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***Abstract*— This paper presents the design of an event-triggering mechanism for the damped linear Schrödinger equation. Localized damping is considered. The absence of any accumulation points of the time updates sequence is proven, ensuring the avoidance of Zeno behavior. The global exponential stability is ensured through some energy estimates exploiting observability inequality. An illustrative example based on the one dimensional Schrödinger equation demonstrates the efficiency of the results.**

Keywords: Schrödinger equation, Event-triggering mechanism, Global exponential stability, Observability inequality

I. Introduction

Event-triggered control is a control technique used to implement aperiodically a feedback law only when some triggering conditions occur. More precisely, as soon as some specific energy condition is met, the feedback controller is updated and the new control value is transmitted to the actuators. This allows to avoid possible waste of ressources (e.g., computation, communication, and energy) [25]. Event-based control is well studied for classical finite dimensional systems but has been investigated only recently for infinite dimensional system e.g. described by partial differential equations (PDE). For instance, in the context of finite dimensional system, we refer to the seminal works [1], [2] or the most recent ones [24], [10], [9] (for linear systems), [19], [12] (for nonlinear systems). In parallel in the context of inifinite dimensional systems, we refer to [21], [7] regarding parabolic systems and [8], [26], [27] regarding hyperbolic ones.

The Schrödinger equation, most known in quantum theory, arises for instance in nonlinear optics for laser beam propagation or in cold atom physics to describe Bose Einstein condensation. Its solution describes the shape of the probability wave function that governs the motion of small particles, and the equation specifies how these waves are altered by external influences [23]. Several control problems for the linear Schrödinger equation have been studied e.g. in [16] and [17] about exact controllability and stabilization problems, discussed through multiplier techniques and construction of energy functionals. On the other hand, backstepping approach is used in [28], [15], [20] to deal with stabilization issues.

In this paper, considering a possibly locally damped Schrödinger equation, we design an event-triggering update mechanism for the damping, aiming at maintaining the exponential stability of the closed-loop system. We also need to avoid the occurrence of infinitely many updates of the control in a bounded time interval which is known as the Zeno effect. Our approach follows an *emulation method*, where only the event-triggering rules have to be designed, as in [19], [8], [7], contrary to the *co-design method* such as in [22], [11] where the joint design of the control law and the event-triggering conditions are tackled.

In order to avoid the risk of Zeno behavior, the majority of the previous works in the event-triggered control literature added some specific term to the triggering condition as in [8, Definition 2], [5, Definition 3], [3] or constructed dynamical event-triggering mechanism as in [9], [6]. When the event-triggering law is built on the comparison between an error term (the difference of the state value at the last triggering instant and the current one) and a proportion of the energy, it was usually added a term exponentially decreasing and depending on the initial condition as in [3], [8], [13]. Some recent exception to these approaches is detailed in [14] for the wave equation. The current paper deals with Schrödinger equation and follows the same route in order to prove the absence of Zeno phenomenon without any extra exponential term in the event-triggering law. Hence, using an observability inequality for the linear Schrödinger equation, the exponential stability of the closed-loop system under state-based event-triggered control is established. Furthermore, following the same reasoning as in [14] the avoidance of Zeno behavior is guaranteed by showing the absence of accumulation points in the sequence of time updates.

The rest of the paper is organized as follows. In Section II we set up the problem and the PDE system under consideration. The main results on the proposed event-triggering mechanism are presented in Section III. The well-posedness of the associated closed-loop system, some useful intermediate result, the avoidance of the Zeno phenomenon and the exponential stability are exposed. Section IV numerically illustrates the theoretical results. Concluding remarks and perspectives are given in Section V.

Notation. Given an open set $\Omega \subset \mathbb{R}^N$, $L^2(\Omega)$ is the Hilbert space of square integrable scalar functions endowed with the norm $\|z\| = (\int_\Omega |z(x)|^2 dx)^{\frac{1}{2}}$. The gradient and the Laplacian of z are denoted $\nabla z = (\partial_{x_1} z, \dots \partial_{x_N} z)$ and $\Delta z = \sum_{i=1}^N \partial^2_{x_i} z$. We define the Sobolev spaces $H^1_0(\Omega) = \{z \in L^2(\Omega), \nabla z \in \left(L^2(\Omega)\right)^N, z = 0 \text{ on } \partial\Omega\}$, with norm $\|z\|_{H^1_0(\Omega)} = \|\nabla z\|$ and $H^2(\Omega) = \{z \in L^2(\Omega), \nabla z \in \left(L^2(\Omega)\right)^N, \partial_{x_j}\partial_{x_i} z \in L^2(\Omega)\}$, the set of all function such that $\int_\Omega \left(|z|^2 + |\nabla z|^2 + |\Delta z|^2\right) dx$ is finite. The dual space

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of the Sobolev space H is H'. We will often write $\int_\Omega g(t)$ instead of $\int_\Omega g(x,t)dx$ to ease the reading. *Im*(z) and *Re*(z) are respectively the imaginary part and real part of $z \in \mathbb{C}$ and its complex conjugate is $\bar{z}$.

II. Problem formulation

Consider a localized damped linear Schrödinger equation

$$\begin{cases} i\partial_t z(x,t) + \Delta z(x,t) = -i\alpha(x)z(x,t) & (x,t) \in \Omega \times \mathbb{R}^+, \\ z(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}^+ \\ z(x,0) = z_0(x) & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$ and $\alpha \in L^\infty(\Omega;\mathbb{R})$ is the damping coefficient. For $x_0 \in \mathbb{R}^N$, set $\Gamma_0 = \{x \in \partial\Omega, (x - x_0) \cdot \nu(x) > 0\}$ where $\nu(x)$ denotes the unit outward normal vector to Ω at $x \in \partial\Omega$ and $\cdot$ denote the scalar product in $\mathbb{R}^N$. Let $\omega \subset \Omega$ be a neighborhood of $\overline{\Gamma}_0$ and assume there exist $\alpha_0, \alpha_1 > 0$ such that

$$\begin{cases} 0 < \alpha < \alpha_1 & \text{a.e. in } \Omega \\ \alpha \geq \alpha_0 & \text{a.e. in } \omega \subset \Omega. \end{cases} \tag{2}$$

We are interested by the implementation of the control term $-\alpha z$, so that the control signal applied to the plant is updated only at certain instants $\{t_k\}_{k\in\mathbb{N}}$, defined by an event-triggering law. We assume that the control action is held constant between two successive updates. Furthermore, differently from classical periodic sampling techniques, the inter-sampling time $t_{k+1} - t_k$ is not assumed to be constant. The closed-loop system can then be described for all $t \in [t_k, t_{k+1})$ as follows[1]:

$$\begin{cases} i\partial_t z + \Delta z = -i\alpha z(t_k), & \text{in } \Omega \times [t_k, t_{k+1}), k \in \mathbb{N} \\ z = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot, 0) = z_0 & \text{in } \Omega \end{cases} \tag{3}$$

where $0 = t_0 < t_1 < \cdots < t_k < t_{k+1}$.

Therefore, we can summarize the problem we intend to solve as the one of designing a simple triggering condition in order to guarantee (i) the well-posedness of the closed-loop system (3), (ii) the avoidance of any Zeno behavior and (iii) the exponential stability of the closed loop.

In this direction, we will strongly exploit and expand for system (3) the results associated to system (1), as the well-posedness and exponential stability widely studied in the literature. For instance, in [4], it is proven that for any initial conditions $z_0 \in L^2(\Omega)$, there exists a unique weak solution to (1) satisfying

$$z \in C^0(\mathbb{R}^+; L^2(\Omega)) \cap C^1(\mathbb{R}^+; (H^2(\Omega) \cap H_0^1(\Omega))'). \tag{4}$$

Furthermore, for initial data $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, we can prove that the solution to (1) satisfies

$$z \in C^0(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbb{R}^+; H_0^1(\Omega)), \tag{5}$$

and the following exponential stability theorem holds.

[1]The dependence in x and t is omitted to simplify.

Theorem 2.1: For any initial condition in $L^2(\Omega)$, there exist $C > 0$ and $\delta > 0$ such that the weak solution z to (1) verifies for all $t > 0$

$$E(t) \leq CE(0)e^{-2\delta t} \tag{6}$$

where the L^2-energy E is defined by

$$E(t) = \frac{1}{2}\|z(t)\|^2. \tag{7}$$

III. Event-triggering strategy

In order to expand the event-triggering strategy developed in the context of finite-dimensional systems (ODE) as for example in [24], [19], [9], let us introduce the following deviation between the last sampled state and the current one:

$$e_k(x,t) = z(x,t) - z(x,t_k) \tag{8}$$

$\forall x \in \Omega$ and $t \in [t_k, t_{k+1})$. In the sequel, we use the shortcut notation $e_k(t)$ or e_k. Therefore, we can characterize the event-triggering law we propose:

$$t_{k+1} = \inf\Big\{t \geq t_k \text{ such that } \|e_k(t)\|^2 > 2\gamma E(t)\Big\} \tag{9}$$

where $\gamma > 0$ is a design parameter. In other words, as soon as the deviation term gets larger than a $\gamma-$proportion of the energy, an update event is generated.

In the following we split the study into three steps i), ii) and iii) previously mentioned.

A. Well-posedness

Let us begin by defining the maximal time T under which the system (3) subjected to the event-triggering law (9) has a solution:

$$\begin{cases} T = +\infty & \text{if } (t_k) \text{ is a finite sequence,} \\ T = \limsup\limits_{k\to+\infty} t_k & \text{if not.} \end{cases} \tag{10}$$

The absence of Zeno behavior will actually be stemming from the proof that $T = +\infty$ since no accumulation point of the sequence $(t_k)_{k\geq 0}$ will therefore be possible.

Leveraging on some regularity of the classical solutions to the Schrödinger equation we prove the following:

Theorem 3.1: Let Ω be an open bounded domain of class C^2. For any initial conditions $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique strong solution to (3) under the event-triggering mechanism (9), satisfying

$$z \in C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)). \tag{11}$$

Proof: The well-posedness on every sampled interval $[t_k, t_{k+1}]$ is proven by induction.

- **Initialization.** On the first time interval $[0, t_1]$, the control system (3) reads simply

$$\begin{cases} i\partial_t z + \Delta z = -\alpha z_0, & \text{in } \Omega \times [0, t_1), \\ z = 0 & \text{on } \partial\Omega \times (0, t_1), \\ z(0) = z_0, & \text{in } \Omega. \end{cases} \tag{12}$$

This is a Schrödinger equation with initial data $z_0 \in H^2(\Omega) \times H_0^1(\Omega)$ and source term $f(t,x) = -i\alpha z_0(x)$. Since

$z_0 \in H_0^1(\Omega)$, $f \in L^1(0,t_1;H^2(\Omega)\cap H_0^1(\Omega))$. (5) allows to deduce that there exists a unique solution satisfying

$$z \in C([0,t_1];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,t_1];H_0^1(\Omega)).$$

• **Heredity.** Let us bring to the forefront that this solution satisfies $z(t_1) \in H^2(\Omega)\cap H_0^1(\Omega)$ so that system (3) considered on the next time interval $[t_1,t_2)$ has an initial condition in $H^2(\Omega)\cap H_0^1(\Omega)$ and a source term $i\alpha z(t_1) \in L^1(t_1,t_2;H^2(\Omega)\cap H_0^1(\Omega))$.

Hence, the same reasoning holds again and the heredity is proved similarly at any step $k \in \mathbb{N}$.

• **Conclusion.** By induction, the following regularity holds for any $k \in \mathbb{N}, z \in C^0([t_k,t_{k+1}];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([t_k,t_{k+1}];H_0^1(\Omega))$. Therefore, from the extension by continuity at the update instants t_k, one can conclude that (3) has a unique solution in the class (11). ■

B. Avoidance of Zeno behavior

In this section, we address the proof of the absence of Zeno behavior, based on the proof that the maximal time of existence of a solution to the closed-loop system can only be $T=+\infty$. Indeed, proving that no accumulation point of the sequence $(t_k)_{k\geq 0}$ is possible, we ensure the absence of infinite updates in finite time.

Before proving that this phenomenon cannot occur, let us show that the natural energy of the closed-loop system, defined in (7), has a useful property stated in the following lemma. From (8), the closed-loop system reads:

$$\begin{cases} i\partial_t z + \Delta z = -i\alpha z + i\alpha e_k, & \text{in } \Omega \times [t_k,t_{k+1}), \\ z = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot,0) = z_0, & \text{in } \Omega. \end{cases} \quad (13)$$

Lemma 1: Under the triggering law (9) there exists a constant $C>0$ such that for all $t \in (0,T)$:

$$E(0)e^{-2Ct} \leq E(t) \leq E(0)e^{2Ct}. \quad (14)$$

Proof: The time-derivative of $E(t)$ along the trajectories of system (13) is given by

$$\dot{E}(t) = \textit{Re}\left(\int_\Omega \bar{z}(t)\partial_t z(t)\right) = \textit{Im}\left(\int_\Omega i\bar{z}(t)\partial_t z(t)\right)$$
$$= -\textit{Im}\int_\Omega \left(\bar{z}(t)\Delta z(t) - i\alpha(x)|z(t)|^2 + i\alpha(x)e_k(t)\bar{z}(t)\right).$$

By the Green's formula (Lemma 4 in Appendix) with $z=0$ on $\partial\Omega$, and since α takes its values in $\mathbb{R}$,

$$\dot{E}(t) = -\int_\Omega \alpha(x)|z(t)|^2 + \textit{Re}\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right). \quad (15)$$

Then, from Cauchy Schwarz's inequality (see Lemma 3 in Appendix) and assumption (2), we deduce

$$\dot{E}(t) \leq \alpha_1 \|e_k(t)\|\|z(t)\|.$$

Thus, by using the event-triggering law:

$$\|e_k(t)\| \leq \sqrt{2\gamma}\,\|z(t)\|, \quad \forall t \in [t_k,t_{k+1}). \quad (16)$$

Using (16) and the definition (7) of the energy E, we get:

$$\begin{aligned} |\dot{E}(t)| &\leq 2\alpha_1 E(t) + \sqrt{2\alpha_1 E(t)}\sqrt{2E(t)} \\ &\leq 2\alpha_1 E(t) + 2\alpha_1\sqrt{\gamma}E(t) \\ |\dot{E}(t)| &\leq 2CE(t) \text{ with } C = \alpha_1(1+\sqrt{\gamma}). \end{aligned} \quad (17)$$

This shows that $-2CE(t) \leq \dot{E}(t) \leq 2CE(t)$. By Gronwall's Lemma on $[t_k,t]$, the second inequality gives $E(t) \leq E(t_k)\exp\left(\int_{t_k}^t 2Cdu\right), \forall t \geq t_k$, that is $E(t) \leq E(t_k)e^{2C(t-t_k)}$. By applying also the Gronwall's Lemma to the first inequality one gets:

$$E(t) \geq E(t_k)e^{-2C(t-t_k)}.$$

Hence,

$$E(t_k)e^{-2C(t-t_k)} \leq E(t) \leq E(t_k)e^{2C(t-t_k)}. \quad (18)$$

Then taking $t=t_{k+1}$, inequality (18) becomes

$$E(t_k)e^{-2C(t_{k+1}-t_k)} \leq E(t_{k+1}) \leq E(t_k)e^{2C(t_{k+1}-t_k)}.$$

Inferring (18) for $E(t_k)$ allows to deduce

$$\begin{aligned} E(t_{k-1})e^{-2C(t_{k+1}-t_{k-1})} &\leq E(t_{k+1}) \\ leE(t_{k-1})&e^{2C(t_{k+1}-t_{k-1})}. \end{aligned}$$

Since $t_0=0$, by induction we get

$$E(0)e^{-2Ct_{k+1}} \leq E(t_{k+1}) \leq E(0)e^{2Ct_{k+1}}.$$

Then inequality (18) yields

$$E(0)e^{-2Ct_k}e^{-2C(t-t_k)} \leq E(t) \leq E(0)e^{2Ct_k}e^{2C(t-t_k)},$$

showing that (14) holds for all $t \in \mathbb{R}^+$. ■

We can now state the main result of this section.

Theorem 3.2: There is no Zeno Phenomenon for the system (3) under the event-triggering mechanism (9). Equivalently, the maximal time defined by (10) is $T=+\infty$.

Proof: Following the same reasoning as in [24], [9], the proof is based on the study of the function φ defined on $[t_k,t_{k+1})$ by $\varphi : t \mapsto \varphi(t) = \dfrac{\|e_k(t)\|^2}{2\gamma E(t)}$.

The function φ is non negative and satisfies, $\forall k \in \mathbb{N}$, $\varphi(t_k^+)=0$ and jumps from $\varphi(t_{k+1}^-)=1$ to $\varphi(t_{k+1}^+)=0$. Of course, we need to assume that $E(t)\neq 0$, $\forall t>0$, recalling that $E(t)=0$ would mean stopping the updates since, then, E remains null. Let us study the time-derivative of φ :

$$\dot{\varphi}(t) = \frac{\textit{Re}\left(\int_\Omega \partial_t e_k(t)\bar{e}_k(t)\right)}{\gamma E(t)} - \frac{\dot{E}(t)\|e_k(t)\|^2}{2\gamma\left(E(t)\right)^2}. \quad (19)$$

We have from (8), $\partial_t e_k = \partial_t z$ a.e. in Ω, and using equation (13) and the Cauchy Schwartz's inequality we get,

$$\forall t \in [t_k,t_{k+1}),\ \textit{Re}\left(\int_\Omega \partial_t e_k(t)\bar{e}_k(t)\right)$$
$$= \textit{Im}\left(\int_\Omega \Delta z(t)\bar{e}_k(t)\right) - \textit{Re}\left(\int_\Omega \alpha z(t)\bar{e}_k(t) + \alpha|e_k(t)|^2\right)$$
$$\leq \|e_k(t)\|\|\Delta z(t)\| + \alpha_1\|e_k(t)\|\|z(t)\| + \alpha_1\|e_k(t)\|^2.$$

Since for any $z_0 \in H^2(\Omega) \times H^1_0(\Omega)$, the closed-loop system (13) under the event-triggering mechanism (9) has a unique solution $z \in C^0([0,T]; H^2(\Omega) \cap H^1_0(\Omega))$, then there exists a constant $C_1 > 0$ such that $\forall t \in [0,T]$,

$$\|\Delta z(t)\| \leq \|\Delta z\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1, \tag{20}$$

where C_1 depends on $\|z_0\|_{H^2(\Omega)} + \|z_0\|_{H^1_0(\Omega)}$. Then using $\|z(t)\|^2 = 2E(t)$ and (16) it follows :

$$\begin{aligned}
&\textit{Re}\left(\int_\Omega \partial_t e_k(t)\bar{e}_k(t)dx\right)/\gamma E(t) \\
&\leq \frac{C_1\sqrt{2\gamma E(t)}}{\gamma E(t)} + \frac{\alpha_1\sqrt{2\gamma E(t)}\sqrt{2E(t)}}{\gamma E(t)} + 2\alpha_1\varphi(t) \\
&\qquad\qquad \leq \frac{C_1\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2\alpha_1}{\sqrt{\gamma}} + 2\alpha_1\varphi(t) \qquad (21)
\end{aligned}$$

On the other hand, using (17) we get:

$$\frac{-\dot{E}(t)\|e_k(t)\|^2}{2\gamma\left(E(t)\right)^2} \leq 2\alpha_1(1+\sqrt{\gamma})\varphi(t). \tag{22}$$

Gathering the terms (21) and (22) we have:

$$\dot{\varphi}(t) \leq \frac{C_1\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2\alpha_1}{\sqrt{\gamma}} + 2\alpha_1(2+\sqrt{\gamma})\varphi(t).$$

Since $\varphi(t) \leq 1$ from the event-triggering law, it follows

$$\dot{\varphi}(t) \leq \frac{C_1\sqrt{2}}{\sqrt{\gamma E(t)}} + \frac{2\alpha_1}{\sqrt{\gamma}} + 2\alpha_1(2+\sqrt{\gamma}),$$

or equivalently, with $A = \frac{2\alpha_1}{\sqrt{\gamma}} + \alpha_1(2+\sqrt{\gamma})$, $B = C_1\sqrt{\frac{2}{\gamma}}$,

$$\dot{\varphi}(t) \leq A + \frac{B}{\sqrt{E(t)}}.$$

Using Lemma 1, one has $\forall t \in [0,T], E(t) \geq E(0)e^{-2CT}$, and then

$$\dot{\varphi}(t) \leq A + \frac{Be^{CT}}{\sqrt{E(0)}}.$$

Therefore, $\forall k \in \mathbb{N}$, integrating on $[t_k, t_{k+1}]$ knowing that $\varphi(t_k) = 0$ and $\varphi(t_{k+1}) = 1$ we obtain:

$$1 \leq \left[A + \frac{Be^{CT}}{\sqrt{E(0)}}\right](t_{k+1} - t_k). \tag{23}$$

Now let $t_k \to T$ as $k \to +\infty$ in (23), then we get a contradiction if $T \neq +\infty$. We therefore need to get $T = +\infty$ leading to the absence of any accumulation points. Hence, the avoidance of Zeno behavior is guaranteed. ∎

C. Exponential stability

Let us now propose sufficient conditions to ensure the exponential stability of system (3)-(9).

Inspired by [16], we start with the following Lemma.

Lemma 2: Consider the solution z to system (13). For any $\tau > 0$ there exist some constant $K_1, K_2 > 0$ such that

$$E(\tau) \leq K_1 \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dxdt + K_2 \int_0^\tau E(t)dt. \tag{24}$$

Proof: Let $\tau > 0$ and let us recall that the time-derivative of $E(t)$ is

$$\dot{E}(t) = -\int_\Omega \alpha(x)|z(t)|^2 + \textit{Re}\left(\int_\Omega \alpha(x)\bar{e}_k(t)z(t)\right).$$

Integrating this relation on $[0,\tau]$, using (2) and the fact that $\int_\Omega \alpha(x)|z(t)|^2 \geq 0$, we get:

$$E(\tau) \leq E(0) + 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt. \tag{25}$$

Let us introduce the variables y and φ such that $z = y + \varphi$ where z is solution to (13) and $y = y(x,t)$ and $\varphi = \varphi(x,t)$ are solution to the following systems

$$\begin{cases} i\partial_t y + \Delta y = -i\alpha z + i\alpha e_k & \text{in } \Omega \times [t_k, t_{k+1}), \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ y(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{26}$$

and

$$\begin{cases} i\partial_t \varphi + \Delta \varphi = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \varphi = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \varphi(\cdot,0) = z_0 & \text{in } \Omega. \end{cases} \tag{27}$$

Besides, for system (27) the following observability inequality is well-known (owing to [16], [18] and thus, relying on the geometrical condition on ω): $\forall \tau > 0, \exists C_0 > 0$ such that,

$$\|\varphi(0)\|^2 \leq C_0 \int_0^\tau \int_\omega |\varphi(x,t)|^2 dxdt.$$

Hence, from (25), assumption (2), and the fact that $\varphi = z - y$ and that for any $a, b \in \mathbb{R}$, $|a-b|^2 \leq 2(a^2+b^2)$, we have:

$$\begin{aligned}
E(\tau) &\leq \frac{1}{2}\|\varphi(0)\|^2 + 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt \\
&\leq \frac{C_0}{2\alpha_0}\int_0^\tau \int_\omega \alpha(x)|\varphi(x,t)|^2 dxdt + 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt \\
&\leq \frac{C_0}{\alpha_0}\int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt + \frac{C_0\alpha_1}{\alpha_0}\|y\|^2_{L^\infty(0,\tau;L^2(\omega))} \\
&\quad + 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt.
\end{aligned}$$

Using classical energy estimate (see [4]), on the Schrödinger equation (26), for a $L^2((0,\tau)\times\Omega)$-source term, there exists $C > 0$ such that

$$\begin{aligned}
&\|y\|^2_{L^\infty(0,\tau;L^2(\omega))} \leq C\|\alpha(e_k - z)\|^2_{L^2(0,\tau;L^2(\Omega))} \\
&\quad \leq C\alpha_1^2 \int_0^\tau \|e_k(t)\|^2 dt + C\alpha_1 \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt.
\end{aligned}$$

From the event-triggering mechanism, at any time $t \in [0,T]$, one has $\|e_k(t)\|^2 \leq 2\gamma E(t)$, so that

$$\begin{aligned}
&\|y\|^2_{L^2(0,\tau;L^2(\omega))} \\
&\quad \leq 2C\alpha_1^2\gamma \int_0^\tau E(t)dt + C\alpha_1 \int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
E(\tau) &\leq \left(\frac{C_0}{\alpha_0} + \frac{C_0 C\alpha_1^2}{\alpha_0}\right)\int_0^\tau \int_\Omega \alpha(x)|z(t)|^2 dt \\
&\quad + \left(2\alpha_1\sqrt{\gamma} + \frac{2C_0C\alpha_1^3\gamma}{\alpha_0}\right)\int_0^\tau E(t)dt.
\end{aligned}$$

Therefore we get inequality (24) with

$$K_1 = \frac{C_0}{\alpha_0}\left(1 + C\alpha_1^2\right); K_2 = 2\alpha_1\sqrt{\gamma}+2C_0C\alpha_1^3\gamma\alpha_0^{-1}. \quad (28)$$

■

Then we can state and prove the following main exponential stability result.

Theorem 3.3: There exists $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$, for any initial condition $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the closed-loop system (3) under the event-triggering mechanism (9) is exponentially stable with decay rate $\delta > 0$. In other words, there exists $K > 0$ such that

$$E(t) \leq KE(0)e^{-2\delta t}, \quad \forall t > 0. \quad (29)$$

Proof: Let us first discuss the case when the damping does not vanish in Ω (corresponding to $\omega = \Omega$). In that case, one obtains from (15), assumption (2) on the damping, the Cauchy-Schwarz's inequality and the event-triggering law (16) that $\dot{E}(t) \leq \left(-2\alpha_0 + 2\alpha_1\sqrt{\gamma}\right)E(t)$. Thus, $\delta = \alpha_0 - \alpha_1\sqrt{\gamma}$ brings $\dot{E}(t) \leq -2\delta E(t)$ and if γ is small enough compared to α_0 and α_1, then $\delta > 0$ and (29) holds.

In the general case, the damping may vanish outside ω which is a neighborhood of $\overline{\Gamma}_0$ and we will need to use Lemma 2. Integrating (15) on $[0, \tau]$, we obtain:

$$E(\tau) - E(0) \leq 2\alpha_1\sqrt{\gamma}\int_0^\tau E(t)dt - \int_0^\tau\int_\Omega \alpha(x)|z(t)|^2 dt. \quad (30)$$

We can rewrite (24) of Lemma 2 as follows

$$-\int_0^\tau\int_\Omega \alpha(x)|z(t)|^2 dxdt \leq -\frac{1}{K_1}E(\tau) + \frac{K_2}{K_1}\int_0^\tau E(t)dt,$$

Combining this last inequality with (30), we get

$$\left(1 + \frac{1}{K_1}\right)E(\tau) \leq E(0) + \left(2\alpha_1\sqrt{\gamma} + \frac{K_2}{K_1}\right)\int_0^\tau E(t)dt.$$

It brings by Gronwall's Lemma,

$$E(\tau) \leq \frac{K_1}{K_1+1}\exp\left[\frac{K_1}{K_1+1}\left(2\alpha_1\sqrt{\gamma} + \frac{K_2}{K_1}\right)\tau\right]E(0),$$

that can be written as $E(\tau) \leq pe^{C_1\tau}E(0)$ with $p = \frac{K_1}{K_1+1}$, $C_1 = \frac{K_1}{K_1+1}\left(2\alpha_1\sqrt{\gamma} + \frac{K_2}{K_1}\right)$.

Next, we use the fact that the linear Schrödinger equation is invariant by translation in time, and this argument applied on the interval $[(n-1)\tau, n\tau]$, for $n = 1, 2, \ldots$, yields (denoting $a = pe^{C_1\tau}$):

$$E(n\tau) \leq aE((n-1)\tau) \leq \cdots \leq a^n E(0) = e^{-n\tau\kappa}E(0),$$

where we set $a^n = \exp\left(-n\tau\frac{1}{\tau}\ln\left(\frac{1}{a}\right)\right)$ and $\kappa = \frac{1}{\tau}\ln\left(\frac{1}{a}\right)$. Note that $\kappa > 0$ if and only if $a < 1$, so that we must have $pe^{\tau C_1} < 1$ which is equivalent to

$$\tau < -\frac{\ln p}{C_1} = \frac{(K_1+1)\ln\left(\frac{K_1+1}{K_1}\right)}{\left(2K_1\alpha_1\sqrt{\gamma} + K_2\right)}.$$

Now, for every positive time t, there exists $n \in \mathbb{N}^*$ such that $(n-1)\tau < t \leq n\tau$. Using (30) and integration on $[(n-1)\tau, t]$ we have:

$$E(t) \leq E((n-1)\tau) + 2\alpha_1\sqrt{\gamma}\int_{(n-1)\tau}^t E(s)ds$$
$$\leq e^{-n\tau\kappa}e^{\tau\kappa}E(0) + 2\alpha_1\sqrt{\gamma}\int_0^t E(s)ds. \quad (31)$$

Since $e^{-n\tau\kappa} \leq e^{-\kappa t}$ for $t \leq n\tau$, and $e^{\tau\kappa} = 1/a$, we get

$$E(t) \leq \frac{1}{a}e^{-\kappa t}E(0) + 2\alpha_1\sqrt{\gamma}\int_0^t E(s)ds.$$

Then by Gronwall's Lemma, it follows:
$E(t) \leq \frac{1}{a}e^{-\kappa t}e^{2\alpha_1\sqrt{\gamma}t}E(0)$ and if $\gamma \leq \frac{\kappa^2}{4\alpha_1^2}$ then

$$2\delta = \kappa - 2\alpha_1\sqrt{\gamma} \geq 0$$

and we obtain $E(t) \leq \frac{1}{a}e^{-2\delta t}E(t)$. The proof of Theorem 3.3 is complete. ■

Remark 3.1: The existence of design parameter γ depends on the domain ω.

- If $\omega = \Omega$, then the design parameter has to satisfy $\gamma \in (0, \frac{\alpha_0^2}{\alpha_1^2})$ where α_0 and α_1 are given in (2).
- If $\omega \subset \Omega$ is a neighborhood of $\overline{\Gamma}_0$, then the design parameter γ is solution to the inequatility $\kappa - 2\alpha_1\sqrt{\gamma} \geq 0$, which gives:

$$\frac{2C_0C\alpha_1^3}{\alpha_0(K_1+1)}\beta^2 + 4\alpha_1\beta + \frac{1}{\tau}\ln\left(\frac{K_1}{K_1+1}\right) \leq 0 \quad (32)$$

where $\beta = \sqrt{\gamma}$, K_1 is given by (28), C_0 is the contant of observability and C is the constant in the classical energy estimate which are detailed in [18, Theorem 2.2 and equation (5.5)]. Since we have $\frac{\alpha_0(K_1+1)}{2C_0C\alpha_1^3}\ln\left(\frac{K_1}{K_1+1}\right) < 0$, then it is guaranteed that (32) admits two opposit sign roots.

IV. Numerical simulation

We consider the one dimensional Schrödinger equation (3) on $\Omega = (0, \pi)$ with initial condition $z(x, 0) = z_0(x) = \sin(x)$, $x \in [0, \pi]$. For numerical simulations, we use the divided differences on a uniform grid for the space variable and the discretization with respect to time was done using the Crank Nicolson scheme.

We stabilize the system under the event-triggering mechanism (9). With respect to (2), we select the damping coefficient $\alpha(x) = 0$ if $x < \pi/10$ and $\alpha(x) = x - \pi/10$ otherwise, so that we can take $\alpha_0 = \pi/10$, $\alpha_1 = 9\pi/10$ and $\omega = (2\pi/10, \pi)$. Using [18, Theorem 2.2 and eq. (5.5)] we select the constants $C_0 = 2.8$ and $C = 0.18$ and we get $K_1 = 14.513$ from (28) and $\gamma \in (0, 0.3416)$ from (32).

A simulation is done with an appropriate $\gamma = 0.1$ and Figure 1 allows to compare the very much alike imaginary part *Imz* of the numerical solution z to the continuous closed-loop systems (1) (top) and the event triggered one (3)-(9)

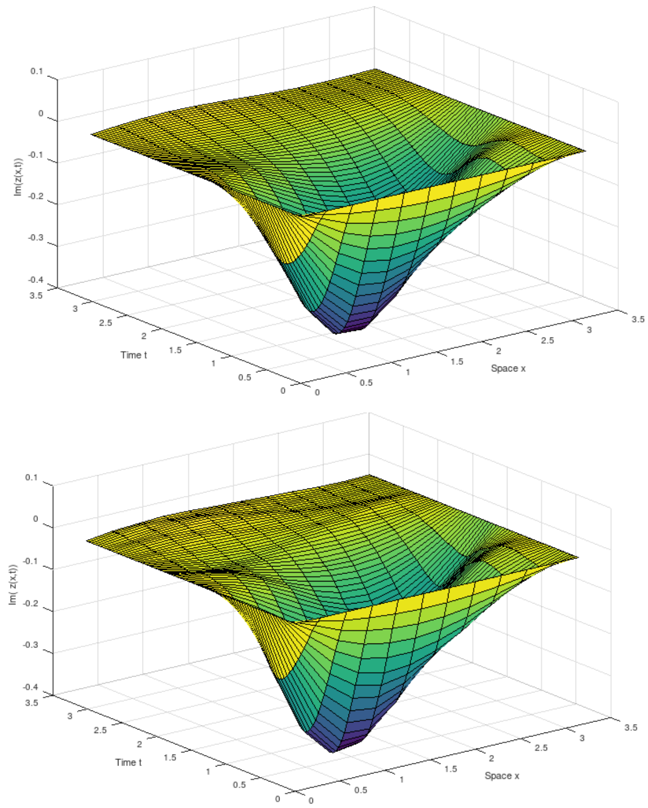

Fig. 1: Imaginary part of the solution: of the closed-loop system (3) under the event-triggering mechanism (9), with $\gamma = 0.1$ (bottom), and of the solution of the continuous closed-loop system (1) (top).

(bottom). It also illustrates the guarantee of the exponential stability of the solution as studied in Theorem 3.3. This is confirmed even more clearly with Figure 2 where we depicted the evolution of the energy of the solution to systems (3) and (1).

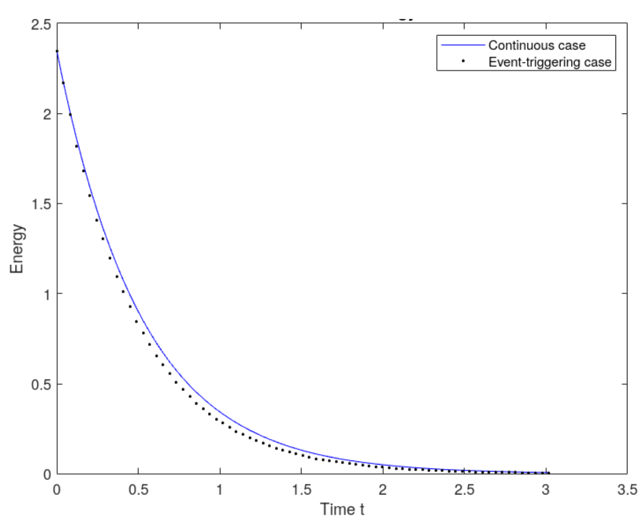

Fig. 2: Time-evolution of the L^2-norm of the solution of the closed-loop systems (3)-(9) (dotted) and (1) (solid line).

V. Conclusion

We considered the problem of exponential stabilization of a damped linear Schrödinger equation under an event-triggering mechanism. Thanks to some regularity of the classical solution to the Schrödinger equation we prove the well-posedness property of the closed loop. We also proved absence of accumulation points in the updates sequence leading to the avoidance of the Zeno behavior. Furthermore, in order to ensure the exponential stability of the closed loop we exploited observability inequality results. Let us mention that we do not know any result proving the exponential stability for periodic sampling.

This paper paves the way for future works. Interesting issues could be to study the presence of input nonlinearity, as saturation, for example.

VI. Acknowledgement

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Appendix

*Lemma 3 (**Cauchy-Schwarz's inequality**):* For any $u, v \in L^2(\Omega)$ it holds

$$\int_\Omega u(x)v(x)dx \leq \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}.$$

*Lemma 4 (**Green's formula**):* Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be a bounded domain with Lipschitz boundary. For all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, n being the outward pointing unit normal vector field, one has

$$\int_\Omega \nabla v \cdot \nabla\, u\, dx = -\int_\Omega v\Delta u\, dx + \int_{\partial\Omega} (n \cdot \nabla v) u\, ds.$$

References

[1] K.E. Åarzén. A simple event-based PID controller. *IFAC Proceedings Volumes*, 32(2):8687–8692, 1999.

[2] K.J. Åström and B. Bernhardsson. Comparison of periodic and event based sampling for first-order stochastic systems. *IFAC Proceedings Volumes*, 32(2):5006–5011, 1999.

[3] L. Baudouin, S. Marx, and S. Tarbouriech. Event-triggered damping of a linear wave equation. *IFAC-PapersOnLine*, 52(2):58–63, 2019.

[4] T. Cazenave, A. Braides, and A. Haraux. *An introduction to semilinear evolution equations*, volume 13. Oxford University Press on Demand, 1998.

[5] N. Espitia, A. Girard, N. Marchand, and C. Prieur. Event-based control of linear hyperbolic systems of conservation laws. *Automatica*, 70:275 – 287, 2016.

[6] N. Espitia, A. Girard, N. Marchand, and C. Prieur. Event-based boundary control of a linear 2×2 hyperbolic system via backstepping approach. *IEEE Transactions on Automatic Control*, 63(8):2686–2693, 2017.

[7] N. Espitia, I. Karafyllis, and M. Krstic. Event-triggered boundary control of constant-parameter reaction-diffusion PDEs: a small-gain approach. In *2020 American Control Conference (ACC)*, pages 3437–3442. IEEE, 2020.

[8] N. Espitia, A. Tanwani, and S. Tarbouriech. Stabilization of boundary controlled hyperbolic PDEs via Lyapunov-based event triggered sampling and quantization. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 1266–1271. IEEE, December 2017.

[9] A. Girard. Dynamic triggering mechanisms for event-triggered control. *IEEE Transactions on Automatic Control*, 60(7):1992–1997, 2015.

[10] W. Heemels, K.H. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. In *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, pages 3270–3285. IEEE, 2012.

[11] W. Heemels, R. Postoyan, M.C.F. Donkers, A.R. Teel, A. Anta, P. Tabuada, and D. Nešic. Periodic event-triggered control. In *Event-based control and signal processing*, pages 105–119. CRC Press/Taylor & Francis, 2015.

[12] L. Hetel, C. Fiter, H. Omran, A. Seuret, E. Fridman, J.P. Richard, and S.I. Niculescu. Recent developments on the stability of systems with aperiodic sampling: An overview. *Automatica*, 76:309–335, 2017.

[13] W. Kang, L. Baudouin, and E. Fridman. Event-triggered control of Korteweg–de Vries equation under averaged measurements. *Automatica*, 123:109315, 2021.

[14] F. Koudohode, L. Baudouin, and S. Tarbouriech. Event-triggering mechanism to damp the linear wave equation. *arXiv preprint arXiv:2107.00292*, 2021.

[15] M. Krstic and A. Smyshlyaev. *Boundary control of PDEs: A course on backstepping designs*. SIAM, 2008.

[16] E. Machtyngier. Exact controllability for the Schrödinger equation. *SIAM Journal on Control and Optimization*, 32(1):24–34, 1994.

[17] E. Machtyngier and E. Zuazua. Stabilization of the Schrodinger equation. *Portugaliae Mathematica*, 51(2):243–256, 1994.

[18] K.D. Phung. Observability and control of Schrödinger equations. *SIAM journal on control and optimization*, 40(1):211–230, 2001.

[19] R. Postoyan, P. Tabuada, D. Nešic, and A. Anta. A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 60(4):982–996, 2014.

[20] B. Ren, J.-M. Wang, and M. Krstic. Stabilization of an ODE–Schrödinger cascade. *Systems & Control Letters*, 62(6):503–510, 2013.

[21] A. Selivanov and E. Fridman. Distributed event-triggered control of diffusion semilinear PDEs. *Automatica*, 68:344–351, 2016.

[22] A. Seuret, C. Prieur, S. Tarbouriech, and L. Zaccarian. LQ-based event-triggered controller co-design for saturated linear systems. *Automatica*, 74:47–54, 2016.

[23] C. Sulem and P.-L. Sulem. *The nonlinear Schrödinger equation: self-focusing and wave collapse*, volume 139. Springer Science & Business Media, 2007.

[24] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9):1680–1685, 2007.

[25] S. Tarbouriech, A. Girard, and L. Hetel. *Control Subject to Computational and Communication Constraints*. Springer, 2018.

[26] J. Wang and M. Krstic. Adaptive event-triggered PDE control for load-moving cable systems. *Automatica*, 129:109637, 2021.

[27] J. Wang and M. Krstic. Event-triggered output-feedback backstepping control of sandwiched hyperbolic PDE systems. *IEEE Transactions on Automatic Control*, pages 1–1, 2021.

[28] J.-M. Wang, B. Ren, and M. Krstic. Stabilization and gevrey regularity of a Schrödinger equation in boundary feedback with a heat equation. *IEEE Transactions on Automatic Control*, 57(1):179–185, 2011.