Even spin minimal model holography

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Abstract: The even spin $\mathcal{W}^e_\infty$ algebra that is generated by the stress energy tensor together with one Virasoro primary field for every even spin $s \geq 4$ is analysed systematically by studying the constraints coming from the Jacobi identities. It is found that the algebra is characterised, in addition to the central charge, by one free parameter that can be identified with the self-coupling constant of the spin 4 field. We show that $\mathcal{W}^e_\infty$ can be thought of as the quantisation of the asymptotic symmetry algebra of the even higher spin theory on AdS_3. On the other hand, $\mathcal{W}^e_\infty$ is also quantum equivalent to the $\mathfrak{so}(N)$ coset algebras, and thus our result establishes an important aspect of the even spin minimal model holography conjecture. The quantum equivalence holds actually at finite central charge, and hence opens the way towards understanding the duality beyond the leading 't Hooft limit.

Keywords: Conformal and W Symmetry, AdS-CFT Correspondence

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Contents

1 Introduction

Higher spin holography is a simplified example of Maldacena's celebrated anti de Sitter / conformal field theory (AdS/CFT) correspondence [1], promising interesting insights into the mechanisms underlying the duality. The bulk theory is a Vasiliev higher spin gauge theory on AdS [2], which has been argued to be dual to the singlet sector of vector-like CFTs [3–6]. Since both theories are in a perturbative regime, the duality can be checked and understood quite explicitly. Although the theories need not be supersymmetric, their symmetry algebras are large enough to constrain the theories considerably [7, 8].

A concrete realisation of this idea was proposed some time ago by Klebanov & Polyakov [9] (and shortly later generalised further in [10]). It relates a higher spin theory on AdS_4 to the singlet sector of the 3-dimensional $\mathrm{O}(N)$ vector model in the large N limit, for a recent review see e.g. [11]. More recently, the lower dimensional version of this duality, connecting a higher spin theory on AdS_3 to a 2-dimensional CFT, was conjectured in [12] (see [13] for a review). One advantage of these low dimensional theories is that they are comparatively well understood while avoiding no-go theorems of the Coleman-Mandula type as in [7]. The 2d CFTs are large N limits of minimal models about which much is known, while the bulk theory can be formulated as a Chern-Simons gauge theory based on $\mathrm{hs}[\mu]$ [14, 15] (see also [16]). Here $\mathrm{hs}[\mu]$ is an infinite-dimensional Lie algebra extending the $\mathfrak{sl}(2)$ gauge algebra of pure gravity on AdS_3. The resulting theory generically contains an infinite tower of higher spin gauge fields. However, for $\mu = N \in \mathbb{N}$, $\mathrm{hs}[\mu]$ reduces to $\mathfrak{su}(N)$, and the higher spin theory truncates to a theory containing higher spin gauge fields of spin $s = 2, \ldots, N$. In addition to the gauge fields, the theory also contains (one or two) complex scalar fields.

The classical asymptotic symmetry algebra of these higher spin theories was determined in [17–19], following the old analysis of Brown & Henneaux [20]. It is described by the non-linear Poisson algebra $\mathcal{W}^{\mathrm{cl}}_\infty[\mu]$, where 'non-linear' means that the Poisson brackets can in general only be expressed in terms of polynomials of the generators, see [21] for a review on $\mathcal{W}$ algebras. This asymptotic symmetry algebra can be realised as a classical Drinfel'd-Sokolov reduction of $\mathrm{hs}[\mu]$ [22, 23]. Furthermore, it was argued to agree [19] with the (semi-classical) 't Hooft limit of the $\mathcal{W}$ algebras of the coset models $\mathcal{W}_{N,k}$, provided one identifies μ with the 't Hooft parameter $\lambda = \frac{N}{N+k}$ that is held constant in the large N, k limit.

In order to make sense of a similar statement for finite N and k, one must first define the quantisation of $\mathcal{W}^{\mathrm{cl}}_\infty[\mu]$. This is non-trivial since the classical algebra is non-linear, and hence a naive replacement of Poisson brackets by commutators does not lead to a consistent Lie algebra (satisfying the Jacobi identity). It was recently shown in [24] how this problem can be overcome (by adjusting the structure constants), and how the resulting quantum algebra $\mathcal{W}_\infty[\mu]$ can be defined for arbitrary central charge; it can therefore be interpreted as the quantum Drinfel'd-Sokolov reduction of $\mathrm{hs}[\mu]$. It was furthermore shown in [24] that $\mathcal{W}_\infty[\mu]$ exhibits an intriguing triality relation that implies, among other things, that we have the equivalence of quantum algebras $\mathcal{W}_{N,k} \cong \mathcal{W}_\infty[\mu = \lambda]$ at the central charge $c = c_{N,k}$ of the minimal models. This relationship can be interpreted as a generalisation of the level-rank dualities of coset algebras proposed in [25, 26].

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In this paper we will extend these results to the higher spin theory on AdS_3 that contains only gauge fields of even spin. This is the natural analogue of the Klebanov & Polyakov proposal, which involves the smallest (or minimal) higher spin theory on AdS_4. For the case of AdS_3, the gauge symmetry can be described by a Chern-Simons theory based on a suitable subalgebra of $\mathrm{hs}[\mu]$, and it was argued in [27, 28] (see also [29]) that it should be dual to the $\mathrm{SO}(N)$ coset theories of the form

$$\frac{\mathfrak{so}(N)_k \oplus \mathfrak{so}(N)_1}{\mathfrak{so}(N)_{k+1}} .$$

While the (classical) asymptotic symmetry algebra of the bulk theory has not yet been determined explicitly, it is clear that it will be described by a classical $\mathcal{W}$ algebra that is generated by one field for every even spin $s = 2, 4, \ldots$. One expects on general grounds that it will be non-linear, and hence the quantisation will exhibit the same subtleties as described above. As a consequence, it is actually simpler to approach this problem by constructing directly the most general *quantum* $\mathcal{W}$ algebra $\mathcal{W}^e_\infty[\mu]$ with this spin content. As in the case of $\mathcal{W}_\infty[\mu]$, one finds that the successive Jacobi identities fix the structure constants of all commutators in terms of a single parameter γ, as well as the central charge. For a suitable identification of γ and μ, we can then think of these algebras as the quantum Drinfel'd-Sokolov reduction of some subalgebra of $\mathrm{hs}[\mu]$, which turns out to be the $\mathrm{hs}^e[\mu]$ algebra of [28]. However, compared to the $\mathcal{W}_\infty[\mu]$ analysis of [24], there is an unexpected subtlety in that there are two natural ways in which one may identify γ and μ at finite c — the two identifications agree in the quasiclassical $c \to \infty$ limit, but differ in their $1/c$ corrections. This reflects the fact that $\mathrm{hs}^e[\mu]$ truncates for $\mu = N$ to either $\mathfrak{sp}(N)$ (if N is even), or $\mathfrak{so}(N)$ (if N is odd), and that the Drinfel'd-Sokolov reduction of these non-simply-laced algebras are Langlands dual (rather than equivalent).

Since the quantum algebra $\mathcal{W}^e_\infty[\mu]$ is the most general $\mathcal{W}$ algebra with the given spin content, we can also identify the $\mathfrak{so}$ and $\mathfrak{sp}$ cosets (or rather their orbifolds) with these algebras. In this way we obtain again non-trivial identifications between quantum $\mathcal{W}^e_\infty$ algebras that explain and refine the holographic conjectures of [27] and [28], see eqs. (4.13) and (4.14) below. Furthermore, there are again non-trivial quantum equivalences between the algebras for different values of μ, which can be interpreted in terms of level-rank dualities of $\mathfrak{so}$ coset models that do not seem to have been noticed before, see eq. (4.11).

The paper is organised as follows. In section 2 we construct the most general quantum $\mathcal{W}^e_\infty$ algebra, and explain how the different structure constants can be determined recursively from the Jacobi identities. We also consider various truncations to finitely generated algebras that have been studied in the literature before (see section 2.2), and explain that the wedge algebra of $\mathcal{W}^e_\infty$ is indeed the $\mathrm{hs}^e[\mu]$ algebra of [28]. Section 3 is devoted towards identifying $\mathcal{W}^e_\infty[\mu]$ as a Drinfel'd-Sokolov reduction of $\mathrm{hs}^e[\mu]$. As in [24] the relation between the two algebras can be most easily analysed by studying some simple representations of the two algebras. It turns out that there is no canonical identification, but rather two separate choices that we denote by $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$, respectively; this nomenclature reflects the origin of this ambiguity, namely that $\mathrm{hs}^e[\mu]$ truncates to either $C_n = \mathfrak{sp}(2n)$ or $B_n = \mathfrak{so}(2n+1)$, depending on whether $\mu = N$ is even or odd.

In section 4 we apply these results to the actual higher spin holography. In particular, we show that the (subalgebras of the) $\mathfrak{so}$ cosets fit into this framework, and hence deduce the precise relation between $\mathcal{WB}_\infty[\mu]$ or $\mathcal{WC}_\infty[\mu]$ and the $\mathfrak{so}$ coset algebras at finite N and k. We comment on the fact that the matching of the partition functions requires that we consider a non-diagonal modular invariant with respect to the orbifold subalgebra of the $\mathfrak{so}$ cosets (see section 4.6). We also explain that the non-trivial identifications among the $\mathcal{W}^e_\infty$ algebras imply a level-rank duality for the $\mathfrak{so}$ cosets themselves, and that also the cosets based on $\mathfrak{sp}(2n)$ and $\mathfrak{osp}(1|2n)$ can be brought into the fold. Finally, we show that, as in [24], only one of the two real scalars in the bulk theory should be thought of as being perturbative.

Section 5 contains our conclusions as well as possible directions for future work. There are two appendices, where some of the more technical material has been collected.

2 The even spin algebra

2.1 Construction

In this section we analyse the most general $\mathcal{W}_\infty$ algebra $\mathcal{W}^e_\infty$ that is generated by the stress energy tensor L and one Virasoro primary field W^s for each even spin $s = 4, 6, \ldots$. As we shall see, the construction allows for one free parameter in addition to the central charge.

The strategy of our analysis is as follows. First we make the most general ansatz for the OPEs of the generating fields W^s with each other. In a second step we then impose the constraints that come from solving the various Jacobi identities. Actually, instead of working directly in terms of modes and Jacobi identities, it is more convenient to do this analysis on the level of the OPEs. Then the relevant condition is that the OPEs are associative.

2.1.1 Ansatz for OPEs

We know on general grounds that the conformal symmetry, i.e. the associativity of the OPEs involving the stress energy tensor L, fixes the coefficients of the Virasoro descendant fields in the OPEs in terms of the Virasoro primary fields. In order to make the most general ansatz we therefore only have to introduce free parameters for the coupling to the Virasoro primary fields. Thus we need to know how many Virasoro primary fields the algebra $\mathcal{W}^e_\infty$ contains. This can be determined by decomposing the vacuum character of $\mathcal{W}^e_\infty$

$$\chi_\infty(q) = \mathrm{Tr}_0 q^{L_0} = \prod_{s\in 2\mathbb{N}} \prod_{n=s}^{\infty} \frac{1}{1-q^n} = \chi_0(q) + \sum_{h=4}^{\infty} d(h)\chi_h(q)\,, \tag{2.1}$$

in terms of the Virasoro characters corresponding to the vacuum representation $\chi_0(q)$, and to a highest weight representation with conformal dimension h

$$\chi_0(q) = \prod_{n=2}^{\infty} \frac{1}{1-q^n}\,, \qquad \chi_h(q) = q^h \prod_{n=1}^{\infty} \frac{1}{1-q^n}\,. \tag{2.2}$$

Note that since we are working at a generic central charge, there are no Virasoro null-vectors. The coefficients $d(h)$ in (2.1) are then the number of Virasoro primary fields of conformal dimension h. Their generating function equals

$$P(q) = \sum_{h=4}^{\infty} d(h) q^h = (1-q)(\chi_{\rm HS}(q) - 1) = q^4 + q^6 + 2q^8 + 3q^{10} + q^{11} + 6q^{12} + \cdots \,, \quad (2.3)$$

where $\chi_{\rm HS}(q) = \chi_\infty(q)/\chi_0(q)$ denotes the contribution of the higher spin fields to the character χ_∞.

The most general ansatz for the OPEs is then

$$\begin{aligned}
W^4 \star W^4 &\sim c_{44}^6 W^6 + c_{44}^4 W^4 + n_4 I \,, \\
W^4 \star W^6 &\sim c_{46}^8 W^8 + a_{46}^8 A^8 + c_{46}^6 W^6 + c_{46}^4 W^4 \,, \\
W^4 \star W^8 &\sim a_{48}^{11} A^{11} + c_{48}^{10} W^{10} + a_{48}^{10,1} A^{10,1} + a_{48}^{10,2} A^{10,2} + c_{48}^8 W^8 + a_{48}^8 A^8 \\
&\qquad + c_{48}^6 W^6 + c_{48}^4 W^4 \,, \\
W^6 \star W^6 &\sim c_{66}^{10} W^{10} + a_{66}^{10,1} A^{10,1} + a_{66}^{10,2} A^{10,2} + c_{66}^8 W^8 + a_{66}^8 A^8 + c_{66}^6 W^6 \\
&\qquad + c_{66}^4 W^4 + n_6 I \,,
\end{aligned} \quad (2.4)$$

where we have only written out the contributions of the Virasoro primaries to the singular part of the OPEs. (As mentioned before, the conformal symmetry fixes the contributions of their Virasoro descendants uniquely.) Furthermore, A^8, $A^{10,1}$, $A^{10,2}$, A^{11} are the composite primary fields at level 8, 10, 11, respectively, as predicted by (2.3). They are of the form

$$\begin{aligned}
A^8 &= \left(W^4\right)^2 + \cdots \,, & A^{10,1} &= W^{4\prime\prime} W^4 - \frac{9(48+c)}{8(64+c)} W^{4\prime} W^{4\prime} + \cdots \,, \\
A^{10,2} &= W^4 W^6 + \cdots \,, & A^{11} &= W^4 W^{6\prime} - \frac{3}{2} W^{4\prime} W^6 + \cdots \,.
\end{aligned} \quad (2.5)$$

Here the ellipses denote Virasoro descendants that have to be added in order to make these fields primary.

2.1.2 Structure constants

Next we want to determine the structure constants appearing in (2.4) by requiring the associativity of the multiple OPEs $W^{s_1} \star W^{s_2} \star W^{s_3}$. Note that in this calculation, we need to work with the full OPEs, rather than just their singular parts. The full OPE is in principle uniquely determined by its singular part, but the actual calculation is somewhat tedious. To do these computations efficiently we have therefore used the Mathematica packages `OPEdefs` and `OPEconf` of Thielemans that are described in some detail in [30, 31].[1]

More explicitly, we start by defining the OPE $W^4 \star W^4$ by the first line of (2.4), which does not contain any composite fields. We can then use this ansatz to compute the composite field A^8, and thus make an ansatz for the OPE $W^4 \star W^6$. At this step, we can already check the associativity of $W^4 \star W^4 \star W^4$, using the built-in-function `OPEJacobi`.

The next step consists of computing the composite fields made from W^4 and W^6, i.e. the remaining composite fields in (2.5). Then we can make an ansatz for the remaining OPEs in (2.4), and check the associativity of $W^4 \star W^4 \star W^6$.

[1] We thank K. Thielemans for providing us with the packages.

It should now be clear how we continue: in each step we first compute all the composite primary fields made of products of fundamental fields whose OPE we have already determined. This then allows us to make an ansatz for the next 'level' of OPEs. Then we can check the associativity of those triple products where all intermediate OPEs are known. Proceeding in this manner, we have computed the constraints arising from the associativity of the OPEs $W^{s_1} \star W^{s_2} \star W^{s_3}$ up to the total level $s_1 + s_2 + s_3 \leq 16$. The resulting relations are (for the sake of brevity we only give the explicit expressions up to total spin $s_1 + s_2 + s_3 \leq 14$ that can be calculated from the OPEs given explicitly in (2.4))

$$
\begin{aligned}
n_4 &= \frac{c(c-1)(c+24)(5c+22)}{12(2c-1)(7c+68)^2}\left(c_{46}^6\right)^2 - \frac{7c(c-1)(5c+22)}{72(2c-1)(7c+68)} c_{46}^6 c_{44}^4 + \frac{c(5c+22)}{72(c+24)}\left(c_{44}^4\right)^2 ,\\
c_{44}^6 c_{46}^4 &= -\frac{8(c-1)(c+24)(5c+22)\left(c^2-172c+196\right)}{(2c-1)^2(7c+68)^3}\left(c_{46}^6\right)^2 + \frac{28(c-1)(5c+22)\left(c^2-172c+196\right)}{3(2c-1)^2(7c+68)^2} c_{44}^4 c_{46}^6\\
&\quad + \frac{4(c-1)(5c+22)(11c+656)}{9(c+24)(2c-1)(7c+68)}\left(c_{44}^4\right)^2 ,\\
c_{46}^8 a_{48}^{11} &= \left(\frac{888}{65c+2580} - \frac{40}{7c+68}\right) c_{46}^6 - \frac{2(13c+918)}{65c+2580} c_{44}^6 a_{46}^8 + \frac{224}{15(c+24)} c_{44}^4 ,\\
c_{48}^8 &= \frac{192-31c}{26c+1032} c_{44}^6 a_{46}^8 + \frac{8(c(33c+1087)+11760)}{(7c+68)(13c+516)} c_{46}^6 - 2c_{44}^4 ,\\
c_{44}^6 c_{46}^8 a_{48}^8 &= \frac{192-31c}{26c+1032}\left(c_{44}^6\right)^2\left(a_{46}^8\right)^2 - \frac{4\left(165c^3+10763c^2+140036c+38568\right)}{3(c+24)(c+31)(55c-6)} c_{44}^4 c_{44}^6 a_{46}^8\\
&\quad + \frac{8(c(33c+1087)+11760)}{(7c+68)(13c+516)} c_{44}^6 a_{46}^8 c_{46}^6 - \frac{896(3c+46)(5c+3)((c-172)c+196)}{(c+31)(2c-1)(7c+68)^2(55c-6)}\left(c_{46}^6\right)^2\\
&\quad + \frac{3136(3c+46)(5c+3)((c-172)c+196)}{3(c+24)(c+31)(2c-1)(7c+68)(55c-6)} c_{44}^4 c_{46}^6 + \frac{448(3c+46)(5c+3)(11c+656)}{9(c+24)^2(c+31)(55c-6)}\left(c_{44}^4\right)^2 ,\\
c_{46}^8 c_{48}^6 &= -\frac{35(c+50)(2c-1)(7c+68)}{3(c+24)(c+31)(55c-6)} c_{44}^4 c_{44}^6 a_{46}^8 + \frac{8\left(25c^3+615c^2-88272c+102332\right)}{3(c+24)(c+31)(55c-6)} c_{44}^4 c_{46}^6\\
&\quad + \frac{16\left(425c^4+15145c^3+233766c^2+6507708c-7565544\right)}{(c+31)(7c+68)(13c+516)(55c-6)}\left(c_{46}^6\right)^2 + \frac{604-4c}{13c+516} c_{44}^6 c_{46}^6 a_{46}^8\\
&\quad + \frac{7840(c+50)(2c-1)(7c+68)}{9(c+24)^2(c+31)(55c-6)}\left(c_{44}^4\right)^2 ,\\
c_{44}^6 c_{46}^8 c_{48}^4 &= -\frac{(c-1)(c+24)(5c+22)\left(65c^4+8637c^3+364470c^2+2897944c+36384\right)}{2(2c-1)(3c+46)(5c+3)(7c+68)^2(13c+516)} c_{44}^6 a_{46}^8\left(c_{46}^6\right)^2\\
&\quad + \frac{7(c-1)(5c+22)\left(65c^4+8637c^3+364470c^2+2897944c+36384\right)}{12(2c-1)(3c+46)(5c+3)(7c+68)(13c+516)} c_{44}^6 a_{46}^8 c_{44}^4 c_{46}^6\\
&\quad - \frac{5(c-1)(c+50)(5c+22)\left(715c^4+90933c^3+2851076c^2+21154896c+6967008\right)}{12(c+24)(c+31)(3c+46)(5c+3)(13c+516)(55c-6)}\left(c_{44}^4\right)^2 c_{44}^6 a_{46}^8\\
&\quad - \frac{32(c-151)(c-1)(c+24)(5c+22)\left(c^2-172c+196\right)}{(2c-1)^2(7c+68)^3(13c+516)}\left(c_{46}^6\right)^3\\
&\quad - \frac{56(c-1)(5c+22)\left(c^2-172c+196\right)\left(20c^3+24807c^2+765640c-185172\right)}{3(c+31)(2c-1)^2(7c+68)^2(13c+516)(55c-6)} c_{44}^4\left(c_{46}^6\right)^2\\
&\quad + \frac{4(c-1)(5c+22)\left(5605c^4-408494c^3-70820464c^2-1703657536c+1312613664\right)}{9(c+24)(c+31)(2c-1)(7c+68)(13c+516)(55c-6)}\left(c_{44}^4\right)^2 c_{46}^6\\
&\quad + \frac{140(c-1)(c+50)(5c+22)(11c+656)}{27(c+24)^2(c+31)(55c-6)}\left(c_{44}^4\right)^3 ,\\
c_{44}^6 c_{66}^{10} &= \frac{3}{4} c_{46}^8 c_{48}^{10} ,\\
\left(c_{44}^6\right)^2 a_{66}^{10,1} &= \frac{3}{4} c_{44}^6 c_{46}^8 a_{48}^{10,1} + \frac{5(c+64)(c+76)(5c+22)(11c+232)}{3(c+24)(c+31)(17c+944)(55c-6)} c_{44}^6 a_{46}^8 c_{44}^4\\
&\quad + \frac{1120(c+64)(11c+656)(47c-614)}{9(c+24)^2(c+31)(17c+944)(55c-6)}\left(c_{44}^4\right)^2
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{7840(c+64)(47c-614)\left(c^2-172c+196\right)}{3(c+24)(c+31)(2c-1)(7c+68)(17c+944)(55c-6)}c_{46}^{6}c_{44}^{4}\\
&-\frac{2240(c+64)(47c-614)\left(c^2-172c+196\right)}{(c+31)(2c-1)(7c+68)^2(17c+944)(55c-6)}\left(c_{46}^{6}\right)^2 ,\\
c_{44}^{6}a_{66}^{10,2} &= \frac{6(c+64)(13c+248)}{(13c{+}516)(17c{+}944)}c_{44}^{6}a_{46}^{8}+\frac{192(c+64)(81c+1274)}{(7c{+}68)(13c{+}516)(17c{+}944)}c_{46}^{6}-\frac{224(c+64)}{(c{+}24)(17c{+}944)}c_{44}^{4}\\
&+\frac{3}{4}c_{46}^{8}a_{48}^{10,2} ,\\
c_{44}^{6}c_{66}^{8} &= \frac{4(4c+61)}{7c+68}c_{46}^{8}c_{46}^{6}-\frac{(11c+656)}{6(c+24)}c_{46}^{8}c_{44}^{4} ,\\
\left(c_{44}^{6}\right)^2 a_{66}^{8} &= -\frac{11c+656}{6(c+24)}c_{44}^{4}c_{44}^{6}a_{46}^{8}+\frac{784((c-172)c+196)}{3(c+24)(2c-1)(7c+68)}c_{44}^{4}c_{46}^{6}-\frac{224((c-172)c+196)}{(2c-1)(7c+68)^2}\left(c_{46}^{6}\right)^2\\
&+\frac{4(4c+61)}{7c+68}c_{46}^{6}c_{44}^{6}a_{46}^{8}+\frac{112(11c+656)}{9(c+24)^2}\left(c_{44}^{4}\right)^2 ,\\
c_{44}^{6}c_{66}^{6} &= \frac{20\left(92c^5+2389c^4+39632c^3+4060c^2-212032c+193984\right)}{(2c-1)^2(7c+68)^3}\left(c_{46}^{6}\right)^2\\
&+\frac{10\left(28c^5-5425c^4-525974c^3+387728c^2+3726976c-3870208\right)}{9(c+24)(2c-1)^2(7c+68)^2}c_{46}^{6}c_{44}^{4}\\
&-\frac{20\left(13c^4-1637c^3-113622c^2+32168c+859328\right)}{27(c+24)^2(2c-1)(7c+68)}\left(c_{44}^{4}\right)^2 ,\\
\left(c_{44}^{6}\right)^2 c_{66}^{4} &= -\frac{8(c-1)(c+24)(5c+22)((c-172)c+196)}{(1-2c)^2(7c+68)^3}\left(c_{46}^{6}\right)^3\\
&+\frac{28(c-1)(5c+22)((c-172)c+196)}{3(1-2c)^2(7c+68)^2}c_{44}^{4}\left(c_{46}^{6}\right)^2+\frac{4(c-1)(5c+22)(11c+656)}{9(c+24)(2c-1)(7c+68)}\left(c_{44}^{4}\right)^2c_{46}^{6} ,\\
\left(c_{44}^{6}\right)^2 n_6 &= -\frac{2(c{-}1)^2c(c{+}24)^2(5c{+}22)^2((c{-}172)c{+}196)}{3(2c-1)^3(7c+68)^5}\left(c_{46}^{6}\right)^4+\frac{(c{-}1)c(5c{+}22)^2(11c{+}656)}{162(c{+}24)^2(2c{-}1)(7c{+}68)}\left(c_{44}^{4}\right)^4\\
&+\frac{14(c-1)^2c(c+24)(5c+22)^2((c-172)c+196)}{9(2c-1)^3(7c+68)^4}c_{44}^{4}\left(c_{46}^{6}\right)^3\\
&-\frac{(c-1)c(5c+22)^2(c(c(17c-13105)+25330)-12092)}{54(2c-1)^3(7c+68)^3}\left(c_{44}^{4}\right)^2\left(c_{46}^{6}\right)^2\\
&-\frac{7(c-1)c(5c+22)^2(c(8c+1161)-1244)}{162(1-2c)^2(c+24)(7c+68)^2}\left(c_{44}^{4}\right)^3c_{46}^{6} .
\end{aligned}
\tag{2.6}
$$

Let us comment on the implications of these results. Of the 23 structure constants that appear in (2.4), 8 remain undetermined by the above relations; for example, a convenient choice for the free structure constants is n_4, n_6, c_{46}^{8}, c_{48}^{10}, as well as a_{46}^{8}, $a_{48}^{10,1}$, $a_{48}^{10,2}$ and c_{44}^{4}. The first 4 of these just account for the freedom to normalise the fields W^4, W^6, W^8 and W^{10}, respectively. The appearance of a_{46}^{8}, $a_{48}^{10,1}$, and $a_{48}^{10,2}$ also has a simple interpretation: it reflects the freedom to redefine W^8 and W^{10} by adding to them composite fields of the same spin. For example, we can set $\hat{a}_{46}^{8}=0$ by redefining $W^8\mapsto\hat{W}^8\equiv W^8+a_{46}^{8}/c_{46}^{8}A^8$, and similarly in the other two cases. Note that this freedom implies that the relations (2.6) must satisfy interesting consistency conditions. For example, if we redefine $\hat{W}^8$ in this manner, the structure constant a_{48}^{11} in the OPE $W^4\star\hat{W}^8$ becomes $\hat{a}_{48}^{11}=a_{48}^{11}+\frac{a_{46}^{8}}{c_{46}^{8}}\frac{2(13c+918)}{65c+2580}c_{44}^{6}$, which then satisfies indeed the third equation of (2.6) with $\hat{a}_{46}^{8}=0$.

Thus, at least up to the level to which we have analysed the Jacobi identities and up to field redefinitions, all structure constants are completely fixed in terms of c and the single fundamental structure constant c_{44}^{4}. Note that for a given choice of n_4, the sign of c_{44}^{4} is not determined since n_4 only fixes the normalisation of W^4 up to a sign. It seems reasonable

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to believe that this structure will continue, i.e. that all remaining structure constants are also uniquely fixed (up to field redefinitions) in terms of the central charge c and

$$\gamma = \left(c_{44}^{4}\right)^{2} . \tag{2.7}$$

The situation is then analogous to what was found for $\mathcal{W}_\infty[\mu]$ in [24], and for $s\mathcal{W}_\infty[\mu]$ in [32]: the resulting algebra depends on one free parameter (in addition to the central charge c), and whenever we want to emphasise this dependence, we shall denote it by $\mathcal{W}_\infty^e(\gamma)$.

In a next step we want to relate $\mathcal{W}_\infty^e(\gamma)$ to the Drinfel'd-Sokolov reduction of $\mathrm{hs}^e[\mu]$. Before doing so, we can however already perform some simple consistency checks on the above analysis.

2.2 Truncations

Since our ansatz is completely general, it should also reproduce the various finite even $\mathcal{W}$ algebras that have been constructed in the literature before [33, 34]. More specifically, we can study for which values of γ, $\mathcal{W}_\infty^e$ develops an ideal such that the resulting quotient algebra becomes a finite $\mathcal{W}$ algebra.

2.2.1 The algebra $\mathcal{W}(2,4)$

The simplest case is the so-called $\mathcal{W}(2,4)$ algebra, which is generated by a single Virasoro primary field W^4 in addition to the stress energy tensor. Thus we need to find the value of γ for which W^6, W^8, etc. lie in an ideal. Imposing c_{46}^4, c_{66}^4, c_{48}^4, and n_6 to vanish we obtain

$$\gamma = \frac{216(c+24)(c^2-172c+196)n_4}{c(2c-1)(7c+68)(5c+22)} . \tag{2.8}$$

The resulting quotient algebra is then in agreement with e.g. [33]. Note that A^8 does not lie in the ideal since the OPE of W^4 with A^8 contains terms proportional to W^4 which are non-vanishing for generic c. Thus we also need to require that $a_{46}^8 = 0$ (but c_{46}^8 need not be zero), which is automatically true by the above conditions. We have also analysed the consistency of the resulting algebra directly, i.e. repeating essentially the same calculation as in [33].

2.2.2 The algebras $\mathcal{W}(2,4,6)$

The next simplest case is the so-called $\mathcal{W}(2,4,6)$ algebra, which should appear from $\mathcal{W}_\infty^e$ upon dividing out the ideal generated by W^8, W^{10}, etc. This requires that we set c_{48}^4, c_{48}^6 and n_8 to zero. Furthermore, since the composite fields A^8, $A^{10,1}$, $A^{10,2}$ and A^{11} have a non-trivial image in the quotient (for generic c), we should expect that also a_{48}^8, $a_{48}^{10,1}$, $a_{48}^{10,2}$ and a_{48}^{11} vanish. Solving eqs. (2.6) together with these constraints then yields the two values for γ

$$\begin{aligned}\gamma = 2n_4\Big[&(18025c^6+1356090c^5+16727763c^4-537533674c^3\\&\quad -5470228116c^2+8831442312c-300564000)\\&\quad \pm(c-1)(5c+22)^2(11c+444)(13c+918)\sqrt{c^2-534c+729}\Big]\\&\times\left[c(2c-1)(3c+46)(4c+143)(5c+3)(5c+22)(5c+44)\right]^{-1} .\end{aligned} \tag{2.9}$$

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Up to a factor of $\frac{1}{2}$, this agrees with two of the four solutions found in [33]; incidentally, they are the ones which were claimed to be inconsistent in [35]. We have again also analysed the consistency of the resulting algebra directly, i.e. by working with an ansatz involving only W^4 and W^6.

As a matter of fact, there are two additional solutions that appear if we enlarge the ideal by also taking A^{11} to be part of it. Then we do not need to impose that $a^{11}_{48} = 0$, and the resulting algebras agree with the other two solutions[2] of [33], i.e. they are characterised by

$$\gamma = -\frac{4(5c^2 + 309c - 14)^2 n_4}{c(c-26)(5c+3)(5c+22)} \tag{2.10}$$

and

$$\gamma = \frac{216(10c^2 + 47c - 82)^2 n_4}{c(4c+21)(5c+22)(10c-7)}\,, \tag{2.11}$$

respectively. In the case of (2.10), the OPEs of W^4 and W^6 with A^{11} show that no additional field of dimension smaller than 11 needs to be included in the ideal. However, in the case of the algebra described by (2.11), the ideal also contains a certain linear combination of $A^{10,1}$ and $A^{10,2}$.

2.3 Identifying the wedge algebra

We expect from the analysis of [28] that $\mathcal{W}^e_\infty(\gamma)$ should arise as the Drinfel'd-Sokolov reduction of the even higher spin algebra. However, as was also explained in [28], it is not clear which higher spin algebra is relevant in this context, and two possibilities, $\mathrm{hs}^e[\mu]$ and $\mathrm{hso}[\mu]$, were proposed. In order to decide which of the two algebras is relevant, it is sufficient to determine the wedge algebra of $\mathcal{W}^e_\infty(\gamma)$, i.e. the algebra that is obtained by restricting the modes to the wedge $|m| < s$, and taking $c \to \infty$. (The reason for this is that restricting to the wedge algebra is in a sense the inverse of performing the Drinfel'd-Sokolov reduction, see [19, 36] for a discussion of this point.) As it turns out, the wedge algebra commutators of $\mathcal{W}^e_\infty(\gamma)$, as obtained from (2.4) together with (2.6), agree with the $\mathrm{hs}^e[\mu]$ commutators (B.1) of appendix B provided we identify $c^4_{44} = \sqrt{\gamma}$ with μ as

$$c^4_{44} = \frac{12}{\sqrt{5}}(\mu^2 - 19) + \mathcal{O}(c^{-1})\,. \tag{2.12}$$

Furthermore, we normalise our fields as

$$\begin{aligned} n_4 &= c(\mu^2-9)(\mu^2-4)\,, & n_6 &= c(\mu^2-25)(\mu^2-16)(\mu^2-9)(\mu^2-4)\,, \\ c^8_{46} &= -8\sqrt{\frac{210}{143}}\,, & c^{10}_{48} &= -20\sqrt{\frac{6}{17}}\,, \end{aligned} \tag{2.13}$$

and take the field redefinition parameters to be $a^8_{46} = a^{10,1}_{48} = a^{10,2}_{48} = 0$. Thus we conclude that the $\mathcal{W}^e_\infty(\gamma)$ algebra can be interpreted as the quantum Drinfel'd-Sokolov reduction of $\mathrm{hs}^e[\mu]$, where μ and γ are related as in (2.12); this will be further elaborated on in section 3.

[2] Incidentally, there is a typo in [33]: the structure constant a^8_{46} in the $\mathcal{W}(2,4,6)$ algebra satisfying (2.11) should be given by $(a^8_{46})^2 = \frac{256(2c-1)(5c+3)^2(3c+46)^2(7c+68)n_6}{(31c-192)^2(c+11)(14c+11)(5c+22)(c+24)n_4}$.

We should mention in passing that $\mathrm{hs}^e[\mu]$ and $\mathrm{hso}[\mu']$ are not isomorphic (even allowing for some general relation between μ and μ'), since they possess different finite-dimensional quotient algebras, see [28]. Thus the above analysis also proves that the wedge algebra of $\mathcal{W}_\infty^e$ is *not* isomorphic to $\mathrm{hso}[\mu]$ for any μ, and hence that $\mathcal{W}_\infty^e$ is *not* the quantum Drinfel'd-Sokolov reduction of $\mathrm{hso}[\mu]$ for any μ.

2.4 Minimal representation

Our next aim is to determine the exact c dependence of (2.12). This can be done using the same trick as in [24] and [32], following the original analysis of [37]. The main ingredient in this analysis is a detailed understanding of the structure of the 'minimal representations' of $\mathcal{W}_\infty^e$. Recall that the duality of [28] suggests that $\mathcal{W}_\infty^e$ possesses two minimal representations whose character is of the form

$$\chi_{\mathrm{min}}(q) = \frac{q^h}{1-q} \prod_{s\in 2\mathbb{N}} \prod_{n=s}^{\infty} \frac{1}{1-q^n} , \tag{2.14}$$

and for which h is finite in the 't Hooft limit. It follows from this character formula that the corresponding representation has (infinitely) many low-lying null-vectors; this will allow us to calculate h as a function of c and γ.

Let us denote the primary field of the minimal representation by P^0. First, we need to make the most general ansatz for the OPEs $W^s \star P^0$. In order to do so we have to enumerate the number of Virasoro primary states in the minimal $\mathcal{W}_\infty^e$ representation. Decomposing (2.14) in terms of irreducible Virasoro characters as

$$\chi_{\mathrm{min}}(q) = \sum_{n=0}^{\infty} d_{\mathrm{min}}(n)\, \chi_{h+n}(q) , \tag{2.15}$$

where $\chi_h(q)$ was defined in (2.2), $d_{\mathrm{min}}(n)$ equals then the multiplicity of the Virasoro primaries of conformal dimension $h+n$. It follows from (2.15) that the corresponding generating function is

$$\sum_{n=0}^{\infty} d_{\mathrm{min}}(n)\, q^n = \prod_{s=2}^{\infty} \prod_{n=2s}^{\infty} \frac{1}{1-q^n} = 1 + q^4 + q^5 + 2q^6 + \cdots . \tag{2.16}$$

Then the most general ansatz for the OPEs $W^4 \star P^0$ and $W^6 \star P^0$ is

$$W^4 \star P^0 \sim w^4 P^0 , \qquad W^6 \star P^0 \sim w^6 P^0 + a^4 P^4 + a^5 P^5 , \tag{2.17}$$

where P^4 and P^5 are the primary fields of conformal dimension $h+4$ and $h+5$, respectively. Note that these fields are unique, as follows from (2.16); explicitly, they are of the form

$$P^4 = W^4 P^0 + \cdots , \qquad P^5 = \frac{h}{4+h} W^{4\prime} P^0 - \frac{4}{4+h} W^4 P^{0\prime} + \cdots , \tag{2.18}$$

where the ellipses stand for Virasoro descendants that are required to make these fields primary. As in [32], the condition that P^0 defines a representation of $\mathcal{W}_\infty^e$ is now equivalent to the constraint that all OPEs $W^{s_1} \star W^{s_2} \star P^0$ are associative. While we cannot test all of these conditions, imposing the associativity of $W^4 \star W^4 \star P^0$ implies already

$$\begin{aligned}
w^4 &= \frac{12h\left(c^2(9-2(h-1)h)+3c(h((49-12h)h-40)+2)-2h(h(12h+5)-14)\right)}{(c(h-2)(2h-3)+h(4h-5))} \\
&\quad\times\frac{n_4}{c(5c+22)c_{44}^4}\,, \\
w^6 &= \frac{8(c-1)(5c+22)h\left(c(h+2)+15h^2-26h+8\right)(c(2h+3)+4h(12h-7))n_4}{3c(c+24)(2c-1)(7c+68)(c(h-2)(2h-3)+h(4h-5))c_{44}^6}\,, \\
a^4 &= \frac{16(5c+22)(4h-9)\left(c(h+2)+15h^2-26h+8\right)}{(c+24)\left((c-7)h+c+3h^2+2\right)(2ch+c+2h(8h-5))c_{44}^6}w^4\,, \\
a^5 &= \frac{20(5c+22)(h-4)(h-1)(c(2h+3)+4h(12h-7))}{(c+24)h\left((c-7)h+c+3h^2+2\right)(2ch+c+2h(8h-5))c_{44}^6}w^4\,,
\end{aligned} \tag{2.19}$$

up to a sign ambiguity of the self-coupling $c_{44}^4 = \pm\sqrt{\gamma}$. Furthermore, the conformal dimension h is determined by the equation

$$\begin{aligned}
\gamma &= \frac{[c^2(-9-2h+2h^2)+3c(-2+40h-49h^2+12h^3)+2h(-14+5h+12h^2)]^2}{[c(1+h)+2-7h+3h^2][(1+2h)c-10h+16h^2][(6-7h+2h^2)c-5h+4h^2]} \\
&\quad\times\frac{144n_4}{c(5c+22)}\,.
\end{aligned} \tag{2.20}$$

Given γ and c, this is a sextic equation for h. We also note that our result is consistent with the one obtained in [37]. Moreover, we have checked that we arrive at the same result using commutators instead of OPEs; this calculation, which is analogous to the one performed in [24] for the algebra $\mathcal{W}_\infty[\mu]$, is presented in appendix A.

We should stress that the above constraints are necessary conditions for the minimal representation to exist, but do not prove that they are actually compatible with the full $\mathcal{W}_\infty^e$ structure. Furthermore, since we have only used the low-lying OPEs, our analysis actually holds for *any* algebra of type $\mathcal{W}(2,4,\dots)$ with no simple field of spin 5, and for any representation whose character coincides with (2.14) up to q^5, see also [37].

3 Drinfel'd-Sokolov reductions

As we have seen in section 2.3, the wedge algebra of $\mathcal{W}_\infty^e(\gamma)$ is $\mathrm{hs}^e[\mu]$, where $\gamma = (c_{44}^4)^2$ is identified with a certain function of μ, see eq. (2.12). Thus we should expect that the quantum $\mathcal{W}_\infty^e[\mu]$ algebras (where we now label $\mathcal{W}_\infty^e$ in terms of μ rather than γ) can be thought of as being the Drinfel'd-Sokolov reduction of $\mathrm{hs}^e[\mu]$. Actually, as we shall shortly see, the situation is a little bit more complicated. The subtlety we are about to encounter is related to the fact that $\mathrm{hs}^e[\mu]$ is in some sense a non-simply-laced algebra.[3]

Since Drinfel'd-Sokolov reductions of infinite-dimensional Lie algebras are complicated, we shall first (as in [24]) consider the special cases when μ is a positive integer. Then

[3] Note, however, the situation is also complicated by the fact that $\mathrm{hs}^e[\mu]$ is infinite dimensional, and the construction of [21] only applies to finite-dimensional Lie algebras. On the other hand, given that things worked nicely [24] for the infinite-dimensional algebra $\mathrm{hs}[\mu]$, we suspect that the infinite-dimensionality of $\mathrm{hs}^e[\mu]$ is not the origin of the subtlety.

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$\mathrm{hs}^e[\mu]$ can be reduced to finite-dimensional Lie algebras; indeed, as was already explained in [28], we have

$$\mathrm{hs}^e[N]/\chi_N = \begin{cases} \mathfrak{so}(N) & \text{for } N \text{ odd} \\ \mathfrak{sp}(N) & \text{for } N \text{ even,} \end{cases} \tag{3.1}$$

where χ_N is the ideal of $\mathrm{hs}^e[\mu]$ that appears for $\mu = N \in \mathbb{N}$. Note that in both cases, the resulting algebra is non-simply-laced, suggesting that $\mathrm{hs}^e[\mu]$ should be thought of as being non-simply-laced itself.

As in [24] we should now expect that the quantum Drinfel'd-Sokolov reduction of $\mathrm{hs}^e[\mu]$ agrees, for $\mu = N$, with the quantum Drinfel'd-Sokolov reduction of $B_n = \mathfrak{so}(2n+1)$ or $C_n = \mathfrak{sp}(2n)$, respectively. The representation theory of these $\mathcal{WB}_n$ and $\mathcal{WC}_n$ algebras is well known, and thus, at least for these integer values of μ, we can compare the conformal dimension of the corresponding minimal representations with what was determined above, see eq. (2.20). This will allow us to deduce an exact relation between γ and $\mu = N$ (for all values of the central charge). Analytically continuing the resulting expression to non-integer μ should then lead to the precise relation between γ and μ, for all values of μ.

3.1 The B_n series approach

According to [21], the Drinfel'd-Sokolov reduction of $\mathfrak{so}(2n+1)$, which we shall denote by $\mathcal{WB}_n$, is an algebra of type $\mathcal{W}(2, 4, \ldots, 2n)$ with central charge

$$c_B = n - 12|\alpha_+\rho_B + \alpha_-\rho_B^\vee|^2 \tag{3.2}$$

and spectrum

$$h_\Lambda = \frac{1}{2}(\Lambda, \Lambda + 2\alpha_+\rho_B + 2\alpha_-\rho_B^\vee)\,, \qquad \Lambda \in \alpha_+P_+ + \alpha_-P_+^\vee\,. \tag{3.3}$$

Here ρ_B and $\rho_B^\vee$ are the $\mathfrak{so}(2n+1)$ Weyl vector and covector, respectively, and P_+ and $P_+^\vee$ are the lattices of $\mathfrak{so}(2n+1)$ dominant weights and coweights, respectively. We work with the convention that the long roots have length squared equal to 2, and $\alpha_\pm$ are defined by

$$\alpha_- = -\sqrt{k_B + 2n - 1}\,, \qquad \alpha_+ = \frac{1}{\sqrt{k_B + 2n - 1}}\,, \tag{3.4}$$

so that $\alpha_+\alpha_- = -1$. Furthermore k_B is the level that appears in the Drinfel'd-Sokolov reduction. Note that the dual Coxeter number of $\mathfrak{so}(2n+1)$ equals $g_B = 2n-1$. Plugging in the expressions for $\alpha_\pm$ into (3.2), the central charge of $\mathcal{WB}_n$ takes the form

$$\begin{aligned} c_B(\mu, k_B) &= -\frac{n[k_B(2n+1) + 4n^2 - 2n][2k_B(n+1) + 4n^2 - 3]}{k_B + 2n - 1} \\ &= \frac{(1-\mu)(k_B\mu + \mu^2 - 3\mu + 2)[k_B(1+\mu) + \mu^2 - 2\mu - 2]}{2(k_B + \mu - 2)}\,, \end{aligned} \tag{3.5}$$

where in the second line we have replaced $n = \frac{\mu-1}{2}$. The minimal representations of $\mathcal{WB}_n$ arise for $\Lambda = \Lambda_+ = \alpha_+\mathrm{f}$, and $\Lambda = \Lambda_- = \alpha_-\mathrm{f}^\vee$, where f is the highest weight of the fundamental $\mathfrak{so}(2n+1)$ representation, and $\mathrm{f}^\vee$ the corresponding coweight. The conformal dimensions of these two representations are

$$h_+ = h_{\Lambda_+} = -\frac{n(k_B+2n-2)}{k_B+2n-1}\,, \qquad h_- = h_{\Lambda_-} = k_B\left(n+\frac{1}{2}\right) + n(2n-1)\,. \tag{3.6}$$

They are both solutions of eq. (2.20), provided $\gamma = \gamma_B(\mu, k_B)$ with γ_B equal to

$$\begin{aligned}
\gamma_B =\,&144\,(240 - 420k_B + 210k_B^2 - 30k_B^3 + 1188\mu - 1520k_B\mu + 773k_B^2\mu - 190k_B^3\mu \\
&+ 19k_B^4\mu - 2138\mu^2 + 2237k_B\mu^2 - 818k_B^2\mu^2 + 107k_B^3\mu^2 + 264\mu^3 + 614k_B\mu^3 \\
&- 703k_B^2\mu^3 + 220k_B^3\mu^3 - 20k_B^4\mu^3 + 1107\mu^4 - 1516k_B\mu^4 + 615k_B^2\mu^4 - 75k_B^3\mu^4 \\
&- 644\mu^5 + 462k_B\mu^5 - 51k_B^2\mu^5 - 12k_B^3\mu^5 + k_B^4\mu^5 + 67\mu^6 + 43k_B\mu^6 - 36k_B^2\mu^6 \\
&+ 4k_B^3\mu^6 + 39\mu^7 - 36k_B\mu^7 + 4k_B\mu^8 + \mu^9 + 6k_B^2\mu^7 - 12\mu^8)^2\, n_4 \,/\, \Big[c_B(\mu-3) \\
&\times (3k_B + k_B\mu - 6 + \mu^2)(8 - 2k_B - 5\mu + k_B\mu + \mu^2)(1 - k_B - 4\mu + k_B\mu + \mu^2) \\
&\times (1 - 3\mu + k_B\mu + \mu^2)(k_B + k_B\mu - 4 - 2\mu + \mu^2)(2k_B - 2 - \mu + k_B\mu + \mu^2) \\
&\times (108 - 54k_B - 74\mu + 25k_B\mu - 5k_B^2\mu - 20\mu^2 + 5k_B\mu^2 + 55\mu^3 - 30k_B\mu^3 \\
&\quad + 5k_B^2\mu^3 - 30\mu^4 + 10k_B\mu^4 + 5\mu^5)\Big]\,,
\end{aligned} \tag{3.7}$$

where we have again replaced $n = \frac{\mu-1}{2}$. For each μ, we therefore obtain a family of $\mathcal{W}^e_\infty$ algebras that depend on k_B; these algebras will be denoted by $\mathcal{WB}_\infty[\mu]$ (where we suppress the explicit k_B dependence). Note that, for fixed μ, these algebras really depend on k_B, rather than just on c_B: for a fixed c and μ, there are always two solutions $k_B^{(i)}$, $i = 1, 2$, for $c = c(\mu, k_B^{(i)})$, see (3.5). However, in general the corresponding γ values do not agree, $\gamma_B(\mu, k_B^{(1)}) \neq \gamma_B(\mu, k_B^{(2)})$, and hence the two solutions for k_B do not lead to isomorphic $\mathcal{W}^e_\infty$ algebras. This is different than what happened for $\mathcal{W}_\infty$ in [24], and closely related to the fact that $\mathrm{hs}^e[\mu]$ is non simply-laced, see below.

By construction, the algebras $\mathcal{WB}_\infty[\mu]$ truncate, for $\mu = 2n+1$, to $\mathcal{WB}_n$. (Note that also $\mathcal{WB}_n$ depends actually on the level k_B, and not just on c.) We have also checked that, for $n = 2$, $\gamma_B(2n+1, k_B)$ agrees with the γ given in eq. (2.8) at $c = c_B(2n+1, k_B)$. Similarly, for $n = 3$, $\gamma_B(2n+1, k_B)$ agrees with the γ of eq. (2.9) at $c = c_B(2n+1, k_B)$. (For $n = 3$, the two algebras corresponding to the two different solutions for k_B correspond to the choice of the branch cut in the square root of eq. (2.9).)

3.2 The C_n series approach

The analysis for the Drinfel'd-Sokolov reduction of $\mathfrak{sp}(2n)$, which we shall denote by $\mathcal{WC}_n$, is essentially identical. Also $\mathcal{WC}_n$ is an algebra of type $\mathcal{W}(2, 4, \ldots, 2n)$, and its central charge equals

$$c_C = n - 12|\alpha_+\rho_C + \alpha_-\rho_C^\vee|^2\,, \tag{3.8}$$

where now ρ_C and $\rho_C^\vee$ are the Weyl vector and covector of $\mathfrak{sp}(2n)$, respectively. The spectrum is described by the analogue of eq. (3.3), where[4]

$$\alpha_- = -\sqrt{k_C+2n+2}\,, \qquad \alpha_+ = \frac{1}{\sqrt{k_C+2n+2}}\,. \tag{3.9}$$

[4]In our conventions, the *short* roots of $C_n = \mathfrak{sp}(2n)$ have length squared equal to 2.

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Expressed in terms of n and k_C, the central charge then takes the form

$$c_C(\mu,k_C) = -\frac{n[k_C(2n+1)+4n^2+4n)][k_C(2n-1)+4n^2-3]}{k_C+2n+2} \tag{3.10}$$
$$= -\frac{\mu[(k_C+2)\mu+k_C+\mu^2][k_C(\mu-1)+\mu^2-3]}{2(k_C+\mu+2)}\,,$$

where we have, in the second line, replaced $n=\frac{\mu}{2}$. The conformal dimensions of the minimal representations are now

$$h_+ = h_{\Lambda_+} = \frac{k_C(1-2n)-4n^2+3}{2k_C+4n+4}\,, \qquad h_- = h_{\Lambda_-} = n(k_C+2n+1)\,, \tag{3.11}$$

and they are both solutions of eq. (2.20) provided $\gamma=\gamma_C(\mu,k_C)$ equals

$$\begin{aligned}\gamma_C =\,& 144(-224-520k_C-340k_C^2-68k_C^3-888\mu-1064k_C\mu-161k_C^2\mu+114k_C^3\mu\\ &+19k_C^4\mu-372\mu^2+687k_C\mu^2+946k_C^2\mu^2+227k_C^3\mu^2+730\mu^3+1390k_C\mu^3\\ &+377k_C^2\mu^3-100k_C^3\mu^3-20k_C^4\mu^3+553\mu^4+134k_C\mu^4-315k_C^2\mu^4-85k_C^3\mu^4\\ &-34\mu^5-326k_C\mu^5-129k_C^2\mu^5+4k_C^3\mu^5+k_C^4\mu^5-111\mu^6-83k_C\mu^6+12k_C^2\mu^6\\ &+4k_C^3\mu^6-19\mu^7+12k_C\mu^7+6k_C^2\mu^7+4\mu^8+4k_C\mu^8+\mu^9)^2 n_4\Big/\Big[c_C(\mu-2)\\ &\times(k_C\mu-5-k_C+\mu^2)(k_C\mu-3k_C-5-2\mu+\mu^2)(k_C\mu+4+3k_C+4\mu+\mu^2)\\ &\times(k_C\mu-2+k_C+2\mu+\mu^2)(k_C\mu+4+2k_C+3\mu+\mu^2)(k_C\mu-1+\mu+\mu^2)\\ &\times(-88-44k_C-44\mu-15k_C\mu-5k_C^2\mu-30\mu^2-25k_C\mu^2-15\mu^3+10k_C\mu^3\\ &\quad+5k_C^2\mu^3+10\mu^4+10k_C\mu^4+5\mu^5)\Big]\,,\end{aligned} \tag{3.12}$$

where we have again replaced $n=\frac{\mu}{2}$. For each μ, we therefore obtain a family of $\mathcal{W}_\infty^e$ algebras that depend on k_C; these algebras will be denoted by $\mathcal{WC}_\infty[\mu]$ (where we suppress as before the explicit k_C dependence). Again, these algebras actually depend on k_C, rather than just c_C. By construction, $\mathcal{WC}_\infty[\mu]$ has the property that it truncates to $\mathcal{WC}_n$ for $\mu=2n$. We have also checked that, for $n=2$, $\gamma_C(2n,k_C)$ agrees with the γ of eq. (2.8) at $c=c_C(2n,k_C)$. Similarly, for $n=3$, $\gamma_C(2n,k_C)$ agrees with the γ of eq. (2.9) at $c=c_C(2n,k_C)$, where again the two solutions for k_C correspond to the two signs in front of the square root in eq. (2.9).

3.3 Langlands duality

Naively, one would have expected that the two quantum algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$ should be equivalent, but this is not actually the case: if we fix μ and c, and determine $k_B^{(i)}$, $i=1,2$, and $k_C^{(j)}$, $j=1,2$, by the requirement that

$$c = c_B(\mu,k_B^{(i)}) = c_C(\mu,k_C^{(j)})\,, \tag{3.13}$$

then none of the four different algebras we obtain from $\mathcal{WB}_\infty[\mu]$ for $k_B=k_B^{(i)}$ and $\mathcal{WC}_\infty[\mu]$ for $k_C=k_C^{(j)}$ are equivalent to one another. Thus there is not a 'unique' quantisation of $\mathcal{W}_\infty^e[\mu]$!

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The two constructions are, however, closely related to one another since we have the identifications

$$\begin{aligned} c_B(\mu+1,k_B) &= c_C(\mu,k_C) \\ \gamma_B(\mu+1,k_B) &= \gamma_C(\mu,k_C) \end{aligned} \qquad \text{when} \quad (k_B+\mu-1)(k_C+\mu+2)=1\,. \tag{3.14}$$

This relation is the 'analytic continuation' of the Langlands duality that relates $B_n = \mathfrak{so}(2n+1)$ and $C_n = \mathfrak{sp}(2n)$. Indeed, the Dynkin diagrams of B_n and C_n are obtained from one another upon reversing the arrows, i.e. upon exchanging the roles of the long and the short roots. Correspondingly, the root system of one algebra can be identified with the coroot system of the other (provided we scale the roots and coroots appropriately — this is the reason for our non-standard normalisation convention for the roots of C_n). It is then manifest from the above formulae that the central charge and spectrum is the same provided we also exchange the roles of α_+ and α_-. In terms of the levels k_B and k_C, this is then equivalent to the requirement that $(k_B+\mu-1)(k_C+\mu+2)=1$ for $\mu=2n$. Thus we can think of $\mathcal{WB}_\infty[\mu+1]$ and $\mathcal{WC}_\infty[\mu]$ to be related by Langlands duality for all μ.

The ambiguity in the definition of the quantum algebra associated with $\mathcal{W}^e_\infty[\mu]$ therefore simply reflects that Langlands duality acts non-trivially on $\mathrm{hs}^e[\mu]$, i.e. that $\mathrm{hs}^e[\mu]$ is non-simply-laced. This is to be contrasted with the case of $\mathcal{W}_\infty[\mu]$ where the two solutions of k for a given μ and c actually gave rise to equivalent $\mathcal{W}_\infty$ algebras, see eq. (2.9) of [24], reflecting the fact that $\mathrm{hs}[\mu]$ can be thought of as being 'simply-laced'.

3.4 Classical limit

In the semiclassical limit of large levels, the two quantum algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$ actually become equivalent. More concretely, if we choose the normalisation of n_4 as in section 2.3, we have in the semiclassical limit

$$c_B \sim -\frac{1}{2}\mu(\mu^2-1)k_B + \mathcal{O}(k_B^0)\,, \qquad c_C \sim -\frac{1}{2}\mu(\mu^2-1)k_C + \mathcal{O}(k_C^0)\ , \tag{3.15}$$

$$\gamma_B \sim \frac{144}{5}(\mu^2-19)^2 + \mathcal{O}(k_B^{-1})\,, \qquad \gamma_C \sim \frac{144}{5}(\mu^2-19)^2 + \mathcal{O}(k_C^{-1})\,. \tag{3.16}$$

In particular, the central charges agree, and the parameter γ is of the form predicted by eq. (2.12), recalling that $\gamma = (c^4_{44})^2$. Thus both quantum algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$ define consistent quantisations of the classical Poisson algebra, and both can be thought of as Drinfel'd-Sokolov reductions of $\mathrm{hs}^e[\mu]$. However, as mentioned before, the $\mathcal{O}(c^{-1})$ corrections in eq. (3.16) are different, reflecting that non-trivial action of Langlands duality as described by eq. (3.14).

3.5 Self-dualities

While the parameters μ and k are well suited for characterising the classical limits of the algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$, they do not directly parametrise the inequivalent $\mathcal{W}^e_\infty$ algebras. (The following discussion is directly parallel to the analogous analysis for the case of $\mathcal{W}_\infty[\mu]$ in [24].) Indeed, as was stressed in section 2.1.2, the parameters distinguishing between different $\mathcal{W}^e_\infty$ algebras are c and γ. It follows from eq. (3.7) that there are 12 different combinations $(\mu_i, k_B^{(i)})$ that give rise to the *same* quantum algebra $\mathcal{WB}_\infty[\mu]$, and likewise for $\mathcal{WC}_\infty[\mu]$, see eq. (3.12). Six of these identifications can be written down simply, while the other six require cubic roots; the simple identifications for $\mathcal{WB}_\infty[\mu]$ relate (μ, k_B) to

$$\begin{aligned}
(\mu_2, k_B^{(2)}) &= \left(\mu^2 + \mu(k_B - 2) + k_B - 1\,,\ 3 - \tfrac{1}{\mu + k_B - 2} - \mu_2\right) \\
(\mu_3, k_B^{(3)}) &= \left(\mu^2 + \mu(k_B - 4) - k_B + 4\,,\ \tfrac{1}{\mu + k_B - 3} + 3 - \mu_3\right) \\
(\mu_4, k_B^{(4)}) &= \left(\tfrac{\mu(\mu + k_B - 3)}{\mu + k_B - 2}\,,\ -\mu - k_B + 5 - \mu_4\right) \\
(\mu_5, k_B^{(5)}) &= \left(\tfrac{2}{\mu + k - 3} + \mu + 1\,,\ \tfrac{1}{\mu + k_B - 2} + 2 - \mu_5\right) \\
(\mu_6, k_B^{(6)}) &= \left(-\tfrac{k_B}{\mu + k_B - 2} - \mu + 2\,,\ 2 - \tfrac{1}{\mu + k_B - 3} - \mu_6\right).
\end{aligned} \tag{3.17}$$

Note that all of these identifications are generated by the two primitive transformations $(\mu, k_B) \mapsto (\mu_2, k_B^{(2)})$ and $(\mu, k_B) \mapsto (\mu_3, k_B^{(3)})$. Similarly, for $\mathcal{WC}_\infty[\mu]$ the simple identifications relate (μ, k_C) to

$$\begin{aligned}
(\mu_2, k_C^{(2)}) &= \left(\mu^2 + \mu(k_C + 2) + k_C\,,\ \tfrac{1}{\mu + k_C + 1} - 1 - \mu_2\right) \\
(\mu_3, k_C^{(3)}) &= \left(\mu^2 + \mu k_C - k_C - 3\,,\ -\tfrac{1}{\mu + k_C + 2} - 1 - \mu_3\right) \\
(\mu_4, k_C^{(4)}) &= \left(-\tfrac{\mu(\mu + k_C + 1)}{\mu + k_C + 2}\,,\ -\mu - k_C - 3 - \mu_4\right) \\
(\mu_5, k_C^{(5)}) &= \left(-\tfrac{2}{\mu + k_C + 1} + \mu - 1\,,\ \tfrac{1}{\mu + k_C + 2} - 2 - \mu_5\right) \\
(\mu_6, k_C^{(6)}) &= \left(-\tfrac{k_C}{\mu + k_C + 2} - \mu\,,\ -\tfrac{1}{\mu + k_C + 1} - 2 - \mu_6\right).
\end{aligned} \tag{3.18}$$

Again, all of these identifications are generated by the two primitive transformations $(\mu, k_C) \mapsto (\mu_2, k_C^{(2)})$ and $(\mu, k_C) \mapsto (\mu_3, k_C^{(3)})$.

4 The coset constructions

It was proposed in [27, 28] that the higher spin theory on AdS_3 based on the even spin algebra — from what we have said above, it is now clear that the relevant algebra is in fact $\mathrm{hs}^e[\lambda]$ — should be dual to the 't Hooft limit of the $\mathfrak{so}(2n)$ cosets

$$\mathcal{WD}_{n,k} = \frac{\mathfrak{so}(2n)_k \oplus \mathfrak{so}(2n)_1}{\mathfrak{so}(2n)_{k+1}}\,, \tag{4.1}$$

where the 't Hooft limit consists of taking $n, k \to \infty$ while keeping the parameter

$$\lambda = \frac{2n}{2n + k - 2} \quad \text{fixed.} \tag{4.2}$$

This therefore suggests that the corresponding quantum $\mathcal{W}^e_\infty$ algebras should be isomorphic. Given that there are two different quantisations of the Drinfel'd-Sokolov reduction of $\mathrm{hs}^e[\mu]$ (see section 3), there should therefore be two identifications, relating $\mathcal{WD}_{n,k}$ to either $\mathcal{WB}_\infty[\lambda]$ or $\mathcal{WC}_\infty[\lambda]$. In this section we want to explain in detail these different relations. As in the case of $\mathcal{W}_\infty[\mu]$ studied in [24], the (correctly adjusted) correspondences will actually turn out to hold even at finite n and k.

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4.1 The D_n cosets

In a first step we need to understand the structure of the $\mathcal{W}$ algebra underlying the cosets (4.1). By the usual formula we find that its central charge equals

$$c_{\mathfrak{so}}(2n,k) = n\left[1 - \frac{(2n-2)(2n-1)}{(k+2n-2)(k+2n-1)}\right]\,. \tag{4.3}$$

In order to determine the spin spectrum of the $\mathcal{W}$ algebra we can use that D_n is simply-laced, and hence that (4.1) is isomorphic [21] to the Drinfel'd-Sokolov reduction of D_n, which we denote by $\mathcal{WD}_n$; this algebra is of type $\mathcal{W}(2,4,\ldots,2n-2,n)$. In the 't Hooft limit, i.e. for $n\to\infty$, the spin spectrum of $\mathcal{WD}_n$ involves all even spins (with multiplicity one), and hence becomes a $\mathcal{W}^{\rm e}_\infty$ algebra, but for finite n, this is not the case because of the additional spin n generator, which we shall denote by V. However, as was already explained in [38, 39], $\mathcal{WD}_n$ possesses an outer $\mathbb{Z}_2$ automorphism σ — this is actually the automorphism that is inherited from the spin-flip automorphism of $\mathfrak{so}(2n)$ — under which the generators of spin $2,4,\ldots,2n-2$ are invariant, while the spin n generator V is odd. Then, the 'orbifold' subalgebra $\mathcal{WD}^\sigma_n$, i.e. the σ-invariant subalgebra of $\mathcal{WD}_n$, has the right spin content. It is generated, in addition to the σ-invariant generators of $\mathcal{WD}_n$ of spin $2,4,\ldots,2n-2$, by the normal ordered product of spin $2n$ $:VV:$, as well as its higher derivatives that are schematically of the form $:V\partial^{2l}V:$, see [39].[5]

These arguments imply that we can generate $\mathcal{WD}^\sigma_n$ by (a subset of) the fields contained in $\mathcal{W}^{\rm e}_\infty$. Hence, $\mathcal{WD}^\sigma_n$ is a quotient of $\mathcal{W}^{\rm e}_\infty$ and we can characterise it again in terms of the central charge c, and the parameter $\gamma=(c^4_{44})^2$. As before, a convenient method to compute γ is by comparing the conformal dimension of the 'minimal' representations. Since $\mathcal{WD}^\sigma_n$ is a subalgebra of $\mathcal{WD}_n$, each representation of $\mathcal{WD}_n$ defines also a representation of $\mathcal{WD}^\sigma_n$. In particular, the 'minimal' representations of $\mathcal{WD}_n$ that are labelled by $(v;0)$ and $(0;v)$ — see [28] for our conventions — are also minimal for $\mathcal{WD}^\sigma_n$, and their conformal dimensions equal

$$h(v;0) = \frac{1}{2}\left[1+\frac{2n-1}{k+2n-2}\right]\,, \qquad h(0;v) = \frac{1}{2}\left[1-\frac{2n-1}{k+2n-1}\right]\,. \tag{4.4}$$

Both solve eq. (2.20) for $\gamma=\gamma_{\mathfrak{so}}(N,k)$, where $N=2n$ and

$$\begin{aligned}\gamma_{\mathfrak{so}} = {}& 144(-224+744k-860k^2+408k^3-68k^4+376N-1336kN+1267k^2N\\ &-386k^3N+19k^4N-124N^2+857kN^2-599k^2N^2+76k^3N^2-52N^3\\ &-252kN^3+94k^2N^3+24N^4+36kN^4)^2 n_4\,/\,\Big[c_{\mathfrak{so}}(2+k)(N-1)(2k-4+N)\\ &\times(k-5+2N)(3k-4+2N)(2k-2+3N)(3k-5+4N)\\ &\times(88-132k+44k^2-132N+73kN+5k^2N+44N^2+10kN^2)\Big]\,.\end{aligned} \tag{4.5}$$

It is interesting that also $h=n$ solves eq. (2.20) for $\gamma=\gamma_{\mathfrak{so}}(2n,k)$, thus implying that also the field V generates a minimal representation of $\mathcal{WD}^\sigma_n$.

[5] The counting of the quasiprimary higher spin states is essentially equivalent to the counting of the higher spin fields of a theory of a single real boson, see e.g. [9]. Note that, as is also explained in [39], the resulting algebra is neither freely generated nor infinitely generated, i.e. there are relations between the $\mathcal{W}^{\rm e}_\infty$ type generators that effectively reduce these generators to a finite set.

4.2 The B_n cosets

A closely related family of cosets is obtained from (4.1) by considering instead the odd $\mathfrak{so}$ algebras, i.e.

$$\mathcal{WB}(0,n)^{(0)} = \frac{\mathfrak{so}(2n+1)_k \oplus \mathfrak{so}(2n+1)_1}{\mathfrak{so}(2n+1)_{k+1}}\,. \tag{4.6}$$

These $\mathcal{W}$ algebras can be identified with the bosonic subalgebra of the Drinfel'd-Sokolov reduction of the superalgebras $\mathfrak{osp}(1|2n)$ or $B(0,n)$, see [21]. The latter is a $\mathcal{W}$ algebra of type $\mathcal{W}(2,4,\ldots,2n,n+\frac{1}{2})$, and we shall denote it by $\mathcal{WB}(0,n)$. Since the field of conformal weight $n+\frac{1}{2}$ is fermionic — we shall denote it by S in the following — the bosonic subalgebra does not include S, but contains instead the normal ordered products $:S\partial^{2l+1}S:$ with $l=0,1,\ldots$ — because $:SS:=0$ we now always have an odd number of derivatives.[6] Thus the generating fields include, in addition to the bosonic generating fields of $\mathcal{WB}(0,n)$ of spin $2,4,\ldots,2n$, fields of spin $2n+2, 2n+4,\ldots$; in particular, $\mathcal{WB}(0,n)^{(0)}$ is therefore again a quotient of $\mathcal{W}^e_\infty$, and can be characterised in terms of γ and c. The analysis is essentially identical to what was done for the $\mathfrak{so}(2n)$ case above — indeed, the central charge, as well as the conformal dimensions of the minimal representations are obtained from (4.3) and (4.4) upon replacing $2n\mapsto 2n+1$, and thus γ is simply $\gamma=\gamma_{\mathfrak{so}}(N,k)$, where $N=2n+1$ and $\gamma_{\mathfrak{so}}$ was already defined in (4.5). Thus these two families of cosets are naturally analytic continuations of one another.

As an additional consistency check we note that the algebra $\mathcal{WB}(0,1)^{(0)}$ is of type $\mathcal{W}(2,4,6)$, see [40], and its structure constants are explicitly known [41]. In section 2.2.2 we have reproduced this algebra as a quotient of $\mathcal{W}^e_\infty$. The corresponding value of γ, given in eq. (2.11), agrees indeed with $\gamma_{\mathfrak{so}(3)}$.

4.3 Level-rank duality

The expressions (4.3) and (4.5) are invariant under the transformation

$$N\mapsto N\,, \qquad k\mapsto -2N-k+3\,. \tag{4.7}$$

For even $N=2n$ this is a consequence of the Langlands self-duality of D_n, which in turn follows from the fact that D_n is simply-laced, implying that the Drinfel'd-Sokolov reduction has the symmetry $\alpha_\pm\mapsto-\alpha_\mp$. As a result, $\mathcal{WD}_n$ actually only depends on c, rather than directly on k. This is reflected in the fact that $\gamma_{\mathfrak{so}}$ can be written as an unambiguous function of N and c as

$$\begin{aligned}\gamma_{\mathfrak{so}} = {}& 72\big(2c^2(N^2-2N-18)+3c(6N^3-49N^2+80N-8)+2N(6N^2+5N-28)\big)^2\\ &\times n_4\Big/\Big[(5c+22)c(c(N^2-7N+12)+2N^2-5N)(c(N+1)+4N^2-5N)\\ &\quad\times(2c(N+2)+3N^2-14N+8)\Big]\,.\end{aligned} \tag{4.8}$$

[6] The counting of the quasiprimary higher spin fields is in this case analogous to that of counting the higher spin fields of a single free fermion.

Note that, in the large c limit, $\mathcal{WD}_n^\sigma$ becomes a classical Poisson algebra, which can be identified with the σ-invariant classical Drinfel'd-Sokolov reduction of D_n. In fact, taking n_4 as in eq. (2.13), it follows from eq. (4.8) that the corresponding γ parameter equals

$$\gamma_{\mathfrak{so}} = \frac{144}{5}(\mu^2 - 19)^2 + \mathcal{O}(c^{-1}), \qquad \text{where} \quad \mu = 2n - 1\,. \tag{4.9}$$

Note that this ties in with the fact that the wedge algebra of $\mathcal{WD}_n^\sigma$ is the σ-invariant subalgebra of $\mathfrak{so}(2n)$, which in turn equals $\mathfrak{so}(2n-1)$. This explains why (4.9) agrees with (3.16) for $\mu = 2n-1$.

Next we observe that eq. (4.8) is a polynomial equation of order 6 in N, with coefficients that are functions of γ and c, and hence there is a six-fold ambiguity in the definition of N. If we parametrise $c = c_{\mathfrak{so}}(N,k)$, then the algebra associated with (N,k) is equivalent to the one associated with

$$(N_2,k_2) = \left(\frac{k+2N-3}{k+N-2}\,,\ \frac{k}{k+N-2}\right), \quad (N_3,k_3) = \left(\frac{k}{k+N-1}\,,\ \frac{2N+k-3}{k+N-1}\right), \tag{4.10}$$

while the other three solutions involve cubic roots. Obviously, we can also replace $k \mapsto -2N-k+3$ without modifying the algebra, see (4.7), and thus, expressed in terms of N and k, there are 12 different pairs (N_i,k_i) that define the same algebra. We should also mention that the third solution above is obtained by applying the map $(N,k) \mapsto (N_2,k_2)$ twice. This fundamental transformation has a nice interpretation in terms of a level-rank type duality rather similar to the one appearing for $\mathfrak{su}(N)$ in [24]:

$$\left(\frac{\mathfrak{so}(N)_k \oplus \mathfrak{so}(N)_1}{\mathfrak{so}(N)_{k+1}}\right)^\sigma \cong \left(\frac{\mathfrak{so}(M)_l \oplus \mathfrak{so}(M)_1}{\mathfrak{so}(M)_{l+1}}\right)^\sigma, \tag{4.11}$$

where

$$k = \frac{N-1}{M-1} - N + 2\,, \qquad l = \frac{M-1}{N-1} - M + 2\,, \tag{4.12}$$

and the superscript σ means that we take the σ-invariant subalgebra if N or M are even integers. Obviously, as a true level-rank duality, this only makes sense if M and N are positive integers. As far as we are aware, this level-rank duality has not been noticed before.

4.4 Holography

With these preparations we can now return to the main topic of this section, the precise relation between the σ-even subalgebra of the $\mathfrak{so}(2n)$ cosets of eq. (4.1), and the quantum algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$. As we have explained before, all three algebras are in general (quotients of) $\mathcal{W}_\infty^e$ algebras, and hence are uniquely characterised in terms of γ and c. By comparing the relations (3.5) and (3.7) for $\mathcal{WB}_\infty[\mu]$ with (4.3) and (4.5) for the $\mathfrak{so}(2n)$ cosets, we conclude that we have the identification

$$\mathcal{WD}_{n,k}^\sigma \cong \mathcal{WB}_\infty[\lambda_B]\,, \quad \text{with} \quad \lambda_B = \frac{2n-2}{k+2n-2}\,, \quad k_B = k + 2n + 1 - \lambda_B\,. \tag{4.13}$$

Similarly, for the case of $\mathcal{WC}_\infty[\mu]$ we find instead from (3.10) and (3.12) that

$$\mathcal{WD}_{n,k}^\sigma \cong \mathcal{WC}_\infty[\lambda_C]\,, \quad \text{with} \quad \lambda_C = \frac{2n}{k+2n-2}\,, \quad k_C = k + 2n - 3 - \lambda_C\,. \tag{4.14}$$

Obviously, using the self-duality relations of the various algebras, see eqs. (3.17), (3.18) and (4.10), there are also other versions of these identifications, but the above is what is relevant in the context of minimal model holography: the above analysis shows that the (σ-even subalgebra of the) $\mathfrak{so}$ cosets[7] are equivalent to the quantum Drinfel'd-Sokolov reduction of the $\mathrm{hs}^e[\lambda_{B/C}]$ algebras with $\lambda_{B/C}$ given above. Note that λ_C agrees exactly with λ given in (4.2) above, see also [28], while for λ_B the difference is immaterial in the 't Hooft limit. These statements are now true even at finite n and k, hence giving the correct quantum version of the even spin holography conjecture.

4.5 The semiclassical behaviour of the scalar fields

With our detailed understanding of the symmetry algebras at finite c, we can now also address the question of whether the duals of the two minimal coset fields of [27, 28] should be thought of as being perturbative or non-perturbative excitations of the higher spin bulk theory. As in the case studied in [24], this issue can be decided by studying the behaviour of their conformal dimensions in the semiclassical limit, i.e. for $c \to \infty$.

Let us consider then the $\mathcal{WD}_n^\sigma$ coset at fixed n. If c takes one of the actual minimal model values, $c = c_{\mathfrak{so}}(2n, k)$ with $k \in \mathbb{N}$, see eq. (4.3), the algebra has the two minimal representations $(v; 0)$ and $(0; v)$, whose conformal dimensions are given in eq. (4.4). Written in terms of n and c (rather than n and k), they take the form

$$h_\pm(n, c) = \frac{1}{2}\left(1 + \frac{n - c \pm \sqrt{(c-n)\,(c-(3-4n)^2 n)}}{4(n-1)n}\right), \tag{4.15}$$

where $h(v; 0) = h_+(n, c)$ and $h(0; v) = h_-(n, c)$. Since we know that the algebra $\mathcal{WD}_n^\sigma$ depends only on c (rather than k), it is then clear that (4.15) are the conformal weights of minimal representations for any value of c.

We are interested in the semiclassical limit, which consists of taking $c \to \infty$ at fixed n. There is obviously an ambiguity in how precisely c is analytically continued, but taking c, say, along the positive real axis to infinity, we read off from (4.15) that

$$h(v; 0) = h_+(n, c) = \frac{1-\mu}{2} + \mathcal{O}(c^{-1}), \tag{4.16}$$

$$h(0; v) = h_-(n, c) = \frac{c}{\mu^2 - 1} + \mathcal{O}(1), \tag{4.17}$$

where $\mu = 2n - 1$, see eq. (4.9). In this limit $h(v; 0)$ remains finite, while $h(0; v)$ is proportional to c. Thus we conclude that only the coset representation $(v; 0)$ corresponds to a perturbative scalar of the higher spin theory based on $\mathrm{hs}^e[\mu]$, while $(0; v)$ describes a non-perturbative excitation. This is directly analogous to what happened in [24].

4.6 The full orbifold spectrum

Now that we have understood the relation between the symmetries in the duality conjecture of [27, 28] we can come back to the comparison of the partition functions that

[7] For $n \in \mathbb{N} + \frac{1}{2}$, the left hand side of eqs. (4.13) and (4.14) should be understood as the chiral algebra of the cosets (4.6). Coset interpretations exist also when n is a negative half-integer, see section 4.7.

was performed in [28]. It was shown there that the spectrum of the charge conjugate modular invariant of the $\mathcal{WD}_{n,k}$ algebra coincides, in the 't Hooft limit, with the bulk 1-loop partition function of a suitable higher spin theory on thermal AdS$_3$.

As we have seen above, at finite n and c, the relevant symmetry algebra is actually not $\mathcal{WD}_{n,k}$, but only the σ-invariant subalgebra $\mathcal{WD}^{\sigma}_{n,k}$. Every representation of $\mathcal{WD}_{n,k}$ defines also a representation of $\mathcal{WD}^{\sigma}_{n,k}$, and hence the charge conjugation (or A-type) modular invariant of the $\mathcal{WD}_{n,k}$ algebra also defines a consistent partition function with respect to $\mathcal{WD}^{\sigma}_{n,k}$. However, from the latter point of view, it is not the charge conjugation modular invariant, but rather of what one may call 'D-type'.

It is then natural to ask whether the charge conjugation (A-type) modular invariant of $\mathcal{WD}^{\sigma}_{n,k}$ also has a bulk interpretation. We shall not attempt to answer this question here, but we shall only show that it leads to a *different* partition function in the 't Hooft limit. Thus, if the charge-conjugation modular invariant of $\mathcal{WD}^{\sigma}_{n,k}$ also has a consistent AdS$_3$ dual, this must be a different theory than the one considered in [27, 28].

In the charge conjugation (A-type) modular invariant of the $\mathcal{WD}^{\sigma}_{n,k}$ algebra, every untwisted representation of $\mathcal{WD}^{\sigma}_{n,k}$ appears once. Obviously, not all representations of $\mathcal{WD}^{\sigma}_{n,k}$ arise as subrepresentations of untwisted $\mathcal{WD}_{n,k}$ representations. In particular, each σ-twisted representation of $\mathcal{WD}_{n,k}$ (for which V is half-integer moded) also leads to an untwisted representation of $\mathcal{WD}^{\sigma}_{n,k}$. Since σ is inherited from the outer automorphism of $\mathfrak{so}(2n)$, these twisted representations of $\mathcal{WD}_{n,k}$ can be described via the cosets

$$\frac{\mathfrak{so}(2n)^{(2)}_k \oplus \mathfrak{so}(2n)^{(2)}_1}{\mathfrak{so}(2n)^{(2)}_{k+1}}\,, \tag{4.18}$$

where $\mathfrak{so}(2n)^{(2)}_k$ is the twisted affine algebra, see e.g. [42] for an introduction. The representations of $\mathfrak{so}(2n)^{(2)}_k$ are labelled by $\mathfrak{so}(2n-1)$ dominant highest weights Ξ, satisfying certain integrability conditions, and the corresponding conformal dimensions equal

$$h_{\mathfrak{so}(2n)^{(2)}_k}(\Xi) = \frac{\mathrm{Cas}(\Xi)}{2(k+2n-2)} + \frac{k(2n-1)}{16(k+2n-2)}\,, \tag{4.19}$$

where Cas is the Casimir of $\mathfrak{so}(2n-1)$. The conformal dimension of the representations of (4.18) can then be obtained from (4.19) by the usual coset formula. In particular, the twisted vacuum, where we take Ξ to be the vacuum representation ($\Xi = 0$) of $\mathfrak{so}(2n-1)$ for all 3 factors in eq. (4.18), has conformal dimension

$$\frac{1}{16}\left[1 - \frac{(2n-1)(2n-2)}{(k+2n-2)(k+2n-1)}\right]. \tag{4.20}$$

This state does not appear in the 1-loop bulk higher spin calculation of [28], and thus the dual of the charge conjugation modular invariant of $\mathcal{WD}^{\sigma}_{n,k}$ must be a different bulk theory than the one considered in [28].

4.7 Other minimal models

Let us close this discussion with a comment about other minimal models one may consider. As we have seen in sections 4.1 and 4.2, the dual of the even higher spin theories on

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AdS can be identified with the cosets of *either* the $\mathfrak{so}$(even) or the $\mathfrak{so}$(odd) algebras. It is then natural to ask how the cosets of the $\mathfrak{sp}$ algebras fit into this picture. Using the field counting techniques of [43] (see also [44]) one can show that the cosets[8]

$$\frac{\mathfrak{sp}(2n)_k \oplus \mathfrak{sp}(2n)_{-1}}{\mathfrak{sp}(2n)_{k-1}} \tag{4.21}$$

possess a $\mathcal{W}^e_\infty$ symmetry in the 't Hooft limit. The essential points of this calculation are (i) that $\mathfrak{sp}(2n)_{-1}$ has a free field realisation in terms of n $\beta\gamma$-systems; and (ii) that the coset vacuum character can be computed by counting $\mathfrak{sp}(2n)$ invariant products of $\beta\gamma$-fields and their derivatives, using standard arguments of classical invariant theory.

It is then natural to ask what $\mathcal{W}^e_\infty$ algebras the cosets (4.21) lead to when analytically continued in n and k. The answer can be schematically formulated as

$$\frac{\mathfrak{sp}(2n)_k \oplus \mathfrak{sp}(2n)_{-1}}{\mathfrak{sp}(2n)_{k-1}} \cong \left(\frac{\mathfrak{so}(-2n)_{-k} \oplus \mathfrak{so}(-2n)_1}{\mathfrak{so}(-2n)_{-k+1}}\right)^\sigma , \tag{4.22}$$

where both cosets stand for the corresponding $\mathcal{W}^e_\infty$ algebras (or their quotients), and the equality means that both the analytically continued central charge and the self-coupling γ agree.

Incidentally, there is an independent check for our claim that the cosets (4.21) are quotients of $\mathcal{W}^e_\infty$. For $n = 1$, the coset (4.21) is known to be of type $\mathcal{W}(2, 4, 6)$, see [45],[9] and its structure constants have been computed explicitly in [46], coinciding with the solution given in eq. (2.10) of section 2.2.2. We also note that the corresponding value of γ agrees indeed with $\gamma_{\mathfrak{so}(-2)}$, as required by (4.22).

The above arguments apply similarly for the cosets

$$\frac{\mathfrak{osp}(1|2n)_k \oplus \mathfrak{osp}(1|2n)_{-1}}{\mathfrak{osp}(1|2n)_{k-1}} , \tag{4.23}$$

for which the emergence of a $\mathcal{W}^e_\infty$ symmetry in the 't Hooft limit can be proven using analogous methods, in particular, noting that $\mathfrak{osp}(1|2n)_{-1}$ has a free field realisation in terms of a single Majorana fermion and n $\beta\gamma$-systems. In this case, the analogue of (4.22) is

$$\frac{\mathfrak{osp}(1|2n)_k \oplus \mathfrak{osp}(1|2n)_{-1}}{\mathfrak{osp}(1|2n)_{k-1}} \cong \frac{\mathfrak{so}(-2n+1)_{-k} \oplus \mathfrak{so}(-2n+1)_1}{\mathfrak{so}(-2n+1)_{-k+1}} . \tag{4.24}$$

5 Conclusions

In this paper we have constructed the quantum $\mathcal{W}^e_\infty$ algebra that is generated by one Virasoro primary field for every even spin, using systematically Jacobi identities. We have seen that, up to the level to which we have evaluated these constraints, the algebra depends only on two parameters: the central charge c and a free parameter γ, which is essentially the self-coupling of the spin 4 field. We have shown that the first few commutators of the wedge algebra of $\mathcal{W}^e_\infty$ agree with those of $\mathrm{hs}^e[\mu]$. This suggests that the dual higher spin

[8] In our conventions, the *short* roots of $\mathfrak{sp}(2n)$ have length squared equal to 2.

[9] We thank the authors of [44] for drawing our attention to this reference.

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theory on AdS_3 should be described in terms of a Chern-Simons theory based on $\mathrm{hs}^e[\mu]$. Furthermore, given the usual relation between wedge algebras and Drinfel'd-Sokolov reductions, $\mathcal{W}_\infty^e$ should be thought of as the quantum Drinfel'd-Sokolov reduction of $\mathrm{hs}^e[\mu]$. As we have explained, there are actually two different quantisations of the classical Drinfel'd-Sokolov reduction of $\mathrm{hs}^e[\mu]$, which we called $\mathcal{WB}_\infty$ and $\mathcal{WC}_\infty$, respectively. We have argued that this ambiguity is closely related to the fact that $\mathrm{hs}^e[\mu]$ is non-simply-laced.

Given that $\mathcal{W}_\infty^e$ describes the most general $\mathcal{W}$ algebra with this spin content, we can identify (quotients of) $\mathcal{W}_\infty^e$ with (orbifolds of) the coset algebras based on $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(2n)$, and $\mathfrak{osp}(1|2n)$. In particular, this proves that the $\mathfrak{so}$ coset algebras of [27, 28] are equivalent, for suitable values of μ, to the quantum algebras $\mathcal{WB}_\infty[\mu]$ and $\mathcal{WC}_\infty[\mu]$. This quantum equivalence is even true at finite n and k, and therefore establishes an important part of the holographic proposals of [27, 28]. We also showed, in close analogy with [24], that only one of the 'scalar' excitations should be thought of as being perturbative, while the other should correspond to a non-perturbative classical solution. It would be interesting to check, following [47, 48], whether the corresponding classical solutions exist and have the appropriate properties. It would also be interesting to study whether the A-type modular invariant of the $\mathcal{WD}^\sigma_{n,k}$ algebra has a bulk interpretation, see section 4.6.

As in the case of the higher spin algebra $\mathrm{hs}[\mu]$ discussed in [24], it would be interesting to reproduce the quantum corrections predicted by the CFT directly from a perturbative bulk calculation. In this context it would be important to understand the systematics of the quantum Drinfel'd-Sokolov reduction for $\mathrm{hs}^e[\mu]$ in more detail. In particular, this should shed some light on which choices have to be made in quantising the bulk theory.

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A Minimal representations using commutators

In section 2.4 we computed the structure constant c_{44}^4 in terms of the conformal dimension h of minimal representations. The calculation was carried out using OPEs. An alternative, but equivalent approach uses commutators rather than OPEs and shall be sketched in this appendix.

We will need the following commutators of $\mathcal{W}_\infty^e$:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{0,m+n}\,,\\
[L_m, W_n^4] &= (3m-n)W^4_{m+n}\\
[W_m^4, W_n^4] &= \frac{1}{2}(m-n)\left(c_{44}^6 W^6_{m+n} + q_{44}^{6,1}Q^{6,1}_{m+n} + q_{44}^{6,2}Q^{6,2}_{m+n} + q_{44}^{6,3}Q^{6,3}_{m+n}\right)\\
&\quad + \frac{1}{36}\left(m^2 - mn + n^2 - 7\right)(m-n)\left(c_{44}^4 W^4_{m+n} + q_{44}^4 Q^4_{m+n}\right)
\end{aligned}$$

$$+\big(3(m^4+n^4)-(2mn+39)(m^2+n^2)+4m^2n^2+20mn+108\big)$$
$$\times\frac{1}{3360}(m-n)q_{44}^2L_{m+n}$$
$$+\frac{1}{5040}m(m^2-1)(m^2-4)(m^2-9)n_4\delta_{0,m+n}\,, \qquad \text{(A.1)}$$

where the composite quasiprimary fields $Q^4, Q^{6,1}, Q^{6,2}$ and $Q^{6,3}$ are given by

$$Q^4 = LL-\frac{3L''}{10}\,, \qquad Q^{6,1}=LW^4-\frac{W^{4\prime\prime}}{6}\,,$$
$$Q^{6,2}=L'L'-\frac{4}{5}L''L-\frac{\partial^4 L}{42}\,, \qquad Q^{6,3}=L(LL)-\frac{1}{3}L'L'-\frac{19}{30}L''L-\frac{\partial^4 L}{36}\,. \qquad \text{(A.2)}$$

Solving the Jacobi identity $[L_m,[W_n^4,W_l^4]]+\text{cycl.}=0$, we find that

$$q_{44}^2=\frac{8}{c}n_4\,,\quad q_{44}^{6,1}=\frac{28}{3(c+24)}c_{44}^4\,, \qquad q_{44}^{6,2}=-\frac{2(19c-524)}{3c(2c-1)(7c+68)}n_4\,,$$
$$q_{44}^{6,3}=\frac{96(72c+13)}{c(2c-1)(5c+22)(7c+68)}n_4\,, \qquad q_{44}^4=\frac{168}{c(5c+22)}n_4\,. \qquad \text{(A.3)}$$

This fixes the structure constants of the Virasoro descendants in terms of their primaries. Similarly, by considering Jacobi identities of higher level, we can reobtain in this manner the relations between structure constants given in section 2.1.2.

Recall from section 2.4 that the defining property of a minimal representation is a character of the form

$$\frac{q^h}{1-q}\prod_{s\in 2\mathbb{N}}^{\infty}\prod_{n=s}^{\infty}\frac{1}{1-q^n}=q^h(1+q+2q^2+3q^3+\dots)\,, \qquad \text{(A.4)}$$

where h is the conformal dimension of the highest weight state Φ.

Thus, at level 1 all the states must be proportional to $L_{-1}\Phi$, at level 2 they are linear combinations of, say, $L_{-1}^2\Phi$ and $L_{-2}\Phi$, and at level 3 of, for instance, $L_{-3}\Phi$, $L_{-2}L_{-1}\Phi$ and $L_{-1}^3\Phi$. Therefore, we can conclude that the representation must have null relations of the form

$$\mathcal{N}_{1W^4}=(W_{-1}^4-\frac{2w^4}{h}L_{-1})\Phi\,, \qquad \text{(A.5)}$$
$$\mathcal{N}_{2W^4}=(W_{-2}^4+aL_{-1}^2+bL_{-2})\Phi\,, \qquad \text{(A.6)}$$
$$\mathcal{N}_{3W^4}=(W_{-3}^4+dL_{-3}+eL_{-2}L_{-1}+fL_{-1}^3)\Phi\,, \qquad \text{(A.7)}$$

where w^4 is the eigenvalue of the zero mode of W^4 on Φ. The coefficient in front of L_{-1} in $\mathcal{N}_{1W^4}$ follows from the condition

$$L_1\mathcal{N}_{1W^4}=0\,. \qquad \text{(A.8)}$$

Similarly, the coefficients a and b in $\mathcal{N}_{2W^4}$ can be determined from the conditions

$$L_1^2\mathcal{N}_{2W^4}=0 \quad \text{and} \quad L_2\mathcal{N}_{2W^4}=0\,, \qquad \text{(A.9)}$$

and d, e and f from $L_3\mathcal{N}_{3W^4} = 0$, $L_2L_1\mathcal{N}_{3W^4} = 0$ and $L_1^3\mathcal{N}_{3W^4} = 0$. The result is

$$\begin{aligned}
a &= -\frac{(5c+16h)w^4}{h(2ch+c+2h(8h-5))}\,, \qquad b = \frac{4(11-8h)w^4}{2ch+c+2h(8h-5)}\,,\\
d &= -\frac{6\big[c(h+3)(2h-3)+2h(h-2)(8h-21)-22\big]w^4}{\big[(c-7)h+c+3h^2+2\big]\big[2ch+c+2h(8h-5)\big]}\,,\\
e &= -\frac{12\big[c(6h(h-1)-2)+h(h(8h-15)+9)\big]w^4}{h\big[(c-7)h+c+3h^2+2\big]\big[2ch+c+2h(8h-5)\big]}\,,\\
f &= -\frac{(5c+22)(c-h)w^4}{h\big[(c-7)h+c+3h^2+2\big]\big[2ch+c+2h(8h-5)\big]}\,. \qquad \text{(A.10)}
\end{aligned}$$

Finally, solving the slightly more involved null relations

$$W_1^4\mathcal{N}_{1W^4} = 0\,, \qquad W_2^4\mathcal{N}_{2W^4} = 0\,, \qquad W_3^4\mathcal{N}_{3W^4} = 0\,, \qquad \text{(A.11)}$$

and plugging in the structure constants (A.3) leads to the same expressions for w^4, w^6 and c_{44}^4 as those obtained in (2.19) and (2.20) by associativity. Here w^6 is the eigenvalue of the zero mode of W^6 on Φ.

B Structure constants of hs$^e[\mu]$

The algebra hs$^e[\mu]$ is a subalgebra of hs$[\mu]$ and the structure constants of the latter are known explicitly, see [49]. We have rescaled the generators of this reference so that the first few commutation relations take the form

$$\begin{aligned}
[L_m, W_n^s] &= ((s-1)m-n)\,W_{m+n}^s\,, \qquad \text{(B.1)}\\
[W_m^4, W_n^4] &= -\frac{20}{\sqrt{7}}P_6^{44}(m,n)W_{m+n}^6 + \frac{12}{\sqrt{5}}\left(\mu^2-19\right)P_4^{44}(m,n)W_{m+n}^4\\
&\quad + 8\left(\mu^4-13\mu^2+36\right)P_2^{44}(m,n)L_{m+n}\,, \qquad \text{(B.2)}\\
[W_m^4, W_n^6] &= -8\sqrt{\frac{210}{143}}P_8^{46}(m,n)W_{m+n}^8 + \frac{14}{\sqrt{5}}\left(\mu^2-49\right)P_6^{46}(m,n)W_{m+n}^6\\
&\quad - \frac{20}{\sqrt{7}}\left(\mu^4-41\mu^2+400\right)P_4^{46}(m,n)W_{m+n}^4\,, \qquad \text{(B.3)}\\
[W_m^6, W_n^6] &= -252\sqrt{\frac{5}{2431}}P_{10}^{66}(m,n)W_{m+n}^{10} + 28\sqrt{\frac{6}{143}}\left(\mu^2-115\right)P_8^{66}(m,n)W_{m+n}^8\\
&\quad - \frac{40}{3\sqrt{7}}\left(\mu^2-88\right)\left(\mu^2-37\right)P_6^{66}(m,n)W_{m+n}^6\\
&\quad + \frac{14}{\sqrt{5}}(\mu^2-49)(\mu^2-25)(\mu^2-16)P_4^{66}(m,n)W_{m+n}^4\\
&\quad + 12(\mu^2-25)(\mu^2-16)(\mu^2-9)(\mu^2-4)P_2^{66}(m,n)L_{m+n}\,, \qquad \text{(B.4)}\\
[W_m^4, W_n^8] &= -20\sqrt{\frac{6}{17}}P_{10}^{48}(m,n)W_{m+n}^{10} - \frac{72}{13\sqrt{5}}\left(277-3\mu^2\right)P_8^{48}(m,n)W_{m+n}^8\\
&\quad - 40\sqrt{\frac{210}{143}}(\mu^2-49)(\mu^2-36)P_6^{48}(m,n)W_{m+n}^6\,, \qquad \text{(B.5)}
\end{aligned}$$

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where $P^{ss'}_{s''}(m,n)$ are the universal polynomials containing the mode dependence of the structure constants in a commutator of quasiprimary fields of a CFT. They are given by

$$P^{ss'}_{s''}(m,n) := \sum_{r=0}^{s+s'-s''-1} \binom{s+m-1}{s+s'-s''-r-1} \frac{(-1)^r (s-s'+s'')_{(r)}(s''+m+n)_{(r)}}{r!(2s'')_{(r)}} ,$$

where we have introduced the Pochhammer symbols $x_{(r)} = \Gamma(x+r)/\Gamma(x)$. When m, n are restricted to the wedge, these universal polynomials are essentially the Clebsch-Gordan coefficients of $\mathfrak{sl}(2)$ [50]. The proportionality factors between the generators T^j_m of [49] and our generators W^s_m are explicitly

$$T^j_m = \sqrt{\frac{(j-m)!(j+m)!}{(2j)!}} W^{j+1}_m . \tag{B.6}$$

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [INSPIRE].

[2] M.A. Vasiliev, *Higher spin algebras and quantization on the sphere and hyperboloid*, *Int. J. Mod. Phys.* **A 6** (1991) 1115 [INSPIRE].

[3] B. Sundborg, *Stringy gravity, interacting tensionless strings and massless higher spins*, *Nucl. Phys. Proc. Suppl.* **102** (2001) 113 [hep-th/0103247] [INSPIRE].

[4] E. Witten, *Spacetime reconstruction*, talk at the *John Schwarz* 60th *birthday symposium*, http://theory.caltech.edu/jhs60/witten/1.html.

[5] A. Mikhailov, *Notes on higher spin symmetries*, hep-th/0201019 [INSPIRE].

[6] E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl. Phys.* **B 644** (2002) 303 [*Erratum ibid.* **B 660** (2003) 403] [hep-th/0205131] [INSPIRE].

[7] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a higher spin symmetry*, arXiv:1112.1016 [INSPIRE].

[8] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry*, arXiv:1204.3882 [INSPIRE].

[9] I. Klebanov and A. Polyakov, *AdS dual of the critical $O(N)$ vector model*, *Phys. Lett.* **B 550** (2002) 213 [hep-th/0210114] [INSPIRE].

[10] E. Sezgin and P. Sundell, *Holography in 4D (super) higher spin theories and a test via cubic scalar couplings*, *JHEP* **07** (2005) 044 [hep-th/0305040] [INSPIRE].

[11] S. Giombi and X. Yin, *The higher spin/vector model duality*, arXiv:1208.4036 [INSPIRE].

[12] M.R. Gaberdiel and R. Gopakumar, *An AdS_3 dual for minimal model CFTs*, *Phys. Rev.* **D 83** (2011) 066007 [arXiv:1011.2986] [INSPIRE].

[13] M.R. Gaberdiel and R. Gopakumar, *Minimal model holography*, arXiv:1207.6697 [INSPIRE].

[14] S. Prokushkin and M.A. Vasiliev, *Higher spin gauge interactions for massive matter fields in 3D AdS space-time*, *Nucl. Phys.* **B 545** (1999) 385 [hep-th/9806236] [INSPIRE].

JHEP01(2013)185

[15] S. Prokushkin and M.A. Vasiliev, *3D higher spin gauge theories with matter*, hep-th/9812242 [INSPIRE].

[16] M.A. Vasiliev, *Higher spin gauge theories: star product and AdS space*, in *The many faces of the superworld*, M.A. Shifman ed., World Scientific Singapore (2000), pg. 533 [hep-th/9910096] [INSPIRE].

[17] M. Henneaux and S.-J. Rey, *Nonlinear W_∞ as asymptotic symmetry of three-dimensional higher spin anti-de Sitter gravity*, *JHEP* **12** (2010) 007 [arXiv:1008.4579] [INSPIRE].

[18] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [arXiv:1008.4744] [INSPIRE].

[19] M.R. Gaberdiel and T. Hartman, *Symmetries of holographic minimal models*, *JHEP* **05** (2011) 031 [arXiv:1101.2910] [INSPIRE].

[20] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, *Commun. Math. Phys.* **104** (1986) 207 [INSPIRE].

[21] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, *Phys. Rept.* **223** (1993) 183 [hep-th/9210010] [INSPIRE].

[22] B. Khesin and F. Malikov, *Universal Drinfeld-Sokolov reduction and matrices of complex size*, *Commun. Math. Phys.* **175** (1996) 113 [hep-th/9405116] [INSPIRE].

[23] A. Campoleoni, S. Fredenhagen and S. Pfenninger, *Asymptotic W-symmetries in three-dimensional higher-spin gauge theories*, *JHEP* **09** (2011) 113 [arXiv:1107.0290] [INSPIRE].

[24] M.R. Gaberdiel and R. Gopakumar, *Triality in minimal model holography*, *JHEP* **07** (2012) 127 [arXiv:1205.2472] [INSPIRE].

[25] A. Kuniba, T. Nakanishi and J. Suzuki, *Ferromagnetizations and antiferromagnetizations in RSOS models*, *Nucl. Phys.* **B 356** (1991) 750 [INSPIRE].

[26] D. Altschuler, M. Bauer and H. Saleur, *Level rank duality in nonunitary coset theories*, *J. Phys.* **A 23** (1990) L789 [INSPIRE].

[27] C. Ahn, *The large-N 't Hooft limit of coset minimal models*, *JHEP* **10** (2011) 125 [arXiv:1106.0351] [INSPIRE].

[28] M.R. Gaberdiel and C. Vollenweider, *Minimal model holography for* SO(2N), *JHEP* **08** (2011) 104 [arXiv:1106.2634] [INSPIRE].

[29] C. Ahn, *The primary spin-4 Casimir operators in the holographic* SO(N) *coset minimal models*, *JHEP* **05** (2012) 040 [arXiv:1202.0074] [INSPIRE].

[30] K. Thielemans, *A Mathematica package for computing operator product expansions*, *Int. J. Mod. Phys.* **C 2** (1991) 787 [INSPIRE].

[31] K. Thielemans, *An algorithmic approach to operator product expansions, W-algebras and W-strings*, Ph.D. thesis, KU Leuven, Leuven Belgium (1994) [hep-th/9506159] [INSPIRE].

[32] C. Candu and M.R. Gaberdiel, *Duality in $N = 2$ minimal model holography*, arXiv:1207.6646 [INSPIRE].

[33] H. Kausch and G. Watts, *A study of W algebras using Jacobi identities*, *Nucl. Phys.* **B 354** (1991) 740 [INSPIRE].

JHEP01(2013)185

[34] R. Blumenhagen et al., *W algebras with two and three generators*, *Nucl. Phys.* **B 361** (1991) 255 [INSPIRE].

[35] A. Kliem, *The construction of W algebras*, Diploma thesis, Universität Bonn, Bonn Germany (1991) [INSPIRE].

[36] P. Bowcock and G. Watts, *On the classification of quantum W algebras*, *Nucl. Phys.* **B 379** (1992) 63 [hep-th/9111062] [INSPIRE].

[37] K. Hornfeck, *Classification of structure constants for W algebras from highest weights*, *Nucl. Phys.* **B 411** (1994) 307 [hep-th/9307170] [INSPIRE].

[38] A. Honecker, *Automorphisms of W algebras and extended rational conformal field theories*, *Nucl. Phys.* **B 400** (1993) 574 [hep-th/9211130] [INSPIRE].

[39] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck and R. Hubel, *Coset realization of unifying W algebras*, *Int. J. Mod. Phys.* **A 10** (1995) 2367 [hep-th/9406203] [INSPIRE].

[40] P. Bouwknegt, *Extended conformal algebras from Kac-Moody algebras*, MIT-CTP-1665, M.I.T., Cambridge U.S.A. (1988) [INSPIRE].

[41] A. Honecker, *A note on the algebraic evaluation of correlators in local chiral conformal field theory*, hep-th/9209029 [INSPIRE].

[42] P. Goddard and D.I. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, *Int. J. Mod. Phys.* **A 1** (1986) 303 [INSPIRE].

[43] C. Candu and M.R. Gaberdiel, *Supersymmetric holography on AdS_3*, arXiv:1203.1939 [INSPIRE].

[44] T. Creutzig, Y. Hikida and P.B. Ronne, *$N = 1$ supersymmetric higher spin holography on AdS_3*, arXiv:1209.5404 [INSPIRE].

[45] J. de Boer, L. Feher and A. Honecker, *A class of W algebras with infinitely generated classical limit*, *Nucl. Phys.* **B 420** (1994) 409 [hep-th/9312049] [INSPIRE].

[46] W. Eholzer, A. Honecker and R. Hubel, *How complete is the classification of W symmetries?*, *Phys. Lett.* **B 308** (1993) 42 [hep-th/9302124] [INSPIRE].

[47] A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, *Conical defects in higher spin theories*, *JHEP* **02** (2012) 096 [arXiv:1111.3381] [INSPIRE].

[48] E. Perlmutter, T. Prochazka and J. Raeymaekers, *The semiclassical limit of W_N CFTs and Vasiliev theory*, arXiv:1210.8452 [INSPIRE].

[49] E. Fradkin and V.Y. Linetsky, *Supersymmetric Racah basis, family of infinite dimensional superalgebras,* $\mathrm{SU}(\infty + 1|\infty)$ *and related 2D models*, *Mod. Phys. Lett.* **A 6** (1991) 617 [INSPIRE].

[50] P. Bowcock, *Quasi-primary fields and associativity of chiral algebras*, *Nucl. Phys.* **B 356** (1991) 367 [INSPIRE].