{n}{7}\right) + \frac{576}{5}c_7(n) \right) q^n.$$

PROOF: This follows from Lemmas 4.2, 4.5 and 4.6.  $\square$

LEMMA 4.7.

$$P(q)P(q^7) = 1 + \sum_{n=1}^{\infty} \left( \frac{24}{5}\sigma_3(n) + \frac{1176}{5}\sigma_3\left(\frac{n}{7}\right) - \frac{144}{7}n\sigma(n) - 144n\sigma\left(\frac{n}{5}\right) - \frac{288}{35}c_7(n) \right) q^n.$$

PROOF: This follows from Lemmas 3.1, 4.1, 4.7 and the identity

$$P(q)P(q^7) = \frac{1}{14}P(q)^2 + \frac{49}{14}P(q^7)^2 - \frac{1}{14}(P(q) - 7P(q^7))^2. \quad \square$$

PROOF OF THEOREM 2. Letting  $q \rightarrow q^7$  in (2.6), we obtain

$$(4.5) \quad P(q^7) = 1 - 24 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{7}\right)q^n.$$

Next

$$\begin{aligned} \sum_{n=1}^{\infty} W_7(n)q^n &= \sum_{n=1}^{\infty} \left( \sum_{m < \frac{n}{7}} \sigma(m)\sigma(n - 7m) \right) q^n \\ &= \left( \sum_{l=1}^{\infty} \sigma(l)q^l \right) \left( \sum_{m=1}^{\infty} \sigma(m)q^{7m} \right) \\ &= \left( \frac{1 - P(q)}{24} \right) \left( \frac{1 - P(q^7)}{24} \right) \\ &= \frac{1}{576} - \frac{1}{576}P(q) - \frac{1}{576}P(q^7) + \frac{1}{576}P(q)P(q^7). \end{aligned}$$

Appealing to (2.6), (4.5) and Lemma 4.7, and equating coefficients of  $q^n$ , we obtain the formula for  $W_7(n)$  in Theorem 2.

We now show that the  $c_7(n)$  are integers. It is easy to check that for all  $n \in \mathbb{N}$  we have

$$\sigma_3(n) \equiv \sigma(n) \pmod{3}$$

and

$$\sigma_3(n) \equiv (2n - 1)\sigma(n) \pmod{4}.$$

Hence

$$\begin{aligned} 7\sigma_3(n) + 343\sigma_3(n/7) + (35 - 30n)\sigma(n) + (35 - 210n)\sigma(n/7) \\ \equiv \sigma_3(n) + \sigma_3(n/7) - \sigma(n) - \sigma(n/7) \equiv 0 \pmod{3} \end{aligned}$$

and

$$\begin{aligned} 7\sigma_3(n) + 343\sigma_3(n/7) + (35 - 30n)\sigma(n) + (35 - 210n)\sigma(n/7) \\ \equiv -\sigma_3(n) - \sigma_3(n/7) - (1 + 2n)\sigma(n) - (1 + 2n)\sigma(n/7) \equiv 0 \pmod{4} \end{aligned}$$

so

$$7\sigma_3(n) + 343\sigma_3(n/7) + (35 - 30n)\sigma(n) + (35 - 210n)\sigma(n/7) \equiv 0 \pmod{12}.$$

Thus

$$e(n) := \frac{7\sigma_3(n) + 343\sigma_3(n/7) + (35 - 30n)\sigma(n) + (35 - 210n)\sigma(n/7)}{12} \in \mathbb{Z}.$$

Hence

$$c_7(n) = e(n) - 70W_7(n) \in \mathbb{Z}. \quad \square$$

Similarly to the proof of (3.5) we can use Theorem 2 to show that

$$(4.6) \quad c_7(7^k) = (-1)^k 7^k, \quad k \in \mathbb{N} \cup \{0\}.$$

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