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Abstract

We provide an event-triggered control technique for a family of linear time-varying continuous-time systems with a constant known pointwise delay in the input. We adopt a subpredictor based prediction technique, and we provide sufficient conditions that ensure that Zeno behavior does not occur. At each time, only delayed measurements are needed to implement the control. Also, the delay can be an arbitrarily large constant. We prove an input-to-state stability property for the closed-loop system, using the theory of cooperative systems. We apply our method to a gyroscopic control problem for a curve tracking dynamics arising in marine robotics.

Key words: Delay, event-trigger, stability, positive system

1 Introduction

Event-triggered control provides the basis for numerous significant recent developments in systems and controls; see, e.g., [3], [17], [35], [37], and references therein. While recently developed computing methods can ease the burden of recalculating control values, the widespread use of shared wired (or shared wireless) networks strongly motivates taking communication, computation, and energy constraints into consideration when designing feedback controls. This motivated the development of event-triggered techniques to reduce the number of control value recomputation times needed to achieve stability properties (compared, e.g., with other sampled based feedback methods), to reduce the communication load in networked systems. When input delays are present, event triggered control can be complicated by the fact that traditional event-triggered controls were not designed to compensate for time delays, and so would not provide the performance guarantees that they would guarantee in the absence of input delays.

On the other hand, many systems have input delays, which frequently arise from transport and delays in sensors; see, e.g., [5], [6], [7], [8], [9], [10], [11], and [30]. Even small input delays can have detrimental effects, so it can be essential to take them into account; see, e.g., [4] for a grid-tied inverter dynamics that is destabilized by a delay of $\tau = 0.001$. To cope with input delays, several stabilization techniques exist in the frequency and time domains, including the celebrated Smith predictor, stabilization that is robust to small delays (which computes bounds on allowable input delays), and the reduction model and prediction approaches from [2] and [20]. See also the subpredictor approach [31] for known arbitrarily large delays. An advantage of subpredictors is that they only use pointwise delays, meaning that at each time t, the control uses state arguments from time $t-\tau$ for a constant delay $\tau > 0$. However, we are not aware of results on event-triggered controls that ensure robustness to small known delays in the input. The reduction model approach was combined with event-triggered control to stabilize systems with an input delay in [33]. See also [18] for a Smith predictor approach. To the best of our knowledge, the problem of designing event-triggered controls under input delays and involving only pointwise delays has not been addressed in the literature.

This motivates the present paper, which builds on recent studies of event-triggered control for systems with delay; see in particular [9], [21], [36], and [39]. We use a subpredictor based prediction approach, which was developed in works such as [26] and [31] in the absence of event-triggering, to stabilize linear continuous time systems with a constant pointwise delay in the input and an additive uncertainty. We use an event-triggered control, which combines aspects of co-design of controls and triggering rules with emulation. We assume that the delay is constant and known, and that a bound for the additive uncertainty term is known. The stability analysis we perform is reminiscent of the one of the recent contribution [28] which did not allow input delays. We rely on comparison systems whose

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stability is analyzed through tools of the theory of cooperative systems. Such tools have been used in notable works such as [1], [12], [14], and [32], but these earlier works did not consider event triggering. We prove that, for the control law we propose, the Zeno behavior is excluded, meaning each finite length time interval contains only finitely many event triggering times. The limitation of our approach resides mostly in the fact that we assume that the delay is known. To the best of our knowledge, no result for unknown input delay with a known upper bound has been established for event-triggered control systems. However, in addition to our allowing arbitrarily large constant delays, a significant novel feature is that only time lagged measurements are needed in the control or event-triggering formula. In Section 2, we present the systems under study. We also introduce the subpredictors. We state and prove our main result in Sections 3-4. Then Section 5 has an application to a marine robotic example. We conclude in Section 6 by summarizing our findings and our ideas for possible follow up research.

For simplicity, we only consider constant, pointwise, and known delays. We found these assumptions on the delay to be suitable for practice, because for instance, one can first apply delay estimation methods such as the notable work [13] or experiments for delay estimation. Also, in the surface marine robotic, power electronics, and other applications that we encountered, the delays are typically constant and pointwise; see, e.g., [4], and see [27] and [38] for sequential predictor methods under time-varying delays but no event triggering, which we conjecture can be extended to event triggering using the approach in the present work, because the present work uses the constantness of the delay primarily in the chain predictor construction.

We use standard definitions and notation, which we simplify when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise. Given a matrix $G = [g_{ij}] \in \mathbb{R}^{r\times s}$, we set $|G| = [|g_{ij}|]$, i.e., the entries of $|G|$ are the absolute values of the corresponding entries of G. When G is time varying with bounded entries, we set $|G|_\infty = [m_{ij}]$, where $m_{ij} = \sup\{|g_{ij}(t)| : t \in \mathbb{R}\}$. We also set $G^+ = [\max\{g_{ij}, 0\}]$ and $G^- = G^+ - G$. A square matrix is called Metzler provided all of its off diagonal entries are nonnegative. Given two matrices $D = [d_{ij}]$ an $E = [e_{ij}]$ of the same size, we write $D < E$ (respectively, $D \le E$) provided $d_{ij} < e_{ij}$ (respectively, $d_{ij} \le e_{ij}$) for all i and j. Also, we write $D \nleq E$ provided there is a pair (i, j) such that $d_{ij} > e_{ij}$. We use similar notation for vectors. Also, 0 denotes the zero matrix, and I is the identity matrix, of the size under consideration.

For each constant $\tau > 0$, we let $C([-\tau, 0])$ denote the set of all continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ for the dimension n under consideration, which will serve as the set of all initial functions for our closed loop systems having the input delay τ, where we assume that the initial times are $t_0 = 0$. A matrix M is called positive provided $0 < M$. We set $\phi_t(s) = \phi(t + s)$ for all ϕ, $t \ge 0$, and $s \le 0$ such that $t + s$ is in the domain of ϕ. For a matrix M in $\mathbb{R}^{n\times n}$, we let D_M denote the diagonal matrix such that all of the diagonal entries of $M - D_M$ are equal to zero. We let $R_M = D_M + (M - D_M)^+$ and $N_M = (M - D_M)^-$. Hence, when M is Metzler, we have $M = R_M$. We let $||\cdot||$ denote the standard Euclidean norm of matrices and vectors, and we let $||\cdot||_\infty$ (respectively, $||\cdot||_{\mathcal{I}}$) denote the sup norm of matrix valued functions in this norm over their domain (respectively, an interval $\mathcal{I}$ in their domain). We use standard notation and properties for input-to-state stability (or ISS, which we also use to abbreviate input-to-state stable). We use the usual definition and properties of fundamental solutions, e.g., from [34, Appendix C]. We use $\Phi_{\mathcal{M}}$ to denote the fundamental solution for a system of differential equations of the form $\dot{z}(t) = \mathcal{M}(t)z(t)$.

2 Studied system and subpredictors

In this part, we present the system we study. Also, we recall a result on prediction of future values of state variables, which can be found in the papers [26] and [31], for instance.

2.1 Studied system

We consider the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t - \tau) + \delta(t) \tag{1}$$

where x is valued in $\mathbb{R}^n$, u is valued in $\mathbb{R}^p$, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R} \to \mathbb{R}^{n\times p}$ are given bounded continuous matrix valued functions, and the unknown function $\delta : [0, +\infty) \to \mathbb{R}^n$ is bounded and piecewise continuous and can represent control or model uncertainty. We assume that the constant delay $\tau \ge 0$ is known and that we know a bounded piecewise continuous function $\bar{\delta} : [0, +\infty) \to \mathbb{R}^n$ such that

$$|\delta(t)| \le \bar{\delta}(t) \tag{2}$$

for all $t \ge 0$. Also, u represents a control that will be specified by our theorem below.

2.2 Subpredictors

This section reviews basic ideas for chain predictors with event triggering, which were presented in [26] and [31] and which we use to prove our main result in the next section. Choose a bounded continuous matrix valued function $L : \mathbb{R} \to \mathbb{R}^{n\times n}$ and an integer $m > 0$ such that with the choices of A and τ from (1), the origin of the system

$$\dot{X}(t) = A(t)X(t) + L(t)X\left(t - \frac{\tau}{m}\right) + \kappa(t) \tag{3}$$

is globally exponentially ISS on $\mathbb{R}^n$ with respect to κ, where κ is an unknown measurable locally bounded function that represents uncertainty, i.e., there are constants $p_i > 0$ for $i = 1$ to 3 such that for all $t \ge s$ and $s \ge 0$, all solutions of (3) satisfy

$$||X(t)|| \le p_1 e^{-p_2(t-s)} \sup_{r\in[s-2\tau/m, s]} ||X(r)|| + p_3 \sup_{r\in[s,t]} ||\kappa(r)||. \tag{4}$$

Later, we specialize (3) to cases where X is a difference between subpredictor values evaluated at suitable times.

For instance, we can take $L(t) = -I - A(t)$ when

$$m > 2\sqrt{2}\tau||I + A||_\infty \max\{||I + A||_\infty, ||A||_\infty\}, \qquad (5)$$

by standard Lyapunov-Krasovskii functional arguments; see Appendix A below. We assume that $m \geq 2$.

We introduce the subpredictors

$$\left\{\begin{array}{rcl} \dot{z}_1(t) &=& A\left(t+\frac{\tau}{m}\right)z_1(t) + B\left(t+\frac{\tau}{m}\right)u\left(t-\frac{m-1}{m}\tau\right) \\ && +L\left(t+\frac{\tau}{m}\right)\left[z_1\left(t-\frac{\tau}{m}\right)-x(t)\right] \\ \dot{z}_2(t) &=& A\left(t+\frac{2\tau}{m}\right)z_2(t) + B\left(t+\frac{2\tau}{m}\right)u\left(t-\frac{m-2}{m}\tau\right) \\ && +L\left(t+\frac{2\tau}{m}\right)\left[z_2\left(t-\frac{\tau}{m}\right)-z_1(t)\right] \\ &\vdots& \\ \dot{z}_m(t) &=& A(t+\tau)z_m(t) + B(t+\tau)u(t) \\ && +L(t+\tau)\left[z_m\left(t-\frac{\tau}{m}\right)-z_{m-1}(t)\right] \end{array}\right. \qquad (6)$$

with initial conditions in $C([-\tau, 0])$, and the operators

$$e_j(t) = z_j(t) - z_{j-1}\left(t + \frac{\tau}{m}\right) \qquad (7)$$

for $j = 1$ to m for all $t \geq 0$, where $z_0(t) = x(t)$.

Then we can use a telescoping sum argument to get

$$z_m(t) = x(t+\tau) + \xi(t) \qquad (8)$$

for all $t \geq 0$, where

$$\xi(t) = \sum_{j=1}^{m} e_j\left(t + \frac{(m-j)\tau}{m}\right), \qquad (9)$$

and then we can evaluate (8) at $t-\tau$ to remove the dependence of (6) on the current values $x(t)$, as follows. Notice that by combining (1) and (6), we get

$$\begin{array}{l} \dot{e}_1\left(t+\frac{(m-1)\tau}{m}\right) = A(t+\tau)e_1\left(t+\frac{(m-1)\tau}{m}\right) \\ +L(t+\tau)e_1\left(t+\frac{(m-2)\tau}{m}\right) - \delta(t+\tau) \end{array} \qquad (10)$$

and

$$\begin{array}{rcl} \dot{e}_j\left(t+\frac{(m-j)\tau}{m}\right) &=& A(t+\tau)e_j\left(t+\frac{(m-j)\tau}{m}\right) \\ && +L(t+\tau)e_j\left(t+\frac{(m-j-1)\tau}{m}\right) \\ && -L(t+\tau)e_{j-1}\left(t+\frac{(m-j)\tau}{m}\right) \end{array} \qquad (11)$$

for $j = 2$ to m for all $t \geq 0$. Bearing in mind the properties of (3), it follows that we have a cascade of ISS systems (10)-(11) having the state value components $(e_1, \ldots, e_m)$.

According to (4), we deduce that there are two constants $c_1 > 0$ and $c_3 > 0$ such that for any initial condition of the system (1) and (6), we have

$$\begin{array}{rcl} ||\xi(t)|| &\leq& c_1 e^{-p_2(t-s)} \sup_{\ell\in[s-2\tau/m,s]} ||\xi(\ell)|| \\ && + c_3 \sup_{\ell\in[s,t+\tau]} ||\delta(\ell)|| \end{array} \qquad (12)$$

when $t \geq s \geq 0$.

Although the dynamics of the first subpredictor z_1 in (6) uses current $x(t)$ values, our control will be computed from delayed values $z_m(t-\tau)$ of the last subpredictor z_m, so only delayed values of x will be needed to compute our control; see (a) in Section 3 below. This makes our control practical for use where only time lagged values are available for use in the control. See also [38] for methods to compute a constant $D_* > 0$ such that the conclusions of this subsection continue to hold if $x(t)$ in the first subpredictor is replaced by a delayed version $x(t-D)$ for any constant delay $D \in [0, D_*]$, which can represent sensor delays. However, for simplicity, we do not consider these sensor delays in this work.

3 Main result

In this section, we introduce an event-triggered control and establish an ISS property for the corresponding closed-loop system. To this end, we first introduce a bounded continuous matrix valued function $K : \mathbb{R} \to \mathbb{R}^{p\times n}$ and the function

$$\omega(t,s) = \Phi_A(t,s) + \int_s^t \Phi_A(t,\ell)B(\ell)K(\ell)\mathrm{d}\ell. \qquad (13)$$

Since $\omega(s,s) = I$ for all real s, and since ω is a C^1 function such that $\sup\{||(\partial\omega/\partial t)(t,s)|| : |t-s| \leq \beta\}$ is bounded for each constant $\beta > 0$, we deduce that there is a constant $\omega_L > 0$ such that $\omega(t,s)$ is invertible for all (t,s) such that $|t-s| \leq \omega_L$. Consequently, one can select constants $\nu > 0$ and $T > 0$ such that $T \geq 2\nu$ and such that $\omega(t,s)$ is invertible when $|t-s| \leq T$. We deduce that the matrices

$$\Gamma = \sup\left\{\left|\omega(t,s)^{-1} - I\right| : 0 \leq s \leq t \leq s+\nu\right\}, \qquad (14)$$

$$\begin{array}{l} \lambda = \sup\left\{\left|\omega(t,s)^{-1}\int_s^t \Phi_A(t,\ell)B(\ell)K(\ell)\mathrm{d}\ell\right| : \right. \\ \qquad \left. 0 \leq s \leq t \leq s+\nu\right\}, \text{ and} \end{array} \qquad (15)$$

$$\varpi = \sup\left\{\left|\omega(t,s)^{-1}\right| : 0 \leq s \leq t \leq s+\nu\right\} \qquad (16)$$

are well defined, where the sup is computed entrywise; see Remark 1 for ways to compute the entries of the matrices in the suprema in (14), (15), and (16). To simplify the notation, we define a matrix valued function

$$H(t) = A(t) + B(t)K(t). \qquad (17)$$

Then, for each $t \geq 0$, the matrix

$$M(t) = R_H(t) + N_H(t) + |B(t)K(t)|\Gamma \qquad (18)$$

is Metzler. We assume that the constant $\nu > 0$ and matrix K specified above are such that the matrix (18) satisfies:

Assumption 1 *There exist a positive vector $V \in \mathbb{R}^n$ and a constant $p > 0$ such that $V^\top M(t) \leq -pV^\top$ for all $t \geq 0$.*

The motivation for Assumption 1 is that it can be viewed as a generalized exponential stability condition on the subset of all solutions of the system $\dot{q}(t) = M(t)q(t)$ that are nonnegative valued, which reduces to a robust decay condition that applies to the subset of all solutions of $\dot{q}(t) = H(t)q(t)$ that are nonnegative valued in the special case where $H(t)$ is Metzler for each t, with $|B(t)K(t)|\Gamma$ viewed as a perturbation term. Its role is analogous to the role played by the asymptotic stability condition on the nominal system in standard emulation-based event-triggered control with no delays. In our proof of our theorem, we use Assumption 1 to build a linear Lyapunov function $\mathcal{V}(\zeta) = V^\top\zeta$ to establish asymptotic stability conditions for a positive system whose state ζ is a difference between upper and lower estimates from an interval observer. Assumption 1 holds if M is a constant Hurwitz and Metzler matrix (e.g., by [16, Lemma 2.3, p.41]), and so also if M has the form $M_0 + \Delta_M(t)$ with M_0 being a constant Hurwitz and Metzler matrix when the sup norm of Δ_M is small enough; see Remark 4 for general conditions under which Assumption 1 is satisfied.

In terms of the function ξ from (9), our event-triggered control and triggering times t_i for (1) are defined by

(a) $u(t-\tau) = K(t)z_m(t_i - \tau)$ if $i \geq 0$ is such that $t \in [t_i, t_{i+1})$,
(b) For each $i \geq 0$ and each $t \in [t_i + \tau, t_{i+1} + \tau)$, we have

$$\begin{aligned}&|x(t-\tau) - x(t_i)| \leq \Gamma|x(t-\tau)| + \lambda|\xi(t_i-\tau)| \\ &+\varpi \int_{\max\{0,t-T-\tau\}}^{t-\tau} e^{|A|_\infty(t-\tau-\ell)}\bar{\delta}(\ell)\mathrm{d}\ell,\end{aligned} \tag{19}$$

(c) $|t - t_i| \leq T$ for each $t \in [t_i, t_{i+1}]$ and $i \geq 0$, and
(d) for each i, either $t_{i+1} - t_i = T$, or for each $\epsilon > 0$, there is a t_c such that $t_c \in (t_{i+1} + \tau, t_{i+1} + \epsilon + \tau)$ and such that

$$\begin{aligned}&|x(t_c-\tau) - x(t_i)| \not\leq \Gamma|x(t_c-\tau)| + \lambda|\xi(t_i-\tau)| \\ &+\varpi \int_{\max\{0,t_c-T-\tau\}}^{t_c-\tau} e^{|A|_\infty(t_c-\tau-\ell)}\bar{\delta}(\ell)\mathrm{d}\ell\end{aligned} \tag{20}$$

where $t_0 = 0 < t_1 < t_2 < \ldots$, which uses the control value $u(t-\tau) = K(t)z_m(-\tau)$, until either time $t_1 = T$, or until some $t_1 \in [0, T)$ that is the infimum of the times when a violation (20) of (19) occurs if such a violation time occurs, and then we repeat the preceding process with $t_0 = 0$ and t_1 replaced by t_1 and t_2 respectively, and then argue inductively, to define the control, event triggering times t_i, and closed loop system at all times $t \geq 0$. Our justification for using $t - \tau$ and $t_c - \tau$ instead of t and t_c in (19)-(20) is that it allows us to determine the event triggering times t_i from measurements of $x(t-\tau)$ and $x(t_i)$, where t_i in (b) and (d) can be viewed as the last event trigger time before $t - \tau$ or $t_c - \tau$. Hence, the three parts of our codesigned control and event-trigger (consisting of the subpredictors (6), the formula $u(t-\tau) = K(t)z_m(t_i - \tau)$ from (a), and the triggering conditions in (b) and (d)) only require time lagged state measurements, which is an advantage for applications where current state measurements are unavailable. Moreover, although $x(t-\tau)$ appears in the triggering rule, the control in part (a) is a sampled one with the sample times t_i being state dependent. This takes resource constraints into account by only changing the control values at the times t_i. If there is uncertainty in these $x(t-\tau)$ measurements, then we can use estimates $\hat{x}(t-\tau)$ of $x(t-\tau)$ in the triggering rules. Then the results of this paper provide ISS with respect to the combined variable $(\bar{\delta}, \bar{\delta}_e)$ if we replace the values $x(t-\tau)$ in (19) and (20) by $\hat{x}(t-\tau)$ and add $(I+\Gamma)\bar{\delta}_e$ to their right sides, when we know a nonnegative vector $\bar{\delta}_e$ such that $|x(t) - \hat{x}(t)| \leq \bar{\delta}_e$ for all $t \geq 0$, i.e., a bound on the estimation error. This follows from using the triangle inequality to obtain $-\bar{\delta}_e + |x(t-\tau) - x(t_i)| \leq |\hat{x}(t-\tau) - x(t_i)|$ and $\Gamma|\hat{x}(t-\tau)| \leq \Gamma|x(t-\tau)| + \Gamma\bar{\delta}_e$ for all $t \geq 0$, and by then collecting the terms involving $\bar{\delta}_e$ on the right sides.

By Condition (c), Condition (d) can be restated as follows: for each index i such that $t_{i+1} \in [t_i, t_i + T)$, we have

$$\begin{aligned}&t_{i+1} = \inf\{s \in [t_i, t_i + T) : |x(s) - x(t_i)| \not\leq \Gamma|x(s)| \\ &+\lambda|\xi(t_i - \tau)| + \varpi \int_{\max\{0,s-T\}}^{s} e^{|A|_\infty(s-\ell)}\bar{\delta}(\ell)\mathrm{d}\ell\Big\},\end{aligned} \tag{21}$$

where the equivalence of the preceding condition and (d) above can be seen by noting that considering values of t such that $t - \tau \in [t_i, t_i + T)$ is equivalent to considering values of $s \in [t_i, t_i + T)$. Later, we show that the constant $\nu > 0$ we chose above is such that $t_{i+1} - t_i \geq \nu$ for all integers $i \geq 0$, so the Zeno phenomenon does not occur.

In terms of the constants in (12), the delay $\tau > 0$ in (1), T and p chosen as above, and any constant $c_* > 0$ such that

$$2p_2(c_*+1)e^{p_2(T+\tau)}(c_1+1) < e^{c_* p_2} - 1, \tag{22}$$

which can be satisfied when $c_* > 0$ is large enough, and using (15) and the constants

$$\begin{aligned}&b = ||V^\top|BK|_\infty(\lambda + I)||, \quad v = \min_i V_i, \\ &T_e = \max\{c_*, T\},\ T_\dagger = c_* + T, \\ &p_\dagger = \tfrac{||V|| + (1+b)c_*^2}{v},\ \ p_* = \min\left\{p, \tfrac{1}{2c_*}\right\}, \\ &\text{and}\ \ c_e = \tfrac{||V^\top|BK|_\infty \varpi e^{|A_\infty|T}||T + ||V|| + c_3(1+b)(c_*+1)}{vp_\star},\end{aligned} \tag{23}$$

where V_i is the ith component of V from Assumption 1, the main result of this paper is as follows:

Theorem 1 *Let Assumption 1 hold. Then the solutions of (1) and (6), with ξ defined by (9) and with the event-triggered control defined by (a)-(d), satisfy*

$$\begin{aligned}&||x(t)|| \leq \\ &p_\dagger e^{-p_\star(t-r)}\left(\sup_{\ell\in[r-T_\dagger, r+\tau]}||x(\ell)|| + \sup_{\ell\in[r-T_\dagger-\tau, r]}||z_m(\ell)||\right) \\ &+c_e \sup_{\ell\in[r-T_e, t+\tau]}||\bar{\delta}(\ell)||\end{aligned} \tag{24}$$

for all $r \geq T_\dagger + \tau$ and $t \geq r$.

Remark 1 *Due to the $p_\dagger e^{-p_\star(t-r)}$ in (24) and the positiveness of $p_\star$, $x(t)$ asymptotically approaches 0 when δ is zero, because in that case, we can take $\bar{\delta} = 0$. Also, $||x(t)||$ approaches the ball of radius $c_e d_*$ asymptotically when the uncertainty is bounded by a constant d_* in the norm $||\cdot||$. Hence, uncertainties with smaller sup norms ensure convergence of the states to arbitrarily small prescribed values.*

In the special case where A, B, and K are all constant, we get $\omega(t,s) = \Omega(t-s)$, where

$$\Omega(r) = e^{Ar} + \int_0^r e^{(r-\ell)A}\mathrm{d}\ell BK \tag{25}$$

is the corresponding function that was used in the undelayed case in [28]. Also, when A, B, and K all have some period $\bar{p} > 0$, then $\omega(t+\bar{p}, s+\bar{p}) = \omega(t,s)$ for all $s \geq 0$ and $t \geq s$, so our invertibility requirement for ω can be checked by checking that $\omega(t,s)$ is invertible for all $s \in [0,\bar{p}]$ and $t \in [s, s+\nu]$. This requires computing $\Phi_A(t,m)$ for all $m \in [s,t]$, $s \in [0,\bar{p}]$, and $t \in [s, s+\nu]$, which can be done *by writing $\Phi_A(t,m) = \alpha_A(t)\beta_A(m)$ for all m and $t \geq m$, where α_A and β_A are the unique solutions of*

$$\dot{\alpha}_A(t) = A(t)\alpha_A(t) \text{ and } \dot{\beta}_A(m) = -\beta_A(m)A(m) \quad (26)$$

that satisfy $\alpha_A(0) = \beta_A(0) = I$, and by then choosing $\Gamma = \sup\{|I - \omega^{-1}(t,s)| : s \in [0,\bar{p}], t \in [s, s+\nu]\}$. Similar reasoning allows us to compute λ and $\bar{\omega}$ from (15)-(16).

Remark 2 *Our control algorithm (a)-(d) is valid in the special case where $\tau = 0$, in which case $\xi = 0$. When $\tau = 0$ and A, B, and K are constant, our proof of the theorem will show that we can remove condition (c) and then not consider the possibility $t_{i+1} - t_i = T$ in (d), and then (a)-(d) is the same as the event-triggered algorithm from [28, Theorem 1] in this special undelayed case.*

Remark 3 *The design of the sequence t_i ensures that for all $i \geq 0$, we have $t_{i+1} - t_i \in [0, T]$. This constraint on this sequence is not very restrictive because, in general, $\omega(t,s)$ is invertible for all t and s satisfying $|t-s| \leq \omega_L$ for a large value of ω_L, which implies that T can be a large constant.*

Remark 4 *Assumption 1 is a stabilizability condition. To evaluate how restrictive it is, let us consider the case where the pair (A,B) is a constant controllable pair. In this case, we can select a constant matrix K such that the corresponding matrix $H = A + BK$ is Hurwitz and such that all of its eigenvalues are real numbers. It follows that one can find a matrix $P \in \mathbb{R}^{n\times n}$ such that PHP^{-1} is Hurwitz and Metzler, by choosing P such that PHP^{-1} is the Jordan canonical form of H. Hence, after a preliminary change of coordinates, we obtain a system for which there is a matrix K such that H is Hurwitz and Metzler. Then the matrix M defined in (18) satisfies $M = H + |BK|\Gamma$. Since Γ can be rendered arbitrarily small by an appropriate choice of $\nu > 0$, we deduce that, in this case, Assumption 1 is satisfied for some constants ν (again by [16, Lemma 2.3, p.41]).*

Remark 5 *In (a)-(d), the function $\bar{\delta}$ is present. Thus Theorem 1 does not apply when no upper bound for $|\delta|$ is known.*

4 Proof of Theorem 1

The proof has four parts. In the first part, we prove that, for the inter-sampling intervals $[t_i, t_{i+1})$ arising from using the control (a)-(d), the Zeno phenomenon does not occur. In the second part, we show that the function $\mathcal{V}(\zeta) = V^\top \zeta$ satisfies an ISS Lyapunov function decay condition along the dynamics for $\zeta(t) = \overline{x}(t) - \underline{x}(t)$, where $(\overline{x}, \underline{x})$ is the vector of states for a framer that we will construct for the state variable $x(t)$, and where $V > 0$ is from Assumption 1. In the third part, we prove an ISS estimate for the functional

$$\eta(\xi_t) = \sup_{\ell \in [t-T-\tau, t]} ||\xi(\ell)|| \quad (27)$$

with δ viewed as the disturbance. In the final part, we combine the ISS estimate for (27) with the ISS decay estimate on $V^\top\zeta$ to obtain the desired ISS estimate in the state of the closed loop system from the theorem.

First part: Inter-sampling intervals. We prove that the Zeno phenomenon does not occur, by proving that $t_{i+1} - t_i \geq \nu$ for all i where $\nu > 0$ was chosen in Section 3.

The solutions of the closed-loop system satisfy the following for all integers $i \geq 0$:

$$\left\{\begin{array}{l} \dot{x}(t) = A(t)x(t) + B(t)K(t)z_m(t_i - \tau) + \delta(t) \\ \text{for all } t \in [t_i, t_{i+1}); \\ |x(t-\tau) - x(t_i)| \leq \Gamma|x(t-\tau)| + \lambda|\xi(t_i - \tau)| \\ +\bar{\omega}\int_{\max\{0,t-T-\tau\}}^{t-\tau} e^{|A|_\infty(t-\tau-\ell)}\bar{\delta}(\ell)\mathrm{d}\ell \\ \text{for all } t \in [t_i + \tau, t_{i+1} + \tau); \text{ and} \\ \text{either } t_{i+1} - t_i = T, \text{ or } |t_{i+1} - t_i| < T \text{ and for} \\ \text{each } \epsilon > 0 \text{ there exists a } t_s \in (t_{i+1}, t_{i+1} + \epsilon) \\ \text{such that } |x(t_s) - x(t_i)| \not\leq \Gamma|x(t_s)| + \lambda|\xi(t_i - \tau)| \\ +\bar{\omega}\int_{\max\{0,t_s-T\}}^{t_s} e^{|A|_\infty(t_s-\ell)}\bar{\delta}(\ell)\mathrm{d}\ell. \end{array}\right. \quad (28)$$

According to the equality (8),

$$\begin{array}{rcl} \dot{x}(\ell) & = & A(\ell)x(\ell) + B(\ell)K(\ell)x(t_i) \\ & & + B(\ell)K(\ell)\xi(t_i - \tau) + \delta(\ell) \end{array} \quad (29)$$

for all $\ell \in [t_i, t_{i+1})$. Here and in the sequel, all inequalities and equalities are for each integer $i \geq 0$ unless otherwise indicated. To show that $t_{i+1} \geq t_i + \nu$, we can first apply variation of parameters to (29) on $[t_i, t-\tau]$ for any $t \in [t_i + \tau, t_i + \tau + T]$ with the initial state $x(t_i)$ to obtain

$$\begin{array}{rcl} x(t-\tau) & = & \Phi_A(t-\tau, t_i)x(t_i) \\ & & + \int_{t_i}^{t-\tau} \Phi^\sharp(t,\ell)\mathrm{d}\ell[x(t_i) + \xi(t_i - \tau)] \\ & & + \int_{t_i}^{t-\tau} \Phi_A(t-\tau,\ell)\delta(\ell)\mathrm{d}\ell \\ & = & \omega(t-\tau, t_i)x(t_i) + \int_{t_i}^{t-\tau} \Phi^\sharp(t,\ell)\mathrm{d}\ell\xi(t_i - \tau) \\ & & + \int_{t_i}^{t-\tau} \Phi_A(t-\tau,\ell)\delta(\ell)\mathrm{d}\ell \end{array} \quad (30)$$

where $\Phi^\sharp(t,\ell) = \Phi_A(t-\tau,\ell)B(\ell)K(\ell)$ and ω was defined in (13).

Since $\omega(r,s)$ is invertible when $|s-r| \leq T$, we deduce that

$$\begin{array}{l} x(t_i) - x(t-\tau) = \left[\omega(t-\tau, t_i)^{-1} - I\right] x(t-\tau) \\ -\omega(t-\tau, t_i)^{-1} \int_{t_i}^{t-\tau} \Phi^\sharp(t,\ell)\mathrm{d}\ell\xi(t_i - \tau) \\ -\,\omega(t-\tau, t_i)^{-1} \int_{t_i}^{t-\tau} \Phi_A(t-\tau,\ell)\delta(\ell)\mathrm{d}\ell. \end{array} \quad (31)$$

Thus, for all $t \in [t_i + \tau, t_i + \nu + \tau]$, we have $|t - \tau - t_i| \leq \nu$ and $t - \tau \geq t_i$, so our choices (14)-(15) of Γ and λ give

$$\begin{array}{l} |x(t_i) - x(t-\tau)| \leq \Gamma|x(t-\tau)| \\ + \left|\omega(t-\tau, t_i)^{-1} \int_{t_i}^{t-\tau} \Phi^\sharp(t,\ell)\mathrm{d}\ell\right| |\xi(t_i - \tau)| \\ + \left|\omega(t-\tau, t_i)^{-1} \int_{t_i}^{t-\tau} \Phi_A(t-\tau, t_i)\delta(\ell)\mathrm{d}\ell\right| \end{array} \quad (32)$$

and so also

$$\begin{array}{l} |x(t_i) - x(t-\tau)| \leq \Gamma|x(t-\tau)| + \lambda\,|\xi(t_i - \tau)| \\ + \left|\omega(t-\tau, t_i)^{-1} \int_{t_i}^{t-\tau} \Phi_A(t-\tau,\ell)\delta(\ell)\mathrm{d}\ell\right|. \end{array} \quad (33)$$

Bearing in mind (2), we deduce that

$$\begin{array}{l} |x(t_i) - x(t-\tau)| \leq \Gamma|x(t-\tau)| + \lambda\,|\xi(t_i - \tau)| \\ + \left|\omega(t-\tau, t_i)^{-1}\right| \int_{t_i}^{t-\tau} e^{|A|_\infty(t-\tau-\ell)}\bar{\delta}(\ell)\mathrm{d}\ell \\ \text{and } |t - \tau - t_i| \leq T \end{array} \quad (34)$$

for all $t \in [t_i+\tau, t_i+\nu+\tau]$. Since $\left|\omega(t-\tau, t_i)^{-1}\right| \leq \overline{\omega}$ for all $t \in [t_i+\tau, t_i+\nu+\tau]$, we conclude that the inequality (19) in (b) holds for all $t \in [t_i+\tau, t_i+\tau+\nu]$, so $t_{i+1} \geq t_i+\nu$. This implies that the Zeno phenomenon does not occur.

Second part: ISS Lyapunov function decay estimate for $\mathcal{V}(\zeta) = V^\top \zeta$. To analyze the stability of the closed loop system (28), we use the approach from [15] and [25]. Hence, our stability analysis relies on framers, which are used as comparison systems. For all integers $i \geq 0$, we can evaluate (29) at $\ell = t$ to write (29) as

$$\dot{x}(t) = H(t)x(t) + \mu(t) + \delta(t), \tag{35}$$

where μ is the function defined by

$$\mu(t) = B(t)K(t)[x(t_i) - x(t)] + B(t)K(t)\xi(t_i - \tau) \tag{36}$$

for all $t \in [t_i, t_{i+1})$ and where $H = A + BK$ is the matrix valued function that we defined in (17).

We introduce the framer

$$\begin{cases} \dot{\overline{x}}(t) = R_H(t)\overline{x}(t) - N_H(t)\underline{x}(t) + \mu(t)^+ + \delta(t)^+ \\ \dot{\underline{x}}(t) = -N_H(t)\overline{x}(t) + R_H(t)\underline{x}(t) - \mu(t)^- - \delta(t)^- \end{cases} \tag{37}$$

with initial conditions

$$\overline{x}(r) = x^+(r) \text{ and } \underline{x}(r) = -x^-(r) \tag{38}$$

with $r \geq T+\tau$. Let $x_\ddagger(t) = -\underline{x}(t)$. This produces the following system:

$$\begin{cases} \dot{\overline{x}}(t) = R_H(t)\overline{x}(t) + N_H(t)x_\ddagger(t) + \mu(t)^+ + \delta(t)^+ \\ \dot{x}_\ddagger(t) = N_H(t)\overline{x}(t) + R_H(t)x_\ddagger(t) + \mu(t)^- + \delta(t)^- \end{cases} \tag{39}$$

Then since $\mu(t)^+ + \delta(t)^+ \geq 0$ and $\mu(t)^- + \delta(t)^- \geq 0$ for all $t \geq 0$, and since the matrix

$$\begin{bmatrix} R_H(t) & N_H(t) \\ N_H(t) & R_H(t) \end{bmatrix} \tag{40}$$

is Metzler for each choice of t, the system (39) is cooperative; see, e.g., [24, Lemma 1]. It follows that

$$\overline{x}(t) \geq 0 \text{ and } x_\ddagger(t) \geq 0 \tag{41}$$

hold for all $t \geq r$ and all solutions of (39). Hence, the corresponding solutions of (37) satisfy

$$\overline{x}(t) \geq 0 \text{ and } \underline{x}(t) \leq 0 \tag{42}$$

for all $t \geq r$. Similarly, since $H = R_H - N_H$, and since $\mu = \mu^+ - \mu^-$ and $\delta = \delta^+ - \delta^-$, one can prove that

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t) \tag{43}$$

for all $t \geq r$, by using (35) and (37) to show the positivity of the dynamics for $(\bar{x} - x, x - \underline{x})$.

Let us introduce $\zeta(t) = \overline{x}(t) - \underline{x}(t)$. Then the inequalities (42) ensure that $\zeta(t) \geq 0$ for all $t \geq r$. Moreover, for all $t \geq r$, our formula (37) for our framer gives

$$\dot{\zeta}(t) = (R_H(t) + N_H(t))\zeta(t) + |\mu(t)| + |\delta(t)| \tag{44}$$

From (19) from (b) (with $t - \tau$ in (19) replaced by t), we deduce that

$$\begin{aligned} |\mu(t)| \leq\ & |B(t)K(t)|\Gamma|x(t)| + |BK|_\infty \Big(\lambda|\xi(t_i-\tau)| \\ & + \overline{\omega}\int_{\max\{0,t-T\}}^t e^{|A|_\infty(t-\ell)}\overline{\delta}(\ell)\mathrm{d}\ell\Big) \\ & + |BK|_\infty|\xi(t_i-\tau)| \end{aligned} \tag{45}$$

for all $t \geq \tau$ with $t \in [t_i, t_{i+1})$. Thus, for all $t \geq r$, we have

$$\begin{aligned} |\mu(t)| \leq\ & |B(t)K(t)|\Gamma|x(t)| \\ & + |BK|_\infty\overline{\omega}\int_{t-T}^t e^{|A|_\infty(t-\ell)}\overline{\delta}(\ell)\mathrm{d}\ell \\ & + |BK|_\infty(\lambda+I)|\xi(t_i-\tau)|. \end{aligned} \tag{46}$$

This inequality and (44) give

$$\begin{aligned} \dot{\zeta}(t) \leq\ & (R_H(t)+N_H(t))\zeta(t) + |B(t)K(t)|\Gamma|x(t)| \\ & + |BK|_\infty\overline{\omega}\int_{t-T}^t e^{|A|_\infty(t-\ell)}\overline{\delta}(\ell)\mathrm{d}\ell \\ & + |BK|_\infty(\lambda+I)|\xi(t_i-\tau)| + \overline{\delta}(t). \end{aligned} \tag{47}$$

According to (42)-(43), we have $\underline{x}(t) - \bar{x}(t) \leq x(t) \leq \bar{x}(t) - \underline{x}(t)$ and so also

$$|x(t)| \leq \zeta(t) \tag{48}$$

for all $t \geq r$, which we use to upper bound the right side of (47) to get

$$\begin{aligned} \dot{\zeta}(t) \leq\ & M(t)\zeta(t) + |BK|_\infty(\lambda+I)|\xi(t_i-\tau)| \\ & + |BK|_\infty\overline{\omega}\int_{t-T}^t e^{|A|_\infty(t-\ell)}\overline{\delta}(\ell)\mathrm{d}\ell + \overline{\delta}(t) \end{aligned} \tag{49}$$

where M is the matrix defined in (18) for all $t \geq r$.

We next use the linear Lyapunov function

$$\mathcal{V}(\zeta) = V^\top \zeta, \tag{50}$$

where V is the vector present in Assumption 1 as before. According to Assumption 1 and the inequality (49), and also using the nonnegative valuedness of ζ, the time derivative of $\mathcal{V}(\zeta)$ along the trajectories of (49) satisfies

$$\begin{aligned} \dot{\mathcal{V}}(t) \leq\ & -p\mathcal{V}(\zeta(t)) + V^\top|BK|_\infty(\lambda+I)|\xi(t_i-\tau)| \\ & + \Delta_1(t) \end{aligned} \tag{51}$$

for all $t \geq r$, where

$$\Delta_1(t) = V^\top|BK|_\infty\overline{\omega}e^{|A|_\infty T}\int_{t-T}^t \overline{\delta}(\ell)\mathrm{d}\ell + V^\top\overline{\delta}(t). \tag{52}$$

Consequently

$$\dot{\mathcal{V}}(t) \leq -p\mathcal{V}(\zeta(t)) + b\sup_{a\in[t-T-\tau,t-\tau]}||\xi(a)|| + \Delta_1(t), \tag{53}$$

where $b = ||V^\top|BK|_\infty(\lambda+I)||$, which is the desired ISS Lyapunov function decay estimation on $\mathcal{V}$.

Third part. ISS estimate for (27). From the inequality (12), it follows that the functional η we defined in (27) satisfies

$$\begin{aligned} \eta(\xi_t) \leq\ & c_1 e^{p_2(T+\tau)}e^{p_2(s-t)}\sup_{\ell\in\left[s-\frac{2\tau}{m},s\right]}||\xi(\ell)|| \\ & + c_3\sup_{\ell\in[s,t+\tau]}||\delta(\ell)|| \end{aligned} \tag{54}$$

for all $s \geq 0$ and $t \geq s+T+\tau$.

Now, consider the case where $t \in [s, s+T+\tau]$ and $s \geq T+\tau$.

Then

$$\begin{aligned}\eta(\xi_t) &\leq \sup_{l\in[s-T-\tau,t]} ||\xi(l)|| \\ &\leq \sup_{l\in[s-T-\tau,s]} ||\xi(l)|| + \sup_{l\in[s,t]} ||\xi(l)|| \\ &\leq \sup_{l\in[s,t]} \left(c_1 e^{p_2(s-l)} \sup_{\ell\in\left[s-\frac{2\tau}{m},s\right]} ||\xi(\ell)|| \right. \\ &\quad \left. + c_3 \sup_{\ell\in[s,l+\tau]} ||\delta(\ell)|| \right) + \sup_{l\in[s-T-\tau,s]} ||\xi(l)||\end{aligned} \tag{55}$$

where the last inequality is a consequence of (12). Since $m \geq 2$, it follows that

$$\begin{aligned}\eta(\xi_t) &\leq \eta(\xi_s) + \sup_{l\in[s,t]} \left(c_1 \sup_{\ell\in\left[s-\frac{2\tau}{m},s\right]} ||\xi(\ell)|| \right. \\ &\quad \left. + c_3 \sup_{\ell\in[s,l+\tau]} ||\delta(\ell)|| \right) \\ &\leq \eta(\xi_s) + c_1 \sup_{\ell\in\left[s-\frac{2\tau}{m},s\right]} ||\xi(\ell)|| \\ &\quad + c_3 \sup_{\ell\in[s,t+\tau]} ||\delta(\ell)|| \\ &\leq (c_1+1)\eta(\xi_s) + c_3 \sup_{\ell\in[s,t+\tau]} ||\delta(\ell)|| \\ &\leq (c_1+1)e^{p_2(T+\tau)}e^{p_2(s-t)}\eta(\xi_s) \\ &\quad + c_3 \sup_{\ell\in[s,t+\tau]} ||\delta(\ell)||,\end{aligned} \tag{56}$$

where the last inequality is a consequence of the fact that $t - s \leq T + \tau$. We deduce that

$$\eta(\xi_t) \leq c_4 e^{p_2(s-t)}\eta(\xi_s) + c_3 \sup_{\ell\in[s,t+\tau]} ||\delta(\ell)|| \tag{57}$$

for all $s \geq T + \tau$ and $t \geq s$ with $c_4 = (c_1 + 1)e^{p_2(T+\tau)}$. Condition (57) is our required ISS estimate for η.

Fourth part: ISS property for closed loop system. This part entails combining (53) and (57) into the ISS estimate as required by the theorem. It uses the functional

$$\mathcal{F}(\zeta(t), \xi_t) = \mathcal{V}(\zeta(t)) + (1 + b)\mathcal{W}(\xi_t), \tag{58}$$

where b is the constant defined in (23), and where

$$\mathcal{W}(\xi_t) = \int_{t-c_*}^{t} \int_a^t \eta(\xi_l) \mathrm{d}l \mathrm{d}a \tag{59}$$

with $c_* > 0$ being any constant satisfying our condition (22). For details, see Appendix B below.

5 Illustration

We illustrate our theorem, using a benchmark two-dimensional (i.e., planar) curve tracking dynamical system from [22], which we studied in our work [29] that was confined to unelayed cases. While simple (insofar that its only control is a steering control), it will illustrate the value of our approach for compensating any positive constant delay while reducing the computational burden as compared to continuous time controls that are not event-triggered. The dynamics are

$$\begin{cases} \dot{d}(t) = -\sin(\phi(t)) \\ \dot{\phi}(t) = \frac{\kappa\cos(\phi(t))}{1+\kappa d(t)} + u(t-\tau) \end{cases} \tag{60}$$

which describes curve tracking by a unit speed marine robot having a gyroscopic steering control u with a constant delay $\tau > 0$, where d is the distance between the marine robot and the closest point (i.e., its projection) on the curve being tracked (assuming as in [22,23] that the closest point is unique at each time), ϕ denotes the heading angle (measuring the difference between the angles of the tangent lines of the path of the marine robot and of the path being traced by the closest point on the curve that is being tracked at each time t), and κ denotes the curvature at each time. We assume for simplicity that the curvature is a positive constant.

Linearization of the system (60) around a trajectory $(d_r(t), \phi_r(t))$ gives

$$\begin{cases} \dot{d}_a(t) = a_{12}(t)\phi_a(t) \\ \dot{\phi}_a(t) = a_{21}(t)d_a(t) + a_{22}(t)\phi_a(t) + u(t-\tau), \end{cases} \tag{61}$$

where

$$\begin{aligned} &a_{12}(t) = -\cos(\phi_r(t)),\ a_{21}(t) = -\tfrac{\kappa^2\cos(\phi_r(t))}{(1+\kappa d_r(t))^2}, \text{ and} \\ &a_{22}(t) = -\tfrac{\kappa\sin(\phi_r(t))}{1+\kappa d_r(t)}. \end{aligned} \tag{62}$$

We consider the reference trajectory (d_r, ϕ_r), where

$$d_r(t) = \tfrac{1}{\kappa}\left(1 + \tfrac{t\Delta_*}{1+t}\right) \text{ and } \phi_r(t) = -\arcsin\left(\tfrac{\Delta_*}{\kappa(1+t)^2}\right). \tag{63}$$

The physical meaning of the constant

$$\Delta_* \in (-\min\{1, \kappa\}, \min\{1, \kappa\}) \tag{64}$$

is that it provides the speed at which the robot is traveling towards the curve that is being tracked (if $\Delta_* < 0$) or away from the curve being tracked (if $\Delta_* > 0$). This contrasts with the undelayed case that was studied in [29], where the corresponding constant Δ_* was required to be nonpositive. Therefore, for this reference trajectory, $(1 + \Delta_*)/\kappa$ is the limit of the distance between the robot and the projection point on the curve being tracked as $t \to +\infty$. This gives

$$\begin{aligned} &a_{12}(t) = \mathcal{A}_1(t) - 1,\ a_{21}(t) = -\tfrac{\kappa^2}{4} + \mathcal{A}_2(t), \\ &\text{and } a_{22}(t) = \mathcal{A}_3(t), \text{ where} \\ &\mathcal{A}_1(t) = 1 - \left(1 - \tfrac{\Delta_*^2}{\kappa^2(1+t)^4}\right)^{1/2}, \\ &\mathcal{A}_2(t) = \kappa^2\left(\tfrac{1}{4} - \tfrac{\sqrt{1-\frac{\Delta_*^2}{\kappa^2(1+t)^4}}}{\left(2+\frac{t\Delta_*}{1+t}\right)^2}\right), \\ &\text{and } \mathcal{A}_3(t) = \tfrac{\Delta_*}{(1+t)^2}\tfrac{1}{2+\frac{t\Delta_*}{1+t}}. \end{aligned} \tag{65}$$

Let us choose $\kappa = 2$. This produces the dynamics

$$\begin{cases} \dot{d}_a(t) = (\mathcal{A}_1(t) - 1)\phi_a(t) \\ \dot{\phi}_a(t) = -(1 - \mathcal{A}_2(t))d_a(t) + \mathcal{A}_3(t)\phi_a + u(t-\tau). \end{cases} \tag{66}$$

The change of coordinates $(x_1(t), x_2(t)) = (d_a(t), d_a(t) - \phi_a(t))$ produces the dynamics $\dot{x}(t) = A(t)x(t) + Bu(t-\tau)$, where

$$A(t) = \begin{bmatrix} \mathcal{A}_1(t) - 1 & 1 - \mathcal{A}_1(t) \\ \mathcal{A}_4(t) & 1 - \mathcal{A}_1(t) + \mathcal{A}_3(t) \end{bmatrix}, \qquad (67)$$

$B = [0, -1]^\top$, and $\mathcal{A}_4 = \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3$.

Using Mathematica calculations, it follows that our requirements of Assumption 1 are satisfied with $V = [1, 2]^\top$, $\delta = 0$, $K = [0, 1.75]$, each entry of Γ being 0.1, and $p = 0.01$ when the parameter in the reference trajectory (63) is $\Delta_* = 0.15$. We obtained the preceding V, p, K, and Γ values by first finding constant choices of V, p, K and Γ for which Assumption 1 is satisfied when $\Delta_* = 0$ (and with a constant H that is both Metzler and Hurwitz), and by then choosing $\Delta_* > 0$ to be a small enough constant so that our requirements are still satisfied with these choices of V, p, K, and Γ. Then the requirement on ν is satisfied with $\nu = 0.026$, so our theorem applies when the number $m \geq 2$ of subpredictors satisfies $m > 2\sqrt{2}\tau||I + A||_\infty \max\{||I + A||_\infty, ||A||_\infty\}$, which in this case gives the requirement $m > 14.9701\tau$, meaning the number of subpredictors should be at least approximately 15 times the value τ of the input delay.

In the top panel of Fig. 1, we plot MATLAB simulations we obtained for the (d_a, ϕ_a) dynamics in closed loop with the event-triggered control from our theorem, using the preceding values with $\tau = 0.1$ with $m = 2$ subpredictors and constant initial functions. Since it shows fast convergence of the error states to zero, it helps to verify our theorem in the special case of the curve tracking dynamics (60). On the other hand, a key point in our method is its ability to compensate for arbitrarily long delays $\tau > 0$. We illustrate this point in the bottom two panels of Fig. 1, which show our MATLAB simulations using the same parameters as the first simulation, except with $\tau = 1.2$ and $m = 21$ subpredictors used in the middle panel; and with $\tau = 3$ and $m = 45$ used in the bottom panel. Each panel shows satisfactory control performance, and therefore helps to illustrate the value of our theorem.

If we replace (63) by the reference trajectory

$$d_r(t) = \tfrac{1}{\kappa}(1 - 0.1\sin(t)) \text{ and } \phi_r(t) = \arcsin\left(\tfrac{\cos(t)}{10\kappa}\right), (68)$$

and if we replace V, p, Γ, and ν by $V = [1, 3.7]^\top$, $p = 0.001$,

$$\Gamma = \begin{bmatrix} 0.6075 & 0.648 \\ 0.081 & 0.243 \end{bmatrix} \qquad (69)$$

and $\nu = 0.2125$, respectively, and keep the rest of the dynamics the same, then simple Mathematica calculations show that the requirements of our theorem are again satisfied for any delay τ when the number of subpredictors m satisfies $m > 14.6989\tau$. This illustrates the possibility of changing the reference trajectory to obtain larger values of the guaranteed lower bound ν on the inter-sampling intervals $t_{i+1} - t_i$, while not increasing the required lower bound on the ratio m/τ of the number of subpredictors to the delay.

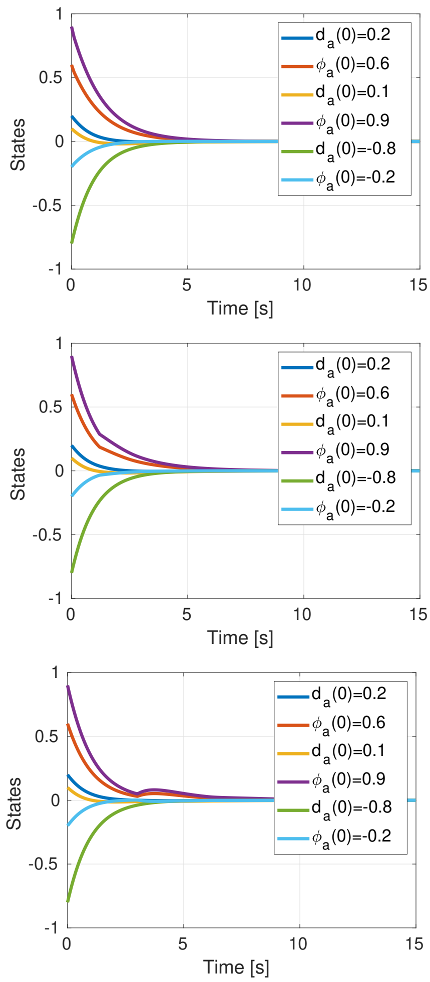

Fig. 1. Curve Tracking with $K = [0, 1.75]$ and Delay $\tau = 0.1$ (Top Panel), $\tau = 1.2$ (Middle Panel), and $\tau = 3$ (Bottom Panel)

6 Conclusion

We proposed a new event-triggered control for time-varying linear continuous time systems with a delay in the input and additive uncertainty terms. It uses a chain predictor, which is a dynamic extension of the original system that involves copies of the original system running on different scales and additional stabilizing terms, and yields ISS with respect to the uncertainty for an arbitrarily large constant delay. This contrasts with traditional exact predictor approaches, which would have instead produced distributed terms in the control that would have made numerical implementations more difficult.

The work was strongly motivated by the ubiquity of input delays, and the benefits of event-triggered control as a way to eliminate the need to continuously (or too frequently) change the control value. Instead of standard Lyapunov function approaches, we used framers and positive systems. A key feature of our approach is that neither the control nor the trigger mechanism require current state measurements. Extensions to systems with a time-varying delay are expected, as well as extensions to output feedback designs. The case where the delay is poorly known, applications to networked systems [40], and comparisons of our approach with alternative approaches that could be based on quadratic Lyapunov functions, will be subjects of further studies too.

Appendix A: Lower bound on m

We show that (3) is globally exponentially stable to 0 on $\mathbb{R}^n$ with the choice $L(t) = -I - A(t)$ when $m > 2\sqrt{2}\tau||L||_\infty \max\{||L||_\infty, ||A||_\infty\}$ and $\tau > 0$. To this end, we rewrite (3) as $\dot{X}(t) = -X(t) + (A(t)+I)(X(t) - X(t-\tau/m)) + \kappa(t)$, which we can use to check that the time derivative of $V_0(X) = \frac{1}{2}||X||^2$ along solutions of (3) satisfies

$$\begin{aligned}\dot{V}_0 \le\ & -||X(t)||^2 + ||X(t)||||L||_\infty \int_{t-\tau/m}^t ||\dot{X}(\ell)||\mathrm{d}\ell \\ & + ||X(t)||||\kappa(t)|| \\ \le\ & -||X(t)||^2 \\ & + \{||X(t)||\} \left\{||L||_\infty \bar{C} \int_{t-2\tau/m}^t ||X(\ell)||\mathrm{d}\ell\right\} \\ & + ||X(t)|||\kappa(t)| + ||X(t)||||L||_\infty \tfrac{\tau}{m}||\kappa||_\infty \\ \le\ & -\tfrac{1}{2}||X(t)||^2 + ||L||_\infty^2 \bar{C}^2 \tfrac{2\tau}{m} \int_{t-2\tau/m}^t ||X(\ell)||^2\mathrm{d}\ell \\ & + \hat{c}||\kappa||_\infty^2\end{aligned} \tag{A.1}$$

for all $t \ge 0$, where $\bar{C} = \max\{||A||_\infty, ||L||_\infty\}$, $\hat{c} = (1 + ||L||_\infty \tau/m)^2$, and the last inequality in (A.1) followed from two applications of Holder's inequality $rs \le \frac{1}{4}r^2 + s^2$ (first with r and s being the terms in curly braces in (A.1), and then applied with $r = ||X(t)||$ and $s = \hat{c}^{1/2}||\kappa||_\infty$) followed by Jensen's inequality. Choosing any constant $\epsilon > 0$ such that $m > 2\sqrt{2}\tau\sqrt{1+\epsilon}||L||_\infty \max\{||L||_\infty, ||A||_\infty\}$, it follows that the time derivative of

$$\begin{aligned}V^\sharp(X_t) =\ & \tfrac{1}{2}||X(t)||^2 \\ & + ||L||_\infty^2 \bar{C}^2 \tfrac{2\tau}{m}(1+\epsilon) \int_{t-2\tau/m}^t \int_\ell^t ||X(s)||^2\mathrm{d}s\mathrm{d}\ell\end{aligned} \tag{A.2}$$

along all solutions of (3) satisfies

$$\begin{aligned}\dot{V}^\sharp \le\ & -\left(\tfrac{1}{2} - ||L||_\infty^2 \bar{C}^2 \left(\tfrac{2\tau}{m}\right)^2 (1+\epsilon)\right) ||X(t)||^2 \\ & - \epsilon||L||_\infty^2 \bar{C}^2 \tfrac{2\tau}{m} \int_{t-2\tau/m}^t ||X(r)||^2\mathrm{d}r + \hat{c}||\kappa||_\infty^2 \\ \le\ & -\epsilon_0 V^\sharp(X_t) + \hat{c}||\kappa||_\infty^2\end{aligned} \tag{A.3}$$

for all $t \ge 0$, where

$$\epsilon_0 = \min\left\{1 - 2||L||_\infty^2 \bar{C}^2 \left(\tfrac{2\tau}{m}\right)^2 (1+\epsilon), \tfrac{m\epsilon}{2\tau(1+\epsilon)}\right\} \tag{A.4}$$

is positive because of our lower bound on m. By applying variation of parameters to (A.3), then lower bounding the left side $V^\sharp(X_t)$ of the result by $\frac{1}{2}|X(t)|^2$, then multiplying through by 2, and then finally taking square roots of both sides, we obtain the desired exponential ISS estimate

$$\begin{aligned}||X(t)|| \le\ & \sqrt{\tfrac{2}{\epsilon_0}\hat{c}}||\kappa||_\infty \\ & + e^{-\epsilon_0(t-s)/2} \left(\tfrac{1}{2} + c_a\right)^{1/2} \sqrt{2}||X||_{[s-2\tau/m,s]}\end{aligned} \tag{A.5}$$

with $c_a = (2\tau/m)^3(||L||_\infty^2 \bar{C})^2(1+\epsilon)$, by the bound $(2\tau/m)^2||X||_{[t-2\tau/m,t]}^2$ on the double integral in (A.2).

Appendix B: Fourth part of proof of Theorem 1

This appendix completes the proof of Theorem 1 by converting a decay estimate on the functional $\mathcal{F}$ from (58) (with the choice (59) of $\mathcal{W}$) into the ISS estimate from the conclusion of the theorem. To this end, first notice that a direct calculation yields

$$\dot{\mathcal{W}}(t) = -\eta(\xi_t) - \int_{t-c_*}^t \eta(\xi_\ell)\mathrm{d}\ell + (c_*+1)\eta(\xi_t) \tag{B.1}$$

for all $t \ge c_* + T + \tau$. On the other hand, (57) gives

$$e^{-p_2 s}\eta(\xi_t) \le c_4 e^{-p_2 t}\eta(\xi_s) + c_3 e^{-p_2 s} \sup_{\ell\in[s,t+\tau]} ||\delta(\ell)|| \tag{B.2}$$

for all $s \ge T + \tau$ and $t \ge s$. Integrating (B.2) gives

$$\begin{aligned}& \int_{t-c_*}^t e^{-p_2 s}\eta(\xi_t)\mathrm{d}s \le c_4 e^{-p_2 t} \int_{t-c_*}^t \eta(\xi_s)\mathrm{d}s \\ & + c_3 \int_{t-c_*}^t e^{-p_2 s} \sup_{\ell\in[s,t+\tau]} ||\delta(\ell)||\mathrm{d}s\end{aligned} \tag{B.3}$$

for all $t \ge c_* + T + \tau$. The inequality (B.3) is equivalent to

$$\begin{aligned}& \tfrac{e^{p_2 c_*}-1}{p_2}\eta(\xi_t) \le c_4 \int_{t-c_*}^t \eta(\xi_s)\mathrm{d}s \\ & + c_3 \int_{t-c_*}^t e^{p_2(t-s)} \sup_{\ell\in[s,t+\tau]} ||\delta(\ell)||\mathrm{d}s.\end{aligned} \tag{B.4}$$

It follows that

$$\begin{aligned}\eta(\xi_t) \le\ & \tfrac{c_4 p_2}{e^{p_2 c_*}-1} \int_{t-c_*}^t \eta(\xi_s)\mathrm{d}s \\ & + c_3 \tfrac{p_2}{e^{p_2 c_*}-1} \int_{t-c_*}^t e^{p_2(t-s)} \sup_{\ell\in[s,t+\tau]} ||\delta(\ell)||\mathrm{d}s \\ \le\ & \tfrac{c_4 p_2}{e^{p_2 c_*}-1} \int_{t-c_*}^t \eta(\xi_s)\mathrm{d}s \\ & + c_3 \tfrac{p_2}{e^{p_2 c_*}-1} \int_{-c_*}^0 e^{p_2(-s)}\mathrm{d}s \sup_{\ell\in[t-c_*,t+\tau]} ||\delta(\ell)|| \\ =\ & \tfrac{c_4 p_2}{e^{p_2 c_*}-1} \int_{t-c_*}^t \eta(\xi_s)\mathrm{d}s + c_3 \sup_{\ell\in[t-c_*,t+\tau]} ||\delta(\ell)||\end{aligned} \tag{B.5}$$

for all $t \ge c_* + T + \tau$. Combining (B.5) and (B.1), we obtain

$$\begin{aligned}\dot{\mathcal{W}}(t) \le\ & -\eta(\xi_t) - \int_{t-c_*}^t \eta(\xi_l)\mathrm{d}l \\ & + (c_*+1)\tfrac{c_4 p_2}{e^{p_2 c_*}-1} \int_{t-c_*}^t \eta(\xi_s)\mathrm{d}s \\ & + c_3(c_*+1) \sup_{\ell\in[t-c_*,t+\tau]} ||\delta(\ell)||\end{aligned} \tag{B.6}$$

for all $t \ge c_* + T + \tau$. Since our condition (22) on c_* and our choice $c_4 = (c_1+1)e^{p_2(T+\tau)}$ imply that

$$(c_*+1)\tfrac{c_4 p_2}{e^{p_2 c_*}-1} \le \tfrac{1}{2}, \tag{B.7}$$

we obtain

$$\begin{aligned}\dot{\mathcal{W}}(t) \le\ & -\eta(\xi_t) - \tfrac{1}{2}\int_{t-c_*}^t \eta(\xi_l)\mathrm{d}l \\ & + c_3(c_*+1) \sup_{\ell\in[t-c_*,t+\tau]} ||\delta(\ell)||.\end{aligned} \tag{B.8}$$

It follows from the bound

$$\mathcal{W}(\xi_t) \le c_* \int_{t-c_*}^t \eta(\xi_l)\mathrm{d}l \tag{B.9}$$

that

$$\begin{aligned}\dot{\mathcal{W}}(t) \le\ & -\eta(\xi_t) - \tfrac{1}{2c_*}\mathcal{W}(\xi_t) \\ & + c_3(c_*+1) \sup_{\ell\in[t-c_*,t+\tau]} ||\delta(\ell)||\end{aligned} \tag{B.10}$$

for all $t \geq c_* + T + \tau$. Recalling our formula (27) for η, and using (53) and (B.10), we then have

$$\begin{aligned}\dot{\mathcal{F}}(t) \;&\leq\; -p\mathcal{V}(\zeta(t)) + b \sup_{a\in[t-T-\tau,t-\tau]} ||\xi(a)|| + \Delta_1(t)\\ &\quad - (1+b)\eta(\xi_t) - \tfrac{1+b}{2c_*}\mathcal{W}(\xi_t)\\ &\quad + c_3(1+b)(c_*+1) \sup_{\ell\in[t-c_*,t+\tau]} ||\delta(\ell)||\\ &\leq -p\left[\mathcal{V}(\zeta(t)) + \tfrac{1+b}{2c_* p}\mathcal{W}(\xi_t)\right] - \eta(\xi_t) + \Delta_1(t)\\ &\quad + c_3(1+b)(c_*+1) \sup_{\ell\in[t-c_*,t+\tau]} ||\delta(\ell)||\\ &\leq -p_\star\mathcal{F}(\zeta(t),\xi_t) + \Delta_2(t)\end{aligned} \tag{B.11}$$

for all $t \geq \max\{r, c_* + T + \tau\}$, where

$$\Delta_2(t) = \Delta_1(t) + c_3(1+b)(c_*+1) \sup_{\ell\in[t-c_*,t+\tau]} ||\overline{\delta}(\ell)|| \tag{B.12}$$

and $p_\star = \min\{p, \frac{1}{2c_*}\}$ is the constant from (23).

Applying variation of parameters to (B.11) then gives

$$\begin{aligned}\mathcal{F}(\zeta(t),\xi_t) \;&\leq\; e^{-p_\star(t-s)}\mathcal{F}(\zeta(s),\xi_s)\\ &\quad + \int_s^t e^{-p_\star(t-\ell)}\Delta_2(\ell)\mathrm{d}\ell\end{aligned} \tag{B.13}$$

for all $t \geq s \geq \max\{r, c_* + T + \tau\}$. Since $\mathcal{V}(\zeta) = V^\top\zeta \geq v(\zeta_1 + \dots \zeta_n) \geq v||\zeta||$ for all $\zeta \in [0,+\infty)^n$, it follows that

$$\begin{aligned}&v||\zeta(t)|| + (1+b)\mathcal{W}(\xi_t)\\ &\leq e^{-p_\star(t-s)}||V||||\overline{x}(s) - \underline{x}(s)||\\ &+(1+b)e^{-p_\star(t-s)}\mathcal{W}(\xi_s) + \int_s^t e^{-p_\star(t-\ell)}\Delta_2(\ell)\mathrm{d}\ell,\end{aligned} \tag{B.14}$$

where $v = \min_i V_i$ is the smallest entry of V as defined in (23). Using (48) and upper bounding the formula (59) using the formula (B.9), we deduce that

$$\begin{aligned}||x(t)|| \;&\leq\; \tfrac{||V||}{v}e^{-p_\star(t-s)}||\overline{x}(s) - \underline{x}(s)||\\ &\quad + \tfrac{(1+b)c_* e^{-p_\star(t-s)}}{v}\int_{s-c_*}^{s} \sup_{\ell\in[a-T-\tau,a]} ||\xi(\ell)||\,\mathrm{d}a\\ &\quad + \tfrac{1}{v}\int_s^t e^{-p_\star(t-\ell)}\Delta_2(\ell)\mathrm{d}\ell\\ &\leq \tfrac{||V||}{v}e^{-p_\star(t-s)}||\overline{x}(s) - \underline{x}(s)||\\ &\quad + \tfrac{(1+b)c_*^2}{v}e^{-p_\star(t-s)} \sup_{\ell\in[s-c_*-T-\tau,s]} ||\xi(\ell)||\\ &\quad + \tfrac{1}{v}\int_s^t e^{-p_\star(t-\ell)}\Delta_2(\ell)\mathrm{d}\ell\end{aligned} \tag{B.15}$$

when $t \geq s \geq \max\{r, c_* + T + \tau\}$. According to (8), we have $\xi(t) = z_m(t) - x(t+\tau)$, so we deduce that

$$\begin{aligned}||x(t)|| \;&\leq\; \tfrac{||V||}{v}e^{-p_\star(t-s)}||\overline{x}(s) - \underline{x}(s)||\\ &\quad + \tfrac{(1+b)c_*^2}{v}e^{-p_\star(t-s)} \sup_{\ell\in[s-c_*-T-\tau,s]} ||z_m(\ell)||\\ &\quad + \tfrac{(1+b)c_*^2}{v}e^{-p_\star(t-s)} \sup_{\ell\in[s-c_*-T,s+\tau]} ||x(\ell)||\\ &\quad + \tfrac{1}{v}\int_s^t e^{-p_\star(t-\ell)}\Delta_2(\ell)\mathrm{d}\ell.\end{aligned} \tag{B.16}$$

Consequently, when $r \geq c_* + T + \tau$, our requirements (38) on the initial conditions give

$$\begin{aligned}||x(t)|| \;&\leq\; \tfrac{||V||}{v}e^{-p_\star(t-r)}||x^+(r) + x^-(r)||\\ &\quad + \tfrac{(1+b)c_*^2}{v}e^{-p_\star(t-r)} \sup_{\ell\in[r-c_*-T-\tau,r]} ||z_m(\ell)||\\ &\quad + \tfrac{(1+b)c_*^2}{v}e^{-p_\star(t-r)} \sup_{\ell\in[r-c_*-T,r+\tau]} ||x(\ell)||\\ &\quad + \tfrac{1}{v}\int_r^t e^{-p_\star(t-\ell)}\Delta_2(\ell)\mathrm{d}\ell\\ &\leq e^{-p_\star(t-r)}\Bigg(c_\sharp \sup_{\ell\in[r-c_*-T,r+\tau]} ||x(\ell)||\\ &\quad + c_\ddagger \sup_{\ell\in[r-c_*-T-\tau,r]} ||z_m(\ell)||\Bigg)\\ &\quad + \tfrac{1}{v}\int_r^t e^{-p_\star(t-\ell)}\Delta_2(\ell)\mathrm{d}\ell\end{aligned} \tag{B.17}$$

for all $t \geq r$, where

$$c_\sharp = \tfrac{||V|| + (1+b)c_*^2}{v} \text{ and } c_\ddagger = \tfrac{(1+b)c_*^2}{v}. \tag{B.18}$$

It follows from our choice of c_e in (23), and from our choices of Δ_1 and Δ_2 in (52) and (B.12), that with the choice $c_5 = ||V|| + c_3(1+b)(c_*+1)$, we have

$$\begin{aligned}||x(t)|| \;&\leq e^{-p_\star(t-r)}\Bigg(c_\sharp \sup_{\ell\in[r-c_*-T,r+\tau]} ||x(\ell)||\\ &\quad + c_\ddagger \sup_{\ell\in[r-c_*-T-\tau,r]} ||z_m(\ell)||\Bigg)\\ &\quad + \tfrac{1}{vp_\star}\Bigg[||V^\top|BK|_\infty\overline{\omega}e^{|A|_\infty T}|| \int_{r-T}^t \overline{\delta}(\ell)\mathrm{d}\ell\\ &\quad + c_5 \sup_{\ell\in[r-c_*,t+\tau]} ||\overline{\delta}(\ell)||\Bigg]\\ &\leq e^{-p_\star(t-r)}\Bigg(c_\sharp \sup_{\ell\in[r-c_*-T,r+\tau]} ||x(\ell)||\\ &\quad + c_\ddagger \sup_{\ell\in[r-c_*-T-\tau,r]} ||z_m(\ell)||\Bigg)\\ &\quad + c_e \sup_{\ell\in[r-\max\{c_*,T\},t+\tau]} ||\overline{\delta}(\ell)||.\end{aligned} \tag{B.19}$$

This allows us to conclude.

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