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Brief paper

Event-based broadcasting for multi-agent average consensus☆

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ABSTRACT

A novel control strategy for multi-agent coordination with event-based broadcasting is presented. In particular, each agent decides itself when to transmit its current state to its neighbors and the local control laws are based on these sampled state measurements. Three scenarios are analyzed: Networks of single-integrator agents with and without communication delays, and networks of double-integrator agents. The novel event-based scheduling strategy bounds each agent's measurement error by a time-dependent threshold. For each scenario it is shown that the proposed control strategy guarantees either asymptotic convergence to average consensus or convergence to a ball centered at the average consensus. Moreover, it is shown that the inter-event intervals are lower-bounded by a positive constant. Numerical simulations show the effectiveness of the novel event-based control strategy and how it compares to time-scheduled control.

1. Introduction

In industrial control and automation, there is a growing interest in wireless networks. Control architectures consisting of wireless sensor and actuator nodes, connected over a shared network, are appealing due to high flexibility and low installation cost. Distributed control over wireless networks also plays a major role in multi-vehicle coordination. However, control over networks with limited resources is a challenging task. In this paper, we show that event-based scheduling is beneficial for cooperative networked control and can facilitate the efficient usage of the shared resources.

The cooperative control task under consideration is average consensus, which recently has received major attention in the field of multi-agent coordination (Olfati-Saber, Fax, & Murray, 2007). The consensus problem for single-integrator agents is addressed in Olfati-Saber and Murray (2004). Distributed control laws for networks with and without communication delays are introduced and convergence is analyzed for directed networks with fixed or switching topology, as well as for undirected networks with communication delays. A consensus protocol for double-integrator agents is proposed in Ren and Atkins (2007). In practice, such control laws have to be implemented on digital platforms. The traditional method is time-scheduled periodic sampling, i.e., measurements are taken periodically according to a constant sampling period and the controllers are updated synchronously. The time-scheduled implementation of single-integrator consensus protocols is investigated in Xie, Liu, Wang, and Jia (2009) and conditions on the sampling period for asymptotic convergence are derived. The case of double-integrator agents is addressed in Ren and Cao (2008).

For single feedback control loops, event-based sampling is proposed in Åström and Bernhardsson (1999) as an alternative to traditional periodic sampling. The idea is to sample and update the controller only when measurements cross certain thresholds, which outperforms periodic sampling for the first-order system under consideration. An event-based PID controller is proposed and tested on a double-tank process in Årzén (1999); the CPU usage is reduced while the performance degradation is only minor. In Miskowicz (2006) it is shown that for some wireless sensor networks, event-based sampling is superior to periodic sampling in terms of the number of samples per time unit. In Tabuada (2007), a triggering mechanism for state feedback controllers is presented. An event is triggered and the control

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law re-computed whenever the norm of the measurement error, i.e., the difference between the plant's current state and the state at the last sampling instant, crosses a threshold depending on the norm of the current state. A certain level of performance is guaranteed, while the inter-execution times are lower bounded by a positive constant. This idea is further developed in Mazo and Cao (2011) and Mazo and Tabuada (2011) for control over wireless sensor and actuator networks, where a decentralized event-based implementation of centralized nonlinear controllers is presented. In Wang and Lemmon (2008) and Wang and Lemmon (2011), events are related to measurement broadcasts in distributed networked control systems. Each subsystem decides itself when to broadcast state measurements and the local controllers are updated whenever they receive new state measurements. The proposed triggering mechanism guarantees asymptotic stability of the entire system. Both Mazo and Tabuada (2011) and Wang and Lemmon (2011) suggest that event-based scheduling reduces the number of transmissions required in networked control systems. First steps towards a comprehensive theory for event-driven systems are taken in Heemels, Sandee, and Van Den Bosch (2008).

Event-based scheduling seems to be suitable for cooperative control of multi-agent systems over networks with limited resources. However, only a few studies have considered this topic. An event-based implementation of the consensus protocol is developed in Dimarogonas, Frazzoli, and Johansson (2012). Following the ideas of Tabuada (2007), the authors present a decentralized event-based strategy to determine the control updates such that the overall system reaches average consensus asymptotically. In Dimarogonas et al. (2012), all agents continuously monitor their neighbors' states. Each agent updates its control law not only at its own event-times, but also whenever one of its neighbors triggers an event.

In this paper, we address the limitations of these prior approaches and propose a novel, event-based control strategy for multi-agent average consensus. In contrast to Dimarogonas et al. (2012), measurement broadcasts are scheduled in an event-based fashion, such that continuous monitoring of the neighbors' states is no longer required. Each agent updates its controller whenever it sends or receives a new measurement. This idea is inspired by Wang and Lemmon (2011), but a major difference is that in Wang and Lemmon (2011) each subsystem is aware of the equilibrium state to be stabilized. In the present work, the consensus point is unknown to the agents, which makes it more challenging to find suitable trigger conditions. Each agent decides based on the difference of its current state and its latest broadcast state, called the measurement error, when it has to send a new value. An event is triggered whenever the norm of the measurement error crosses a certain threshold. We propose trigger functions that realize exponentially decreasing thresholds in time with nonnegative offset. This guarantees asymptotic convergence to average consensus or to a ball centered at the consensus. Furthermore, we derive a positive lower bound on the inter-event times. We analyze time-delays in the communication links in networks of single-integrators and show that the approach is valid in this case. We also propose an event-based implementation of the double-integrator consensus protocol, following the same ideas. Time-delays in networks of double-integrators can be treated analogously and are therefore not discussed in this paper. Comparisons to the maximum sampling periods for periodic control (see Ren & Cao, 2008; Xie et al., 2009) show the benefit of our strategy in terms of load on the communication medium.

The rest of the paper is organized as follows. Section 2 contains mathematical preliminaries as well as the problem statement. Section 3 presents the novel event-based control strategy for single-integrator agents. Communication links subject to time-delays are addressed in Section 4. Section 5 discusses double-integrator agents. In Section 6, the novel control strategy is illustrated in numerical examples and compared to the time-scheduled approach. Section 7 concludes this paper.

2. Background and problem statement

This section reviews some facts from algebraic graph theory (Godsil & Royle, 2001), and describes the problem setup.

2.1. Preliminaries

In the following, $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced 2-norm for matrices, respectively.

A scalar continuous function $\gamma(r)$ defined for $r \in [0, a[$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$, and it is said to belong to class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\gamma(r) \to \infty$ as $r \to \infty$. A scalar continuous function $\beta(r, s)$ defined for $r \in [0, a[$, $s \in [0, \infty[$ is said to belong to class $\mathcal{KL}$ if for each fixed s it belongs to class $\mathcal{K}$ and for each fixed r it is decreasing in s and $\beta(r, s) \to 0$ as $s \to \infty$, cf., Khalil (2002).

A dynamical system with state x and input w is called input-to-state stable (ISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that $\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|w_{[0,t[}\|_\infty)$ for all $t \geq 0$, see Sontag (1989). For a signal w, $\|\cdot\|_\infty$ denotes the $\mathcal{L}_\infty$-norm: $\|w_{[0,s[}\|_\infty = \sup_{0\leq t<s} \|w(t)\|$. For linear systems, ISS is equivalent to global asymptotic stability of the unforced system (Khalil, 2002).

2.2. Graph theory

Consider a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ consisting of a set of vertices (or nodes) $\mathcal{V} = \{1, \ldots, N\}$ and edges $\mathcal{E}$. If there is an edge (i, j) between nodes i and j, then i and j are called adjacent, i.e., $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : i, j \text{ adjacent}\}$. $\mathcal{G}$ is called undirected if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. The adjacency matrix A is defined by $a_{ij} = 1$ if i and j are adjacent and $a_{ij} = 0$ otherwise. A path from i to j is a sequence of distinct nodes, starting from i and ending with j, such that each pair of consecutive nodes is adjacent. If there is a path from i to j, then i and j are called connected. If all pairs of nodes in $\mathcal{G}$ are connected, then $\mathcal{G}$ is called connected. The distance $d(i, j)$ between two nodes is the number of edges of the shortest path from i to j. The diameter d of $\mathcal{G}$ is the maximum distance $d(i, j)$ over all pairs of nodes. The degree matrix D of $\mathcal{G}$ is the diagonal matrix with elements d_i equal to the cardinality of node i's neighbor set $N_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The Laplacian matrix L of $\mathcal{G}$ is defined as $L = D - A$. For undirected graphs, L is symmetric and positive semi-definite, i.e., $L = L^T \geq 0$. The row sums of L are zero. Thus, the vector of ones $\mathbf{1}$ is an eigenvector corresponding to eigenvalue $\lambda_1(\mathcal{G}) = 0$, i.e., $L\mathbf{1} = \mathbf{0}$. For connected graphs, L has exactly one zero eigenvalue, and the eigenvalues can be listed in increasing order $0 = \lambda_1(\mathcal{G}) < \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_N(\mathcal{G})$. The second eigenvalue $\lambda_2(\mathcal{G})$ is called the algebraic connectivity.

The next lemma follows from the results in Olfati-Saber and Murray (2004) and will be useful in the remainder of this paper.

Lemma 2.1. *Suppose L is the Laplacian of an undirected, connected graph $\mathcal{G}$. Then, for all $t \geq 0$ and all vectors $v \in \mathbb{R}^N$ with $\mathbf{1}^T v = 0$, it holds that*

$$\|e^{-Lt} v\| \leq e^{-\lambda_2(\mathcal{G})t} \|v\|.$$

Proof. See Seyboth, Dimarogonas, and Johansson (2011). □

2.3. Problem statement

We consider multi-agent systems consisting of N agents and a communication graph $\mathcal{G}$. We impose the following standing assumption.

Assumption 2.2. Graph $\mathcal{G}$ is undirected and connected.

Sections 3 and 4 address single-integrator agents

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{V}, \tag{1}$$

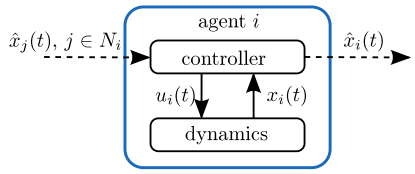

Fig. 1. Single agent in the event-based control setup.

where $x_i(t) \in \mathbb{R}$ is the state and $u_i(t) \in \mathbb{R}$ its control input. In Section 5, double-integrator agents

$$\dot{x}_i(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t), \quad i \in \mathcal{V}, \tag{2}$$

are covered, where $x_i(t) = [\xi_i(t), \zeta_i(t)]^T \in \mathbb{R}^2$. In both cases, the agents are coordinated in a distributed fashion, i.e., $u_i(t)$ depends only on information from neighbors $j \in N_i$. The communication over each edge may be subject to a constant time-delay $\Delta \geq 0$.

Each agent consists of a controller and dynamics as shown in Fig. 1. The controller of agent i monitors its own state $x_i(t)$ continuously. Based on local information, it decides when to broadcast its current state over the network. The latest broadcast state of agent i given by $\hat{x}_i(t) = x_i(t_k^i)$, $t \in [t_k^i, t_{k+1}^i[$, where $t_0^i, t_1^i, \ldots$ is the sequence of event times of agent i. Whenever agent i transmits its state or receives a new state value from one of its neighbors, it re-computes its control u_i immediately.

Problem. Find a triggering rule that determines, based on local information, when agent i has to trigger and broadcast a new state value to its neighbors, such that all agents' states converge to the average of their initial conditions.

3. Single-integrator agents

In this section, it is assumed that there is no time-delay in the communication, i.e., $\Delta = 0$. With the stack vectors $x = [x_1, \ldots, x_N]^T$ and $u = [u_1, \ldots, u_N]^T$, the multi-agent system (1) can be written as $\dot{x}(t) = u(t)$, $x(0) = x_0 \in \mathbb{R}^N$. Recall that the continuous distributed control law

$$u_i(t) = -\sum_{j \in N_i} \big(x_i(t) - x_j(t)\big) \tag{3}$$

globally asymptotically solves the average consensus problem, i.e., $x_i(t) \to 1/N \sum_{i \in \mathcal{V}} x_i(0)$ for all $i \in \mathcal{V}$ as $t \to \infty$, see Olfati-Saber and Murray (2004). The closed-loop system can be written as $\dot{x}(t) = -Lx(t)$. We propose the event-based implementation of (3), given by

$$u_i(t) = -\sum_{j \in N_i} \big(\hat{x}_i(t) - \hat{x}_j(t)\big), \tag{4}$$

or in stack vector form $u(t) = -L\hat{x}(t)$. Note that agent i does not use its true state $x_i(t)$ but the last broadcast value $\hat{x}_i(t)$. Otherwise $u(t)$ might have nonzero average and the agents would not reach average consensus.

The novel triggering strategy works as follows. We define a trigger function $f_i(\cdot)$ for each agent which takes values in $\mathbb{R}$ and depends on local information only, i.e., on time t and the true and broadcast states $x_i(t)$ and $\hat{x}_i(t)$, respectively. In general it is conceivable that $f_i(\cdot)$ also depends on the neighbors' broadcast states $\hat{x}_j(t)$, $j \in N_i$, but this is not pursued here. An event for agent i is triggered as soon as the trigger condition

$$f_i\left(t, x_i(t), \hat{x}_i(t)\right) > 0 \tag{5}$$

is fulfilled. Consequently, the sequence of event-times $0 \leq t_0^i \leq t_1^i \leq t_2^i \leq \cdots$ for agent i is defined iteratively as $t_{k+1}^i = \inf\left\{t : t > t_k^i, f_i(t, x_i(t), \hat{x}_i(t)) > 0\right\}$, where t_0^i is the first instant when (5) is fulfilled. The problem formulated in Section 2 is to find a suitable $f_i(\cdot)$, such that the closed-loop system does not exhibit Zeno behavior (Johansson, Egerstedt, Lygeros, & Sastry, 1999), i.e., there is no trajectory with an infinite number of events in finite time, and reaches average consensus.

For each $i \in \mathcal{V}$ and $t \geq 0$, we define the measurement error $e_i(t) = \hat{x}_i(t) - x_i(t)$. With $e = [e_1, \ldots, e_N]^T$, the closed-loop system is described by

$$\dot{x}(t) = -L\hat{x}(t) = -L(x(t) + e(t)). \tag{6}$$

Let $a(t) = (1/N)\mathbf{1}^T x(t)$ be the average of all states. The derivative of $a(t)$ is $\dot{a}(t) = -(1/N)\mathbf{1}^T u(t) = 0$ since $u(t) = L\hat{x}(t)$ has zero average, i.e., $\mathbf{1}^T u(t) = \mathbf{0}^T$. Thus, $a(t) = a(0) = a$ for all $t \geq 0$ and the state $x(t)$ can be decomposed according to $x(t) = a\mathbf{1} + \delta(t)$, where $\delta(t)$ is the disagreement vector of the multi-agent system, i.e., $\mathbf{1}^T \delta(t) = 0$, following the notation of Olfati-Saber and Murray (2004).

Remark 3.1. From Lemma 2.1 it follows that the rate of convergence for (1) with (3) is at least $\lambda_2(\mathcal{G})$, because $\|\delta(t)\| = \|\exp(-Lt)\delta(0)\| \leq \exp(-\lambda_2(\mathcal{G})t)\|\delta(0)\|$. This has also been shown in Olfati-Saber and Murray (2004) using Lyapunov methods.

The original idea of event-based control in Åström and Bernhardsson (1999) is to trigger an event and close the feedback loop whenever the state deviates from the equilibrium and crosses a predefined threshold c_0, say. Applied to the present problem, this yields the trigger condition $|e_i(t)| > c_0$, i.e., $f_i(e_i(t)) = |e_i(t)| - c_0$. Here, we also consider more general time-dependent functions $f_i(t, e_i(t)) = |e_i(t)| - h_i(t)$ with thresholds $h_i(t) : \mathbb{R}_0^+ \mapsto \mathbb{R}^+$ of the form $h_i(t) = c_0 + c_1 \exp(-\alpha t)$. The behavior of the closed-loop system is characterized in the following theorem.

Theorem 3.2. *Consider the multi-agent system (1) with control law (4). Suppose the trigger function is given by*

$$f_i(t, e_i(t)) = |e_i(t)| - \left(c_0 + c_1 e^{-\alpha t}\right) \tag{7}$$

with constants $c_0 \geq 0$ and $c_1 \geq 0$, $c_0 + c_1 > 0$, and $0 < \alpha < \lambda_2(\mathcal{G})$. Then, for all initial conditions $x_0 \in \mathbb{R}^N$, the closed-loop system does not exhibit Zeno behavior. Moreover, the disagreement vector δ of the closed-loop system converges to a ball centered at the origin with radius

$$r = \|L\|\sqrt{N}c_0/\lambda_2(\mathcal{G}). \tag{8}$$

Proof. From (6) and $\delta(t) = x(t) - a\mathbf{1}$, it follows that $\dot{\delta}(t) = -L\delta(t) - Le(t)$ and therefore

$$\delta(t) = e^{-Lt}\delta(0) - \int_0^t e^{-L(t-s)} Le(s) ds. \tag{9}$$

The disagreement vector is bounded by $\|\delta(t)\| \leq \|\exp(-Lt)\delta(0)\| + \int_0^t \|\exp(-L(t-s))Le(s)\| ds$. Since $Le(t)$ has zero average, Lemma 2.1 can be applied, i.e.,

$$\|\delta(t)\| \leq e^{-\lambda_2 t}\|\delta(0)\| + \int_0^t e^{-\lambda_2(t-s)}\|Le(s)\| ds, \tag{10}$$

where $\lambda_2 = \lambda_2(\mathcal{G})$. Since $\|Le(t)\| \leq \|L\|\|e(t)\|$ and the trigger condition enforces $|e_i(t)| \leq c_0 + c_1 \exp(-\alpha t)$, it follows that

$$\begin{aligned} \|\delta(t)\| &\leq e^{-\lambda_2 t}\|\delta(0)\| \\ &\quad + \|L\|\sqrt{N}\int_0^t e^{-\lambda_2(t-s)}(c_0 + c_1 e^{-\alpha s}) ds \\ &= e^{-\lambda_2 t}\left(\|\delta(0)\| - \|L\|\sqrt{N}\left(\frac{c_0}{\lambda_2} + \frac{c_1}{\lambda_2 - \alpha}\right)\right) \\ &\quad + e^{-\alpha t}\frac{\|L\|\sqrt{N}c_1}{\lambda_2 - \alpha} + \frac{\|L\|\sqrt{N}c_0}{\lambda_2}. \end{aligned} \tag{11}$$

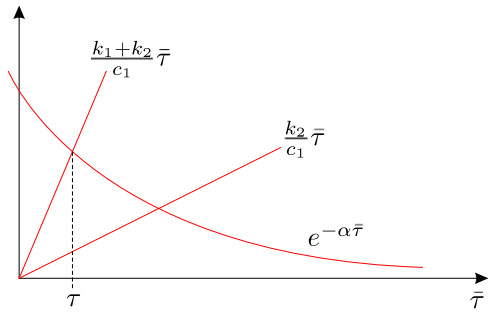

Fig. 2. Solution of the implicit equation for τ.

In order to exclude Zeno behavior, we show that the inter-event times are lower-bounded by a positive constant τ, say. Assume that i triggers at time $t^* \geq 0$. Then $e_i(t^*) = 0$ and $f_i(t^*, e_i(t^*)) \leq 0$. Between two events the time-derivative of $e_i(t)$ is given by $\dot{e}_i(t) = -\dot{x}_i(t) = -u_i(t)$ and thus

$$|e_i(t)| \leq \int_{t^*}^{t} |u_i(s)|\, ds \tag{12}$$

for t between t^* and the next event time. Observe that

$$\begin{aligned} |u_i(t)| &\leq \|u(t)\| = \|L(x(t)+e(t))\| = \|L(\delta(t)+e(t))\| \\ &\leq \|L\| \left(\|\delta(t)\| + \|e(t)\|\right) \\ &\leq \|L\| \left(\|\delta(t)\| + \sqrt{N}(c_0 + c_1 e^{-\alpha t})\right) \end{aligned} \tag{13}$$

for all $i \in \mathcal{V}$. Inequality (13) with (11) gives an upper bound on agent i's control input. Using that, we derive a lower bound $\tau(t^*)$ on the inter-event interval, i.e., the time before f_i crosses zero again. Two different cases are considered ($c_0 \neq 0$ and $c_0 = 0$):

Case 1 Assume $c_0 \neq 0$. From (11) follows that $\|\delta(t)\| \leq \|\delta(0)\| + \left(\|L\|\sqrt{N}c_1\right)/(\lambda_2-\alpha) + \left(\|L\|\sqrt{N}c_0\right)/\lambda_2 = \tilde{\delta}$ for all $t \geq 0$, and with (13), $|u_i(t)| \leq \|L\| \left(\tilde{\delta} + \sqrt{N}(c_0 + c_1)\right) = \tilde{u}$, where $\tilde{\delta}$ and $\tilde{u}$ are positive constants. With (12) it follows that $|e_i(t)| \leq (t - t^*)\tilde{u}$ for $t \geq t^*$ and before the next event time. The next event is triggered as soon as (7) crosses zero, which does not happen before $|e_i(t)| = c_0$, and thus not before $(t - t^*)\tilde{u} = c_0$. Thus, a lower bound on the inter-event times is given by $\tau(t^*) = c_0/\tilde{u}$. This bound holds for all event times t^* and all agents i.

Case 2 Assume $c_0 = 0$. Then inequality (13) with (11) yields an upper bound on $u_i(t)$ which depends on t^*, i.e., $|u_i(t)| \leq \exp(-\lambda_2 t^*)k_1 + \exp(-\alpha t^*)k_2$ for positive constants $k_1 = \|L\| \|\delta(0)\|$ and $k_2 = \|L\|\sqrt{N}c_1(1 + \|L\|/(\lambda_2 - \alpha))$. With (12), it follows that

$$|e_i(t)| \leq \left(e^{-\lambda_2 t^*}k_1 + e^{-\alpha t^*}k_2\right)(t - t^*). \tag{14}$$

The next event will not be triggered before $|e_i(t)| = c_1 \exp(-\alpha t)$. Thus, a lower bound on the inter-event intervals is given by $\tau = t - t^*$ that solves the equation $[\exp((\alpha - \lambda_2)t^*)k_1 + k_2]\tau = c_1 \exp(-\alpha\tau)$. For $\alpha < \lambda_2$ the term in brackets is upper bounded by $k_1 + k_2$ and lower bounded by k_2. For all $t^* \geq 0$ the solutions $\tau(t^*)$ are greater or equal to τ given by $(k_1 + k_2)\tau = c_1 \exp(-\alpha\tau)$, which is strictly positive, as illustrated in Fig. 2.

Since there is a positive lower bound τ on the inter-event intervals in both cases, there are no accumulation points in the event sequences so Zeno behavior is excluded and t diverges. Since the measurement error $e(t)$ is piecewise continuous and the right-hand side of (6) is globally Lipschitz in δ, existence and uniqueness of the solution is guaranteed (Khalil, 2002). Consequently $\|\delta(t)\|$ converges exponentially to a ball with radius (8) as $t \to \infty$. □

Remark 3.3. The graph $\mathcal{G}$ is assumed to be undirected in this paper. However, it is easy to see that Theorem 3.2 extends to strongly connected and balanced directed graphs.

Remark 3.4. The disagreement dynamics are asymptotically stable for $e(t) = 0$ and therefore ISS w.r.t. $e(t)$. This is consistent with Theorem 3.2, which shows that $\|\delta(t)\|$ is bounded for bounded $\|e(t)\|$ and converges to zero asymptotically if $e(t)$ vanishes. The ISS property is exploited in the analysis of the delayed case in Section 4.

Remark 3.5. The radius (8) can be chosen arbitrarily small since it scales with c_0. For $c_1 = 0$ the density of events is independent of c_0 for large t, but for small t the inter-event intervals are short if c_0 is small. This motivates why $c_1 > 0$ might be suitable in practice.

Remark 3.6. Theorem 3.2 states that (8) vanishes for $c_0 = 0$. The closed-loop system reaches average consensus asymptotically in this case. The condition $\alpha < \lambda_2(\mathcal{G})$ is intuitive, because the states should converge faster than the threshold decreases. However, if $c_0 > 0$, a positive lower bound on the inter-event times exists for $\alpha > \lambda_2(\mathcal{G})$ as well, cf., Case 1 in the proof of Theorem 3.2. Consequently, knowledge of $\lambda_2(\mathcal{G})$ is not necessary if $c_0 > 0$.

The class of time-dependent trigger functions can be extended to $f_i(t, e_i(t)) = |e_i(t)| - h_i(t)$ with suitable thresholds $h_i : \mathbb{R}_0^+ \to \mathbb{R}^+$ as stated in the next corollary.

Corollary 3.7. *Consider the multi-agent system (1) with control law (4). Suppose the trigger function is given by*

$$f_i(t, e_i(t)) = |e_i(t)| - h_i(t)$$

with $c_l \exp(-\alpha t) \leq h_i(t) \leq c_u \exp(-\alpha t)$, $i \in \mathcal{V}$, where $0 < c_l < c_u$ and $0 < \alpha < \lambda_2(\mathcal{G})$. Then, for all initial conditions $x_0 \in \mathbb{R}^N$, the closed-loop system does not exhibit Zeno behavior. Moreover, the disagreement vector δ of the closed-loop system converges to the origin asymptotically.

Proof. With $\|e(t)\| \leq \sqrt{N}c_u \exp(-\alpha t)$ and (10), it follows that $\|\delta(t)\| \leq \exp(-\lambda_2 t)\|\delta(0)\| + \left(\|L\|\sqrt{N}c_u\right)/(\lambda_2 - \alpha)(\exp(-\alpha t) - \exp(-\lambda_2 t))$. Assume that agent i triggers at time $t^* \geq 0$. Analogously to (14), the following bound is obtained: $|e_i(t)| \leq \left(\exp(-\lambda_2 t^*)k_1 + \exp(-\alpha t^*)k_2\right)(t - t^*)$ for $t > t^*$ with $k_1 = \|L\|\|\delta(0)\|$ and $k_2 = \|L\|\sqrt{N}c_u(1 + \|L\|/(\lambda_2 - \alpha))$. The next event is not triggered before $|e_i(t)| = c_l \exp(-\alpha t)$. The same argument as in the proof of Theorem 3.2 shows that the inter-event times are lower-bounded by the positive solution τ of $(k_1 + k_2)\tau = c_l \exp(-\alpha\tau)$. □

Remark 3.8. Corollary 3.7 enlarges the class of suitable trigger functions compared to Theorem 3.2 and shows that the agents do not need to share the same trigger function. However, in order to choose an appropriate trigger function, each agent has to be aware of $\lambda_2(\mathcal{G})$. This assumption can partly be avoided by the following observation. In Mohar (1991), a lower bound on $\lambda_2(\mathcal{G})$ in terms of N and diameter d is given: $\lambda_2(\mathcal{G}) \geq 4/(Nd) \geq 4/(N(N-1))$. If α is chosen smaller than this bound, then obviously $\alpha < \lambda_2(\mathcal{G})$. Therefore it is sufficient that each agent is aware of N or an upper bound thereof. A more advanced method to resolve this issue is through distributed estimation of $\lambda_2(\mathcal{G})$, see Yang et al. (2010). In Aragues, Shi, Dimarogonas, Sagues, and Johansson (2012) it is indicated how such an estimation can be used in order to adapt the trigger condition.

4. Single-integrator agents with delayed communication

From a practical point of view, the effect of communication delays needs to be accounted for. The consensus problem in presence of delays and with continuous feedback is treated, e.g., in Bliman and Ferrari-Trecate (2008), Lestas and Vinnicombe (2007), Münz, Papachristodoulou, and Allgöwer (2010) and Olfati-Saber and Murray (2004). We show that the event-based control strategy is still applicable in this case since the closed-loop system with delays is still ISS. Assuming a delay of $\Delta \geq 0$ in all channels, the control law is $u(t) = -L\hat{x}(t-\Delta)$ and we obtain

$$\dot{\delta}(t) = -L\delta(t-\Delta) - Le(t-\Delta). \tag{15}$$

The ISS concept has been extended to time-delay systems in Pepe and Jiang (2006), Teel (1998) and is therefore suitable for the analysis of this system.

Lemma 4.1. *System (15) is ISS w.r.t. $e(t-\Delta)$, i.e., there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $t \geq 0$,*

$$\|\delta(t)\| \leq \beta(\|\delta(0)\|, t) + \gamma\left(\|e_{[-\Delta,t-\Delta[}\|_\infty\right), \tag{16}$$

if and only if $0 \leq \Delta < \pi/(2\lambda_N(\mathcal{G}))$.

Proof. From Pepe and Jiang (2006, Proposition 2.5), it follows that (15) is ISS w.r.t. $e(t-\Delta)$ if and only if the unforced system $\dot{\delta}(t) = -L\delta(t-\Delta)$ is asymptotically stable. This is the case if and only if $0 \leq \Delta < \pi/(2\lambda_N(\mathcal{G}))$, see Olfati-Saber et al. (2007, Theorem 4). □

Since an upper bound on $\|e(t)\|$ is enforced by the triggering mechanism, it follows from Lemma 4.1 that $\|\delta(t)\|$ converges to a ball around the origin if $\Delta \in [0, \pi/(2\lambda_N(\mathcal{G}))[$. This ball scales with the bound on $\|e(t)\|$. It remains to show that Zeno behavior is excluded. For the sake of simplicity we restrict ourselves to trigger functions with constant threshold.

Theorem 4.2. *Consider the multi-agent system (1) with control law $u(t) = -L\hat{x}(t-\Delta)$, $\Delta \in [0, \pi/(2\lambda_N(\mathcal{G}))[$. Suppose the trigger function is given by $f_i(t, e_i(t)) = |e_i(t)| - c_0$ with $c_0 > 0$. Then, for all initial conditions $x(0) = x_0 \in \mathbb{R}^N$, $x(s) = 0$ for $s \in [-\Delta, 0[$, the closed-loop system does not exhibit Zeno behavior. Moreover, the disagreement vector δ of the closed-loop system converges to a ball centered at the origin.*

Proof. From Lemma 4.1, we know that there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (16) holds. The event-based control strategy assures $\|e_{[-\Delta,t-\Delta[}\|_\infty \leq \sqrt{N}c_0$. The control $|u_i(t)|$ is therefore bounded by $|u_i(t)| \leq \|u(t)\| = \|L\delta(t-\Delta) + Le(t-\Delta)\| \leq \|L\|\left(\|\delta(t-\Delta)\| + \sqrt{N}c_0\right) \leq \|L\|\left(\beta(\|\delta(0)\|, 0) + \gamma\left(\sqrt{N}c_0\right) + \sqrt{N}c_0\right) = \tilde{u}$ for all $t \geq 0$. Analogously to the proof of Theorem 3.2, it can be concluded that the inter-event times are lower bounded by $\tau = c_0/\tilde{u}$. The disagreement $\delta(t)$ converges to a ball with radius $r = \gamma\left(\sqrt{N}c_0\right)$. □

Remark 4.3. A similar result can be obtained for more general trigger functions such as (7), as long as α is picked smaller than the rate of convergence of the time-delay system $\dot{\delta}(t) = -L\delta(t-\Delta)$. Estimates of the convergence rate and also the ISS gains β and γ in (16) may be obtained using an LMI approach Fridman, Dambrine, and Yeganefar (2008).

Remark 4.4. In Bliman and Ferrari-Trecate (2008) and Lestas and Vinnicombe (2007), it is shown that in the continuous feedback case consensus is achieved under non-identical symmetric delays $\Delta_{ij} = \Delta_{ji} \leq \Delta$ in the links, if $\Delta < \pi/(2\lambda_N(\mathcal{G}))$. Using the results from Pepe and Jiang (2006), it can be concluded that the disagreement dynamics are ISS with respect to additive disturbances in this case. Consequently, the assumption of identical delays in the links can be relaxed to $\Delta_{ij} = \Delta_{ji} \leq \Delta$.

5. Double-integrator agents

In this section, agents with double-integrator dynamics (2) are addressed. We assume that there is no time-delay, i.e., $\Delta = 0$. We consider the distributed control law proposed in Ren and Atkins (2007) given by

$$u_i(t) = -\sum_{j\in N_i}\left(\xi_i(t) - \xi_j(t)\right) - \mu\sum_{j\in N_i}\left(\zeta_i(t) - \zeta_j(t)\right) \tag{17}$$

with $\mu > 0$. With stack vectors $\xi = [\xi_1, \ldots, \xi_N]^T$ and $\zeta = [\zeta_1, \ldots, \zeta_N]^T$, the closed-loop dynamics can be written as

$$\begin{bmatrix}\dot{\xi}\\ \dot{\zeta}\end{bmatrix} = \Gamma\begin{bmatrix}\xi\\ \zeta\end{bmatrix}, \quad \text{where } \Gamma = \begin{bmatrix}0 & I\\ -L & -\mu L\end{bmatrix}.$$

In Ren and Atkins (2007) it is shown that consensus is reached asymptotically if and only if Γ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. In case of undirected connected $\mathcal{G}$, all eigenvalues of L are real, and thus, by Ren and Atkins (2007, Lemma 4.2), it follows that consensus is achieved for all $\mu > 0$. Define the initial averages $a = 1/N\sum_{i\in\mathcal{V}}\xi_i(0)$ and $b = 1/N\sum_{i\in\mathcal{V}}\zeta_i(0)$. Then, for all $i \in \mathcal{V}$, it holds that $\xi_i(t) \to a + bt$, $\zeta_i(t) \to b$ as $t \to \infty$.

Analogously to the single-integrator case, the broadcast states are described by $\hat{\xi}_i(t) = \xi_i(t_k^i)$ and $\hat{\zeta}_i(t) = \zeta_i(t_k^i)$, $t \in [t_k^i, t_{k+1}^i[$, with corresponding stack vectors $\hat{\xi}$ and $\hat{\zeta}$. As event-based implementation of (17), we propose

$$u(t) = -L\left(\hat{\xi}(t) + \mathrm{diag}(t-t_k^1, \ldots, t-t_k^N)\hat{\zeta}(t) + \mu\hat{\zeta}(t)\right). \tag{18}$$

Note that this control law incorporates a first-order hold instead of a zero-order hold for the position $\xi_i(t)$, i.e., $\xi_i(t)$ is approximated by $\hat{\xi}_i(t) + (t-t_k^i)\hat{\zeta}_i(t)$ in the interval $t \in [t_k^i, t_{k+1}^i[$. In order to express (18) in terms of (17), we define the measurement errors

$$e_\xi(t) = \hat{\xi}(t) + \mathrm{diag}(t-t_k^1, \ldots, t-t_k^N)\hat{\zeta}(t) - \xi(t)$$
$$e_\zeta(t) = \hat{\zeta}(t) - \zeta(t).$$

This yields $u(t) = -L\left(\xi(t) + \mu\zeta(t) + e_\xi(t) + \mu e_\zeta(t)\right)$. Consequently, the closed-loop dynamics of the second-order multi-agent system (2) with control law (18) are

$$\begin{bmatrix}\dot{\xi}\\ \dot{\zeta}\end{bmatrix} = \Gamma\begin{bmatrix}\xi\\ \zeta\end{bmatrix} - \begin{bmatrix}0 & 0\\ L & L\end{bmatrix}e(t), \tag{19}$$

with overall measurement error $e = [e_\xi^T \mu e_\zeta^T]^T$. It can easily be verified that the average velocity b of all agents remains constant over time and the average position is $a + bt$. Thus, the state vector can be decomposed as

$$\xi(t) = a\mathbf{1} + bt\mathbf{1} + \delta_\xi(t)$$
$$\zeta(t) = b\mathbf{1} + \delta_\zeta(t), \tag{20}$$

such that the disagreement vectors $\delta_\xi(t)$ and $\delta_\zeta(t)$ have zero average, i.e., $\mathbf{1}^T\delta_\xi(t) = \mathbf{1}^T\delta_\zeta(t) = 0$. From (19) and (20) we obtain

$$\begin{bmatrix}\dot{\delta}_\xi\\ \dot{\delta}_\zeta\end{bmatrix} = \Gamma\begin{bmatrix}a\mathbf{1} + bt\mathbf{1} + \delta_\xi\\ b\mathbf{1} + \delta_\zeta\end{bmatrix} - \begin{bmatrix}0 & 0\\ L & L\end{bmatrix}e(t) - \begin{bmatrix}b\mathbf{1}\\ 0\end{bmatrix}.$$

With $\delta = [\delta_\xi^T \delta_\zeta^T]^T$ the disagreement dynamics are

$$\dot{\delta}(t) = \Gamma\delta(t) - \begin{bmatrix}0 & 0\\ L & L\end{bmatrix}e(t). \tag{21}$$

The following lemma is a counterpart to Lemma 2.1 for the double-integrator case, based on the results in Ren and Atkins (2007).

Lemma 5.1. *Suppose L is the Laplacian of an undirected, connected graph $\mathcal{G}$ and $\mu > 0$. Then, for all $t \geq 0$ and all vectors $v \in \mathbb{R}^{2N}$ with $[\mathbf{1}^T\mathbf{0}^T]v = [\mathbf{0}^T\mathbf{1}^T]v = 0$, it holds that*

$$\|e^{\Gamma t}v\| \leq e^{\mathrm{Re}(\lambda_3(\Gamma))t}c_V\|v\|,$$

where $c_V = \|V^{-1}\|\|V\|$.

Proof. The matrix Γ has exactly two zero eigenvalues and only one linearly independent eigenvector corresponding to the zero eigenvalues (Ren & Atkins, 2007). The eigenvalues are denoted by $0 = \lambda_1(\Gamma) = \lambda_2(\Gamma) > \mathrm{Re}(\lambda_3(\Gamma)) \geq \cdots \geq \mathrm{Re}(\lambda_{2N}(\Gamma))$. It can be verified that $v_1 = 1/\sqrt{N}[\mathbf{1}^T\mathbf{0}^T]^T$ and $v_2 = 1/\sqrt{N}[\mathbf{0}^T\mathbf{1}^T]^T$ are a left eigenvector and generalized left eigenvector of Γ, respectively, corresponding to eigenvalue zero. Using a non-singular matrix V consisting of v_1^T, v_2^T, and normalized eigenvectors v_j^T corresponding to eigenvalues $\lambda_j(\Gamma), j = 3, \ldots, 2N$, matrix Γ can be transformed to Jordan normal form J. Therefore it holds that $\exp(\Gamma t) = V^{-1}\exp(Jt)V$. It can be verified that $\|\exp(\Gamma t)v\| \leq \exp(\mathrm{Re}(\lambda_3(\Gamma))t)\|V^{-1}\|\|V\|\|v\|$ for all vectors $v \in \mathbb{R}^{2N}$ with $[\mathbf{1}^T\mathbf{0}^T]v = [\mathbf{0}^T\mathbf{1}^T]v = 0$, since $v_1^Tv = v_2^Tv = 0$. □

Theorem 5.2. *Consider the multi-agent system (2) with control law (18). Suppose the trigger function is given by*

$$f_i(t, e_{\xi,i}(t), e_{\zeta,i}(t)) = \left\|\begin{bmatrix} e_{\xi,i}(t) \\ \mu e_{\zeta,i}(t)\end{bmatrix}\right\| - \left(c_0 + c_1 e^{-\alpha t}\right) \tag{22}$$

with constants $c_0 \geq 0$ and $c_1 \geq 0$, $c_0 + c_1 > 0$, and $0 < \alpha < |\mathrm{Re}(\lambda_3(\Gamma))|$. Then, for all initial conditions $\xi_0, \zeta_0 \in \mathbb{R}^N$, the closed-loop system does not exhibit Zeno behavior. Moreover, the disagreement vector δ of the closed-loop system converges to a ball centered at the origin with radius

$$r = c_0 c_V \|L\|\sqrt{2N}/|\mathrm{Re}(\lambda_3(\Gamma))|. \tag{23}$$

Proof. The analytical solution of the disagreement dynamics (21) is given by

$$\delta(t) = e^{\Gamma t}\delta(0) - \int_0^t e^{\Gamma(t-s)}\begin{bmatrix} 0 & 0 \\ L & L\end{bmatrix} e(s)ds.$$

Lemma 5.1 yields $\|\delta(t)\| \leq \exp(\mathrm{Re}(\lambda_3(\Gamma))t)c_V\|\delta(0)\| + c_V\sqrt{2}\|L\| \int_0^t \exp(\mathrm{Re}(\lambda_3(\Gamma))(t-s))\|e(s)\|ds$. The trigger function enforces $\|e(t)\| \leq \sqrt{N}(c_0 + c_1\exp(-\alpha t))$ and therefore

$$\begin{aligned}\|\delta(t)\| &\leq e^{\mathrm{Re}(\lambda_3(\Gamma))t}c_V\|\delta(0)\| \\ &\quad + c_V\sqrt{2N}\|L\|\int_0^t e^{\mathrm{Re}(\lambda_3(\Gamma))(t-s)}\left(c_0 + c_1 e^{-\alpha s}\right)ds \\ &\leq k_1 + k_2 e^{-\alpha t} + k_3 e^{\mathrm{Re}(\lambda_3(\Gamma))t}\end{aligned} \tag{24}$$

with positive constants $k_1 = c_0 c_V\sqrt{2N}\|L\|/|\mathrm{Re}(\lambda_3(\Gamma))|$, $k_2 = c_1 c_V\sqrt{2N}\|L\|/|\mathrm{Re}(\lambda_3(\Gamma)) + \alpha|$, $k_3 = c_V\|\delta(0)\|$. Note that $\mathrm{Re}(\lambda_3(\Gamma)) < -\alpha < 0$.

Next, Zeno behavior is excluded. Assume that i triggers at time $t^* \geq 0$. Observe that for $t \geq t^*$ and before the next event time,

$$\left\|\begin{bmatrix} e_{\xi,i}(t) \\ \mu e_{\zeta,i}(t)\end{bmatrix}\right\| \leq \int_{t^*}^t \left\|\begin{bmatrix} \dot{e}_{\xi,i}(s) \\ \mu \dot{e}_{\zeta,i}(s)\end{bmatrix}\right\| ds \leq \int_{t^*}^t \|\dot{e}(s)\|ds.$$

In this interval the time-derivative of $e(t)$ is given by

$$\dot{e}(t) = \begin{bmatrix} \dot{e}_\xi(t) \\ \mu\dot{e}_\zeta(t)\end{bmatrix} = \begin{bmatrix} \hat{\zeta}(t) - \zeta(t) \\ -\mu u(t)\end{bmatrix} = \begin{bmatrix} e_\zeta(t) \\ -\mu u(t)\end{bmatrix}$$

and therefore $\|\dot{e}(t)\| \leq 1/\mu\|e(t)\| + \mu\|u(t)\|$. The control $u(t)$ is bounded by $\|u(t)\| = \|[-L - \mu L]\delta(t) - [LL]e(t)\| \leq \sqrt{1+\mu^2}\|L\|\|\delta(t)\| + \sqrt{2}\|L\|\|e(t)\|$. With (24) and $\|e(t)\| \leq \sqrt{N}(c_0 + c_1\exp(-\alpha t))$, $\|u(t)\| \leq \sqrt{1+\mu^2}\|L\|(k_1 + k_2\exp(-\alpha t) + k_3\exp(\mathrm{Re}(\lambda_3(\Gamma))t)) + \sqrt{2N}\|L\|(c_0 + c_1\exp(-\alpha t))$. Therefore it holds that $\|\dot{e}(t)\| \leq \left(1/\mu + \mu\sqrt{2}\|L\|\right)\sqrt{N}(c_0 + c_1\exp(-\alpha t)) + \mu\sqrt{1+\mu^2}\|L\|(k_1 + k_2\exp(-\alpha t) + k_3\exp(\mathrm{Re}(\lambda_3(\Gamma))t))$. Two different cases are considered ($c_0 \neq 0$ and $c_0 = 0$):

Case 1 Assume $c_0 \neq 0$. Then $\|\dot{e}(t)\| \leq \left(1/\mu + \mu\sqrt{2}\|L\|\right)\sqrt{N}(c_0 + c_1) + \mu\sqrt{1+\mu^2}\|L\|(k_1 + k_2 + k_3) = C = const$. An upper bound on the measurement error for $t \geq t^*$ and before the next event time is given by $\|[e_{\xi,i}(t), \mu e_{\zeta,i}(t)]^T\| \leq \int_{t^*}^t \|\dot{e}(s)\|ds \leq (t - t^*)C$. The next event will not be triggered before (22) crosses zero, and not before $(t - t^*)C = c_0$. Thus, a positive lower bound τ on the inter-event times is given by $\tau = c_0/C$.

Case 2 Assume $c_0 = 0$. Then $k_1 = 0$ and it holds that $\|\dot{e}(t)\| \leq \left(1/\mu + \mu\sqrt{2}\|L\|\right)\sqrt{N}c_1\exp(-\alpha t^*) + \mu\sqrt{1+\mu^2}\|L\|(k_2\exp(-\alpha t^*) + k_3\exp(\mathrm{Re}(\lambda_3(\Gamma))t^*))$. Denote this bound by $C(t^*)$ since it depends on t^*. The measurement error is bounded by $\|[e_{\xi,i}(t), \mu e_{\zeta,i}(t)]^T\| \leq \int_{t^*}^t \|\dot{e}(s)\|ds \leq (t - t^*)C(t^*)$. The next event will not be triggered before (22) crosses zero, i.e., before $(t - t^*)C(t^*) = c_1\exp(-\alpha t)$. This leads to the implicit equation $c_1\exp(-\alpha\tau) = \mu\sqrt{1+\mu^2}\|L\|(k_2 + k_3\exp((\mathrm{Re}(\lambda_3(\Gamma)) + \alpha)t^*))\tau + \left(1/\mu + \mu\sqrt{2}\|L\|\right)\sqrt{N}c_1\tau$ for $\tau = t - t^*$. Note that $\mathrm{Re}(\lambda_3(\Gamma)) + \alpha < 0$ by assumption. By the same graphical argument as in the proof of Theorem 3.2, it can be concluded that a lower bound on the inter-event times is given by the positive constant τ, which solves the implicit equation for $t^* = 0$.

Since there is a positive lower bound on the inter-event times in both cases, Zeno behavior of the closed-loop system is excluded. Since $e(t)$ is piecewise continuous and the right-hand side of (21) is globally Lipschitz in δ, existence and uniqueness of the solution is guaranteed (Khalil, 2002). Therefore $\|\delta(t)\|$ converges exponentially to the ball $\|\delta\| \leq k_1$ as $t \to \infty$, which is equivalent to (23). □

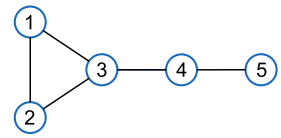

Fig. 3. Communication graph $\mathcal{G}$ of the multi-agent system.

6. Simulations

In this section, the theoretical results are illustrated through simulations and the event-based control strategy is compared to traditional time-scheduled control. A network of five agents with $\mathcal{G}$ as in Fig. 3 is considered.

6.1. Single-integrator agents

The initial conditions $x(0)$ are chosen such that all modes of the system are excited, i.e., if v_i is the normalized eigenvector corresponding to $\lambda_i(\mathcal{G})$, $i \in \mathcal{V}$, the initial conditions are set to $x(0) = (v_2 + \cdots + v_N)/\|v_2 + \cdots + v_N\|$. Fig. 4 shows the simulation result for trigger functions (7). As expected, the states converge close to consensus. Due to the exponentially decreasing threshold in (7), the events are not dense for small times t. Note that c_0 can be chosen arbitrarily small without increasing the event density.

The effectiveness of event-based control becomes clear in the following comparison. In the time-scheduled implementation of (3), the controllers are updated periodically according to a constant sampling period τ_s, i.e.,

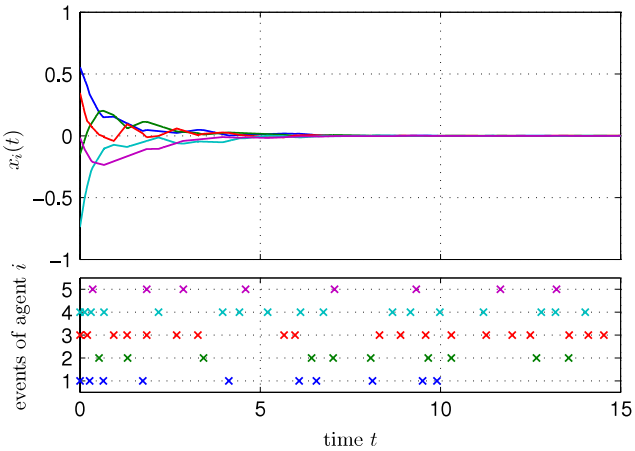

Fig. 4. Simulation result for single-integrator agents and trigger functions (7) with $c_0 = 0.0001$, $c_1 = 0.2499$, and $\alpha = 0.9\lambda_2(\mathcal{G}) = 0.4669$.

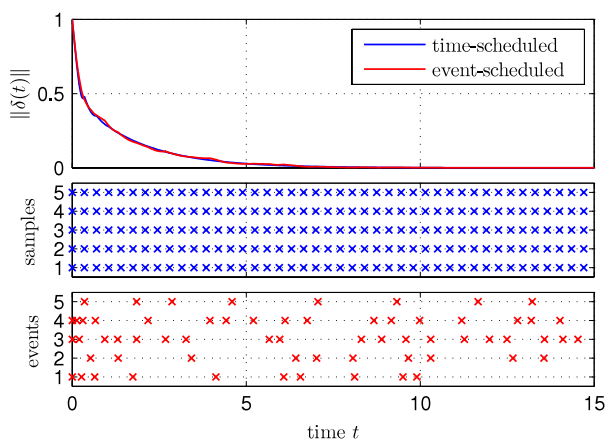

Fig. 5. Comparison of time-scheduled control with constant sampling period $\tau_s = 0.35$ and event-scheduled control.

$$u(t) = -Lx(t_k), \quad t \in [t_k, t_{k+1}[\tag{25}$$

where $t_{k+1} = t_k + \tau_s$, $t_0 = 0$. In Xie et al. (2009) it is shown that (25) globally asymptotically solves the average consensus problem if and only if $0 < \tau_s < 2/\lambda_N(\mathcal{G})$. We set $\tau_s = 0.35$, which yields similar performance to event-based control. Fig. 5 shows a comparison of the control strategies. The average inter-event time over all agents and over time resulting from event-based control is $\tau_{\text{avg}} = 1.2139$, which is more than three times higher than τ_s. The maximum stabilizing sampling period is $2/\lambda_N(\mathcal{G}) = 0.4796$. Consequently, time-scheduled control with average sampling period τ_{avg} would render the system unstable. This illustrates the superiority of event-based control in terms of load on the communication medium.

6.2. Double-integrator agents

Simulation results for double-integrator agents with random initial conditions are shown in Fig. 6. The results are consistent with Theorem 5.2, as the overall system converges to a small region around the average consensus in both velocity and position coordinates. The time-scheduled implementation of (17) is investigated in Ren and Cao (2008). If (17) is updated periodically with constant sampling period τ_s, then, for given μ, consensus is achieved if and only if $\tau_s < \min\{2\mu, 2/(\mu\lambda_N(\mathcal{G}))\}$. Analogously to the single-integrator case, numerical examples show that the average sampling period resulting from event-based control is significantly bigger than the maximum stabilizing sampling period for time-scheduled control. In the present example, the average sampling period is $\tau_{\text{avg}} = 0.7874$, while $2/(\mu\lambda_N(\mathcal{G})) = 0.2398$. Thus, event-based control also outperforms time-scheduled control in terms of network load for the double-integrator case.

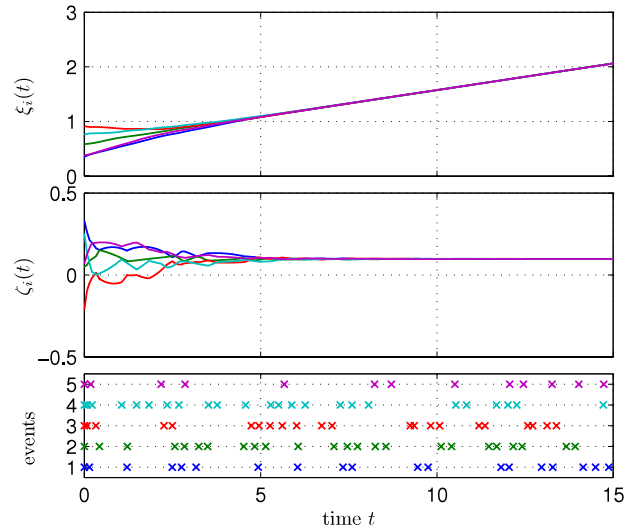

Fig. 6. Simulation result for double-integrator agents and trigger functions (22) with $c_0 = 0.0001$, $c_1 = 0.2499$, $\alpha = 0.9|\text{Re}(\lambda_3(\Gamma))| = 0.4669$, and $\mu = 2$.

7. Conclusions

In this article, a novel event-based control strategy for distributed multi-agent coordination is proposed. Each agent decides based on local information, mainly on the difference between its actual state and its latest broadcast state, when a new measurement has to be transmitted over the network. Time-dependent trigger conditions with exponentially decreasing thresholds on the measurement errors guarantee asymptotic convergence to average consensus, while Zeno behavior is excluded. Small non-vanishing thresholds help to cope with numerical problems or measurement noise, while the convergence to an arbitrarily small region around the consensus point is guaranteed. The proposed strategy extends to networks subject to time-delays as well as groups of double-integrator agents. The effectiveness is illustrated in simulations, including comparisons to traditional time-scheduled control, which demonstrate the superiority of the novel approach in terms of load on the communication medium.

Future work will address networks with time-varying topologies and study the effects of disturbances acting on individual agents.

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