Event-Triggered Broadcasting across Distributed Networked Control Systems

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***Abstract*— This paper examines event-triggered broadcasting of state information in distributed control systems implemented over wireless communication networks. Event-triggering requires a subsystem to only broadcast its state information when the local state error exceeds a given threshold. The paper designs an event triggering scheme that assures asymptotic stability of the entire networked system. The results apply to networks of linear time-invariant systems. We derive lower bounds on the estimated time to next broadcast and present simulation results showing that event triggering allows a subsystem to adjust its broadcast frequency to the amount of activity in its immediate neighborhood. These results are significant because they show how one might stabilize distributed control systems over ad hoc wireless networks without necessarily requiring a high degree of synchronzation within the communication network.**

I. Introduction

A networked dynamical system consists of numerous loosely coupled systems. These networked systems are found throughout our national infrastructure with specific examples being the electrical power grid and transportation networks. In recent years, it has become popular to refer to such networked systems as *cyber-physical systems*. Increased demands on such infrastructure due to demographic shifts and greater regulatory burdens have made it increasingly difficult to reliably manage these networks in a cost effective manner. There is, therefore, a compelling national need to develop more robust and cost effective methods for controlling such networked systems.

It is impractical to control such large-scale systems in a centralized manner. Centralized control algorithms would require state information from all subsystems before computing the control action. This centralization requires a very powerful communication network to transport state information in a timely manner and it requires extremely detailed models of subsystem interactions. Both of these requirements can greatly limit the scalability of centralized approaches to networked control systems.

For this reason, many researchers have begun investigating either decentralized or distributed approaches to networked control. Decentralized control strategies only use a subsystem's local state data to control the given subsystem. Such local controls can be effective provided the degree of coupling between subsystems is weak. Note that such decentralized approaches have no run-time communication requirements since we rely heavily on a priori system models to assure the robustness of the decentralized control law.

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If such coupling is not weak, then we must resort to distributed feedback control laws. In a distributed networked control system, a given subsystem uses its state and the states of its immediate neighbors to determine its control action. Because it uses feedback from neighboring subsystems, decentralized control can assure asymptotic stability with a higher degree of subsystem coupling. Decentralized controllers must exchange information between nearest neighbors, so that some communication effort is required. Provided the neighborhood size is relatively small, then the communication effort required will scale well as the system size increases.

The use of wireless communication raises important issues regarding the impact that such communication has on the control system's performance. Wireless communication can only broadcast data in discrete packets. Moreover, the wireless media is a resource that is usually accessed in a mutually exclusive manner by neighborhood subsystems. This means that the throughput capacity of such networks is limited. So one important issue in the implementation of such distributed control systems is to identify methods for more effectively using the limited network bandwidth available for transmitting state information.

This paper addresses this issue through the use of an event-triggered feedback scheme. Event-triggering has the subsystem broadcast its state information when its local "error" signal exceeds a given threshold. Using a Lyapunov analysis similar to that suggested by Tabuada et al. [1], we show that an event-triggering rule based only on the subsystem's local state error can guarantee the asymptotic stability for the entire group. The analysis is valid for linear time-invariant subsystems that have full access to their local state. We establish bounds on the "time to next broadcast" and use simulation results to demonstrate how the approach adapts the broadcast rate to variations in a subsystem's external disturbance environment.

II. Prior Work

This paper deals with event-triggering in distributed control of loosely coupled systems that may be spread over a wide spatial domain. Such systems arise naturally in the control of geographically distributed systems. In [2] it was shown that optimal controllers with a quadratic objective possess an inherent degree of spatial localization. This suggests that it should be possible to effectively regulate the behavior of distributed systems using local interactions between spatially adjacent subsystems. One approach to distributed control builds upon model predictive control [3] [4]. These methods are appropriate for finite-dimensional systems. But there are a large number of flow control applications [5] [6] which may be more appropriately viewed as infinite-dimensional networked systems. Significant progress was made toward this goal in an approach that modelled system coupling using linear fractional transformations [7] [8]. More recent work has used integrator backstepping to extend this approach to networks of nonlinear systems [9].

In all of this prior work, it is assumed that subsystem controllers can communicate at will. In practice, however, such communication takes place over a digital network which means that information is transmitted in discrete time, rather than continuous-time. Moreover, all real networks have bandwidth limitation that can cause delays in message delivery [10]. Such delays can have a major impact on overall system stability. Early work in the study of networked control systems derived bounds on the maximum admissible time interval (MATI) that a message can be delayed while still maintaining closed loop system stability [11]. This work led to scheduling methods [12] that were able to assure the MATI was not violated. All of this early work in networked control systems confined its attention to communication networks that are traditionally found in industrial applications such as CAN (control area network) buses.

In recent years there has been considerable interest in developing distributed controllers in which the communication infrastructure is realized over an ad hoc wireless network. This is usually found in sensor network applications [13]. The problem faced in using wireless networks is that throughput capacity is limited[14]. As network density increases, the throughput seen by an individual agent asymptotically approach zero. There is, therefore, great interest in being able to develop networked control systems which are extremely frugal in their use of network bandwidth.

One approach for reducing the bandwidth requirements within a networked control system is to reduce the frequency with which agents communicate. The basic intuition behind this approach is that when a system is at its equilibrium point, there is little need for it to communicate with its neighbor. In fact, if we consider recent work in quantized feedback control [15], it is apparent that the transmission rates required to assure closed loop stability are well below those usually used in real-life computer controlled systems. It should therefore be possible to adaptively adjust transmission rates to the needs of the system in a way that only uses channel resources when the system has been perturbed away from its equilibrium point.

This then is the motivation of this paper. Namely we want to adaptively adjust agent broadcasts in a manner that is sensitive to what is currently happening within the system. One approach for doing this is to use *event-triggered* broadcasts. Event-triggering has a subsystem broadcast its state information only when "needed". In this case "needed" means that some measure of the agent's state error is above a specified threshold. Event-triggering was originally proposed in [16] and has appeared under a number of names that include interrupt-based feedback [17], Lebesgue sampling [18], asynchronous sampling [19], state-triggered feedback [1], and self-triggered feedback [20]. All of this prior work, however, has focused on using event-triggered feedback in single processor real-time systems. This paper uses the approach presented in [1] to design event triggering rules that allow agents to adapt their broadcasts to the current activity level in the system.

III. Problem Statement

This section formally presents the assumed system model and establishes some of the necessary mathematical notation.

Notational Conventions: If $V : \Re^n \to \Re$ is a function then its directional derivative with respect to the differential equation $\dot{x} = f(x)$ is

$$L_f V = \frac{\partial V}{\partial x} f(x)$$

If $x \in \Re^n$, then we let $\|x\|_2$ denote the Euclidean 2-norm of this vector. If $A \in \Re^{n\times m}$ is a real matrix we let $\|A\|$ denote the matrix gain induced with respect to the Euclidean 2-norm. We let $\mathcal{N}$ denote the set $\{1, 2, \cdots, N\}$ of N integers and we let $|\mathcal{N}|$ denote the number of elements in that set.

The system under study is a group of N linear time-invariant systems. The local state of the ith subsystem (also called an *agent*) is a function $x_i : \Re \to \Re^{n_i}$ where n_i is the local state space dimension and $i \in \mathcal{N} = \{1, 2, \cdots, N\}$. This function satisfies the linear differential equation

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + \sum_{n\in\mathcal{N}_i} H_{ij} x_j(t) \\ x_i(0) &= x_{i0} \end{aligned} \qquad (1)$$

where $x_{i0} \in \Re^{n_i}$ is the initial state and $\mathcal{N}_i \subset \mathcal{N}$ is the set of *neighbors* for agent i. We assume that the neighborhood sets $\mathcal{N}_i$ are such that $i \notin \mathcal{N}_i$. We further assume that being in a neighborhood is a symmetric relation in the sense that $j \in \mathcal{N}_i$ if and only if $i \in \mathcal{N}_j$. The signal $u_i : \Re \to \Re^{m_i}$ is the local control signal generated by agent i's controller where m_i is the dimension of the control set. $A_i \in \Re^{n_i\times n_i}$, $B_i \in \Re^{n_i\times m_i}$, and $H_{ij} \in \Re^{n_i\times n_j}$ are matrices of appropriate dimension.

For each $i \in \mathcal{N}$ we assume there exist $K_i \in \Re^{m_i\times n_i}$, $P_i \in \Re^{n_i\times n_i}$, and $Q_i \in \Re^{n_i\times n_i}$ such that

$$A_{K_i}^T P_i + P_i A_{K_i} \leq -Q_i \qquad (2)$$

where

$$A_{K_i} = A + B_i K_i \qquad (3)$$

Note that this inequality is equivalent to requiring that the function $V_i : \Re^{n_i} \to \Re$ define as $V_i(x_i) = x_i^T P_i x_i$ is a control Lyapunov function for the *decoupled* system,

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t).$$

We're interested in controls that are generated in a distributed manner so that

$$u_i(t) = K_i \hat{x}_i(t) + \sum_{i\in\mathcal{N}_i} L_{ij} \hat{x}_j(t) \qquad (4)$$

where K_i is the state feedback gain satisfying the Lyapunov equation 2, $L_{ij} \in \Re^{m_i \times n_j}$ is a set of *decoupling* gains, and $\hat{x}_j(t)$ is the measured state for the jth agent available at time t.

Note that we distinguish between the measured feedback state and the actual state of the agent. This is because a subsystem can only broadcast its state information at discrete times. We model this discrete transmission by associating a monotone increasing sequence of *broadcast times*, $\{b_j[k]\}_{k=0}^{\infty}$, with the jth agent. The broadcast times are increasing in the sense that $b_j[k] < b_j[k+1]$ for all k. The time $b_j[k]$ denotes the kth consecutive time instant when the jth agent broadcasts its local state x_j to all of its neighbors in $\mathcal{N}_j$.

The "measured" states used by an agent i in equation 4 are the functions $\hat{x}_j : \Re \to \Re^{n_j}$ where $j \in \mathcal{N}_i$ and

$$\hat{x}_j(t) = x_j(b_j[k]) \tag{5}$$

for $t \in [b_j[k], b_j[k+1])$ and all $k = 0, \cdots, \infty$. The measured state, therefore, is a sampled version of the neighbor's state trajectory where the sampling instants are the broadcast times. For simplicity we assume that all neighbors receive the broadcasted state without any delay so that $\hat{x}_j(t)$ is accessible to any subsystem i that lies in $\mathcal{N}_j$.

IV. Event Triggering for Asymptotic Stability

This section derives the event-triggering rule that assures the entire system is asymptotically stable. The first lemma characterizes the directional derivative of the function

$$V_i(x_i) = x_i^T P_i x_i \tag{6}$$

where P_i satisfies the Lyapunov equation 2. We use lemma 4.1 to characterize the directional derivative of the function $V : \Re^{\sum_i n_i} \to \Re$ defined as

$$V(x_1, x_2, \cdots, x_N) = \sum_i x_i^T P_i x_i. \tag{7}$$

which is used in theorem 4.2 to establish a condition for event triggering.

Lemma 4.1: Consider the system in equation 1 where

1) the control u_i is the distributed control in equation 4 using measured states defined by equation 5,
2) P_i, K_i, and Q_i satisfy the Lyapunov equation 2
3) and $e_i(t) = \hat{x}_i(t) - x_i(t)$ is the error between the measured state and the actual state.

The directional derivative of $V_i(x_i) = x_i^T P_i x_i$ satisfies the inequality

$$\begin{aligned}\frac{\partial V_i}{\partial x_i}\dot{x}_i \leq\ & -\left(\underline{\lambda}(Q_i) - (|\mathcal{N}_i| + 1)\delta\right)\|x_i\|_2^2 \\ & + \frac{\|P_i B_i K_i\|^2}{\delta}\|e_i\|_2^2 + \sum_{j \in \mathcal{N}_i} \frac{2\|P_i B_i L_{ij}\|^2}{\delta}\|e_j\|_2^2 \\ & + \sum_{j \in \mathcal{N}_i} \frac{2\|P_i(B_i L_{ij} - H_{ij})\|^2}{\delta}\|x_j\|_2^2,\end{aligned} \tag{8}$$

for all $i \in \mathcal{N}$ where δ is any positive real constant and where $\underline{\lambda}(Q_i)$ is the minimum eigenvalue of Q_i

Proof: A direct computation shows that

$$\begin{aligned}\frac{\partial V_i}{\partial x_i}\dot{x}_i =\ & x_i^T P_i \left(A_i x_i + B_i K_i \hat{x}_i\right) \\ & + x_i^T P_i \left(\sum_{j \in \mathcal{N}_i} B_i L_{ij}\hat{x}_j + H_{ij} x_j\right) \\ & + \text{transposed terms}\end{aligned} \tag{9}$$

Note that $\hat{x}_i = x_i + e_i$ so we can rewrite equation 9 in terms of x_i and e_i to obtain

$$\begin{aligned}\frac{\partial V_i}{\partial x_i}\dot{x}_i =\ & x_i^T P_i (A_{K_i} x_i + B_i K_i e_i) \\ & + \sum_{j \in \mathcal{N}_i} x_i^T P_i \Delta_{ij} x_j + x_i^T P_i B_i L_{ij} e_j \\ & + \text{transposed terms}\end{aligned} \tag{10}$$

where $\Delta_{ij} = B_i L_{ij} + H_{ij}$ and $A_{K_i} = A_i + B_i K_i$.

We would like to rewrite the cross terms in equation 10 in terms of signal norms. This can be done through the following inequality,

$$\|\delta z - R y\|_2^2 \geq 0 \tag{11}$$

where $z \in \Re^n$, $y \in \Re^m$ and $R \in \Re^{n \times m}$, and δ is any positive real constant. If we expand equation 11, move the cross term to the righthand side and divide through by δ we obtain

$$\delta\|z\|_2^2 + \frac{\|Ry\|_2^2}{\delta} \geq 2 z^T R y \tag{12}$$

Inequality 12 can be used to rewrite equation 10 as

$$\begin{aligned}\frac{\partial V_i}{\partial x_i}\dot{x}_i \leq\ & -x_i^T Q_i x_i + \delta\|x_i\|_2^2 + \frac{\|P_i B_i K_i\|^2}{\delta}\|e_i\|_2^2 \\ & + \sum_{j \in \mathcal{N}_i}\left(\frac{\delta}{2}\|x_i\|_2^2 + \frac{2\|P_i \Delta_i\|^2}{\delta}\|x_j\|_2^2\right) \\ & + \sum_{j \in \mathcal{N}_i}\left(\frac{\delta}{2}\|x_i\|_2^2 + \frac{2\|P_i B_i L_{ij}\|^2}{\delta}\|e_j\|_2^2\right)\end{aligned} \tag{13}$$

where δ is any positive real constant. Collecting the terms in $\|x_i\|_2^2$ and recognizing that

$$-x_i^T Q_i x_i \leq -\underline{\lambda}(Q_i)\|x_i\|_2^2 \tag{14}$$

yields equation 8. ■

Given the characterization of the V_i's directional derivative in equation 8, we can now state and prove the following theorem regarding the asymptotic stability of the entire system. This theorem presumes the decoupling gains, L_{ij}, were chosen to satisfy the *matching condition*, $B_i L_{ij} = -H_{ij}$, which essentially assures perfect decoupling of the subsystems.

Theorem 4.2: Assume that the matching condition,

$$B_i L_{ij} = -H_{ij}, \tag{15}$$

holds for all i and j. Under the assumptions of lemma 4.1, the networked system in equations 1 under the control in 4 is asymptotically stable, if

$$\beta_i \|e_i(t)\|_2^2 \leq \rho_i \|x_i(t)\|_2^2 \tag{16}$$

for all $i \in \mathcal{N}$ and all t, where

$$\beta_i = \frac{\|P_i B_i K_i\|^2}{\delta} + \sum_{j\in\mathcal{N}_i} \frac{2\|P_j B_j L_{ji}\|^2}{\delta} \quad (17)$$

$$\rho_i < \underline{\lambda}(Q_i) - (|\mathcal{N}_i| + 1)\delta \quad (18)$$

$$\delta = \min_{i\in\mathcal{N}} \left\{ \frac{\underline{\lambda}(Q_i)}{|\mathcal{N}_i| + 1} \right\}. \quad (19)$$

Proof: Consider the candidate Lyapunov function

$$V(x_1, \cdots, x_N) = \sum_{i\in\mathcal{N}} V_i(x_i).$$

Using lemma 4.1, its directional derivative may be written as

$$\frac{\partial V}{\partial x}\dot{x} \leq -\sum_{i\in\mathcal{N}} \left(\rho_i \|x_i\|_2^2 - \frac{\|P_i B_i K_i\|^2}{\delta}\|e_i\|_2^2 \right) + \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}_i} \frac{\|P_i B_i L_{ij}\|^2}{\delta}\|e_j\|_2^2 \quad (20)$$

Recall that neighborhood membership is a symmetric relation, so that $j \in \mathcal{N}_i$ whenever $i \in \mathcal{N}_j$. Due to this symmetry we can redistribute the terms in the second line of equation 20 to group together terms indexed by $\|e_i\|$ and obtain

$$\frac{\partial V}{\partial x}\dot{x} \leq -\sum_{i\in\mathcal{N}} \left(\rho_i \|x_i\|_2^2 - \frac{\|P_i B_i K_i\|^2}{\delta}\|e_i\|_2^2 \right) + \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}_i} \frac{\|P_j B_j L_{ji}\|^2}{\delta}\|e_i\|_2^2. \quad (21)$$

where we used equation 18 to help simplify.

Collecting terms in $\|e_i\|$ in equation 21 we can rewrite this as

$$\frac{\partial V}{\partial x}\dot{x} \leq -\sum_{i\in\mathcal{N}} \left(\rho_i \|x_i\|_2^2 - \beta_i \|e_i\|_2^2 \right) \quad (22)$$

where we used equation 17 to simplify. By the assumption in equation 16, we see that the righthand side of equation 22 is negative if the requirement on δ (see equation 19) is satisfied, which is sufficient to establish the asymptotic stability of the equilibrium point. ■

Theorem 4.2 is interesting because the error condition in equation 16 is only dependent on what the ith subsystem can directly measure. In other words, if all agents cooperate in the sense of broadcasting their states so that the threshold condition in equation 16 is always satisfied, we can assure the entire system's asymptotic stability.

The inequality in equation 16 can be used as the basis for event-triggering the broadcast of an agent's state. Note that the inequality is trivially satisfied for the ith agent at broadcast time $t = b_i[k]$. So if we trigger the next broadcast, $b_i[k+1]$ any time before equation 16 is violated and if we can guarantee this behavior across all agents in the system, then we are assured the entire networked system is asymptotically stable.

The matching condition assumed in theorem 4.2 is exceptionally restrictive. The following theorem relaxes this assumption.

Theorem 4.3: Assume that the hypotheses in lemma 4.1 are true and assume that for all j

$$W_i \equiv \sum_{j\in\mathcal{N}_i} \|P_j(B_j L_{ji} - H_{ji})\|^2 \leq \frac{\underline{\lambda}(Q_i)}{8(1 + |\mathcal{N}_i|)} \quad (23)$$

for all $i \in \mathcal{N}$ The networked system in equations 1 under the control in 4 is asymptotically stable, if

$$\beta_i \|e_i(t)\|_2^2 \leq \alpha_i \|x_i(t)\|_2^2 \quad (24)$$

for all $i \in \mathcal{N}$ and all t, where

$$\beta_i = \frac{\|P_i B_i K_i\|^2}{\delta} + \sum_{j\in\mathcal{N}_i} \frac{2\|P_j B_j L_{ji}\|^2}{\delta} \quad (25)$$

$$\alpha_i < \underline{\lambda}(Q_i) - (1 + |\mathcal{N}_i|)\delta - \frac{2W_i}{\delta} \quad (26)$$

$$\delta < \min_i \left\{ \frac{\underline{\lambda}(Q_i)}{|\mathcal{N}_i| + 1} \left(\frac{1}{2} + \sqrt{\frac{1}{4} - 2(1 + |\mathcal{N}_i|)W_i} \right) \right\} \quad (27)$$

Proof: The proof of this theorem is similar to that for theorem 4.2. We again consider the candidate Lyapunov function

$$V(x_1, \cdots, c_N) = \sum_{i\in\mathcal{N}} V_i(x_i). \quad (28)$$

From lemma 4.1, the directional derivative of V becomes

$$\frac{\partial V}{\partial x}\dot{x} \leq -\sum_{i\in\mathcal{N}} \left(\rho_i \|x_i\|_2^2 - \frac{\|P_i B_i K_i\|^2}{\delta}\|e_i\|_2^2 \right) + \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}_i} \frac{2\|P_i B_i L_{ij}\|^2}{\delta}\|e_j\|_2^2 + \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}_i} \frac{2\|P_i(B_i L_{ij} - H_{ij})\|_2}{\delta}\|x_j\|_2^2 \quad (29)$$

where ρ_i was defined in equation 18.

Since the neighborhood relation is symmetric, we can redistribute the terms in the second and third lines of equation 29 to obtain

$$\begin{aligned} \frac{\partial V}{\partial x}\dot{x} &\leq \sum_{i\in\mathcal{N}} - \left(\rho_i \|x_i\|_2^2 - \frac{\|P_i B_i K_i\|^2}{\delta}\|e_i\|_2^2 \right) \\ &\quad + \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}_i} \frac{2\|P_j B_j L_{ji}\|^2}{\delta}\|e_i\|_2^2 \\ &\quad + \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}_i} \frac{2\|P_j(B_j L_{ji} - H_{ji})\|_2}{\delta}\|x_i\|_2^2 \\ &= -\sum_{i\in\mathcal{N}} \alpha_i \|x_i\|_2^2 + \sum_{i\in\mathcal{N}} \beta_i \|e_i\|_2^2 \end{aligned} \quad (30)$$

where α_i and β_i are defined in equations 26 and 25, respectively. W_i was defined in equation 23.

We need to verify that the first term in equation 30 is negative definite. This will happen if

$$\alpha_i = \underline{\lambda}(Q_i) - (|\mathcal{N}_i| + 1)\delta - 2\frac{W_i}{\delta} > 0$$

which we can can rewrite as the quadratic inequality

$$(|\mathcal{N}_i| + 1)\delta^2 - \underline{\lambda}(Q_i)\delta + 2W_i < 0$$

The δ that satisfy this inequality have the form

$$\delta < \frac{\underline{\lambda}(Q_i)}{|\mathcal{N}_i|+1}\left(\frac{1}{2}+\sqrt{\frac{1}{4}-2(|\mathcal{N}_i|+1)W_i}\right)$$

which yields equation 27. However to be an admissible solution we also require δ to be positive. A simple substitution of the assumption in equation 23 shows that δ is positive if the inequality in equation 23 is true. ■

Theorem 4.3 relaxes the matching condition of theorem 4.2. In this case, then we require that there exists symmetric matrices P_i and Q_i as well as control gains gains K_i and L_{ij} such that

$$A_{K_i}^T P_i + P_i A_{K_i} \leq -Q_i \quad (31)$$

$$\sum_{j\in\mathcal{N}_i} \|P_j(B_j L_{ji} - H_{ji})\|^2 \leq \frac{\underline{\lambda}(Q_i)}{8(|\mathcal{N}_i|+1)} \quad (32)$$

One traditional way of interpreting these equations is to assume that P_i and Q_i are fixed. We would then use equations 31 and 32 to determine the control gains K_i and decoupling gains, L_{ij}. An alternative approach assumes we select K_i to stabilize the decoupled systems with a given level of robust stability. We would then use equations 31 and 32 to determine the matrices P_i and gains L_{ji}. In this particular case we can view V_i as robust control Lyapunov functions [21] for the networked system.

V. Broadcast Period

This section presents preliminary results bounding the time between broadcasts when the matching condition holds. We define the *broadcast period* of agent i as

$$B_i[k] = b_i[k+1] - b_i[k]. \quad (33)$$

The main result of this section shows that agent i can communicate its expected "time" to its next broadcast in a rather simple manner that is a function of the states in the agent's neighborhood. This means that broadcast frequency is really a function of the activity level in an agent's neighborhood. Moreover, these results show that the time between consecutive broadcasts by the ith agent should be bounded away from zero.

To bound the time between broadcasts, however, we first need the following weaker version of theorem 4.2. A similar corollary can be established under the relaxed form of the matching condition in equation 23.

Corollary 5.1: Consider the networked control system in equation 1 using the control in equation 4. If the matching condition holds under the assumptions of lemma 4.1 and the sequence of agent broadcasts can ensure that

$$(\beta_i + \rho_i)\|e_i(t)\|_2^2 \leq \rho_i \|\hat{x}_i(t)\|_2^2 \quad (34)$$

for all i and all $t \in [b_i[k], b_i[k+1])$, then the networked system is asymptotically stable.

Proof: For notational simplicity let x_b denote $x_i(b_i[k])$, then the condition in corollary 5.1 can be rewritten as

$$\beta_i \|e_i(t)\|_2^2 \leq \rho_i(\|x_b\|_2^2 - \|e_i(t)\|_2^2) \quad (35)$$

Note that

$$\begin{aligned} \|x_i(t)\|_2^2 &= \|x_b - (x_b - x_i(t))\|_2^2 \\ &\geq \|x_b\|_2^2 - \|e_i(t)\|_2^2 \end{aligned} \quad (36)$$

Using equation 36 in equation 35 yields the event-triggering relation (equation 16) in theorem 4.2, so we can immediately conclude the entire system is asymptotically stable. ■

Corollary 5.1 is clearly a weaker condition than that used in theorem 4.2. But we can use it to bound the broadcast period of a given agent. In particular, let's assume that the hypotheses of theorem 4.2 hold and let's further require that an agent broadcasts its state whenever the condition in corollary 5.1 is about to be violated.

Let's assume that agent i broadcasts its state at time r_0. Between this broadcast and the next broadcast by agent i, it is quite possible that agent i will *receive* broadcasts from any of its neighbors. Let r_m denote the mth time when agent i received a neighbor's message. We may therefore order these times as $r_0 < r_1 < r_2 < \cdots$.

We now study the behavior of the state error e_i between any two consecutive times r_m and r_{m+1}. To simplify notation we let

$$z_i(t) = \|e_i(t)\|_2.$$

We can show that

$$\begin{aligned} \dot{z}_i &\leq \|\dot{e}_i\|_2 = \|\dot{x}_i\|_2 \\ &= \left\| A_i x_i + B_i K_i \hat{x}_i + \sum_{j\in\mathcal{N}_i} (B_i L_{ij}\hat{x}_j + H_{ij} x_j) \right\|_2 \end{aligned} \quad (37)$$

Since $e_i = \hat{x}_i - x_i$ we can rewrite the right hand side of inequality 37 in terms of e_i and $\hat{x}_i$ to obtain

$$\begin{aligned} \dot{z}_i &\leq \left\| A_{K_i}\hat{x}_i - B_i K_i e_i - \sum_{j\in\mathcal{N}_i} H_{ij} e_j \right\|_2 \\ &\leq \|A_{K_i}\hat{x}_i\|_2 + \|B_i K_i\|\|e_i\|_2 + \sum_{j\in\mathcal{N}_i} \|H_{ij}\|\|e_j\|_2 \end{aligned} \quad (38)$$

where we used the fact that $B_i L_{ij} + H_{ij} = 0$ (i.e. the matching condition).

By the event-triggering rule in corollary 5.1, agent j only broadcasts if it is about to violate the inequality

$$\|e_j(t)\|_2 \leq \gamma_j \|\hat{x}_j(t)\|_2 \quad (39)$$

for any j where

$$\gamma_j = \sqrt{\frac{\rho_j}{\beta_j + \rho_j}}. \quad (40)$$

Between any two times (say r_m and r_{m+1}) when a message is received (or broadcast) by agent i, we know the measured state $\hat{x}_j$ is constant for any $j \in \mathcal{N}_i$. Therefore equation 38 can be reduced to

$$\dot{z}_i(t) \leq \alpha z_i(t) + \mu \quad (41)$$

for any $t \in [r_m, r_{m+1})$ where

$$\begin{aligned} \alpha &= \|B_i K_i\| \\ \mu &= \|A_{K_i} \hat{x}_i(t)\|_2 + \sum_{j \in \mathcal{N}_i} \gamma_j \|H_{ij}\| \|\hat{x}_j\|_2 \end{aligned}$$

Note that μ is constant between any two consecutive receptions. Moreover, note that it is a function of the system state $x_j(r_m)$ at time r_m. We can therefore solve the differential inequality in equation 41 to show that

$$z_i(t) \leq e^{\alpha(t-r_m)} z_i(r_m) + \frac{\mu}{\alpha}(e^{\alpha(t-r_m)} - 1) \tag{42}$$

for $t \in [r_m, r_{m+1})$.

Now it is, of course, possible that agent i may broadcast its state before it receives the next message at time r_{m+1}. This will happen at a time T that satisfies

$$z_i(T) \geq \gamma_i \|x_i(r_0)\|_2 \tag{43}$$

We can use our expression for $z_i(t)$ in equation 42 to solve for T in equation 43. This yields

$$T - r_m \geq \frac{1}{\alpha} \ln \left(1 + \frac{\gamma_i \|x_i(r_0)\|_2 - z_i(r_m)}{z_i(r_m) + \mu/\alpha} \right). \tag{44}$$

If there were a finite number, M, of received messages between consecutive broadcasts of agent i, then clearly the broadcast period can be bounded as

$$B_i = T - r_M + \sum_{k=1}^{M} (r_k - r_{k-1})$$

This sum must be finite as long as μ remains bounded, so we can readily conclude that the time between consecutive broadcast of the same agent must be bounded strictly away from zero.

VI. Simulation Results

This section presents simulation results demonstrating event triggering in a networked control system. The system under study is a collection of three inverted pendulums (figure 1) whose pendulum arms are coupled together by springs. The basic system matrices for the three pendulums are

$$\begin{aligned} A_i &= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} - \frac{k_i}{m\ell^2} & 0 \end{bmatrix} \\ B_i &= \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} \end{aligned}$$

where $g = 10$ is gravitational acceleration, $\ell = 2$ is the length of the pendulum, $m = 1$ is the mass of the pendulum bob, and $k_1 = k_3 = 5$ and $k_2 = 10$ are spring constants. The coupling matrices, H_{ij}, have the form

$$H_{ij} = \begin{bmatrix} 0 & 0 \\ \frac{k}{m\ell^2} & 0 \end{bmatrix}$$

where $k = 0$ if $i = j$ or $(i,j) \in \{(1,3),(3,1)\}$. Otherwise $k = 5$.

A local set of control gains, K_i, were obtained to place the decoupled system's poles at -1 and -2. This resulted in

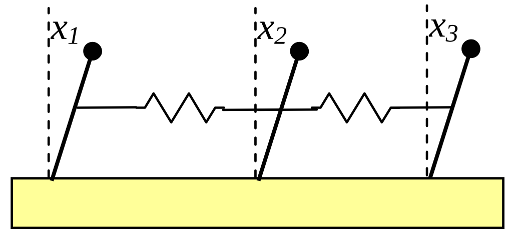

Fig. 1. Network of three inverted pendulums

$K_i = \begin{bmatrix} -23 & -12 \end{bmatrix}$ for $i = 1$ and 3. The gain for agent 2 was $K_2 = \begin{bmatrix} -18 & -12 \end{bmatrix}$. In this problem the matching condition, $B_i L_{ij} = -H_{ij}$, can be used if we select $L_{ij} = \begin{bmatrix} 5 & 0 \end{bmatrix}$.

The candidate control Lyapunov function V_i for agent i was chosen to be $x_i^T P_i x_i$ where $P_i = \begin{bmatrix} 1.25 & .25 \\ .25 & .25 \end{bmatrix}$ for all i. The matrices P_i were obtained by solving the following Lyapunov equation

$$(A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) = -I$$

where I is a 2×2 identity matrix.

With this setup we computed the coefficients β_i and ρ_i in the event-triggering inequality 16. Our simulation then triggered agent i to broadcast its state whenever

$$-0.5\|x_i\|_2^2 + \beta_i \|e_i\|_2^2 > 0$$

where $\beta_1 = \beta_3 = 32.7177$ and $\beta_2 = 24.2812$. These values were obtained for a δ that was one half of its maximum possible value in equation 18.

The simulation results are shown in figure 2 where the initial states were $x_{10} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $x_{20} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and $x_{30} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The simulation ran for 16 seconds, with a large disturbance being applied to the third system halfway through the simulation. The top plot in figure 2 is the state time history for all three inverted pendulums. Note that the system is stable. The bottom plot in figure 2 is the history of broadcast periods generated by the event-triggering inequality. Note that the broadcast periods vary considerably over those intervals when the state has been perturbed away from its equilibrium point. This shows that our event triggering scheme indeed adjusts broadcast periods in response to what is happening in the plant. We computed the average broadcast periods, $\overline{B}_i$, for the three inverted pendulums simulated in figure 2. The average periods for agents 1 to 3, respectively, were 0.0929, 0.1263, and 0.0913. The average of these three periods is 0.1037.

Let's now compare the performance of the event-triggered system against a periodically triggered system. To make the comparison fair, we assumed that each agent attempts to broadcast its state at a period which is one third of the average broadcast rate (0.1037) generated by event-triggered system. We assume that only one agent can "broadcast"

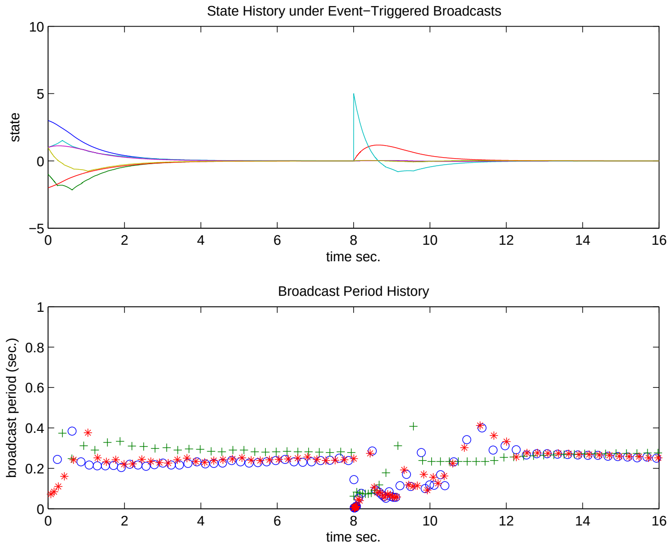

Fig. 2. Event-triggered broadcast simulation results

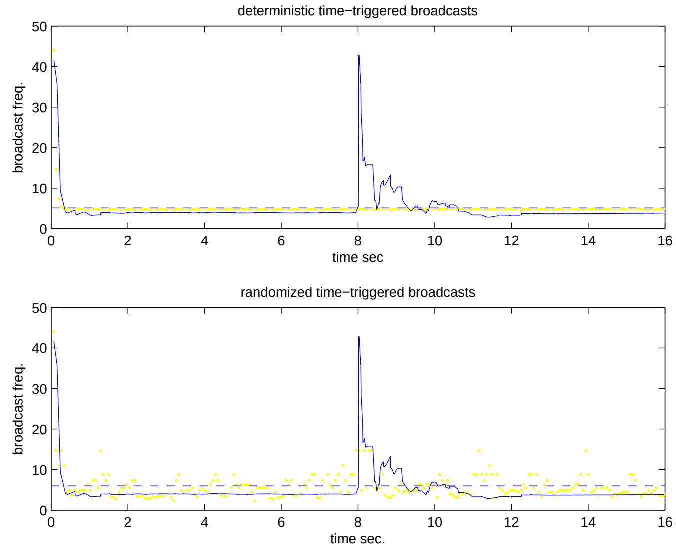

Fig. 3. Periodically-Triggered Broadcasts versus Event-Triggered Broadcasts

at a time, so this one third rate averages to a rate that is comparable to that found in the event-triggered simulation. We considered two cases. The first case assumed that agents access the channel in a sequential fashion, where agent 1 first broadcasts, then agent 2 broadcast, and then agent 3 broadcasts. This is the type of media access we'd find in time-slotted networks using a conflict-resolution algorithm to determine broadcast order. The second case assumed that agents compete for access to the channel as might be found in networks using carrier sense media access (CSMA) protocols. In this case, the probability of an agent accessing the medium is 1/3.

Figure 3 plots the the inverse of the broadcast period (what we refer to as broadcast frequency) for both of these cases. The solid blue line in both plots of figure 3 is the broadcast frequency generated by the event-triggered system. The yellow dots represent the broadcast frequency generated by the periodically triggered broadcast system. The dashed line shows the average broadcast interval of the time-triggered system. The top figure plots data for the time-triggered system with sequential access to the channel. As expected, the average broadcast period is equal to the average broadcast period. This average frequency is slightly higher than the lowest broadcast frequencies generated by the event-triggered system. A similar result is seen in the bottom plot of figure 3. This plot shows broadcast frequencies generated by a randomized time-triggered system. As expected the average broadcast frequency is somewhat higher than the lowest broadcast frequencies generated by the event-triggered system. In both cases, we see that event-triggering allows the system to reduce the amount of channel access during periods of low system activity.

VII. Summary

This paper presented an event-triggering approach to broadcasting state data in distributed control systems implemented over ad hoc wireless networks. Broadcasts are triggered in a decentralized manner, so that all agents make their broadcast decisions solely on the basis of their own measured states. Information from neighboring subsystems is used to adjust the event-triggering level. This approach therefore allows a subsystem to adjust its broadcast rate to the amount of activity in its immediate neighborhood. We were able to bound the time between broadcast events and simulation results supported our contention that event-triggering provides an effective means of adapting broadcast rates in sensor-actuator networks.

The work presented in this paper is preliminary in nature. There are a number of important issues that will need to be addressed in our future work. Some of these issues are itemized below.

- It would be valuable to see how we can take advantage of the relaxed matching condition in controller synthesis. As noted above, we can use the conditions in theorem 4.3 to design both the decoupling gains, L_{ij}, and robust control Lyapunov functions for the networked systems. Precisely how such distributed controllers can be synthesized is a topic for future study.
- The current work restricts its attention to linear time-invariant systems. It would be valuable to extend this to networks of nonlinear systems. We believe this may be possible for nonlinear systems that are affine in the controls. Once again the matching condition becomes a major concern in such analyses.
- This paper did not address the issue of message collisions. In practice, such collisions will delay the delivery of messages in a way that can adversely effect system stability. Our recent work [22] in self-triggered feedback control, however, suggests it may be possible to find practical bounds on these delays as a function of the broadcast period. Bounding such delays as was done

in [22] may help in analyzing the impact message collisions have on overall system stability.

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