Event-based Synchronization of Linear Discrete-time Dynamical Networks

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Abstract

This paper investigates the problem of event-based synchronization of linear discrete-time dynamical networks. Leader-following and leaderless synchronizations are achieved by a distributed event-trigger strategy. It is shown that feedback control updating is unnecessary until an event is triggered. The combinational-state variables and the Riccati equation are used to construct a Liapunov function and to design the event-triggering conditions. Numerical examples are provided to illustrate the theoretical results.

Keywords: Event-triggered control, discrete-time synchronization, leader-following / leaderless networks, Riccati equation.

1   Introduction

Cooperative control of dynamical networks, which are modeled as multi-agent systems (MAS), originates from two parallel studies: synchronization of circuit systems [1] and consensus problems inspired by biological distributed behaviors [2]. It has received much research interest in the last two decades [3]–[25]. Previous works on cooperative control focused on first-order and second-order systems [4]–[11]. Recently, research attention is more on MAS with higher-order dynamics [12]–[22] and nonlinear systems [23]–[25]. For first-order and second-order systems, there is no essential difference between the cooperative control of continuous-time MAS and that of discrete-time MAS. While for the general linear systems, the network coupling issue leads to more difficulties in analysis of discrete-time MAS [18]–[22] than the continuous-time counterpart [12]–[17].

Event-based sampling and control have been studied since the late 1990s [26, 27]. This has led to the gradually forming event-triggered control (ETC), which can prevent unnecessary control updates with respect to the traditional periodic control. The ETC theory is first systematically studied in [28], which is based on the Liapunov stability theory. The event-trigger strategy is applied to sensor/actuator networks and generalized to a decentralized form in [29]. In addition, the distributed ETC is analyzed in [30] as well. And the ETC over noisy feedback channels is studied in [31]. Recently, the periodic event-triggered control (PETC) has been proposed in [32, 33]. This

scheme combines the advantages of both ETC and traditional periodic control. In ETC, the event-triggering conditions need to be checked all the time. In PETC, traditional periodic sampling is preserved while the updating of feedback control is event-triggered. Consequently, the event-triggering conditions only need to be checked at sampling instants. Thus, the PETC system can be modeled as the discrete-time ETC [34].

In the past several years, event-based cooperative control has attracted much research interest [35]–[42]. The problems on first-order and second-order systems are resolved in [35]–[39]; and a kind of linear MAS with the system matrices satisfying that $\text{rank}(AB) = \text{rank}(A)$, which contains the single-integrator and double-integrator as special cases, is considered in [40]. For the MAS with general linear dynamics, the problems on the continuous-time system [41, 42] and the sampled system [43] have been resolved. In [41], the algebraic Riccati equation (ARE) is used to design the controller matrix. In [42], the combinational-state variables [38] are used for the Liapunov stability analysis and the event-triggering conditions design. In the case of single-integrator or double-integrator, the event-triggering function can be state-dependent [36, 39] and state-independent [37]. The advantage of the state-dependent method is that the asymptotic convergence can be achieved and the convergence rate is independent from any external signal. Furthermore, the combinational-state approach [38] can help the analysis and design for higher-order systems [42].

In this paper, the event-based leader-following and leaderless synchronizations of linear discrete-time MAS are concerned. An event-trigger strategy is proposed for the feedback control updating. Each follower agent is associated with an event-trigger to detect the event that the value of some event-triggering function becomes positive. When such an event is detected by any one of the agents, the control is updated for each agent using the local information obtained at that event time. The combinational-state variables, which are combinations of each agent's neighbors' states rather than the agents' own states [38], are utilized to design the event-triggering functions and to perform the Liapunov stability analysis; the modified algebraic Riccati equation (MARE) is used to compute the controller matrix and the quadratic matrix for Liapunov function as well. Comparing to the LMI approach in [43] for sampled higher-order MAS, the solvability condition of MARE is straightforward to be checked.

The paper is organized as follows. In Section 2, the problem of event-based discrete-time synchronization is formulated and some preliminaries in graph theory are reviewed. The main results of leader-following and leaderless synchronizations are presented in Section 3. Numerical examples are provided in Section 4. Finally, conclusion is drawn in Section 5.

Nomenclature: Throughout this paper, $\mathbb{R}^p$ and $\mathbb{R}^{p \times q}$ represent the p-dimensional real vector space and the set of all $p \times q$ real matrices, respectively. For $x \in \mathbb{R}^p$, $\|x\|$ denotes its Euclidian norm; and $\|x\|_\infty \triangleq \max_i |x_i|$. For a sequence $\{x(t)\}$, x^+ denotes the time-shift operator defined as $x^+(t) \triangleq x(t+1)$. For $X \in \mathbb{R}^{p \times p}$, its eigenvalues are denoted by $\lambda_1(X)$, $\lambda_2(X)$,..., $\lambda_p(X)$ satisfying that $|\lambda_1(X)| \leq ... \leq |\lambda_p(X)|$; and $\rho(X) = |\lambda_p(X)|$ denotes its spectral radius. For $M \in \mathbb{R}^{p \times q}$, M^T denotes its transpose and $\|M\| \triangleq \sqrt{\rho(M^T M)}$ denotes its spectral norm. A square matrix A is said to be Schur if $\rho(A) < 1$. A matrix pair (A, B) is stabilizable if there exists some matrix F such that $(A + BF)$ is Schur, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. The $p \times p$ identity matrix is denoted by I_p; $\mathbf{1}_p \triangleq [1 \ ... \ 1]^T \in \mathbb{R}^p$; I denotes an identity matrix with compatible dimension; and $\text{diag}\{\cdot\}$

denotes a diagonal matrix. $X \succ Y$ (respectively, $X \succeq Y$) means that $(X - Y)$ is positive definite (respectively, positive semi-definite). Properties of the Kronecker product are reviewed.

Lemma 1. [44] If $A = [a_{ij}] \in \mathbb{R}^{p \times p}$ and $B \in \mathbb{R}^{q \times q}$, then the Kronecker product of A and B, denoted by $A \otimes B$, is defined to be the partitioned matrix $[a_{ij}B] \in \mathbb{R}^{pq \times pq}$. The Kronecker product has the following properties: 1) $(A \otimes B)^T = A^T \otimes B^T$; 2) $(A \otimes B)(C \otimes D) = AC \otimes BD$, where matrix multiplication has a higher priority than "$\otimes$"; and 3) If $A, B \succ 0$, then $A \otimes B \succ 0$.

2 Problem Statement

2.1 Event-based Synchronization Problem

Consider a group of N agents, labeled as $1, 2, ..., N$, with general linear dynamics. The dynamics of the agents are described by

$$x_i^+ = Ax_i + Bu_i, \ i = 1, 2, ..., N, \tag{1}$$

where (A, B) is stabilizable; $x_i \in \mathbb{R}^n$ is the state of agent i; $u_i \in \mathbb{R}^m$ is the control input acting on agent i; and superscript " $+$ " denotes the time-shift operator defined as $x_i^+(t) \triangleq x_i(t+1)$, $t = 0, 1, 2, ...$ Denote $x = [x_1^T x_2^T ... x_N^T]^T$ and $u = [u_1^T u_2^T ... u_N^T]^T$ for notational convenience. The motion of the leader, labeled as $N + 1$, is described by

$$x_{N+1}^+ = Ax_{N+1}. \tag{2}$$

The problem of event-based leader-following synchronization for the agents and leader described above is as follows: For each agent i, design some distributed event-triggering condition to generate an event-triggered updating time sequence $\{t_0, t_1, ...\}$, which only depends on local communication among neighboring agents and the leader; and design some feedback law u_i, which uses only local information at the updating time t_k, $k = 0, 1, ...,$ such that the synchronization

$$\lim_{t \to \infty} \|x_i(t) - x_{N+1}(t)\| = 0, \ \forall i = 1, 2, ..., N, \tag{3}$$

is achieved. While the problem of event-based leaderless synchronization is to achieve the following synchronization:

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \ \forall i, j = 1, 2, ..., N, \tag{4}$$

for multi-agent system (1). The special leader-following case where the leader's state x_{N+1} approaches a fixed point and the special leaderless case where each agent's state converges to a common point are usually called consensus problems.

2.2 Graph Theory

The communication network consisting of N agents is described by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. In this graph, the set of vertices $\mathcal{V} = \{1, 2, ..., N\}$ represents the agents in the group and the set of edges $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : i \neq j\}$, containing unordered pairs of vertices, represents neighboring

relations among the agents. Vertices i and j are said to be adjacent if $(i,j) \in \mathcal{E}$. As usual, define the adjacency matrix $\mathcal{A} \triangleq (a_{ij})$ of graph $\mathcal{G}$ as $a_{ij} = a_{ji} = 1$ if $(i,j) \in \mathcal{E}$, and $a_{ij} = a_{ji} = 0$ otherwise. The Laplacian matrix of graph $\mathcal{G}$ with adjacency matrix $\mathcal{A}$ is given by $L \triangleq D - \mathcal{A} \succeq 0$, where the in-degree matrix D is a diagonal matrix with the i-th diagonal element $d_i \triangleq \sum_{j=1}^{N} a_{ij}$. As such, $\lambda_1(L) = 0$ with an eigenvector $\mathbf{1}_N$. Moreover, $d_i > 0$ and $\lambda_2(L) > 0$ if $\mathcal{G}$ is connected [45].

Let $\overline{\mathcal{G}}$ be an extended graph generated by the leader and the undirected graph $\mathcal{G}$ consisting of N agents, and the matrix $H \triangleq \text{diag}\{h_1, h_2, ..., h_N\}$ be defined as $h_i = 1$ if agent i is a neighbor of the leader, and $h_i = 0$ otherwise; denote $\mathscr{L} \triangleq L + H$. The neighboring set of agent i is defined as $\mathcal{N}(i) \triangleq \{j|a_{ij} \neq 0\} \bigcup \{N+1\}$ if $h_i \neq 0$, and as $\mathcal{N}(i) \triangleq \{j|j \leq N, a_{ij} \neq 0\}$ otherwise. The neighboring set of the leader is defined as $\{i|h_i \neq 0\}$.

Lemma 2. [8] For an extended graph $\overline{\mathcal{G}}$ containing a spanning tree with the leader being the root vertex, $\mathscr{L} \succ 0$.

Lemma 3. (i) If an extended graph $\overline{\mathcal{G}}$ contains a spanning tree with the leader being the root vertex, then

$$\min_{\omega \in \mathbb{R}} \max_{i=1,...,N} |1 - \omega \lambda_i(\mathscr{L})| = \frac{\lambda_N(\mathscr{L}) - \lambda_1(\mathscr{L})}{\lambda_N(\mathscr{L}) + \lambda_1(\mathscr{L})} < 1,$$

$$\arg\min_{\omega} \max_{i=1,...,N} |1 - \omega \lambda_i(\mathscr{L})| = \frac{2}{\lambda_1(\mathscr{L}) + \lambda_N(\mathscr{L})}.$$

(ii) [20] If an undirected graph $\mathcal{G}$ is connected, then

$$\min_{\omega \in \mathbb{R}} \max_{i=2,...,N} |1 - \omega \lambda_i(L)| = \frac{\lambda_N(L) - \lambda_2(L)}{\lambda_N(L) + \lambda_2(L)} < 1, \quad \arg\min_{\omega} \max_{i=2,...,N} |1 - \omega \lambda_i(L)| = \frac{2}{\lambda_2(L) + \lambda_N(L)}.$$

Proof. The results in (ii) are exactly those in [20, (14)]; and (i) can be similarly obtained. □

Accordingly, in a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, the set of vertices $\mathcal{V} = \{1, 2, ..., \tilde{N}\}$; the set of edges $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V} : i \neq j\}$; the adjacency matrix $\mathcal{A} = (a_{ij})$ as $a_{ij} > 0$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise; and the Laplacian matrix $L = D - \mathcal{A}$, where $D = \text{diag}\{d_1, ..., d_{\tilde{N}}\}$ with $d_i = \sum_{j=1}^{\tilde{N}} a_{ij}$.

3 Main Results

Theoretical results are presented in this section. The event-trigger strategy is first described in Section 3.1. Then, the control protocol for leader-following synchronization is designed in Section 3.2. The leader-following synchronization is established in Section 3.3. The leaderless synchronization is studied in Section 3.4.

3.1 Event-trigger Strategy

In this subsection, the event-triggered mechanism is described to generate the updating time sequence $\{t_k\}$.

Algorithm 1. *Event-based updating:*

Step 1. At every time step $t \geq 0$, all agents and the leader broadcast their states to their neighboring agents. The initial time is denoted as the first event time: $t_0 \triangleq 0$. At the beginning of each updating process, $t = t_k$, $k \geq 0$, all agents have received the state information from their neighboring agents, and the feedback control input u_i is updated for each agent i using the local information. Some distributed event-triggering function $f_i(t)$ satisfying that $f_i(t_k) \leq 0$ will be designed using only local information for each agent i.

Step 2. For $t \geq t_k + 1$, the next updating event is triggered at instant $t_{k+1} \geq t_k + 1$ if $f_j(t_{k+1}) > 0$ for some agent j and $f_i(t) \leq 0$ for all $t \in [t_k, t_{k+1})$ and all agents i; if no such an event $f_i(t) > 0$ occurs for any agent i and any time $t \geq t_k + 1$, denote $t_{k+1} \triangleq +\infty$. The feedback control input will be designed later in the form of $u_i(t) = g\left(t, x_i(t_k), a_{ij}x_j(t_k)\right)$, $t \in [t_k, t_{k+1})$.

Step 3. When a finite t_{k+1} is triggered, a new updating cycle will begin, then go to Step 1 and reset $f_i(t_{k+1}) \leq 0$. Thus, $f_i(t) \leq 0$ holds for each agent i all the time.

3.2 Event-based Control Protocol

For a Schur A, zero control input can achieve synchronization. For a singular A, there exists a T such that $T^{-1}AT = \operatorname{diag}\{0, \tilde{A}\}$, where $\tilde{A}$ is nonsingular. Accordingly, denote $T^{-1}B = [B_1^T \ B_2^T]^T$, and $T^{-1}x_i = X_i = [X_i^1 \ X_i^2]^T$. For $i = 1, ..., N$, one has that $(X_i^1)^+ = B_1 u_i$ and $(X_i^2)^+ = \tilde{A}X_i^2 + B_2 u_i$. Then, X_i^2 can be treated as the new state variables for the state-space model of MAS (1) and (2), and the control input u_i can be computed using only X_i^2. It is straightforward that synchronization with respect to X_i^2 implies synchronization with respect to x_i. Thus, Assumption 1 below is made without loss of generality.

Assumption 1. *The pair (A, B) is stabilizable, and A is nonsingular and not Schur.*

The following assumptions are important for the event-based leader-following synchronization.

Assumption 2. *The extended graph $\overline{\mathcal{G}}$ consisting of the N agents and the leader is fixed and contains a spanning tree rooted at the leader.*

Assumption 3. *For the extended graph $\overline{\mathcal{G}}$, $\gamma > \gamma_c$, where*

$$\gamma \triangleq \frac{4\lambda_1(\mathcal{L})\lambda_N(\mathcal{L})}{(\lambda_1(\mathcal{L}) + \lambda_N(\mathcal{L}))^2} \in (0, 1], \tag{5}$$

and γ_c is defined in Lemma 4 below.

Lemma 4. [46] Let $\varepsilon > 0$, $R \in \mathbb{R}^{m \times m}$ and $R \succ 0$. Assume that the pair (A, B) is stabilizable and A is not Schur. For the modified algebraic Riccati equation (MARE)

$$P = g_{\tilde{\gamma}}(P) \triangleq A^T PA - \tilde{\gamma}A^T PB(B^T PB + R)^{-1}B^T PA + \varepsilon I, \tag{6}$$

there exists a critical value $\gamma_c \in [0, 1)$ satisfying that

$$\gamma_c \triangleq \inf \left\{ \tilde{\gamma} \mid \exists \ P \succeq 0 \ s.t. \ P \succ g_{\tilde{\gamma}}(P) \right\} \geq 1 - 1/(\rho(A))^2.$$

For any $\tilde{\gamma} \in (\gamma_c, 1]$, MARE (6) has a unique positive semi-definite solution $P(\varepsilon)$, which is positive definite. An method for numerically calculating γ_c is available in [46, Corollary 1].

Assumption 4. *For the extended graph* $\overline{\mathcal{G}}$,

$$\|B^T P_0 B\| \leq \frac{\lambda_N(\mathcal{L}) + \lambda_1(\mathcal{L})}{\lambda_N(\mathcal{L}) - \lambda_1(\mathcal{L})},$$

where $P_0 \triangleq \lim_{\varepsilon \to 0} P(\varepsilon) \succeq 0$, *and* $P(\varepsilon)$ *is the positive definite solution to MARE (6) with* $\tilde{\gamma} = \gamma$ *defined in (5).*

The event-based control in this paper is based on the *combinational-state variables* $v_i(t)$ and the *combinational-error variables* $w_i(t)$, which will be defined in the following. For $t \in [t_k, t_{k+1})$, $k \geq 0$, define the *error variables* as

$$e_i(t) \triangleq A^{t-t_k} x_i(t_k) - x_i(t), \quad i = 1, ..., N, N+1; \tag{7}$$
$$e(t) \triangleq [e_1^T e_2^T ... e_N^T]^T.$$

For $i = 1, ..., N$, the *combinational-state variables* and the *combinational-error variables* are respectively defined as

$$v_i(t) \triangleq h_i(x_i(t) - x_{N+1}(t)) + \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)); \tag{8}$$

$$w_i(t) \triangleq h_i(e_i(t) - e_{N+1}(t)) + \sum_{j=1}^{N} a_{ij}(e_i(t) - e_j(t)), \tag{9}$$

where $w_i(t)$ depends on the neighboring information $a_{ij}e_j(t)$. One has that $v(t) \triangleq [v_1^T v_2^T ... v_N^T]^T = (\mathcal{L} \otimes I_n)x - H\mathbf{1}_N \otimes x_{N+1}$ and $w(t) \triangleq [w_1^T w_2^T ... w_N^T]^T = (\mathcal{L} \otimes I_n)e - H\mathbf{1}_N \otimes e_{N+1}$. Denote $\tilde{x}_i(t) \triangleq x_i - x_{N+1}$, $\tilde{x}(t) \triangleq [\tilde{x}_1^T \tilde{x}_2^T ... \tilde{x}_N^T]^T$; $\tilde{e}_i(t) \triangleq e_i - e_{N+1}$, $\tilde{e}(t) \triangleq [\tilde{e}_1^T \tilde{e}_2^T ... \tilde{e}_N^T]^T$. In fact, $e_{N+1} \equiv 0$ and $\tilde{e} = e$. Thus,

$$v(t) = (\mathcal{L} \otimes I_n)\tilde{x}(t), \qquad w(t) = (\mathcal{L} \otimes I_n)\tilde{e}(t); \tag{10}$$
$$v(t) = \left(I_N \otimes A^{t-t_k}\right)v(t_k) - w(t), \ t \in [t_k, t_{k+1}). \tag{11}$$

The design of the control protocol for synchronization of multi-agent systems (1) and (2) is performed in three steps.

Algorithm 2. *Event-based Leader-following Synchronization:*

Step 1. Find $P(\varepsilon) \succ 0$ to solve the MARE

$$P = A^T P A - \gamma A^T P B (B^T P B + I)^{-1} B^T P A + \varepsilon I, \tag{12}$$

where $\varepsilon > 0$ is the MARE parameter to be designed, and γ is defined in (5). The existence of $P(\varepsilon)$, as well as a numerically computing method, is referred to [46].

Step 2. Denote that
$$K(\varepsilon) \triangleq -\omega(B^T P(\varepsilon)B + I)^{-1}B^T P(\varepsilon)A, \tag{13}$$

where

$$\omega \triangleq \frac{2}{\lambda_1(\mathcal{L}) + \lambda_N(\mathcal{L})}. \tag{14}$$

For brevity, $P(\varepsilon)$ and $K(\varepsilon)$ are denoted as P and K, respectively in the sequel. By Assumption 1, $\|K^T K\| = \|K\|^2 > 0$. For agents $i = 1, 2, ..., N$, design a feedback law using $x_i(t_k)$ and $a_{ij} x_j(t_k)$ as

$$u_i(t) = K A^{t-t_k} v_i(t_k) = K(v_i(t) + w_i(t)), \quad t \in [t_k, t_{k+1}), \ k \geq 0, \tag{15}$$

where v_i is defined in (8).

Step 3. The updating time sequence $\{t_k\}$ is generated by Algorithm 1, where the event-triggering function $f_i(t)$ for agent i is designed as

$$f_i(t) = \frac{1}{\theta} \varrho_2(\varepsilon, \theta) \|w_i(t)\|^2 - \sigma \left(\varepsilon - \theta \varrho(\varepsilon) \right) \|v_i(t)\|^2 \tag{16}$$

with the parameter σ satisfying $\sigma \in (0,1)$; $\varrho(\varepsilon) \triangleq \min\{\max_i\{2d_i + h_i\} \cdot \|S(\varepsilon)\|, \|\Theta(\varepsilon)\|\}$, where

$$S(\varepsilon) \triangleq -A^T P B K - \lambda_1(\mathscr{L}) K^T B^T P B K, \quad \Theta(\varepsilon) \triangleq -\mathscr{L} \otimes A^T P B K - \mathscr{L}^2 \otimes K^T B^T P B K; \tag{17}$$

the parameter θ to be determined later such that

$$0 < \theta \leq \frac{\varepsilon}{2\varrho(\varepsilon)}; \tag{18}$$

and $\varrho_2(\varepsilon, \theta) \triangleq \min\{\max_i\{2d_i + h_i\} \cdot \|S_2(\varepsilon, \theta)\|, \|\Theta_2(\varepsilon, \theta)\|\}$, where

$$S_2(\varepsilon, \theta) \triangleq -A^T P B K - (1-\theta)\lambda_1(\mathscr{L}) K^T B^T P B K, \quad \Theta_2(\varepsilon, \theta) \triangleq \theta \mathscr{L}^2 \otimes K^T B^T P B K + \Theta(\varepsilon). \tag{19}$$

Remark 1. Noting that the combinational-state variable $v_i(t)$ in (8) and the combinational-error variable $w_i(t)$ in (9) depend on the states $x_j(t)$ and the error variables $e_j(t)$, respectively of the neighbors of agent i, the event-triggering function $f_i(t)$ in (16) for the event-based control updating algorithm requires local information exchange at each time step. This is similar to the results in [39] for sampled single-integrator dynamics. In [38], the inherent minimum inter-event time for continuous-time event-triggered systems is used to design an event-trigger strategy that can prevent continuous information broadcasting. But this method is inapplicable to discrete-time systems. In [37], the event-based broadcasting is studied for first-order and second-order systems, where the convergence rate depends on an external signal since that the event-triggering function is state-independent. In [43], the event-triggered broadcasting strategy is proposed for sampled higher-order systems through the LMI approach, which is subject to a feasibility problem that is not straightforward. Just like in [19, 22], the feasibility of the Riccati design in this paper can be easily checked by verifying Assumptions 3 and 4. Assumption 3 guarantees the synchronizability of the network [19, 22]; and Assumption 4 is devoted to ensuring the existence of a feasible ε for Algorithm 2. It is noted in [47] that if $\rho(A) \leq 1$, which contains the single, double, and higher-order integrator dynamics as special cases, then $\gamma_c = 0$ and $P_0 = 0$; thus, Assumptions 3 and 4 hold for any network graph satisfying Assumption 2.

3.3 Event-based Leader-following Synchronization

An equivalent condition for synchronization is established in the following lemma.

Lemma 5. Let Assumption 2 hold. Then, the synchronization in the sense of (3) is equivalent to $\lim_{t \to \infty} \|v(t)\| = 0$.

Proof. The synchronization in the sense of (3) is equivalent to $\lim_{t\to\infty} \|\tilde{x}(t)\| = 0$. Due to Assumption 2, $\mathscr{L} \otimes I_n$ is positive definite, which can be verified using Lemmas 2 and 1. By (10), one has that $\lim_{t\to\infty} \|\tilde{x}(t)\| = 0$ is equivalent to $\lim_{t\to\infty} \|v(t)\| = 0$. $\qquad\square$

The main result of event-based leader-following synchronization is presented in Theorem 1 below.

Theorem 1. *Consider a multi-agent system consisting of N agents with general linear dynamics (1) and a leader with dynamics (2). Let Assumptions 1, 2, 3 and 4 hold. Then, Algorithms 1 and 2 can achieve exponential synchronization of the multi-agent system. That is, for any $\sigma \in (0,1)$ in event-triggering function (16), there exist an MARE parameter $\varepsilon = \varepsilon^* > 0$ and a corresponding θ in (18) such that $\lim_{t\to\infty} \|x_i(t) - x_{N+1}(t)\| = 0$ exponentially for any $i = 1, ..., N$.*

Proof. Step 1. Closed-loop Dynamics.

For $t \in [t_k, t_{k+1})$, combining (1), (7) and (15), and applying the Kronecker product, one has

$$u(t) = (\mathscr{L} \otimes K)(\tilde{x} + \tilde{e}) = (I_N \otimes K)(v(t) + w(t)), \tag{20}$$

and the closed-loop dynamics are described by

$$\begin{aligned} x^+ &= (I_N \otimes A)x + (I_N \otimes B)u; \\ \tilde{x}^+ &= (I_N \otimes A)\tilde{x} + (I_N \otimes B)u. \end{aligned} \tag{21}$$

Then, since $e(t) = (I_N \otimes A^{t-t_k})x(t_k) - x(t)$, one has that

$$\begin{aligned} e^+ &= (I_N \otimes A)e - (I_N \otimes B)u; \\ \tilde{e}^+ &= (I_N \otimes A)\tilde{e} - (I_N \otimes B)u. \end{aligned} \tag{22}$$

By (10), it is straightforward to verify that

$$v^+ = (\bar{A} + \mathscr{\bar{L}}\bar{B}\bar{K})v + \mathscr{\bar{L}}\bar{B}\bar{K}w, \tag{23}$$

$$w^+ = (\bar{A} - \mathscr{\bar{L}}\bar{B}\bar{K})w - \mathscr{\bar{L}}\bar{B}\bar{K}v, \tag{24}$$

where $\bar{A} \triangleq I_N \otimes A$, $\mathscr{\bar{L}} \triangleq \mathscr{L} \otimes I_n$, $\bar{B} \triangleq I_N \otimes B$, and $\bar{K} \triangleq I_N \otimes K$.

Step 2. Liapunov Analysis.

For the stability analysis of (23), the following quadratic Liapunov function is used:

$$V(v(t)) \triangleq v^T (I_N \otimes P) v = v^T \bar{P} v, \tag{25}$$

where $\bar{P} \triangleq I_N \otimes P$. By (23) and (25), one can evaluate $\Delta V(t) \triangleq V(v(t+1)) - V(v(t))$, which is the variation of V along the discrete-time trajectories of v, as follows:

$$\begin{aligned} \Delta V(t) =& v^T \left(\bar{A}^T \bar{P} \bar{A} - \bar{P} + 2\bar{A}^T \bar{P} \mathscr{\bar{L}}\bar{B}\bar{K} + \bar{K}^T \bar{B}^T \mathscr{\bar{L}} \bar{P} \mathscr{\bar{L}}\bar{B}\bar{K}\right) v + w^T \bar{K}^T \bar{B}^T \mathscr{\bar{L}} \bar{P} \mathscr{\bar{L}}\bar{B}\bar{K} w \\ &+ 2v^T \left(\bar{A}^T \bar{P} \mathscr{\bar{L}}\bar{B}\bar{K} + \bar{K}^T \bar{B}^T \mathscr{\bar{L}} \bar{P} \mathscr{\bar{L}}\bar{B}\bar{K}\right) w \\ =& v^T \left(I_N \otimes \left(A^T PA - P\right) + 2\mathscr{L} \otimes A^T PBK + \mathscr{L}^2 \otimes K^T B^T PBK\right) v \\ &+ w^T \left(\mathscr{L}^2 \otimes K^T B^T PBK\right) w + 2v^T \left(\mathscr{L} \otimes A^T PBK + \mathscr{L}^2 \otimes K^T B^T PBK\right) w. \end{aligned} \tag{26}$$

The positive definiteness of $\mathscr{L}$ implies that there exists an orthogonal matrix $U \in \mathbb{R}^{N \times N}$ such that $\mathscr{L} = U^T \Lambda U$, $\Lambda \triangleq \text{diag}\{\lambda_1(\mathscr{L}), \lambda_2(\mathscr{L}), ..., \lambda_N(\mathscr{L})\}$, where $\lambda_i(\mathscr{L}) > 0$. Denote $\xi(t) \triangleq (U \otimes I_n)v(t)$ and $\xi = [\xi_1^T \xi_2^T ... \xi_N^T]^T$ with $\xi_i(t) \in \mathbb{R}^n$. By (12), one has that

$$
\begin{aligned}
& v^T \left(I_N \otimes \left(A^T P A - P \right) + 2\mathscr{L} \otimes A^T P B K + \mathscr{L}^2 \otimes K^T B^T P B K \right) v \\
&= \xi^T (I_N \otimes (A^T P A - P) + \Lambda \otimes 2A^T P B K + \Lambda^2 \otimes K^T B^T P B K)\xi \\
&= -\sum_{i=1}^{N} \xi_i^T \Phi(\lambda_i(\mathscr{L}))\xi_i,
\end{aligned}
\tag{27}
$$

where $\Phi(\phi) \triangleq -A^T P A + P - 2\phi A^T P B K - \phi^2 K^T B^T P B K$. The proof of the following claim is given in Appendix A.

Claim 1.

$$
-\sum_{i=1}^{N} \xi_i^T \Phi(\lambda_i(\mathscr{L}))\xi_i \le -\varepsilon \xi^T \xi = -\varepsilon v^T v.
\tag{28}
$$

Step 3. MARE Parameter Setting.

Next, the MARE parameter ε will be set such that

$$
\Theta(\varepsilon) = -\mathscr{L} \otimes A^T P B K - \mathscr{L}^2 \otimes K^T B^T P B K \succeq 0,
\tag{29}
$$

which is equivalent to $(U\mathscr{L}^{-\frac{1}{2}} \otimes I_n)\Theta(\varepsilon)(\mathscr{L}^{-\frac{1}{2}}U^T \otimes I_n) = -I_N \otimes A^T P B K - \Lambda \otimes K^T B^T P B K \succeq 0$. Then, it is sufficient to make $-A^T P B K \succeq \lambda_N(\mathscr{L})K^T B^T P B K$, which is guaranteed by setting $\varepsilon = \varepsilon^*$ such that

$$
\|B^T P(\varepsilon^*)B\| \le \frac{\lambda_N(\mathscr{L}) + \lambda_1(\mathscr{L})}{\lambda_N(\mathscr{L}) - \lambda_1(\mathscr{L})}.
\tag{30}
$$

This can be verified by noting that (30) implies $(\frac{\lambda_1(\mathscr{L})+\lambda_N(\mathscr{L})}{2\lambda_N(\mathscr{L})} (B^T P B + I)) \succeq B^T P B$, which further implies $(\frac{1}{\lambda_N(\mathscr{L})\omega} (B^T P B + I)^{-1}) \succeq (B^T P B + I)^{-1}B^T P B(B^T P B + I)^{-1}$. In the remaining proof, ε is set as ε^* following (30) such that (29) holds.

By (29) and noting the fact that

$$
\alpha a^T Q a + \frac{1}{\alpha}b^T Q b \ge \pm 2a^T Q b
\tag{31}
$$

for $Q \succeq 0$ and $\alpha > 0$, one obtains that

$$
-2v^T \Theta(\varepsilon^*)w \le \theta v^T \Theta(\varepsilon^*)v + \frac{1}{\theta}w^T \Theta(\varepsilon^*)w,
\tag{32}
$$

where θ satisfying (18) is the parameter in (16). Then, by (26), (27), (28) and (32), one has

$$
\Delta V(t) \le -\varepsilon v^T v + \theta v^T \Theta(\varepsilon^*)v + \frac{1}{\theta}w^T \Theta_2(\varepsilon^*, \theta)w,
\tag{33}
$$

where $\Theta_2(\varepsilon^*, \theta) = \theta \mathscr{L}^2 \otimes K^T B^T P B K + \Theta(\varepsilon^*)$.

Step 4. Event-triggering Conditions.

The proof of the following claim is given in Appendix A.

Claim 2. The variation of V is upper-bounded as follows:

$$\Delta V(t) \leq \frac{1}{\theta}\varrho_2(\varepsilon^*,\theta)\|w\|^2 - (\varepsilon^* - \theta\varrho(\varepsilon^*))\|v\|^2 = \Sigma_{i=1}^N f_i(t) - (1-\sigma)(\varepsilon^* - \theta\varrho(\varepsilon^*))\|v\|^2, \qquad (34)$$

where $f_i(t)$ and σ are defined in (16) and satisfy that

$$\sum_{i=1}^N f_i(t) = \frac{1}{\theta}\varrho_2(\varepsilon^*,\theta)\|w\|^2 - \sigma(\varepsilon^* - \theta\varrho(\varepsilon^*))\|v\|^2.$$

With the event-triggering function (16), by Claim 2, the event-trigger strategy in Algorithm 1 enforces that $\sum_{i=1}^N f_i(t) \leq 0$ and $\Delta V(t) \leq 0$ for all $t \in [t_k, t_{k+1}]$.

Step 5. Exponential Synchronization.

At the beginning of each updating process, $t = t_k$, $k \geq 0$, if the network is not in synchrony, $v(t_k) \neq 0$, then by (11), for all $t \in [t_k, t_{k+1})$, one has $w + v = (I_N \otimes A^{t-t_k})v(t_k) \neq 0$ since that $(I_N \otimes A^{t-t_k})$ is nonsingular, which results from Assumption 1; meanwhile, the event-trigger strategy in Algorithm 1 enforces that $\frac{1}{\theta}\varrho_2(\varepsilon^*,\theta)\|w\|^2 \leq \sigma(\varepsilon^* - \theta\varrho(\varepsilon^*))\|v\|^2$. Thus, $v(t) \neq 0$, $\forall t \in [t_k, t_{k+1})$. Setting the MARE parameter $\varepsilon = \varepsilon^*$ following (30), one has that for all $t \geq 0$, $\Delta V(t) \leq -(1-\sigma)(\varepsilon^* - \theta\varrho(\varepsilon^*))\|v\|^2 < 0$.

If $v(t_k) = 0$ for some $k \geq 0$, that is, synchronization is achieved in finite time, then it is straightforward to verify that $u(t) \equiv 0$, $w(t) \equiv 0$ and $v(t) \equiv 0$ for all $t \geq t_k$. Consequently, no event will occur again and $t_{k+1} = +\infty$.

By the Liapunov stability theory (see [48]), one obtains the exponential convergence $\lim_{t\to\infty} \|v(t)\| = 0$, which contains the possible finite-time synchronization at some updating time t_k as a special case. Applying Lemma 5, one has that $\lim_{t\to\infty} \|x_i(t) - x_{N+1}(t)\| = 0$ exponentially for each agent i. This completes the proof of Theorem 1. $\qquad\square$

Remark 2. (i) The MARE parameter is theoretically set as $\varepsilon = \varepsilon^*$ specified in (30). The existence of an ε^* is guaranteed by Assumption 4. To numerically determine an ε^* fulfilling (30), the method of bisection [17] can be applied.

(ii) The demonstrated exponential synchronization is based on a quadratic Liapunov function which is constructed through the combinational-state approach [38] and the Riccati design method [41, 19, 22]. In [37], the stability analysis for first-order and second-order systems is performed directly instead of Liapunov analysis, but the convergence rate depends on an external signal. In [43], the Liapunov function is constructed through the LMI approach, which is subject to a feasibility problem that is not straightforward. While like in [19, 22], the feasibility of the MARE approach in this paper can be easily checked.

(iii) From (29), (17) and (19), one has $\Theta_2 \succeq \Theta \succeq 0$ and $S_2 \succeq S \succeq 0$ since $I_N \otimes S \succeq (\mathscr{L}^{-1} \otimes I_n)\Theta \succeq 0$. Therefore, $\varrho_2(\varepsilon, \theta)$ is a decreasing function of θ and $\varrho_2(\varepsilon, \theta) \geq \varrho(\varepsilon)$. To reduce the number of event-triggered updates, by (16), the parameter θ for the event-triggering conditions can be set such that the value of $\left(\frac{1}{\varrho_2(\varepsilon^*,\theta)} \cdot (\varepsilon^* - \varrho(\varepsilon^*)\theta)\theta\right)$ is maximized. One has that $\theta \leq \frac{\varepsilon^*}{2\varrho(\varepsilon^*)}$, which explains why (18) is required. In addition, it is noted that the parameter tuning for θ is independent of the parameter σ.

(iv) From (34), the smaller the parameters σ and θ are, the faster the synchronization will be. On the other side, by (16), smaller σ and θ will lead to more event-triggered updates. So in the setting of the parameters σ and θ, one needs to take into account of the trade-off between the speed of convergence and the number of event-triggered updates.

3.4 Event-based Leaderless Synchronization

The following assumptions are important for the event-based leaderless synchronization.

Assumption 5. *The communication network graph is fixed and connected.*

Assumption 6. *For the undirected graph $\mathcal{G}$, $\gamma > \gamma_c$, where*

$$\gamma = \frac{4\lambda_2(L)\lambda_N(L)}{(\lambda_2(L) + \lambda_N(L))^2} \in (0, 1].$$ (35)

and γ_c is defined in Lemma 4.

Assumption 7. *For the undirected graph $\mathcal{G}$,*

$$\|B^T P_0 B\| \leq \frac{\lambda_N(L) + \lambda_2(L)}{\lambda_N(L) - \lambda_2(L)},$$

where $P_0 \triangleq \lim_{\varepsilon \to 0} P(\varepsilon) \succeq 0$, and $P(\varepsilon)$ is the positive definite solution to MARE (6) with $\tilde{\gamma} = \gamma$ defined in (35).

The *combinational-state variables* and the *combinational-error variables* are redefined for the leaderless case as

$$v_i(t) \triangleq \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t));$$ (36)

$$w_i(t) \triangleq \sum_{j=1}^{N} a_{ij}(e_i(t) - e_j(t)), \quad i = 1, ..., N.$$ (37)

Denoting $v(t) \triangleq [v_1^T v_2^T ... v_N^T]^T$ and $w(t) \triangleq [w_1^T w_2^T ... w_N^T]^T$, one has that $(\mathbf{1}_N^T \otimes I_n)v = 0$ due to $\Sigma_{i=1}^N v_i = 0$; and

$$v(t) = (L \otimes I_n)x(t), \qquad w(t) = (L \otimes I_n)e(t);$$ (38)

$$v(t) = (I_N \otimes A^{t-t_k})v(t_k) - w(t), \ t \in [t_k, t_{k+1}).$$ (39)

The design of the control protocol for synchronization of multi-agent system (1) is performed in three steps.

Algorithm 3. Event-based Leaderless Synchronization:

Step 1. For the MARE parameter $\varepsilon > 0$ that is to be designed, find $P \succ 0$ to solve MARE (12) with γ redefined in (35).

Step 2. For agents $i = 1, 2, ..., N$, construct a feedback law using $x_i(t_k)$ and $a_{ij}x_j(t_k)$ as

$$u_i(t) = KA^{t-t_k}v_i(t_k) = K(v_i(t) + w_i(t)), \quad t \in [t_k, t_{k+1}), \ k \geq 0,$$ (40)

where v_i is defined in (36) and K is the same as (13), that is,

$$K = -\omega(B^T PB + I)^{-1}B^T PA \tag{41}$$

but with

$$\omega \triangleq \frac{2}{\lambda_2(L) + \lambda_N(L)}. \tag{42}$$

Step 3. The updating time sequence $\{t_k\}$ is generated by Algorithm 1, where the event-triggering function $f_i(t)$ for agent i is designed as

$$f_i(t) = \frac{1}{\theta}\varrho_2(\varepsilon, \theta)\|w_i(t)\|^2 - \sigma\left(\varepsilon - \theta\varrho(\varepsilon)\right)\|v_i(t)\|^2 \tag{43}$$

with the parameter σ satisfying $\sigma \in (0, 1)$; $\varrho(\varepsilon)$ redefined as $\varrho(\varepsilon) \triangleq \min\{2\max_i\{d_i\}\cdot\|S(\varepsilon)\|, \|\Theta(\varepsilon)\|\}$, where $S(\varepsilon) \triangleq -A^T PBK - \lambda_2(L)K^T B^T PBK$, $\Theta(\varepsilon) \triangleq -L \otimes A^T PBK - L^2 \otimes K^T B^T PBK$; the parameter θ to be determined later such that

$$0 < \theta \le \frac{\varepsilon}{2\varrho(\varepsilon)}; \tag{44}$$

and $\varrho_2(\varepsilon, \theta)$ redefined as $\varrho_2(\varepsilon, \theta) \triangleq \min\{2\max_i\{d_i\}\cdot\|S_2(\varepsilon, \theta)\|, \|\Theta_2(\varepsilon, \theta)\|\}$, where

$$S_2(\varepsilon, \theta) \triangleq -A^T PBK - (1-\theta)\lambda_2(L)K^T B^T PBK, \quad \Theta_2(\varepsilon, \theta) \triangleq \theta L^2 \otimes K^T B^T PBK + \Theta(\varepsilon).$$

An equivalent condition for synchronization based on the combinational-state variables [38] is established in the following lemma, the idea for which is from [42].

Lemma 6. [42] Let Assumption 5 hold. Then, the synchronization in the sense of (4) is equivalent to $\lim_{t\to\infty}\|v(t)\| = 0$.

Proof. See Appendix B. $\square$

The main result of event-based leaderless synchronization is presented in the following theorem.

Theorem 2. *Consider a multi-agent system consisting of N agents with general linear dynamics (1). Let Assumptions 1, 5, 6 and 7 hold. Then, Algorithms 1 and 3 can achieve exponential synchronization of the multi-agent system. That is, for any $\sigma \in (0, 1)$ in event-triggering function (43), there exist an MARE parameter $\varepsilon = \varepsilon^* > 0$ and a corresponding θ in (44) such that $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$ exponentially for any $i, j = 1, 2, ..., N$.*

Proof. For $t \in [t_k, t_{k+1})$, combining (1), (7) and (40), one has that

$$u(t) = (L \otimes K)(x + e) = (I_N \otimes K)(v(t) + w(t)), \tag{45}$$

and the closed-loop dynamics are described by

$$x^+ = (I_N \otimes A)x + (I_N \otimes B)u; \tag{46}$$

$$e^+ = (I_N \otimes A)e - (I_N \otimes B)u. \tag{47}$$

By (38), it is straightforward to verify that

$$v^+ = (\bar{A} + \bar{L}\bar{B}\bar{K})v + \bar{L}\bar{B}\bar{K}w, \tag{48}$$

$$w^+ = (\bar{A} - \bar{L}\bar{B}\bar{K})w - \bar{L}\bar{B}\bar{K}v, \tag{49}$$

where $\bar{A} = I_N \otimes A$, $\bar{L} \triangleq L \otimes I_n$, $\bar{B} = I_N \otimes B$, and $\bar{K} = I_N \otimes K$.

For the stability analysis of (48), the same Liapunov function is used as in (25), that is, $V(v(t)) = v^T \bar{P} v = v^T (I_N \otimes P(\varepsilon)) v$. Similar to (26), one has that

$$
\begin{aligned}
\Delta V(t) =& v^T \left(\bar{A}^T \bar{P} \bar{A} - \bar{P} + 2\bar{A}^T \bar{P}\bar{L}\bar{B}\bar{K} + \bar{K}^T \bar{B}^T \bar{L}\bar{P}\bar{L}\bar{B}\bar{K} \right) v + w^T \bar{K}^T \bar{B}^T \bar{L}\bar{P}\bar{L}\bar{B}\bar{K} w \\
&+ 2v^T \left(\bar{A}^T \bar{P}\bar{L}\bar{B}\bar{K} + \bar{K}^T \bar{B}^T \bar{L}\bar{P}\bar{L}\bar{B}\bar{K} \right) w \\
=& v^T \left(I_N \otimes \left(A^T P A - P \right) + 2L \otimes A^T P B K + L^2 \otimes K^T B^T P B K \right) v \\
&+ w^T \left(L^2 \otimes K^T B^T P B K \right) w + 2v^T \left(L \otimes A^T P B K + L^2 \otimes K^T B^T P B K \right) w. \tag{50}
\end{aligned}
$$

The positive semi-definiteness of L implies that there exists an orthogonal matrix $U \in \mathbb{R}^{N \times N}$ such that $L = U^T \Lambda U$, $\Lambda \triangleq \mathrm{diag}\{0, \lambda_2(L), ..., \lambda_N(L)\}$. The first row of U is a left-eigenvector of L, with all elements being $\sqrt{N}/N$. Denote $\xi(t) \triangleq (U \otimes I_n)v(t)$ and $\xi = [\xi_1^T \xi_2^T ... \xi_N^T]^T$ with $\xi_i(t) \in \mathbb{R}^n$ and $\xi_1 = 0$. By (12) and similar to Theorem 1,

$$
\begin{aligned}
&v^T \left(I_N \otimes \left(A^T P A - P \right) + 2L \otimes A^T P B K + L^2 \otimes K^T B^T P B K \right) v \\
=\ & \xi^T (I_N \otimes (A^T P A - P) + \Lambda \otimes 2A^T P B K + \Lambda^2 \otimes K^T B^T P B K)\xi \leq -\varepsilon \xi^T \xi = -\varepsilon v^T v.
\end{aligned}
$$

Next, set the MARE parameter as $\varepsilon = \varepsilon^*$ such that

$$\|B^T P(\varepsilon^*)B\| \leq \frac{\lambda_N(L) + \lambda_2(L)}{\lambda_N(L) - \lambda_2(L)}. \tag{51}$$

Then, one has that

$$\Theta(\varepsilon) = -L \otimes A^T P B K - L^2 \otimes K^T B^T P B K \succeq 0.$$

The remainder of the proof of Theorem 2 is similar to that of Theorem 1. ☐

Remark 3. (i) Although the Riccati design works for any homogeneous network dynamics satisfying Assumptions 1, 2, 3 and 4, it may result in more updates than the LMI approach [43, 42].

(ii) The results in Theorems 1 and 2 cannot be directly extended to the case of heterogeneous agents since that: 1) the controller in (15) or (40) becomes meaningless if the agent dynamics are not identical; 2) the properties such as (26) and (50) do not hold for heterogeneous networks and the Liapunov stability analysis becomes more difficult.

(iii) The proposed event-based synchronization algorithms can be extended to directed graphs. Noting that for a directed graph $\mathcal{G}$ containing a spanning tree, there exists a positive diagonal matrix $\Sigma \succ 0$ such that $L\Sigma + \Sigma L^T \succeq 0$ [49]. This matrix $(L\Sigma + \Sigma L^T)$ can be used, instead of the Laplacian matrix L, for Liapunov stability analysis.

4　Numerical Examples

In this section, simulation results are presented to illustrate the theoretical results.

(a) The leader-following network graph.   (b) The leaderless network graph.

Figure 1: The communication network topologies.

4.1   Leader-following Synchronization

Example 1. Simulations are performed on a dynamical network consisting of one leader and four follower agents with the following sampled triple-integrator dynamics: $x_{i1}^{+} = x_{i1} + 0.3x_{i2} + 0.045x_{i3} + 0.0045u_i$, $x_{i2}^{+} = x_{i2} + 0.3x_{i3} + 0.045u_i$, $x_{i3}^{+} = x_{i3} + 0.3u_i$, $i = 1, ..., 4$. That is, the state-space matrices for the follower dynamics are as follows:

$$A = \begin{bmatrix} 1 & 0.3 & 0.045 \\ 0 & 1 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0045 \\ 0.045 \\ 0.3 \end{bmatrix}.$$

The communication topology is shown in Figure 1(a). One has $\lambda_1(\mathscr{L}) = 0.3820$ and $\lambda_N(\mathscr{L}) = 3.6180$. By (5) and (14), $\gamma = 0.3455$ and $\omega = 0.5$. The initial values of the components x_{i1}, x_{i2} and x_{i3} of follower agent states are randomly chosen from the cube $[-4, 4] \times [-4, 4] \times [-4, 4]$ for $i = 1, 2, 3, 4$; while for the leader, x_{51}, x_{52} and x_{53} are randomly chosen from the cube $[-1, 1] \times [-1, 1] \times [-1, 1]$. To numerically solve MARE (12) by applying [46, Theorem 6], CVX is used, which is a package for solving convex programs [50]. The MARE parameter is set as $\varepsilon = \varepsilon^* = 9.45 \times 10^{-3}$ such that (30) holds:

$$\|B^T P(\varepsilon^*)B\| = 1.2360 < \frac{\lambda_N(\mathscr{L}) + \lambda_1(\mathscr{L})}{\lambda_N(\mathscr{L}) - \lambda_1(\mathscr{L})} = 1.2361.$$

The corresponding controller in (13) is obtained as $K = [-0.0553, -0.3322, -0.9703]$. The parameter σ in (16) is chosen as $\sigma = 0.8$. The other parameters for the event-triggering functions in Algorithm 2 are obtained as follows: $\|\Theta(\varepsilon^*)\| = 4.0292$, $\|S(\varepsilon^*)\| = 4.2194$, $\max_i\{2d_i + h_i\} = 4$, then $\varrho(\varepsilon^*) = \min\{\max_i\{2d_i + h_i\} \cdot \|S(\varepsilon^*)\|, \|\Theta(\varepsilon^*)\|\} = 4.0292$; since $\frac{\varepsilon^*}{2\varrho(\varepsilon^*)} = 0.0012$ is small, for any $\theta \leq 0.0012$, one has $\varrho_2(\varepsilon^*, \theta) \approx \varrho(\varepsilon^*)$, then the parameter θ is set as $\theta = \frac{\varepsilon^*}{2\varrho(\varepsilon^*)} = 0.0012$ such that $(\frac{1}{\varrho_2(\varepsilon^*,\theta)} \cdot (\varepsilon^* - \varrho(\varepsilon^*)\theta)\,\theta)$ is almost maximized; and $\|\Theta_2(\varepsilon^*, \theta)\| = 4.0322$, $\|S_2(\varepsilon^*, \theta)\| = 4.2200$, then $\varrho_2(\varepsilon^*, \theta) = 4.0322$. The event-triggering condition for agent i is eventually obtained as

$$f_i = 3.4385 \times 10^3 \|w_i\|^2 - 0.0038\|v_i\|^2 > 0,$$

or equivalently, $\|w_i\|/\|v_i\| > 0.0010$. This theoretically obtained trigger threshold value of 0.001 is very small. One set of simulation data is shown in Figure 2. The practical synchronization is achieved within the first 150 steps. Within the first 300 steps, there are 292 feedback updates.

In practice, the trigger threshold value can be tuned to be larger than the theoretically obtained value of 0.001. It is verified that with the event-triggering condition $\|w_i\|/\|v_i\| > 0.014$ for agent i, the practical synchronization is always achieved within the first 150 steps in 100 times of simulations;
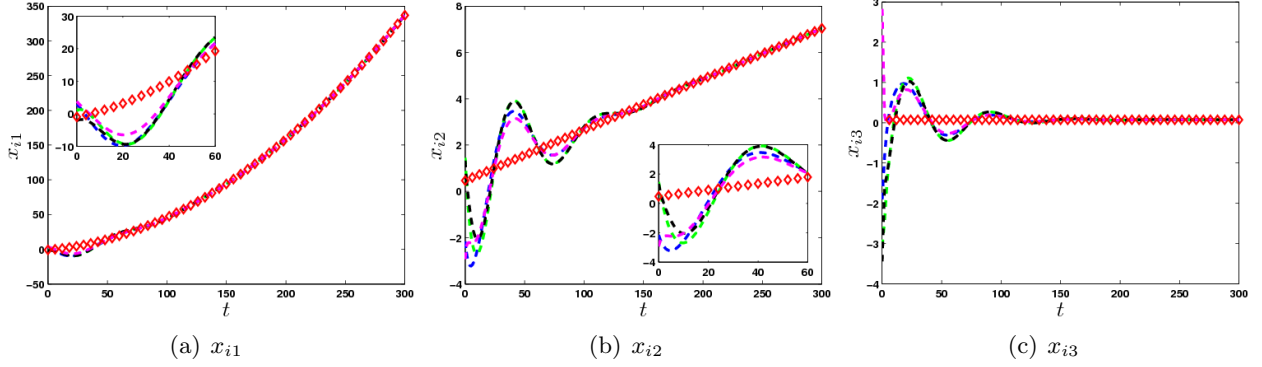
14

Figure 2: Leader-following synchronization: (a) The 1st, (b) the 2nd, and (c) the 3rd state components, where the followers are represented by dashed lines and the leader is marked by diamonds.

while the number of feedback updates in the first 300 steps varies from 190 to 220. And with the event-triggering condition $\|w_i\|/\|v_i\| > 0.14$ for agent i, the practical synchronization can always be achieved within the first 150 steps in 100 times of simulations as well; while the number of feedback updates in the first 300 steps is about 150 on average.

4.2 Leaderless Synchronization

Example 2. [43] Simulations are performed on a dynamical network consisting four agents with the same sampled dynamics with [43, Example 2] but subject to multiplicative noise: $x_{i1}^+ = x_{i1} + 0.0593x_{i2} + 0.0489 \times (1 + n)u_i$, $x_{i2}^+ = 0.9763x_{i2} + 0.0296 \times (1 + n)u_i$, $i = 1, ..., 4$, where $n(t)$ is a white Gaussian noise sequence. That is, the state-space model for the dynamics are as follows:

$$x_i^+ = Ax_i + B(1+n)u_i, \quad A = \begin{bmatrix} 1 & 0.0593 \\ 0 & 0.9763 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0489 \\ 0.0296 \end{bmatrix}.$$

The communication topology is shown in Figure 1(b), which is a line graph. One has $\lambda_2(L) = 0.3820$ and $\lambda_N(L) = 3.6180$. By (35) and (42), $\gamma = 0.5$ and $\omega = 0.5$. The initial values of the components x_{i1} and x_{i2} of agent states are randomly chosen from the square $[-4, 4] \times [-4, 4]$ for $i = 1, 2, 3, 4$. The MARE parameter is set as $\varepsilon = \varepsilon^* = 110.97$ such that (51) holds:

$$\|B^T P(\varepsilon^*) B\| = 1.4141 < \frac{\lambda_N(L) + \lambda_2(L)}{\lambda_N(L) - \lambda_2(L)} = 1.4142.$$

The corresponding controller in (41) is obtained as $K = [-4.7941, -2.2112]$. The parameter σ in (16) is chosen as $\sigma = 0.8$. The other parameters for the event-triggering functions in Algorithm 3 are obtained as follows: $\|\Theta(\varepsilon^*)\| = 111.4927$, $\|S(\varepsilon^*)\| = 111.4893$, $2\max_i\{d_i\} = 4$, then $\varrho(\varepsilon^*) = \min\{2\max_i\{d_i\} \cdot \|S(\varepsilon^*)\|, \|\Theta(\varepsilon^*)\|\} = 111.4927$; since $\frac{\varepsilon^*}{2\varrho(\varepsilon^*)} = 0.4977$ is small, for any $\theta \leq 0.4977$, one has $\varrho_2(\varepsilon^*, \theta) \approx \varrho(\varepsilon^*)$, then the parameter θ is set as $\theta = 0.4977$ such that $\left(\frac{1}{\varrho_2(\varepsilon^*, \theta)} \cdot (\varepsilon^* - \varrho(\varepsilon^*)\theta) \theta\right)$ is almost maximized; and $\|\Theta_2(\varepsilon^*, \theta)\| = 228.6695$, $\|S_2(\varepsilon^*, \theta)\| = 122.9803$, then $\varrho_2(\varepsilon^*, \theta) = 228.6695$. The event-triggering condition for agent i is eventually obtained as

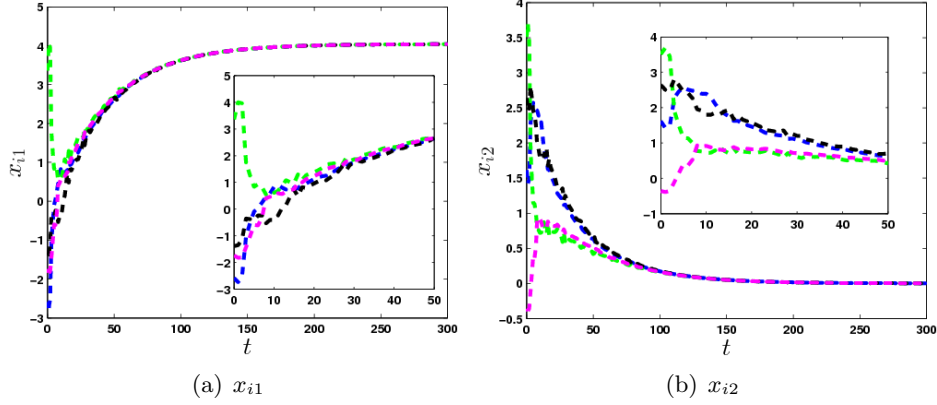$$f_i = 459.4932\|w_i\|^2 - 44.3880\|v_i\|^2 > 0,$$

15

Figure 3: Leaderless consensus: (a) The 1st and (b) the 2nd state components of the four agents.

or equivalently, $\|w_i\|/\|v_i\| > 0.3108$. This theoretically obtained trigger threshold value of 0.3108 is not small. One set of simulation data is shown in Figure 3. The practical consensus is achieved within the first 150 steps. Within the first 300 steps, there are only 109 feedback updates. This example shows that the proposed event-trigger strategy can also work under some non-ideal conditions, such as the case with system disturbance.

5 Conclusion

The event-based leader-following and leaderless synchronizations of discrete-time linear dynamical networks have been established using the distributed event-triggering conditions based on the combinational-state variables and Riccati equation. Future work may include the event-based control of dynamical networks with switching directed graphs, the distributed event-based broadcasting, the event-based stochastic synchronization of dynamical networks with noises, and the event-based synchronization of heterogeneous networks.

6 Acknowledgments

A   Proof of Claim 1 and Claim 2

A.1   For Claim 1

Pre- and post-multiplying both sides of the inequality $B^T PB + I \succ B^T PB$ by $(B^T PB + I)^{-1}$, one has that $(B^T PB + I)^{-1} \succ (B^T PB + I)^{-1} B^T PB (B^T PB + I)^{-1}$. Then, it can be obtained that $-\omega A^T PBK \succeq K^T B^T PBK$, where $\omega = 2/(\lambda_1(\mathscr{L}) + \lambda_N(\mathscr{L}))$. Therefore, $\Phi(\phi)$ is bounded from below as

$$
\begin{aligned}
\Phi(\phi) &\succeq \Psi(\psi(\phi)), \quad \psi(\phi) \triangleq 1 - (1 - \phi\omega)^2, \\
\Psi(\psi) &\triangleq P - A^T PA + \psi A^T PB (B^T PB + I)^{-1} B^T PA.
\end{aligned}
\tag{52}
$$

Denote $\psi_i \triangleq 1 - (1 - \lambda_i(\mathscr{L})\omega)^2$. Using Lemma 3, one has

$$
\min_{i=1,...,N} \psi_i = 4\lambda_N(\mathscr{L})\lambda_1(\mathscr{L})/(\lambda_N(\mathscr{L}) + \lambda_1(\mathscr{L}))^2 = \gamma, \ \forall \ i = 1, ..., N,
$$

where γ is defined in (5). Combining (12) and (52), one has $\Phi(\lambda_i(\mathscr{L})) \succeq \Psi(\psi_i) \succeq \Psi(\gamma) = \varepsilon I$. Then, (28) is obtained. □

A.2   For Claim 2

On one hand, $v^T \Theta(\varepsilon^*)v \leq \|\Theta(\varepsilon^*)\| v^T v$. On the other hand, $v^T \Theta(\varepsilon^*)v \leq v^T(\mathscr{L} \otimes S(\varepsilon^*))v$, where $S(\varepsilon^*) = -A^T PBK - \lambda_1(\mathscr{L})K^T B^T PBK$; and

$$
\begin{aligned}
v^T(\mathscr{L} \otimes S(\varepsilon^*))v &= \sum_{i=1}^N v_i^T S(\varepsilon^*)\left(h_i v_i + \sum_{j=1}^N a_{ij}(v_i - v_j)\right) \\
&= \sum_{i=1}^N (d_i + h_i) v_i^T S(\varepsilon^*) v_i - \sum_{i,j=1}^N a_{ij} v_i^T S(\varepsilon^*) v_j.
\end{aligned}
$$

By (31), one has that

$$
-\sum_{i,j=1}^N a_{ij} v_i^T S(\varepsilon^*) v_j \leq \sum_{i,j=1}^N \frac{a_{ij}}{2}\left(v_i^T S(\varepsilon^*) v_i + v_j^T S(\varepsilon^*) v_j\right) = \sum_i d_i v_i^T S(\varepsilon^*) v_i.
$$

Thus, $v^T \Theta(\varepsilon^*)v \leq \sum_i^N (2d_i + h_i) v_i^T S(\varepsilon^*) v_i \leq \max_i\{2d_i + h_i\}\|S(\varepsilon^*)\| v^T v$. As a result, $\theta v^T \Theta(\varepsilon^*)v \leq \theta \varrho(\varepsilon^*) v^T v$ with $\varrho(\varepsilon^*) = \min\{\max_i\{2d_i + h_i\} \cdot \|S(\varepsilon^*)\|, \|\Theta(\varepsilon^*)\|\}$. Similarly, $\frac{1}{\theta} w^T \Theta_2(\varepsilon^*, \theta) w \leq \frac{1}{\theta} \varrho_2(\varepsilon^*, \theta) w^T w$, with $\varrho_2(\varepsilon^*, \theta) = \min\{\max_i\{2d_i + h_i\} \cdot \|S_2(\varepsilon^*, \theta)\|, \|\Theta_2(\varepsilon^*, \theta)\|\}$. Therefore, by (33), one obtains (34). □

B   Proof of Lemma 6

For the proof of Lemma 6, Lemma 7 below, which is related to a directed graph, is first reviewed. The idea for Lemma 7 is from [41], and a proof is provided here.

Lemma 7. [41] For a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $\mathcal{V} = \{1, 2, ..., \tilde{N}\}$, denote L_{ii} as the matrix generated by deleting the i-th row and the i-th column of L; and $a_i \triangleq (a_{i1}, ..., a_{i(i-1)}, a_{i(i+1)}, ..., a_{i\tilde{N}})$ for each $i \in \mathcal{V}$. Then, the eigenvalues of the matrix $\left(L_{ii} + \mathbf{1}_{\tilde{N}-1} a_i\right)$ are $\lambda_2(L), \lambda_3(L), ..., \lambda_{\tilde{N}}(L)$.

Proof. Denote E_{ij} as the $\tilde{N} \times \tilde{N}$ matrix with all elements being zero except the entry at i-th row and j-th column being 1; and denote $P(i, j(k)) = I_{\tilde{N}} + k E_{ij}$. One has that $\det(P(i, j(k))) = 1$. Let the characteristic polynomial of L be denoted by $f(s) = \det(sI_{\tilde{N}} - L) = s \prod_{i=2}^{\tilde{N}}(s - \lambda_i(L))$. Denote $M_1(s) = (sI_{\tilde{N}} - L) \prod_{i=2}^{\tilde{N}} P(i, 1(1))$ and $M_2(s) = \prod_{i=2}^{\tilde{N}} P(i, 1(-1)) M_1(s)$. On one hand, $\det(M_2(s)) = f(s)$. On the other hand, $M_1(s)$ is obtained by adding all other columns to the first column of $(sI_{\tilde{N}} - L)$; and $M_2(s)$ is obtained by subtracting the first row from all other rows of $M_1(s)$. Then, $M_2(s)$ is a block upper-triangular matrix with two diagonal blocks: the first block is the scalar s and the second block is the matrix $\left(sI_{\tilde{N}-1} - \left(L_{11} + \mathbf{1}_{\tilde{N}-1} a_1\right)\right)$. As a result, $\det(M_2(s)) = s \cdot \det\left(sI_{\tilde{N}-1} - \left(L_{11} + \mathbf{1}_{\tilde{N}-1} a_1\right)\right) = s \prod_{i=2}^{\tilde{N}}(s - \lambda_i(L))$, and Lemma 7 is concluded for the case of $i = 1$. For general case of i, considering the matrix

$$\overline{M}_2(s) = \prod_{j \neq i} P(j, i(-1)) \left(\left(sI_{\tilde{N}} - L\right) \prod_{j \neq i} P(j, i(1))\right)$$

instead of $M_2(s)$, one can similarly obtain the result given in Lemma 7. □

Applying Lemma 7 to the extended graph $\overline{\mathcal{G}}$, one can easily obtain a proof of Lemma 2 that is different from the one in [8]. Now, the proof of Lemma 6 is as follows.

Proof of Lemma 6: Taking $\tilde{N} = N$ and letting $\mathcal{G}$ be undirected and connected, one has that $L_{ii} + \mathbf{1}_{\tilde{N}-1} a_i \succ 0$. Denote x^i as the vector generated by deleting x_i from x, and v^i as the vector generated by deleting v_i from v. Through straightforward algebraic manipulation, one obtains that

$$\left(L_{ii} + \mathbf{1}_{\tilde{N}-1} a_i\right) \left(\mathbf{1}_{\tilde{N}-1} \otimes x_i - x^i\right) = \mathbf{1}_{\tilde{N}-1} \otimes v_i - v^i, \tag{53}$$

from which one has $x_i = x_j, \forall j \neq i \Leftrightarrow v_i = v_j, \forall j \neq i$. If $\lim_{t \to \infty} \|v(t)\| = 0$, by (53), one obtains (4); if (4) holds, by (36), one has $\lim_{t \to \infty} \|v(t)\| = 0$. This completes the proof of Lemma 6. □

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