EVEN ALLOCATIONS FOR GENERALISED RATIONING PROBLEMS*

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ABSTRACT

We study an extension of the standard rationing problem, consisting of the allocation of utility losses. We assume neither linearly transferable utilities nor risk averse agents. As a consequence, the utility possibility sets need not be convex or smooth. This problem is referred to as the *generalised rationing problem*. We introduce the notion of *even allocations* as a solution concept that extends the random arrival rule to this general scenario. Moreover, we show that, when the feasible set is convex, this solution can be characterised by a suitable reformulation of the axioms that define the Nash bargaining solution.

Key words: Rationing Problems, Non-Transferable Utilities, Nash Bargaining Solution, Clarke Cones.

1 Introduction

A *rationing problem* describes a situation in which a given amount of a divisible good must be allocated among a group of agents, when there is not enough to satisfy their demands. This problem is usually formalised as a triple (N, t, x), where N is a finite set of agents, t is a positive real number that represents the amount of resources to be divided, and $x = (x_i)_{i \in N}$ is an n-vector in $\mathbb{R}^n_+$ that specifies the agents' claims, with $\sum_{i \in N} x_i > t$. Bankruptcy problems and cost-sharing problems are the best known examples of these situations. In the bankruptcy problem there is a given amount of commodity which is insufficient to cover the claims of all the agents. The cost-sharing problem refers to the realization of a public project whose benefits exceed the associated costs. This model can also be interpreted as a particular tax problem in which t is the amount of taxes to be collected and x corresponds to the agents' gross income vector. A *solution* to a rationing problem is a procedure, or rule, to allocate costs that meets some ethical and operational criteria. Different rationing situations may recommend the choice of different properties and therefore the use of different solutions.[1]

A characteristic feature of this model is that the only information available refers to the number of agents, their corresponding claims, and the amount of the good to be distributed. No explicit reference is made to the space of preferences or utilities, contrary to the standard social choice problems. This can be rationalised in terms of linear transferable utilities, as in coalitional TU games. In that case one can interpret claims and outcomes as utility values, and the available amount of the good as the aggregate utility. A solution would thus correspond to a special social choice function, that in some cases can be associated with explicit solution functions of TU games (e.g. O'Neill [15], Young [21], Curiel, Maschler & Tijs [5], Aumann & Maschler [1], Herrero, Maschler & Villar [9]). Dagan & Volij ([6]) also formulate the bankruptcy problem as a bargaining problem with transferable utilities.

This paper deals with the analysis of the rationing problem in a more general context. More specifically we introduce the following extensions: (i) utilities are not assumed to be linearly transferable; (ii) agents may or may not be risk averse; and (iii) no smoothness assumption is required. Extension (i) amounts to formulate the rationing problem in the theoretical framework of axiomatic bargaining à la Nash ([14]). Namely, utility functions are assumed to be cardinal and non-comparable, and the only information used to

[1] See e.g. Young ([22], Ch. 6), Moulin ([11] Ch. 6, [12], [13] Ch. 2), Herrero & Villar ([10]), and Thomson ([19]) for a discussion.

solve the problem is given in utility terms. Extension (ii) implies that the utility possibility set may be non-convex. Finally, extension (iii) says that no assumption is made on the differentiability of the Pareto frontier.

We will use the term *generalised rationing problem* to refer to this collective choice problem. Given are a finite set of n agents, a convex and comprehensive feasible set of alternatives in $\mathbb{R}^n$, and a claims point also in $\mathbb{R}^n$. The claims point is not feasible and Pareto dominates some feasible Pareto optimal point. The interpretation is that the feasible set contains the cardinal and non-comparable utility allocations from which a choice must be made, while the claims point is a distinguished utility vector on which the choice can be made to depend. The framework is fairly abstract and admits a number of alternative interpretations. For example, c could be the status quo utilities prior to the choice, or the agents' aspiration levels, or the benefits associated to the provision of a new public facility.

A *solution* on a given domain of generalised rationing problems is a mapping that selects a non-empty subset of feasible utility vectors for each problem in the domain (i.e. a social choice correspondence).[2] We propose a solution concept that associates to each problem the set of *even allocations*. These allocations are the maximisers of a weighted utilitarian social welfare function with the following characteristic: the weights are chosen so that all agents' weighted utilities at the solution point are equal. This implies that at the solution point any marginal movement in favour of one agent decreases his loss, in percentage terms, by exactly the amount by which the loss of the other agents increases. This notion can be given two alternative interpretations. One is an extension of the random arrival rule. The other is an application of the principles of the Nash bargaining solution to this context. Indeed, we show that, when the utility possibility set is convex, the even allocation correspondence is the unique minimal (in the order of set inclusion) solution that satisfies (suitable generalisations of) affine invariance, symmetry and contraction consistency.

Note that the generalised rationing problem is not a bargaining problem because the agents are not sharing a surplus but a loss (measured from the claims point); or, more formally, the point on which the choice depends is not dominated by any feasible alternative. It is not a "bargaining problem with claims", as analyzed by Chun & Thomson ([3]) and Herrero ([8]), because there is no disagreement point defined. Finally, it is not a classical rationing,

[2] While in the rationing literature one usually focusses on choice functions, choice correspondences for bargaining problems were first introduced by Thomson ([18]). Social choice correspondences have recently received some attention regarding the robustness of the Gibbard-Satterthwaite theorem (see for instance Barberà, Dutta & Sen [2] and the references provided there).

bankruptcy or cost-sharing problem because we work in utility space and the Pareto frontier will be typically nonlinear.

2 The model

We are interested in a social choice problem consisting of the allocation of utility losses among a group of agents with non-transferable utilities. Let $N = \{1, 2, ..., n\}$ stand for a collection of agents. Each agent $i \in N$ is characterised by a pair $(u_i,\ c_i)$, where u_i is a von Neumann-Morgenstern utility function, defined on some suitable (commodity) space, and c_i is a reference utility value. The reference vector $c = (c_i)_{i \in N}$ can be interpreted as an expression of rights, needs, demands, or aspirations which are deemed to be relevant for the the decision on how to allocate the losses. This type of problem can be summarised in a utility possibility set $S \subset \mathbb{R}^n$, and a point $c \in \mathbb{R}^n$. The set $S \subset \mathbb{R}^n$ describes the collection of utility allocations which are feasible, while the vector $c \in \mathbb{R}^n \setminus S$ denotes the reference vector, with $c >> s$ for some $s \in S$.[3] A choice must be made out of the feasible set of utility allocations S depending on the distinguished reference utility vector c. To fix ideas we can think of this situation as a case in which agents in N have to share the cost of some collective project whose benefits are measured by c.

2.1 Preliminary definitions

A **generalised rationing problem** (or a **problem**, for short) is a pair (S, c), with $S \subset \mathbb{R}^n$, $c \in \mathbb{R}^n \setminus S$. The set of **admissible** utility allocations**,** denoted by $\mathcal{A}(S, c)$, is defined as follows:

$$\mathcal{A}(S, c) = \{s \in S \ / \ s \leq c\}$$

This set is made of those utility allocations in which agents obtain utilities which are bounded above by the reference vector c. Moreover, we define the (weak) Pareto frontier of the set of admissible allocations, as follows:

$$\mathcal{PA}(S, c) = \{s \in \mathcal{A}(S, c) \ / \ s' >> s \Longrightarrow s' \notin \mathcal{A}(S, c)\}$$

One more element is to be defined. For a given problem (S, c) the point $w_i(S, c)$ describes the maximum value of agent i's utility when $u_j = c_j$ for all $j \neq i$. That is, $w_i(S, c)$ represents agent i's *worst admissible outcome.* One

[3] Vector notation: $s >> t$ means $s_i > t_i$ for all i. $s > t$ means $s_i \geq t_i$ for all i with strict inequality for some i.

can interpret the point $w(S,c) = [w_1(S,c), ..., w_n(S,c)]$ as the dual of the reference vector $c = (c_1, ..., c_n)$.

We concentrate on a family Σ of problems that satisfies some elementary restrictions. We assume $c = \mathbf{0}$ for all the problems in that family, for the sake of simplicity in exposition (i.e. we take the origin of the utility space as the reference vector). Note that there is no loss of generality in this choice as long as the only equivalent utility representations that are possible in this context are those of the form $u'_i = \alpha_i u_i$, for some scalar $\alpha_i > 0$, $i \in N$.

Definition 1 *The family Σ of **generalised rationing problems** consists of all those problems $(S, \mathbf{0})$ such that:*
(i) $S \subset \mathbb{R}^n$ is closed and comprehensive.
(ii) $\mathbf{0} \notin S$.
(iii) For all $i \in N$, $w_i(S, \mathbf{0}) > -\infty$.

Closedness of S is related to the continuity of utility functions with respect to the underlying variables. Comprehensiveness means that if $s \in S$ and $s' \in \mathbb{R}^n$ is such that $s' \leq s$, then $s' \in S$. It is related to the monotonicity of the utility functions and implies that the relevant boundary of the utility possibility set is downward sloping and thus coincides with the set of weakly efficient utility allocations. Part (ii) of the definition stipulates that there is a net loss to be allocated. Finally, part (iii) says that agents' admissible losses are bounded. From a geometrical viewpoint it implies that $\mathcal{PA}(S, \mathbf{0})$ intersects all axes of $\mathbb{R}^n$. Note that these properties ensure that $\mathcal{PA}(S, \mathbf{0})$ is a non-empty compact subset of $-\mathbb{R}^n_+$.

Definition 2 *A **solution** to a generalised rationing problem is a correspondence $\phi : \Sigma \to \mathbb{R}^n$ such that that $\emptyset \neq \phi(S, \mathbf{0}) \subset \mathcal{PA}(S, \mathbf{0})$ for all $(S, \mathbf{0}) \in \Sigma$.*

Points in $\phi(S, \mathbf{0})$ represent sensible compromises in the allocation of utility losses that is chosen in the Pareto frontier of the set of admissible allocations. Note that the way in which this notion is defined implies that $s_i \leq c_i = 0$ for all $i \in N$, whenever $s \in \phi(S, \mathbf{0})$. Also observe that the solution is defined as a set-valued mapping rather than as a function.

Now we introduce two sub-families of the generalised rationing problem that will play an auxiliary role in the ensuing discussion. They correspond to hyperplane and convex problems, denoted by $\sum_H$ and $\sum_C$, respectively, which are defined as follows:

Definition 3 *The family Σ_H of **hyperplane rationing problems** consists of all those problems $(S, \mathbf{0}) \in \sum$ such that $\mathcal{PA}(S, \mathbf{0})$ is a subset of a hyperplane. The family Σ_C of **convex rationing problems** consists of all those problems $(S, \mathbf{0}) \in \sum$ such that S is a convex set.*

Hyperplane rationing problems can be associated with problems in which utilities are linerarly transferable (standard bankruptcy or cost-sharing problems). Convex rationing problems are those in which utilities are not linearly transferable but agents are risk averse. Clearly, $\sum_H \subset \sum_C \subset \sum$.

2.2 Even allocations for hyperplane and convex problems

Let $(H, \mathbf{0})$ be a hyperplane problem. For each problem $(H, \mathbf{0}) \in \sum_H$, all $i \in N$, $w_i(H, \mathbf{0}) \in \mathbb{R}$ tells us the minimum value of agent i's utility when $u_j = 0$ for all $j \neq i$.

Definition 4 *An **even allocation** for a hyperplane problem $(H, \mathbf{0}) \in \sum_H$ is a point*

$$e(H, \mathbf{0}) = \frac{1}{n}(w_1(H, \mathbf{0}), ..., w_n(H, \mathbf{0}))$$

An even allocation picks the feasible utility point which yields the expected utilities of the lottery assigning all agents equal probabilities of getting their full claim $c_i = 0$ and also equal probabilities of getting $w_i(H, \mathbf{0})$. This is a well-known method of fair division with linear utilities (random priority). In the more specific environment of estate division problems, in which the utility possibility frontier has slope -1, this solution corresponds to the equal loss solution. From a geometrical point of view, the even solution selects the utility distribution that corresponds to the centre of gravity of the strong Pareto frontier. Note that this point is uniquely defined for each problem in $\sum_H$.

Remark 1 *If the hyperplane problem is not normalised by setting $c = \mathbf{0}$, an even allocation is a utility distribution that gives to each agent:*

$$e_i(H, \mathbf{0}) = \frac{1}{n} w_i(H, \mathbf{0}) + \frac{n-1}{n} c_i$$

Let now $(S, \mathbf{0}) \in \sum_C$ denote a convex problem, normalised to $\mathbf{0}$. We say that a hyperplane rationing problem $(H, \mathbf{0})$ is an **even support** of $(S, \mathbf{0})$ at $s^* \in \mathcal{PA}(S, \mathbf{0})$, if $\mathcal{PA}(H) = T \cap -\mathbb{R}^n_+$, where T is a supporting hyperplane to S at a point $s^* = e(H, \mathbf{0})$. We say, by extension, that s^* is an **even allocation** of the convex rationing problem $(S, \mathbf{0})$.

The notion of even support allows us to extend the concept of even allocation to situations where the agents are risk-averse and the feasible utility

space does not have a linear frontier, and where as a consequence the lottery-equivalent method of division yields Pareto-dominated outcomes. Note that each problem $(S, \mathbf{0}) \in \sum_C$ may have a multiplicity of even allocations.

Let $(S, \mathbf{0})$ be a convex problem and $e(S, \mathbf{0})$ an even allocation. The marginal rates of substution between agents at $e(S, \mathbf{0})$ are identical to those corresponding to the even allocation in an associated hyperplane rationing problem. This means, taking $n = 2$ for simplicity, that any marginal movement in favour of one agent decreases his loss, *in percentage terms*, by exactly the amount by which the loss of the other agent increases. One can imagine that the agents can appeal against the planner's decision by using arguments of the following type: 'if you make the other agent pay a bit more it will only cost little to him but it will improve my welfare a lot'. The property just described invalidates such objections, at least 'locally'. When the even allocation is unique the objection holds globally. Note that, obviously, we are expressing ourselves in terms of percentages, because utility comparisons of any other type are forbidden.

3 Solving the generalised rationing problem

This section applies the former ideas to the family Σ of generalised rationing problems in which the feasible set S is not assumed to be convex. This case corresponds to von Neumann-Morgenstern agents whose risk attitudes are not restricted. The key issue is, of course, finding a suitable extension of the notion of even allocations that is applicable to this general context. We shall discuss first the way of defining even allocations for generalised rationing problems and then prove that even allocations so defined exist.

3.1 The approach

We now provide an alternative interpretation of the notion of even allocations as solutions to the convex rationing problem, that will help us to solve the general case. Even allocations in the family Σ_C of problems can be viewed as the maximisers of a weighted utilitarian social welfare function, with a weighting system such that the social marginal worth of agent i is inversely proportional to his utility level. This becomes apparent if we re-write the notion of even allocation in the following terms:

Definition 5 *Let $(S, \mathbf{0})$ be a convex problem. A point $s^* \in \mathcal{PA}(S, \mathbf{0})$ is an* ***even allocation*** *if there exists a vector of weights $p^* \in \mathbb{R}^n_+$, with $\sum_{i=1}^n p_i^* = 1$, such that:*

(i) $p^* s^* \geq p^* s$ *for all* $s \in S$.
(ii) $p_i^* s_i^* = p_j^* s_j^*$ *for all* $i, j \in N$.

Part (i) of the definition describes the tangency of the associated hyperplane problem. The hyperplane on which lies the Pareto frontier of $(H, \mathbf{0})$ has normal p^*. Therefore, p^* is *perpendicular* to S at the boundary point $s^* \in S$. Vector p^* can be interpreted as an endogenous social weighting system and s^* is a maximiser of a weighted utilitarian social welfare function with endogenous weights. Part (ii) introduces an equity requirement that tells us about the choice of these weights. It postulates that the coefficients p_i^*, p_j^* associated with any two agents $i, j \in N$ are inversely proportional to their utilities. Hence, we give more weight in social welfare to those agents with smaller utilities (which is reminiscent of Sen's (1973) minimal equity axiom).

Let $(S, \mathbf{0})$ be a convex problem and s^* an even allocation. Let $(H, \mathbf{0})$ be an even support of S at s^* and let p^* denote the corresponding weighting system that satisfies (i) and (ii) above. Then, it follows that:

$$\frac{p_i^*}{p_j^*} = \frac{w_j(H, \mathbf{0})}{w_i(H, \mathbf{0})}, \quad \forall \ i, j \in N$$

How can we define the notion of even support to a comprehensive, not necessarily convex nor smooth sharing problem? Note that we may fail to ensure the existence of allocations with the properties (i) and (ii) in the definition above because now not all points in the boundary of S can be supported by a hyperplane. Or, put in other words, for a given $s' \in \mathcal{PA}(S, \mathbf{0})$ we cannot ensure the existence of a perpendicular vector p', that is, a vector such that $p's' \geq p's$ for all $s \in S$. We need a more general notion of tangent plane or perpendicular vector. The Clarke normal cone provides such an object, as briefly explained in the next subsection.

3.2 Pause: A pinch of cones

The *Clarke normal cone* is a mathematical concept that provides a suitable extension of the notion of perpendicular vectors (or, equivalently, *tangent planes*) that can be applied without reference to either smoothness or convexity. In order to define this concept, let us introduce two preliminary notions [the reader is referred to Clarke ([4], ch. 2) and Villar (2000, ch. 5) for a detailed discussion of all these notions].

For $S \subset \mathbb{R}^n$, let $\phi : S \to \mathbb{R}^n$ be a set valued mapping, and let s' be a point in S. The **Limsup** of ϕ at s' is given by:

$$\underset{s \to s'}{Lim \sup}\, \phi(s') \equiv \{p = \lim p^\nu \ / \ \exists \{s^\nu\} \subset S, \{s^\nu\} \to s' \text{ and } p^\nu \in \phi(s^\nu) \ \forall \nu\}$$

In words: By Limsup of ϕ at s' we denote the set of all points which are limits of sequences of points $p^\nu \in \phi(s^\nu)$, when $s^\nu \to s'$. Observe that when ϕ is a closed correspondence $\underset{y \to y^*}{Lim \sup}\, \phi(s') = \phi(s')$, for each $s' \in S$. When this is not so, the Limsup may be thought of as an operator which "closes the graph" of ϕ.

A vector $p \in \mathbb{R}^n$ is *perpendicular* to a closed set S at s, if s is the point in S at minimum distance of p, that is, if $d_S(p + s) =\| p \|$ (i.e. if the distance between $(p + s)$ and S is precisely the norm of p). Let $S \subset \mathbb{R}^n$ be closed, and let $s \in S$. The **cone of vectors which are perpendicular** to S at s', denoted by $\perp_S(s')$, is given by:

$$\perp_S(s') = \{p = \lambda(s - s'), \lambda \geq 0, s \in \mathbb{R}^n \text{ and } d_S(s) =\| s - s' \|\}$$

When S is a closed convex set $\perp_S(s')$ is a non-degenerate convex cone for each s' in the boundary of S. Moreover, $\perp_S(s')$ is precisely the set of vectors $p \in \mathbb{R}^n$ for which $ps' \geq ps$, for all $s \in S$.

When S is not a convex set there may be points $s \in S$ for which no perpendicular vector exists. A natural way to circumvent this problem is by making use of the Limsup operator. The notion of Clarke cone at $s' \in S$ is now easy to understand: it consists of the convex hull of $\underset{s \to s'}{Lim \sup} \perp_S (s)$. Formally:

Definition 6 *Let S be a closed subset of $\mathbb{R}^n$ and $s' \in S$. Then, the **Clarke Normal Cone** $\mathbb{N}_S(s')$ to S at s' is given by*

$$\mathbb{N}_S(s') \equiv Co\{\underset{s \to s'}{Lim \sup} \perp_S (s)\}$$

By this definition the Clarke Normal Cone at a point s' is the convex cone generated by the vectors perpendicular to S at s', and the limits of vectors which are perpendicular to S in a neighbourhood of s'.

The following properties are worth reminding:

- **Claim:** Let S be a nonempty, closed and comprehensive set in $\mathbb{R}^n$. Then: (i) $\mathbb{N}_S(s)$ is a cone in $\mathbb{R}^n_+$ with vertex zero, $\forall\, s \in S$. (ii) The correspondence $\mathbb{N}_S : S \to \mathbb{R}^n$ which associates $\mathbb{N}_S(s)$ to each $s \in S$, is closed. (iii) $\mathbb{N}_S(s) = \{\mathbf{0}\}$ if and only if $s \in intS$.

The set of hyperplanes generated by vectors in $\mathbb{N}_S(s)$ is also a cone, called the tangent cone $\mathbb{T}_S(s)$. That is, for each p "perpendicular" (in the sense of Clarke) to S at the boundary point s, there exists a hyperplane T with normal p which is "tangent" to S at the boundary point s.

3.3 Even allocations for generalised rationing problems

We can now easily adapt the notion of even allocation to this general context:

Definition 7 *A point $s^* \in S$ is an **even allocation** for the generalised rationing problem $(S, \mathbf{0}) \in \sum$ if there exists a vector of weights $p^* \in \mathbb{R}^n_+ - \{0\}$, with $\sum_{i=1}^n p_i^* = 1$, such that:*
(i) $p^* \in \mathbb{N}_S(s^*)$.
(ii) $p_i^* s_i^* = p_j^* s_j^*$ for all $i, j \in N$.

An even allocation for a problem $(S, \mathbf{0}) \in \sum$ is thus a point that admits an even support, in this more general sense.

The next result tells us about the existence of even allocations:

Proposition 1 *Every generalised rationing problem $(S, \mathbf{0}) \in \sum$ has an even allocation.*

Proof.

Let $(S, \mathbf{0})$ be a problem in $\sum$ and let $\mathbb{P}$ denote unit simplex in $\mathbb{R}^n$, that is, $\mathbb{P} = \{q \in \mathbb{R}^n_+ \ / \ \sum_{i=1}^n q_i = 1\}$. Recall that $\mathcal{PA}(S, \mathbf{0})$ is a closed compact set that is made of the upper boundary of a comprehensive set. Under these conditions $\mathcal{PA}(S, \mathbf{0})$ is homeomorphic to $\mathbb{P}$ [e.g. Villar ([20], Prop. 5.2)]. That is, there exists a continuous function $h : \mathcal{PA}(S, \mathbf{0}) \to \mathbb{P}$, such that, for each $s \in \mathcal{PA}(S, \mathbf{0})$, there exists a unique $z \in \mathbb{P}$ such that $z = h(s)$, and for each $z \in \mathbb{P}$ there exists a unique $s \in \mathcal{PA}(S, \mathbf{0})$ such that $s = h^{-1}(z)$, with h^{-1} continuous.

Let $\mathbb{N}_S$ denote the Clarke normal cone, and let $\widehat{\mathbb{N}}_S$ denote the restriction of this correspondence to the set $\mathcal{PA}(S, \mathbf{0})$. Note that $\widehat{\mathbb{N}}_S$ is an upper hemicontinuous correspondence with nonempty, closed and covex values, when S is a closed comprehensive set, with $\{\mathbf{0}\} \neq \widehat{\mathbb{N}}_S(s) \subset \mathbb{R}^n_+$. Now define a new correspondence $\varphi : \mathbb{P} \to \mathbb{P}$ as follows:

$$\varphi(z) = \widehat{\mathbb{N}}_S(h^{-1}(z)) \bigcap \mathbb{P}$$

By construction, φ is an upper hemicontinuous correspondence with nonempty, compact and convex values.

Let $\gamma : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be defined as follows:

$$\gamma(p,z) = \begin{cases} \{z\} + \left\{ \left(\frac{p_i h_i^{-1}(z)}{ph^{-1}(z)} \right)_{i=1}^{n} \right\} - \{\frac{1}{n}\mathbf{1}\} & \text{if } ph^{-1}(z) \neq 0 \\ \{z\} + \mathbb{P} - \{\frac{1}{n}\mathbf{1}\} & \text{otherwise} \end{cases}$$

Here again γ is an upper hemicontinuous correspondence with nonempty, compact and convex values, as the vector $\left(\frac{p_i h_i^{-1}(z)}{ph^{-1}(z)} \right)_{i=1}^{n}$ is a continuous function of (p,z) whenever $ph^{-1}(z) \neq 0$, and becomes the whole image set otherwise.

Finally, construct a mapping $\beta : \mathbb{P} \times \mathbb{P} \to \mathbb{P} \times \mathbb{P}$ as follows:

$$\beta(p,z) = \varphi(z) \times \gamma(p,z)$$

By construction, this is an upper hemicontinuous correspondence with nonempty, compact and convex values, that applies a nonempty convex set over itself. Therefore, Kakutani's fixed point theorem ensures the existence of $(p^*, z^*) \in \mathbb{P} \times \mathbb{P}$, such that $p^* \in \varphi(z^*)$, $z^* \in \gamma(p^*, z^*)$. The first relation establishes that p^* is a normal vector to S at $s^* = h^{-1}(z^*)$. Therefore, $p^* >> \mathbf{0}$ and it follows from the second relation that:

$$\frac{p_i^* h_i^{-1}(z^*)}{p^* h^{-1}(z^*)} = \frac{p_i^* s_i^*}{p^* s^*} = \frac{1}{n}, \quad \text{for all } i \in N$$

This in turn amount to saying that $p_i^* s_i^* = \frac{1}{n} p^* s^*$ for all i, which shows that s^* is an even allocation. ■

4 Even allocations on Σ_C and the Nash solution

Even allocations constitute a solution to the generalised rationing problem that satisfies some appealing ethical properties and are applicable to a large family of problems. We shall now argue that, when the utility possibility set is convex, even allocations are characterised by the same principles underlying the Nash barganing solution. First we show that there is no single-valued solution to the convex rationing problem that satisfies the Nash's axioms, nor there exists a multi-valued solution for a certain standard extension of Nash's Independence of Irrelevant Alternatives. Then we shall prove that the solution $\phi : \Sigma_C \to \mathbb{R}^n$ that associates to each problem the set of even allocations is the unique minimal (in the order of set inclusion) solution that satisfies a suitable extension of those axioms.

4.1 Impossibility results

We now introduce those axioms that are adaptations of standard properties in the bargaining framework. We use the following notation. A *positive transformation* is a function $\tau : \mathbb{R}^n \to \mathbb{R}^n$ such that there exist numbers $\alpha_i > 0$, $i = 1, 2, ..., n$, with $\tau_i(x) = \alpha_i x_i$. Given $X \subset \mathbb{R}^n$ and a positive transformation τ, denote $\tau(X) = \{y \in \mathbb{R}^n \ / \ y = \tau(x) \text{ for some } x \in X\}$. Given a generalised rationing problem $(S, \mathbf{0})$ and a transformation τ, denote $\tau(S, \mathbf{0}) = [\tau(S), \mathbf{0}]$.

The first axiom postulates that solutions must be independent of positive transformations. This translates the underlying assumption of cardinal non-comparable utility functions, as in the original bargaining problem formulated by Nash. Formally:

Axiom 1 (Positive Invariance) *For all positive transformations* $\tau : \phi(\tau(S), \mathbf{0}) = \tau(\phi(S, \mathbf{0}))$.

The next axiom introduces an equity restriction. It states that if agents cannot be distinguished in a problem, they cannot be distinguished in a solution. To simplify notation let $\mathbf{1}$ denote the unit vector in $\mathbb{R}^n$, that is, $\mathbf{1} = (1, 1, ..., 1)$.

Axiom 2 (Symmetry) *For all* $(S, \mathbf{0}) \in \Sigma$, *if* S *is symmetric with respect the* $\lambda\mathbf{1}$ *line, then* $\{\lambda'\mathbf{1}\} \in \phi(S, \mathbf{0})$ *for some scalar* λ'.

Our final axioms are alternative standard extensions of Nash's "independence of irrelevant alternatives" to the multi-valued solution case.

Axiom 3 (Contraction Consistency) *For all* $(S, \mathbf{0}), (T, \mathbf{0}) \in \Sigma$ *with* $S \subset T$ *and* $\phi(T, \mathbf{0}) \cap S \neq \emptyset : \phi(T, \mathbf{0}) \cap S \subset \phi(S, \mathbf{0})$.

Axiom 4 (Strong Contraction Consistency) *For all* $(S, \mathbf{0}), (T, \mathbf{0}) \in \Sigma$ *with* $S \subset T$ *and* $\phi(T, \mathbf{0}) \cap S \neq \emptyset : \phi(T, \mathbf{0}) \cap S = \phi(S, \mathbf{0})$.

These properties can be interpreted as follows: Take a given problem and suppose that the utility possibility set is reduced, without the reference vector $c = \mathbf{0}$ being altered. Suppose furthermore that a subset of the original solution is still part of the reduced set. Then, that subset of the original solution must be the solution in the reduced problem (strong contraction consistency), or that subset of the original solution must be part of the solution in the reduced problem (contraction consistency).

For single-valued mappings these axioms correspond precisely to the Nash bargaining axioms. Yet assuming single-valued solutions and convex utility

possibility sets does not yield some solution analogous to the Nash Bargaining Solution, but rather an impossibility result.[4]

Proposition 2 *Let Σ_C denote the class of convex rationing problems. Then,*

(i) There is no single-valued mapping f defined on Σ_C that satisfies invariance, symmetry and contraction consistency.

(ii) There is no (multi-valued) solution ϕ defined on Σ_C that satisfies invariance, symmetry and strong contraction consistency.

Proof.

(i) Let $f : \Sigma_C \to \mathbb{R}^n$ be a function that satisfies the axioms of invariance, symmetry and contraction consistency. Construct a family of convex rationing problems $\langle (S_i, \mathbf{0}) \rangle_{i=1,...,n+1}$ in Σ_C by means of $n+1$ hyperplanes, which are defined as follows. For $i = 1, 2, ..., n$, H^i is the hyperplane going through the points $-\mathbf{e}^1, -\mathbf{e}^2, ..., -\mathbf{e}^{i-1}, (-n)\,\mathbf{e}^i, -\mathbf{e}^{i+1}, ..., -\mathbf{e}^n$, and H^{n+1} is the hyperplane going through the points $-\mathbf{e}^i$, $i = 1, ..., n$. Let S^i be the comprehensive hull of $H^i \bigcap \mathbb{R}^n_-$. By Symmetry, $f\left(S^{n+1}, \mathbf{0}\right) = \left(-\frac{1}{n}\right)\mathbf{1}$. Note that the problem $(S^i, \mathbf{0})$ can be obtained from $(S^{n+1}, \mathbf{0})$ by applying the linear transformation

$$\tau_i(x) = (x_1, x_2, ..., x_{i-1}, nx_i, x_{i+1}, ..., x_n)$$

Therefore, invariance implies:

$$f\left(S^i, \mathbf{0}\right) = \left(-\frac{1}{n}, -\frac{1}{n}, ..., -1, -\frac{1}{n}, ..., -\frac{1}{n}\right)$$

with the -1 in i^{th} place. We have $f\left(S^i, \mathbf{0}\right) \in S^j$ for all $i \neq j = 1, ..., n$, since for example $-\mathbf{e}^i \in S^j$ Pareto dominates $f\left(S^i, \mathbf{0}\right)$. Now define $S^* = \bigcap_{i=1}^n S^i$ and consider the problem $(S^*, \mathbf{0})$. Contraction consistency, applied when viewing each of the $(S^i, \mathbf{0})$ as the 'large' problem, implies that $f\left(S^*, \mathbf{0}\right) = f\left(S^i, \mathbf{0}\right)$, for all i, a contradiction.

(ii) The same argument above provides a counter-example for a correspondence that fails to satisfy strng contraction consistency. ■

This proposition shows that invariance, symmetry, and contraction consistency are incompatible with single-valued solutions, even when we consider convex problems. And also that the same incompatibility applies to multi-valued solutions if we use the stronger version of contraction consistency. The

[4] A closely related result appears in Roth ([16]), in a discussion of ideal point dependent solutions.

same argument can be easily adapted to show that there are similar incompatibilities when we substitute symmetry by "strong claims boundedness".[5]

The results in Proposition 2 imply that no solution satisfying symmetry and invariance can be derived by the maximisation of a Social Welfare Ordering (for in that case it would satisfy strong contraction consistency). In this sense choice mappings on the domain under study are intrinsically more complex objects than bargaining solutions. The way out of these impossibility results is to both allow multi-valued solutions and require only the weaker extension of independence of irrelevant alternatives.

4.2 Characterisation

Let $\mathcal{E}(S,c)$ denote the set of even allocations of a convex problem (S,c) in Σ_C. Our next result shows that even allocations correspond precisely to the outcome of the unique minimal solution that satisfies invariance, symmetry and contraction consistency. Formally:

Proposition 3 *There is one and only one minimal (in the order of set inclusion) solution ϕ defined on Σ_C satisfying invariance, symmetry and contraction consistency. It is the solution that associates to each convex problem $(S,\mathbf{0}) \in \Sigma_C$ the set of even allocations, $\phi(S,\mathbf{0}) = \mathcal{E}(S,\mathbf{0})$.*

Proof.

First observe that the mapping defined in the statement is a solution. This follows directly from the definition and Proposition 1 above. Now we show that all even allocations must be in the solution mapping.

Let $(S,\mathbf{0}) \in \Sigma_C$ and let s be an even allocation. Therefore,

$$s = e(H,\mathbf{0}) = \frac{1}{n}(w_1(H,\mathbf{0}), w_2(H,\mathbf{0}), ..., w_n(H,\mathbf{0})$$

for some hyperplane problem $(H,\mathbf{0})$. Let $Ch(H)$ denote the comprehensive hull of $H \bigcap -\mathbb{R}^n_+$. Clearly $S \subseteq Ch(H)$ and s is an even allocation of $Ch(H)$. Define the linear transformation τ by

$$\tau(x) = \left(\frac{1}{w_1(H,\mathbf{0})}x_1, \frac{1}{w_2(H,\mathbf{0})}x_2, ..., \frac{1}{w_n(H,\mathbf{0})}x_n\right)$$

Observe that, for any linear transformation, the transformation of an even allocation remains an even allocation of the transformed problem: in fact

[5] Strong claims boundedness states that for all $(S,c) \in \sum$, all $i \in N$, $f_i(S,c) < c_i$ (that is to say, all agents have to suffer some utility reduction).

both the property of being the support point of a hyperplane and the property of being the centre of gravity of a polygon hold for any linear transformation. So in particular $\tau(s)$ is an even allocation of $\tau(Ch(H))$. Therefore, since $\tau_i(w_1(H, \mathbf{0})) = -1$, we have:

$$\tau(s) = -\frac{1}{n}\mathbf{1}$$

By Symmetry, $\tau(s) \in \phi(\tau(ch(H)), \mathbf{0})$. Contraction Consistency then implies $\tau(s) \in \phi(\tau(S), \mathbf{0})$, and the proof is concluded by an application of invariance. ■

This result points out that the solution mapping that associates to each problem (S, c) the set of even allocations may be regarded as a suitable 'translation' of the Nash Bargaining Solution (NBS) to this setting. In fact, the NBS can also be characterised by the described tangent hyperplane 'splitting' property. Note that a selection from the minimisers of the product of the losses from c, another affine invariant and contraction consistent solution, does not satisfy symmetry because it must assign to at least one player its claim in full.

Remark 2 *This approach bears some resemblance with the analysis of bargaining solutions defined on a domain which includes non-convex problems. However, in that setting there exist single-valued solutions satisfying both contraction consistency and affine invariance, as well as multivalued solutions satisfying both strong contraction consistency and affine invariance. As shown by Zhou ([23]) and by Denicolò and Mariotti ([7]) these solutions must all be selections from the set of maximisers of the Nash product*[6] *on* $\mathbb{R}^n_{++}$. *On the contrary in our setting the maximisation on convex problems of the corresponding non-concave value function in* $-\mathbb{R}^n_{++}$ *generates a conflict.*

5 An illustration: sharing a cost in utility space

Let us illustrate the nature of the results presented above with a simple cost-sharing model. Consider a society given by a set $N = \{1, 2, ..., n\}$ of agents and a commodity space with two commodities, an indivisible public project and a divisible private good (e.g. money). Each agent is characterized by

[6] Denicolò and Mariotti ([7]) in particular exploit the fact that (single-valued) bargaining solutions satisfying Contraction Consistency and defined on particular domains which include non-convex problems define a Social Welfare Ordering on $\mathbb{R}^n_{++}$.

a von Neumann-Morgenstern utility function $u_i : \{0,1\} \times \mathbb{R} \to \mathbb{R}$, and an initial endowment ω_i, where $u_i(1, y_i)$ and $u_i(0, y_i)$ describe agent i's utility when consuming y_i units of the private consumption good, with or without the public project. Assume that $u_i(1, y_i) > u_i(0, y_i)$ for all y_i, and also that u_i is monotone and continuous in y_i, and also that $\omega_i = (0, z_i)$, as usual. The question is how to finance the provision of the public project by individual contributions $\mathbf{t} = (t_1, t_2, ..., t_n)$, such that $\sum_{i \in N} t_i \geq \gamma$, where γ represents the cost of the public project measured in units of the private good.

Let $c_i = u_i(1, z_i)$ denote agent i's individual (gross) utility benefit, and S is the set of utility vectors $[u_i(1, z_i - t_i)]_{i \in N}$ with $z_i - t_i \geq 0$ and $\sum_{i \in N} t_i \geq \gamma$. We normalise utilities so that $u_i(1, z_i) = 0$ for all $i \in N$, so that the cost sharing problem can be formalised as a generalised ratioing problem in $(S, \mathbf{0}) \in \sum$. Under the assumptions established, S is closed and comprehensive. Moreover, the set $\mathcal{A}(S, \mathbf{0})$ of admissible allocations corresponds to the set of points $[u_i(1, z_i - t_i)]_{i \in N}$ with $z_i - t_i \geq 0$ for all i, and $\sum_{i \in N} t_i \geq \gamma$. The set $\mathcal{PA}(S, \mathbf{0})$ is the subset of $\mathcal{A}(S, \mathbf{0})$ which satisfies $\sum_{i \in N} t_i = \gamma$. A solution is a point $[u_i(1, z_i - t_i^*)]_{i \in N}$ in the set $\mathcal{PA}(S, \mathbf{0})$. Therefore, solving a problem in this case amounts to finding a suitable vector $\mathbf{t}^* \in \mathbb{R}^n_+$ of individual contributions.

An even allocation entails the choice of a vector t^* of contributions and a vector p^* in the unit simplex $\mathbb{P}$ such that:

(i) $\sum_{i \in N} p_i^* u_i(1, z_i - t_i^*)_{i \in N} \geq \sum_{i \in N} p_i^* u_i(1, z_i - t_i)_{i \in N}$, for all vectors $t \in \mathbb{R}^n_+$ such that $\sum_{i \in N} t_i = \gamma$.

(ii) $p_i^* u_i(1, z_i - t_i^*) = p_j^* u_j(1, z_j - t_j^*)$ for all $i, j \in N$.

Therefore, the solution chooses a vector of contributions t^* that maximizes a weighted sum of utilities over the utility possibility set S, with respect to an endogenous weighting system in which the social marginal worth of an agent is inversely proportional to his utility.

It is easy to see that, when utilites are differentiable, the weight ratio between any two agents corresponds to inverse of their marginal utilities. More precisely, [7]

$$\frac{\partial u_i / \partial t_i}{\partial u_j / \partial t_j} = \frac{u_i(1, z_i - t_i^*)}{u_j(1, z_j - t_j^*)} = \frac{p_j^*}{p_i^*} \quad \text{for all } i, j \in N$$

That is, p^* is a vector that equalizes the ratios of total utilities with that of marginal utilities for every two agents.

[7] This follows from the F.O.C. when we take p^* as given and consider the maximization of the weighted utilitarian social welfare utility function $\sum_{i=1}^n p_i^* u_i(1, z_i - t_i)$, subject to $\sum_{i \in N} t_i = \gamma$.

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