Event-triggered control through the eyes of a hybrid small-gain theorem

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***Abstract*— A unifying design perspective is presented for emulation-based (dynamic) event-triggered state-feedback control of nonlinear systems. The main component of this new approach is to interpret event-triggered controlled systems as the interconnection of hybrid dynamical systems and to analyse the overall system using a hybrid small gain theorem. Based on this new perspective, we unify several event-triggered schemes that were previously proposed in the literature under one umbrella. Moreover, the design approach offers great flexibility and can be used for the development of novel event-triggered schemes and systematic modification and improvement of existing triggering strategies. We illustrate via simulations that these novel and/or modified event-triggered controllers can lead to a further reduction in the required number of transmissions, while still guaranteeing stability.**

***Index Terms*— Event-triggered control, hybrid systems, networked control systems, small-gain theorem.**

I. Introduction

Event-triggered control (ETC) is a class of sampled-data schemes in which the loop is closed whenever a predefined state- or output-dependent condition is satisfied [1]–[3]. As such, ETC is a natural generalisation of classical sampled-data control [4], since state-dependent sampling is used instead of periodic sampling. Reasons for considering ETC are manifold, and we emphasise the motivation arising from the emerging resource-aware control applications with packed-based communication networks. In this context, the energy consumption, communication bandwidth, and computational power are limited, and thus ETC controllers have been proposed in the literature as an alternative to (periodic) time-triggered controllers in order to decrease the communication load, while preserving appropriate performance and robustness guarantees [3]. The ETC approach has also been experimentally verified on various settings including mobile robots [5] and vehicle platooning [6].

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The benefits of ETC were recognised a long time ago [1], and since then, many different triggering strategies have been proposed in the literature, see e.g., [7]–[19] and the references therein. However, the relationship between these various approaches, the intuition behind them, and their advantages/disadvantages are hard to understand as the underlying design tools and philosophy appear to be different for each particular scheme. Additionally, the stability proofs for available strategies are often similar, and thus repeated for each different scheme, highlighting the need of a unifying analysis approach. Lastly, it is often unclear how the design flexibility within each approach affects the required transmission intervals and system performance. The recent studies [20], [21] for classes of ETC systems shed some light on this issue, but a systematic design framework remains elusive.

The first objective of this paper is to show that a large class of seemingly unrelated ETC schemes can be unified within one design perspective. As a consequence, valuable insights are revealed, which were not previously observed in the literature. Particularly, it provides clear viewpoints on the essential differences and similarities of previously proposed ETC strategies, and adds design flexibility to each existing approach, which allows for a systematic modification of current schemes. In fact, our second main objective is to use this unifying perspective to redesign existing and generate novel ETC schemes. Consequently, the proposed approach enables a clearer and systematic design methodology that may be used to increase inter-event times, as we illustrate via a numerical example. We foresee other performance objectives such as convergence speed and robustness may be addressed under the same small-gain perspective.

As commonly adopted in the ETC literature, see e.g., [7]–[18], we will use an *emulation approach*. That is, a (potentially dynamic) continuous-time controller is first designed to robustly stabilize the continuous-time plant (ignoring the packet-based nature of communication). This controller is then implemented via an event-triggering rule, while preserving important stability properties. For the analysis, we model the resulting closed loop as a hybrid system using the formalism of [22], decompose it as the interconnection of two subsystems, and consider the interconnection from a small-gain perspective. Although hybrid systems have been used to study ETC techniques in, e.g., [13], [13] does not explicitly use a small-gain theorem to analyse them, which limits the understanding of similarities and differences between the studied ETC techniques. Moreover, [13] does not consider the redesign of existing techniques. We emphasise that some of these event-triggering rules might involve auxiliary state variables, see e.g., [11]–[14], with the aim of potentially reducing the number of transmissions. Then, at the core of our unifying view, we propose a specific interconnection that can deal with these auxiliary variables and cover many techniques in the literature. To study stability of this interconnection, we adopt a recently proposed hybrid small-gain theorem [23], which ensures overall stability provided a small-gain condition involving the two subsystems is satisfied. Within this perspective, the analysis boils down to designing the ETC rule (and auxiliary variables, if present) to ensure that the small-gain conditions hold. Equivalently, in hybrid systems terminology, the triggering rule synthesis then consists in shaping the flow and the jump sets (together with the design of the auxiliary variable dynamics) to enforce the small-gain theorem conditions.

We highlight that [23] provides a general tool to study stability of hybrid systems using a small-gain theorem, and its main contributions are unrelated to ETC. Only a particular ETC scheme is studied as an application of this tool in [23, Section V.C]. Specifically, it demonstrates how to interpret the static event-triggering rule in [7] with a small-gain analysis. However, [23] adopts a modified version of the triggering in [7] so that it directly fits the small-gain theorem (we do not need such modifications in the present paper). Compared to [23], we significantly generalise the applications of the small-gain perspective by covering many ETC techniques in the literature (not just the one in [7]). Additionally, we propose a novel interconnection that can deal with *dynamic* ETC techniques, and we show how to systematically redesign the existing techniques.

More in detail, in this paper, we demonstrate that the small-gain perspective underlies various important schemes that have been proposed in the ETC literature, such as, the well-known relative threshold strategy from [7], the dynamic triggering strategy in [11], the fixed-threshold strategy found in [8], [24]–[26], the decreasing threshold on the network-induced error used in [13], [27]–[29], and the decreasing threshold on the Lyapunov function from [14]. It is important to note that, many crucial steps are needed in order to step from [23] and be able to unify many ETC schemes under one umbrella. For instance, in every studied technique, the way we decouple the system becomes important; sometimes the auxiliary variable is coupled with the plant/controller system, and sometimes with the network-induced error system. Additionally, for some dynamic techniques such as the one in [14], new hybrid models for the auxiliary variable are introduced to be able to apply the small-gain theorem. Moreover, for some of the strategies, the proposed approach allows us to provide stronger conclusions. More importantly, the small-gain view not only unifies many influential strategies in the ETC literature under the same umbrella, but it also clarifies how to systematically redesign them to create novel and more general strategies. Particularly, the analysis reveals that *all* considered techniques have one of the gains equal to zero (i.e., cascade-like interconnection), which is a key factor that opens the door for redesigns that can potentially enlarge the inter-transmission times of the original techniques. In this context, a relevant contribution of our work is that we explicitly show how to redesign and combine some of the original approaches listed above, all done by utilising the same small-gain perspective.

Relevant to the work herein, the past work [30] provides a small-gain approach to event-triggered control of nonlinear systems. However, [30] only applies to static triggering rules, i.e., those that relate the state of the controlled system and the network-induced error. As we mentioned above, our results cover a more general class of dynamic ETC rules that use auxiliary variables in their triggering thresholds. Moreover, the small-gain view in [30] is different to the one presented here, since we model the closed loop as a hybrid system and we explicitly provide the dynamics of the network-induced error. This implies that the corresponding small-gain conditions have to hold in adequately defined sets which respect the system dynamics, and thus the system gains have a clear meaning.

Some preliminary results of this work have been presented in [31]. We emphasise that this manuscript is a significant extension of [31]. Particularly, we study many more ETC strategies that were not considered in [31], thus highlighting the applicability of the small-gain perspective. In addition, we generalise the redesigned techniques in [31], and we also present novel ones.

II. Preliminaries

Let $\mathbb{Z}_{>0} := \{1,2,\ldots\}$, $\mathbb{Z}_{\geq 0} := \{0,1,2,\ldots\}$, $\mathbb{R} := (-\infty,\infty)$ and $\mathbb{R}_{\geq 0} := [0,\infty)$. Given a non-empty closed set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the distance of x to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} := \min_{y\in\mathcal{A}} |x-y|$. For a matrix $A \in \mathbb{R}^{n\times m}$ and its singular values λ_i, $i \in \{1,\cdots,n\}$, $|A| := \max\{\lambda_1,\cdots,\lambda_n\}$ is its induced 2-norm. For any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, (x,y) stands for $[x^\top, y^\top]^\top$. Given a real, symmetric matrix P, we denote its maximum and minimum eigenvalue as $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$, respectively. The notation $\mathbb{I}$ stands for the identity map from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$, if it is continuous, zero at zero and strictly increasing and it is of class-$\mathcal{K}_\infty$ if, in addition, it is unbounded. We write $\gamma \in \mathcal{K}_\infty \cup \{0\}$ when function γ is either of class-$\mathcal{K}_\infty$ or it is identically equal to zero. A function $\gamma : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$, if it is continuous, $\gamma(\cdot,r)$ is of class-$\mathcal{K}$ for each $r \in \mathbb{R}_{\geq 0}$, and, for each $s \in \mathbb{R}_{\geq 0}$, $\gamma(s,\cdot)$ is decreasing to zero. For $x, v \in \mathbb{R}^n$ and locally Lipschitz $U : \mathbb{R}^n \to \mathbb{R}$, $U^\circ(x;v)$ is the Clarke derivative of the function U at x in the direction v, i.e., $U^\circ(x;v) := \limsup_{y\to x,\lambda\downarrow 0} \frac{U(y+\lambda v)-U(y)}{\lambda}$. This notion will be useful as we will be working with locally Lipschitz Lyapunov functions, which are not differentiable everywhere, and it reduces to the standard directional derivative $\langle \nabla U(x), v\rangle$ when U is continuously differentiable. For a set $S \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $T_S(x)$ is the tangent cone to S at x, if it is the set of all vectors $v \in \mathbb{R}^n$ for which there exist $x_i \in S$, $\tau_i > 0$ with $x_i \to x$, $\tau_i \to 0$ as $i \to \infty$ such that $v = \lim_{i\to\infty}(x_i - x)/\tau_i$.

We will model closed-loop ETC systems as hybrid systems of the form (1), see [22], for which the jump times will correspond to triggering instants. In particular, a hybrid system

$\mathcal{H}$, as considered here, is given by

$$\mathcal{H}\left\{\begin{array}{ll}\dot{q}=\mathcal{F}(q), & q\in C,\\ q^{+}=\mathcal{G}(q), & q\in D,\end{array}\right. \tag{1}$$

where $q\in\mathbb{R}^n$ is the state, $C,D\subseteq\mathbb{R}^n$ are the flow and the jump sets, which are assumed to be closed, and $\mathcal{F},\mathcal{G}:\mathbb{R}^n\to\mathbb{R}^n$ are continuous functions. Hence, system (1) satisfies the hybrid basic conditions in [22]. For more information about the notion of solutions for $\mathcal{H}$, see [22]. We just recall that a solution is *maximal* if it cannot be extended and it is *complete* if its domain is unbounded. We focus on the following stability property.

Definition 1: A closed set $\mathcal{A}\subseteq\mathbb{R}^n$ is *uniformly globally asymptotically stable (UGAS)* for system $\mathcal{H}$, if there exists $\beta\in\mathcal{KL}$ such that, for any solution φ and $(t,j)\in\operatorname{dom}\varphi$, it holds that $|\varphi(t,j)|_{\mathcal{A}}\leq\beta(|\varphi(0,0)|_{\mathcal{A}},t+j)$, and all maximal solutions are complete. □

We use [13, Definition 3] to characterize the existence of a strictly positive amount of time between any two successive transmissions, which is essential for ETC.

Definition 2: System $\mathcal{H}$ has *a uniform semi-global dwell time outside $\mathcal{A}\subseteq\mathbb{R}^n$, where $\mathcal{A}$ is strongly forward invariant*[1] *for system $\mathcal{H}$*, if for any $\Delta>0$, there exists $\tau(\Delta)>0$ such that for each solution φ with $|\varphi(0,0)|_{\mathcal{A}}\leq\Delta$ and all $(s,i),(t,j)\in\operatorname{dom}\varphi$ with $s+i\leq t+j$, $\varphi(t,j)\notin\mathcal{A}\Rightarrow j-i\leq(t-s)/\tau(\Delta)+1$. System $\mathcal{H}$ has a *uniform semi-global dwell time* if for any $\Delta>0$, there exists $\tau(\Delta)>0$ such that for each solution φ with $|\varphi(0,0)|\leq\Delta$ and for all $(s,i),(t,j)\in\operatorname{dom}\varphi$ with $s+i\leq t+j$, it holds that $j-i\leq(t-s)/\tau(\Delta)+1$. □

III. Hybrid system model

In this section, we present a hybrid system model for the considered ETC setting, like in [13]. In this context, we adopt the emulation approach [32], [33], which is explained next. Consider Fig. 1, where the nonlinear plant has the form

$$\dot{x}_{\mathrm{p}}=f_{\mathrm{p}}(x_{\mathrm{p}},u), \tag{2}$$

with $x_{\mathrm{p}}\in\mathbb{R}^{n_{\mathrm{p}}}$ the state and $u\in\mathbb{R}^{n_u}$ the control input, $n_{\mathrm{p}},n_u\in\mathbb{Z}_{>0}$. The first step in the emulation approach is to design a controller that robustly stabilises the plant (2) in the absence of a communication (packet-based) network. This will be formalised in SA1 further below. We assume that the designed controller is nonlinear and has a continuous-time model of the form

$$\begin{aligned}\dot{x}_c&=f_{\mathrm{c}}(x_{\mathrm{c}},x_{\mathrm{p}}),\\ u&=g_{\mathrm{c}}(x_{\mathrm{c}},x_{\mathrm{p}}),\end{aligned} \tag{3}$$

where $x_{\mathrm{c}}\in\mathbb{R}^{n_{\mathrm{c}}}$ is the state of the controller, $n_{\mathrm{c}}\in\mathbb{Z}_{>0}$. We can also cover static controllers by writing $u=g_{\mathrm{c}}(x_{\mathrm{p}})$ in (3). The functions f_{p} and f_{c} are assumed to be continuous, and g_{p} and g_{c} are assumed to be continuously differentiable and, without loss of generality, zero at zero.

In the second step of emulation, we implement the designed controller (3) over a network as per Fig. 1. At each

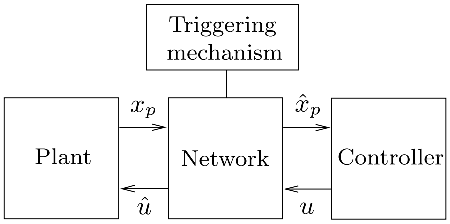

Fig. 1: Block diagram of the considered ETC setting.

transmission instant t_j, $j\in\mathcal{I}\subseteq\mathbb{Z}_{\geq0}$, the controller receives the plant measurements, updates its knowledge accordingly, sends the control input to the actuators, and the latter update the signal applied to the plant. Consequently, plant (2) has no longer access to u, but to its networked version $\hat{u}$, and controller (3) has access to $\hat{x}_{\mathrm{p}}$, the networked version of x_{p}. We assume ideal packed-based communication in the sense that $\hat{x}_{\mathrm{p}}(t_j^+)=x_{\mathrm{p}}(t_j)$ and $\hat{u}(t_j^+)=u(t_j)$, for any $j\in\mathcal{I}$. Between two successive transmission instants, $\hat{x}_{\mathrm{p}}$ and $\hat{u}$ are governed by the dynamics of the holding devices given by

$$\begin{aligned}\dot{\hat{x}}_{\mathrm{p}}&=\hat{f}_{\mathrm{p}}(x_{\mathrm{p}},x_{\mathrm{c}},\hat{x}_{\mathrm{p}},\hat{u}),\\ \dot{\hat{u}}&=\hat{f}_{\mathrm{c}}(x_{\mathrm{p}},x_{\mathrm{c}},\hat{x}_{\mathrm{p}},\hat{u}),\end{aligned} \tag{4}$$

where $\hat{f}_{\mathrm{p}}$ and $\hat{f}_{\mathrm{c}}$ are continuous functions. Note that zero-order-hold devices correspond to $\hat{f}_{\mathrm{p}}=0$, $\hat{f}_{\mathrm{c}}=0$, for instance. We allow $\hat{f}_{\mathrm{p}}$ and $\hat{f}_{\mathrm{c}}$ to depend on $x_{\mathrm{p}},x_{\mathrm{c}},\hat{x}_{\mathrm{p}}$, and $\hat{u}$ for the sake of generality, which allows us to cover cases such as the model-based techniques in [8].

The transmission instants t_j, $j\in\mathcal{I}$, are formally defined by an underlying event-triggering rule, which is to be designed. The main goal of ETC is to synthesize these rules to communicate according to the system needs while guaranteeing stability and satisfactory levels of performance, by closing the loop whenever a predefined state-dependent triggering condition is satisfied. In general, the triggering conditions may depend on x_{p}, x_{c}, u, and some auxiliary variable $\eta\in\mathbb{R}^{n_\eta}$, $n_\eta\in\mathbb{Z}_{\geq0}$, that can be modelled as

$$\begin{aligned}\dot{\eta}&=\tilde{h}(x_{\mathrm{p}},x_{\mathrm{c}},\hat{x}_{\mathrm{p}},\hat{u},\eta),\ t\in(t_j,t_{j+1}),\\ \eta(t_j^+)&=\tilde{\ell}\big(x_{\mathrm{p}}(t_j),x_{\mathrm{c}}(t_j),\hat{x}_{\mathrm{p}}(t_j),\hat{u}(t_j),\eta(t_j)\big),\end{aligned} \tag{5}$$

where $\tilde{h},\tilde{\ell}$ are continuous functions.

Before presenting the hybrid model for the ETC system in Fig. 1, we introduce some useful variables. Let $x:=(x_{\mathrm{p}},x_{\mathrm{c}})\in\mathbb{R}^{n_x}$ be the augmented state of the plant and controller, where $n_x:=n_{\mathrm{p}}+n_{\mathrm{c}}$. Let $e:=(e_{x_{\mathrm{p}}},e_u)\in\mathbb{R}^{n_e}$, $n_e:=n_{\mathrm{p}}+n_u$, denote the network-induced error, where $e_{x_{\mathrm{p}}}:=\hat{x}_{\mathrm{p}}-x_{\mathrm{p}}$ and $e_u:=\hat{u}-u$. Note that $e(t_j^+)=0$ for every $j\in\mathcal{I}$. The error e can be treated as an input to system (2)–(3) and this observation plays a key role in the sequel. Indeed, from the definitions of x, e, and (2)–(3), we can write

$$\dot{x}=f(x,e), \tag{6}$$

where $f(x,e):=\big(f_{\mathrm{p}}(x_{\mathrm{p}},g_{\mathrm{c}}(x_{\mathrm{c}},x_{\mathrm{p}}+e_{x_{\mathrm{p}}})+e_u),f_{\mathrm{c}}(x_{\mathrm{c}},x_{\mathrm{p}}+e_{x_{\mathrm{p}}})\big)$. Consequently, we can model the overall closed-loop

[1] A set $\mathcal{A}\subseteq\mathbb{R}^n$ is strongly forward invariant for system $\mathcal{H}$, if for each solution φ to system (1), $\varphi(t,j)\in\mathcal{A}$ for some $(t,j)\in\operatorname{dom}\varphi$ implies that $\varphi(t',j')\in\mathcal{A}$ for any $(t',j')\in\operatorname{dom}\varphi$ with $t+j\leq t'+j'$.

ETC system in Fig. 1 as the following hybrid system, which we denote by $\mathcal{H}_\star$,

$$\mathcal{H}_\star \begin{cases} \left.\begin{aligned} \dot{x} &= f(x,e) \\ \dot{e} &= g(x,e) \\ \dot{\eta} &= h(x,e,\eta) \end{aligned}\right\} & (x,e,\eta) \in C, \\ \left.\begin{aligned} x^+ &= x \\ e^+ &= 0 \\ \eta^+ &= \ell(x,e,\eta) \end{aligned}\right\} & (x,e,\eta) \in D, \end{cases} \tag{7}$$

where f, g, h and ℓ are continuous functions, with f defined below (6), and g, h and ℓ determined from (2)–(5). We emphasise that the specific dynamics for η and the specific definition for the flow set C and jump set D will be determined by the underlying ETC mechanism as we will illustrate in Section V. In this context, $\mathcal{H}_\star$ in (7) denotes the resulting hybrid model corresponding to each ETC strategy and we thus take $\star \in \{A, B, \ldots, G\}$, where each letter corresponds to a particular strategy considered in Section V.

IV. A SMALL-GAIN PERSPECTIVE

Here we present the main analytical tools that define our unifying perspective. Different from other hybrid system approaches for ETC such as [13], our unifying view is based on small-gain arguments. To that end, we will propose a novel decomposition of the closed loop (7). Let us write η in (5) as $\eta = (\eta_1, \eta_2)$, where $\eta_1 \in \mathbb{R}^{n_{\eta_1}}$, $\eta_2 \in \mathbb{R}^{n_{\eta_2}}$, and $n_{\eta_1}, n_{\eta_2} \in \mathbb{Z}_{\geq 0}$ satisfy $n_{\eta_1} + n_{\eta_2} = n_\eta$. This decomposition of η is useful and provides generality for the upcoming analysis, since we will associate this auxiliary variable with the x–system for some triggering methodologies and with the e–system for others. Therefore, the core of the unifying perspective is to interpret system (7) as a feedback interconnection of the (x, η_1)–system and the (e, η_2)–system, as illustrated by Fig. 2. We can then re-write the hybrid model (7) as

$$\begin{aligned} &\mathcal{H}_1 \begin{cases} (\dot{x}, \dot{\eta}_1) = \mathcal{F}_1(q), & q \in C, \\ (x^+, \eta_1^+) = \mathcal{G}_1(q), & q \in D, \end{cases} \\ &\mathcal{H}_2 \begin{cases} (\dot{e}, \dot{\eta}_2) = \mathcal{F}_2(q), & q \in C, \\ (e^+, \eta_2^+) = \mathcal{G}_2(q), & q \in D, \end{cases} \end{aligned} \tag{8}$$

where $q := (x, e, \eta) \in \mathbb{R}^{n_q}$, $n_q := n_x + n_e + n_\eta$, $\mathcal{F}_1(q) := \big(f(x,e), h_1(q)\big)$, $\mathcal{G}_1(q) := \big(x, \ell_1(q)\big)$, $\mathcal{F}_2(q) := \big(g(x,e), h_2(q)\big)$, $\mathcal{G}_2(q) := \big(0, \ell_2(q)\big)$, $\big(h_1(q), h_2(q)\big) := h(q)$ and $\big(\ell_1(q), \ell_2(q)\big) := \ell(q)$.

The feedback interconnection (8) provides the starting point for our unifying small-gain view. By means of a hybrid small-gain theorem (see Theorem 1 below), we can state conditions that ensure the stability of the overall system (8). Therefore, the ETC design focuses on shaping the flow and jump sets C and D of system (8) (equivalently of system (7)) and defining the flow and jump dynamics of η to guarantee satisfaction of these conditions.

Before presenting the hybrid small-gain theorem, we formalise the robustness property that is ensured by the emulation-based controller introduced in (3). Particularly, the controller (3) is designed to ensure the following *input-to-state stable* (ISS) condition holds with respect to network-induced errors.

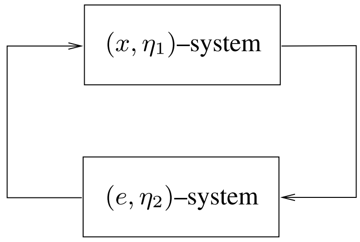

Fig. 2: Block diagram of the decomposition

Standing Assumption 1 (SA1): There exist a continuously differentiable function $V : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, and functions $\underline{\alpha}_V, \overline{\alpha}_V, \alpha_V, \gamma \in \mathcal{K}_\infty$ such that the following hold.

(i) For all $x \in \mathbb{R}^{n_x}$, $\underline{\alpha}_V(|x|) \leq V(x) \leq \overline{\alpha}_V(|x|)$.

(ii) For all $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$, $\langle \nabla V(x), f(x,e) \rangle \leq -\alpha_V(|x|) + \gamma(|e|)$. □

SA1 implies that system (6) is ISS with respect to input e, which acts an additive measurement and input disturbance on the closed-loop system (2)–(3); see [34] for a formal definition on ISS. This assumption is natural in the ETC context and it has been adopted in a wide variety of works in the ETC literature, see e.g., [7], [11]–[13], [35].

We are now ready to present the hybrid small-gain theorem, which is a crucial technical tool for our unifying perspective. This theorem is a tailored version from [23, Theorem III.3] to study stability of a closed set $\mathcal{A} \subseteq \mathbb{R}^{n_q}$ for system (8). The proof is omitted since it follows similar lines to the proof of [23, Theorem III.3]. Given the system decomposition depicted in Fig. 2, we write $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, with $\mathcal{A}_1 \subseteq \mathbb{R}^{n_x + n_{\eta_1}}$ and $\mathcal{A}_2 \subseteq \mathbb{R}^{n_e + n_{\eta_2}}$.

Theorem 1: Suppose that, for any $i \in \{1, 2\}$, there exist locally Lipschitz functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$, $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $\chi_i \in \mathcal{K}_\infty \cup \{0\}$, and positive definite functions $\alpha_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that the following hold.

(i) For all $q \in C \cup D \cup \mathcal{G}(D)$, $\underline{\alpha}_1\left(|(x, \eta_1)|_{\mathcal{A}_1}\right) \leq V_1(x, \eta_1) \leq \overline{\alpha}_1\left(|(x, \eta_1)|_{\mathcal{A}_1}\right)$, and $\underline{\alpha}_2\left(|(e, \eta_2)|_{\mathcal{A}_2}\right) \leq V_2(e, \eta_2) \leq \overline{\alpha}_2\left(|(e, \eta_2)|_{\mathcal{A}_2}\right)$.

(ii) For all $q \in C$ with $\mathcal{F}(q) := (\mathcal{F}_1(q), \mathcal{F}_2(q)) \in T_C(q)$,

$$\begin{aligned} &V_1(x, \eta_1) \geq \chi_1\big(V_2(e, \eta_2)\big) \\ &\qquad \Rightarrow V_1^\circ((x, \eta_1); \mathcal{F}_1(q)) \leq -\alpha_1(|(x, \eta_1)|_{\mathcal{A}_1}), \\ &V_2(e, \eta_2) \geq \chi_2\big(V_1(x, \eta_1)\big) \\ &\qquad \Rightarrow V_2^\circ((e, \eta_2); \mathcal{F}_2(q)) \leq -\alpha_2(|(e, \eta_2)|_{\mathcal{A}_2}). \end{aligned}$$

(iii) For all $q \in D$ and $i \in \{1, 2\}$, $V_i(\mathcal{G}_i(q)) \leq V_i(q)$.

(iv) The small-gain condition $\chi_1 \circ \chi_2(s) < s$ holds for all $s > 0$.

Then, there exist $\overline{\alpha}_U, \underline{\alpha}_U, \rho \in \mathcal{K}_\infty$ and positive definite functions $\alpha_U : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the following hold for $U := \max\{V_1, \rho(V_2)\}$.

(a) For all $q \in C \cup D \cup \mathcal{G}(D)$, $\underline{\alpha}_U\left(|q|_{\mathcal{A}}\right) \leq U(q) \leq \overline{\alpha}_U\left(|q|_{\mathcal{A}}\right)$.

(b) For all $q \in C$ such that $\mathcal{F}(q) \in T_C(q)$, $U^\circ(q; \mathcal{F}(q)) \leq -\alpha_U(|q|_{\mathcal{A}})$.

(c) For all $q \in D$, $U\big(\mathcal{G}(q)\big) \leq U(q)$, where $\mathcal{G}(q) := (\mathcal{G}_1(q), \mathcal{G}_2(q))$. □

Items (i)–(iii) of Theorem 1 imply that the (x, η_1)–system is ISS with respect to input (e, η_2) with the gain χ_1, and the (e, η_2)–system is ISS with respect to input (x, η_1) with a gain χ_2. Item (iv) of Theorem 1 is the small-gain condition. In Theorem 1, a max-type Lyapunov function is constructed for system (8) when the (x, η_1)–system and the (e, η_2)–system are ISS and the small-gain condition is satisfied. We will show in Section V that the conditions of Theorem 1 are verified by various ETC schemes available in the literature. Indeed, for these schemes, we show that the design of the flow and jump sets C and D, and of the η–dynamics in (7), enforces ISS properties for the (x, η_1)–system and the (e, η_2)–system as well as the small-gain condition in item (iv) of Proposition 1. Particularly, we will see that verifying conditions of Theorem 1 follow similar arguments for all the studied techniques. For instance, when verifying item (ii) for the (x, η_1)–system, the definition of C will naturally lead to $\chi_1 = 0$ for every triggering scheme when computing the decrease on V_1. In fact, this is an important observation since it opens the door for the redesign of available triggering techniques so that the small-gain condition holds with non-zero χ_1. This can lead to enlarging the flow set and thereby potentially reducing the number of transmissions and the minimum inter-event times, as will be illustrated in a numerical case study in Section VI. On the other hand, for the (e, η_2)–system, to verify item (ii), we will essentially prove the ISS property vacuously, i.e. we select χ_2 such that $V_2(e, \eta_2) \geq \chi_2(V_1(x, \eta_1))$ never holds (outside the origin). Lastly, item (iii) will often hold trivially since $e^+ = 0$ and $x^+ = x$.

Using Theorem 1 and extra conditions on the hybrid time domains of its solutions, see [13, Theorem 1], we can state *uniform global asymptotic stability* (UGAS) of set $\mathcal{A}$ for system (8), as formalised next.

Theorem 2: Consider system (8) and a given closed set $\mathcal{A} \subseteq \mathbb{R}^{n_q}$. Suppose that the following hold.

1) The conditions (i)–(iv) of Theorem 1 hold.
2) System (8) has a uniform semi-global dwell time outside set $\mathcal{A}$.
3) Maximal solutions are complete.

Then, set $\mathcal{A}$ is UGAS for system (8). □

Formally, the proposed ETC design approach boils down to designing the flow and jump sets C and D, and the flow and jump dynamics of the auxiliary variable η (i.e., functions h and ℓ in (7)), so that the conditions in Theorems 1 and 2 are satisfied. We emphasise that this perspective is general enough to encompass various popular ETC schemes available in the literature, which can all be reinterpreted using this small-gain view. Additionally, the proposed small-gain view not only can recover available results, but it also can be used for the development of new triggering conditions, and systematic modification and improvement of existing triggering strategies, which can potentially generate longer inter-event times.

V. Main results

In this section, we illustrate how to apply the unifying small-gain perspective by revisiting previous event-triggering techniques proposed in the literature. For each ETC technique, we first specify the corresponding hybrid model (7) with flow and jump sets C, D, and η–dynamics when relevant. Then, we state UGAS by showing that all conditions of Theorem 2 are satisfied. Lastly, we show how to modify these previous ETC techniques in order to enlarge the flow set C and shrinking the jump set D, while maintaining the UGAS property. All the proofs are deferred to the Appendix to avoid breaking the flow of exposition.

A. The relative threshold strategy in [7]

We start with the well-known technique proposed by Tabuada in [7], which has been exploited and extended in various other contexts, see e.g., [10], [12], [16], [23], [36]–[42]. We will refer to it throughout as the *relative threshold technique*.

1) Model: The relative threshold technique in [7] does not require an additional auxiliary variable η, i.e., $n_\eta = 0$. Therefore, the resulting hybrid model here is (7) with state $q := (x, e)$ only. Next we define the flow and jump sets. The triggering rule in [7] corresponds to $\gamma(|e|) \geq \sigma\alpha_V(|x|)$, where $\sigma \in (0, 1)$ is a design parameter, and $\alpha_V, \gamma \in \mathcal{K}_\infty$ come from SA1. This leads to the flow and jump sets

$$\begin{aligned} C &:= \{q : \gamma(|e|) \leq \sigma\alpha_V(|x|)\}, \\ D &:= \{q : \gamma(|e|) \geq \sigma\alpha_V(|x|)\}, \end{aligned} \tag{9}$$

see also [13], [35] for similar hybrid models for the relative threshold technique. We use $\mathcal{H}_A$ to denote the hybrid model resulting from the relative threshold strategy described above, i.e., system (7) without η and flow/jump sets as per (9).

We assume the following on the functions f, g, α_V, and γ.

Assumption 1: The functions f, g in (7), and $\alpha_V^{-1}, \gamma \in \mathcal{K}_\infty$ from SA1 are locally Lipschitz. □

The Lipschitz conditions in Assumption 1 are stated to guarantee the existence of a uniform semi-global dwell time outside the attractor specified below, which is needed to state UGAS via Theorem 2. This is formalised in Corollary 1 below.

2) Analysis: Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ with

$$\mathcal{A}_1 := \{x \in \mathbb{R}^{n_x} : x = 0\}, \ \mathcal{A}_2 := \{e \in \mathbb{R}^{n_e} : e = 0\}. \tag{10}$$

In the following, we show that set $\mathcal{A}$ is UGAS for system $\mathcal{H}_A$ via Theorem 2. To that end, we first need to show conditions (i)–(iv) of Theorem 1 hold. In fact, we show in Proposition 1 in the appendix that the conditions of Theorem 1 hold with gains χ_1 and χ_2 given by

$$\chi_1(s) = 0, \ \chi_2(s) = (1 + \varepsilon)\left[\gamma^{-1} \circ \sigma\alpha_V \circ \underline{\alpha}_V^{-1}(s)\right]^2, \tag{11}$$

where $\varepsilon > 0$ can take any value, and $\gamma, \alpha_V, \underline{\alpha}_V$ come from SA1. To state UGAS via Theorem 2, it remains to show items 2) and 3), that is, the dwell-time property and the completeness of maximal solutions. Thanks to Assumption 1, as shown in [7, Theorem III.1] and [13, Theorem 4], system $\mathcal{H}_A$ has a uniform semi-global dwell time outside $\mathcal{A}$. Moreover, according to [13, Theorem 4], all maximal solutions to system $\mathcal{H}_A$ are complete. Therefore, we can use Theorem 2 to state the following.

Corollary 1: Consider system $\mathcal{H}_A$ and suppose Assumption 1 holds. Then, set $\mathcal{A}$ defined by (10) is UGAS. □

We can conclude that the relative threshold technique in [7] fits our small-gain setting. More importantly, this approach reveals that the gain χ_1 in (11), related to the ISS property of the x-system, is equal to zero. This suggests that we can modify the triggering condition, i.e., the flow and jump sets, in such a way that the ISS property of the x-system holds with a non-zero gain, while still preserving the small-gain condition and thus stability. By doing so, we enlarge the flow set C, and shrink the set D, which may help generating longer average and minimum inter-event times, as we will illustrate later on with simulations in Section VI. We formalise this redesign in the next subsection.

Remark 1: Note that [23] already studied the relative threshold technique [7] with a hybrid Lyapunov small-gain theorem. The difference here is that we consider exactly the same condition as in the original paper [7], and not a modified one as used in [23]. Specifically, [23] considers $V_2(e) \geq \chi_2(V_1(x))$ as triggering rule, which immediately fits the small-gain theorem. We consider the exact condition proposed in [7], i.e., $\gamma(|e|) \geq \sigma\alpha_V(|x|)$. □

Remark 2: The relative threshold technique has also been studied in [10] using small-gain theorems, but the networked control system is modelled as a continuous-time system of the form (6), as opposed to a hybrid system as in (7). In [10], system (6) is assumed to be ISS, i.e., $|x(t)| \leq \max\{\beta(|x(0)|, t), \gamma_e(|e|_{[0,t]})\}$ holds for all $t \geq 0$, some $\beta \in \mathcal{KL}$ and $\gamma_e \in \mathcal{K}$, and the event-triggering condition is developed with this trajectory-based gain function γ_e. In contrast, we use the ISS-Lyapunov function-based gain functions χ_1, χ_2 from Theorem 1. Besides the aforementioned differences, these two approaches indeed have the same rationale, while noting that system (7) is ISS if and only if it admits an ISS-Lyapunov function [34]. However, we can explicitly study how the event-triggering conditions impact the performance of the system using Lyapunov function analysis, e.g., through the involvement of α_V, which depicts the decay rate of V, see item (ii) of SA1. □

3) Redesign: We now formalise the redesign foreshadowed in the discussion below Corollary 1. Particularly, we modify the original triggering condition (9) by redefining the flow and jump sets C and D as

$$\begin{aligned} C &:= \{(x, e) : (1-\delta)\gamma(|e|) \leq \sigma\alpha_V(|x|)\}, \\ D &:= \{(x, e) : (1-\delta)\gamma(|e|) \geq \sigma\alpha_V(|x|)\}, \end{aligned} \tag{12}$$

where $\delta\gamma(|e|)$ is the newly introduced term enlarging the flow set, and shrinking the jump set, compared to (9). We use $\mathcal{H}_A^r$ to denote the redesign of system $\mathcal{H}_A$, i.e., system (7) with redesigned flow and jump sets (12). To design δ, we rely on the next assumption, which is satisfied when SA1 holds with $\underline{\alpha}_V, \overline{\alpha}_V, \alpha_V$ all quadratic or all linear for instance.

Assumption 2: There exists $c > 0$ such that for any $s \geq 0$, $\alpha_V \circ \underline{\alpha}_V^{-1}(s) \leq c\alpha_V \circ \overline{\alpha}_V^{-1}(s)$. □

The main objective here is to design δ so that $\mathcal{H}_A^r$ is UGAS. Specifically, δ is designed so that conditions of Theorem 2 are satisfied with $\chi_1 \neq 0$. The details can be found in the proof of Proposition 2 in the Appendix. We thus have that δ has to be chosen in the interval $\left(0, \frac{1-\sigma}{1-\sigma+\sigma c}\right)$. As a result, Theorem 1 holds with χ_1 and χ_2 given by

$$\begin{aligned} \chi_1(s) &= \overline{\alpha}_V \circ \alpha_V^{-1} \circ \frac{\delta}{\nu(1-\sigma)}\gamma(s^{1/2}), \\ \chi_2(s) &= \left[\gamma^{-1} \circ (1+\varepsilon)\frac{\sigma}{1-\delta}\alpha_V \circ \underline{\alpha}_V^{-1}(s)\right]^2, \end{aligned} \tag{13}$$

for all $s \geq 0$, where $\nu \in (0, 1)$ and $\varepsilon > 0$ can be any constants. All the details can be found in the proof of Proposition 2 in the Appendix. As a consequence, we derive the UGAS property for $\mathcal{H}_A^r$ by following the analysis in Section V-A.2.

Corollary 2: Consider system $\mathcal{H}_A^r$ and suppose that Assumptions 1 and 2 hold. Then, set $\mathcal{A}$ defined by (10) is UGAS. □

B. The dynamic triggering strategy in [11]

We note that the relative threshold technique studied in the previous section did not use the auxiliary variable η. We now study an event-triggering rule which does depend on the dynamics of η, with $n_\eta = 1$. This dynamic technique was introduced in [11], and it has been extended afterwards to other (distributed) ETC settings, see e.g., [12], [29], [43].

1) Model: The dynamic technique in [11] employs η which satisfies purely continuous dynamics given by $\dot{\eta} = -\beta(\eta) + \sigma\alpha_V(|x|) - \gamma(|e|)$, where $\eta \in \mathbb{R}_{\geq 0}$, $\beta \in \mathcal{K}_\infty$ is a designed function, and $\alpha_V, \gamma \in \mathcal{K}_\infty$ are as per SA1, and $\sigma \in (0, 1)$. Therefore, the η–dynamics in (7) reduce here to

$$\begin{aligned} \dot{\eta} &= -\beta(\eta) + \sigma\alpha_V(|x|) - \gamma(|e|), & q \in C, \\ \eta^+ &= \eta, & q \in D, \end{aligned} \tag{14}$$

where $q := (x, e, \eta)$. Moreover, the triggering rule in [11] corresponds to $\eta + \theta(\sigma\alpha_V(|x|) - \gamma(|e|)) \leq 0$, where $\theta \in \mathbb{R}_{\geq 0}$ is a design parameter. This leads to the flow and jump sets, for $\theta > 0$,

$$\begin{aligned} C &:= \{q : \eta + \theta(\sigma\alpha_V(|x|) - \gamma(|e|)) \geq 0, \eta \in \mathbb{R}_{\geq 0}\}, \\ D &:= \{q : \eta + \theta(\sigma\alpha_V(|x|) - \gamma(|e|)) \leq 0, \eta \in \mathbb{R}_{\geq 0}\}, \end{aligned} \tag{15}$$

and for $\theta = 0$ we define

$$C := \{q : \eta \geq 0\}, \ D := \{q : \eta = 0, \ \sigma\alpha_V(|x|) \leq \gamma(|e|)\}, \tag{16}$$

where the last inequality in (16) is used to avoid Zeno phenomenon when $\theta = 0$. We then use $\mathcal{H}_B$ to denote the resulting hybrid system for the dynamic triggering strategy, i.e., system (7) with η–dynamics (14) and flow/jump sets (15). Note that the relative threshold rule in (9) can be seen as the limit case of the dynamic rule (15) when $\theta \to +\infty$.

2) Analysis: Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ with

$$\begin{aligned} \mathcal{A}_1 &:= \{(x, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} : (x, \eta) = (0, 0)\}, \\ \mathcal{A}_2 &:= \{e \in \mathbb{R}^{n_e} : \theta e = 0\}. \end{aligned} \tag{17}$$

Note for $\theta > 0$ we have $\mathcal{A}_2 = \{e \in \mathbb{R}^{n_e} : e = 0\}$ and for $\theta = 0$, $\mathcal{A}_2 = \mathbb{R}^{n_e}$. As before, we want to show that the set $\mathcal{A}$ is UGAS via Theorem 2. For this strategy, we divide the analysis in two cases: $\theta > 0$ and $\theta = 0$. For $\theta > 0$, we have from Proposition 3 in the Appendix, that the conditions of Theorem 1 hold with

$$\chi_1(s) = 0, \ \chi_2(s) = \left[\gamma^{-1} \circ (1+\varepsilon)\overline{\chi}_2(s)\right]^2, \tag{18}$$

for all $s \geq 0$, where $\overline{\chi}_2(s) = \frac{s}{\theta} + \sigma\alpha_V \circ \underline{\alpha}_V^{-1}(s)$, for any $\varepsilon > 0$. For $\theta = 0$, we have from Proposition 4 in the Appendix that conditions of Theorem 1 hold with $\chi_1 = \chi_2 = 0$.

It remains to show the dwell time property and that maximal solutions are complete. First, from [11, Proposition 2.3], we know that $\mathcal{H}_B$ admits a uniform semi-global dwell time outside $\mathcal{A}$ when Assumption 1 holds. Lastly, we show that all maximal solutions are complete in Lemma 1 in the Appendix. Therefore, from Theorem 2, we can state the following.

Corollary 3: Consider system $\mathcal{H}_B$ and suppose Assumption 1 holds. Then, set $\mathcal{A}$ defined by (17) is UGAS. □

We note that in [11], asymptotic stability of $(x, \eta) = 0$ was shown for $\theta \in \mathbb{R}_{\geq 0}$. Therefore, with Corollary 3, we not only recover the results from [11], but also derive stronger conditions for the case $\theta > 0$, since we show that $(x, \eta, e) = 0$ is UGAS.

Remark 3: The dynamic triggering technique analysed in this section has also been studied with a trajectory-based small-gain analysis in [29], see Remark 2 for more details about such design. □

3) Redesign: Note that our approach reveals that $\chi_1 = 0$, just like it did for the relative threshold technique in the previous section. Consequently, as in Section V-A, we can exploit this to redesign the original dynamic technique and thus enlarge the flow set C, and shrink the set D, with the hope of obtaining larger inter-event times. Particularly, we modify the flow dynamics of η, as well as the flow and the jump sets, so that χ_1 is no longer zero, while still ensuring the small-gain condition and thus stability. In particular, we modify the dynamics of η to

$$\begin{aligned} &\dot{\eta} = -\beta(\eta) + \sigma\alpha_V(|x|) - \gamma(|e|) + \delta(\gamma(|e|)),\ q \in C, \\ &\eta^+ = \eta,\ q \in D, \end{aligned} \tag{19}$$

where $\beta, \delta \in \mathcal{K}_\infty$ are to be designed, and δ is such that $\mathbb{I} - \delta \in \mathcal{K}_\infty$. The main difference with the original η–dynamics (14) is that we have added the term $\delta(\gamma(|e|))$ in the flow map of η, which slows down the decrease of η and may thus help in reducing the number of transmissions. We modify the flow and the jump sets accordingly as

$$\begin{aligned} C &:= \{q : \eta + \theta(\sigma\alpha_V(|x|) - (\mathbb{I} - \delta) \circ \gamma(|e|)) \geq 0, \eta \geq 0\}, \\ D &:= \{q : \eta + \theta(\sigma\alpha_V(|x|) - (\mathbb{I} - \delta) \circ \gamma(|e|)) \leq 0, \eta \geq 0\}, \end{aligned} \tag{20}$$

where we consider the case $\theta > 0$ for simplicity. Henceforth, we denote the redesign of $\mathcal{H}_B$ by $\mathcal{H}_B^r$, i.e., system (7) with redesigned η–dynamics (19) and flow/jump sets (20).

Exactly as before, the design of δ boils down to ensuring that the conditions of Theorem 1 hold, which we formalise in Proposition 5 in the Appendix. Particularly, we take

$$\delta(s) < \min\left\{\nu\varrho\left(\frac{s}{2(1+\varepsilon)}\right), s/2\right\} \tag{21}$$

where $\varrho(s) := \tilde{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ a_2^{-1}(s)$, $a_2(s) := \frac{s}{\theta} + \sigma\alpha_V(\underline{\alpha}_V^{-1}(s))$, $\overline{\alpha}_1(s) = \overline{\alpha}_V(s) + s$, and $\tilde{\alpha}_1(s) := \min\{(1-\sigma)\alpha_V(s/2), \beta(s/2)\}$. As a result, Theorem 1 holds with

$$\begin{aligned} \chi_1(s) &= \overline{\alpha}_1 \circ \tilde{\alpha}_1^{-1} \circ \frac{1}{\nu}\delta(\gamma(s^{1/2})), \\ \chi_2(s) &= \left[\gamma^{-1} \circ (1+\varepsilon)(\mathbb{I} - \delta)^{-1} \circ a_2(s)\right]^2, \end{aligned} \tag{22}$$

for some $\nu \in (0, 1)$ and $\varepsilon > 0$.

Remark 4: We note that there always exists a $\delta \in \mathcal{K}_\infty$ that satisfies (21) and $\mathbb{I} - \delta \in \mathcal{K}_\infty$. Particularly, we can choose

$$\delta(s) := \ell \min\left\{\int_0^s \min\left\{\nu\frac{\mathrm{d}\overline{\varrho}(s/(2(1+\varepsilon)))}{\mathrm{d}s}, \frac{1}{2}\right\}\mathrm{d}s, \frac{s}{2}\right\},$$

with $\ell \in (0, 1)$ and $\overline{\varrho} \in \mathcal{K}_\infty$ is any continuously differentiable lower bound of ϱ. Then, this function satisfies $0 < \frac{\mathrm{d}\delta(s)}{\mathrm{d}s} < \frac{1}{2}$ and $\delta(s) < \min\left\{\nu\varrho\left(\frac{s}{2(1+\varepsilon)}\right), s/2\right\}$ for any $s \geq 0$. □

It remains to ensure both the dwell time property and completeness of maximal solutions. Note that, for any given initial condition, the first jump generated by $\mathcal{H}_B^r$ occurs later than the one generated by $\mathcal{H}_B$. Using this argument, we can prove that $\mathcal{H}_B^r$ has a semiglobal dwell-time outside $\mathcal{A}$. Lastly, completeness of maximal solutions follows similarly to the proof of Lemma 1. We can thus state the following stability result for the redesigned dynamic technique.

Corollary 4: Consider system $\mathcal{H}_B^r$ and suppose Assumptions 1 holds. Then, set $\mathcal{A}$ defined by (17) with $\theta > 0$ is UGAS. □

C. Fixed threshold on network-induced error

Different to the above two strategies that take into account both x and e in their triggering conditions, we now study a strategy that generates a transmission when $|e|$ is greater than or equal to a fixed value [35]. This mechanism is considered in numerous studies, see e.g., [8], [24]–[26], [13, Section V.D].

1) Model: The *fixed threshold strategy*, also called *absolute event-triggering*, consists in generating a transmission whenever $|e| \geq d'$ for some tunable parameter $d' > 0$. We can equivalently write this condition as $\gamma(|e|) \geq \gamma(d') =: d$ where $\gamma \in \mathcal{K}_\infty$ comes from SA1, and consider $d > 0$ as the tunable parameter. We note that this strategy does not need the auxiliary variable η, i.e., $n_\eta = 0$. As a result, the hybrid model for this case corresponds to (7), without η, and with flow and jump sets defined as

$$C := \{q : \gamma(|e|) \leq d\}, \quad D := \{q : \gamma(|e|) \geq d\}, \tag{23}$$

where $q := (x, e)$. As it is clear by now, we use $\mathcal{H}_C$ to denote the hybrid system corresponding to the fixed threshold strategy, i.e., system (7) with flow/jump sets (23).

2) Analysis: Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ with

$$\begin{aligned} \mathcal{A}_1 &:= \left\{x \in \mathbb{R}^{n_x} : V(x) \leq \overline{\alpha}_V \circ \alpha_V^{-1}(2d)\right\}, \\ \mathcal{A}_2 &:= \{e \in \mathbb{R}^{n_e} : \gamma(|e|) \leq d\}, \end{aligned} \tag{24}$$

with $V, \overline{\alpha}_V, \alpha_V, \gamma$ from SA1. Note that the attractor does not impose x (and e) to be equal to zero as in the previous ETC strategies, but a more general set whose "size" depends on d. From Proposition 6 in the Appendix, the conditions of Theorem 1 are verified with $\chi_1 = \chi_2 = 0$.

Note that system $\mathcal{H}_C$ admits a uniform semi-global dwell time and its maximal solutions are complete, as shown in the proof of [13, Theorem 5]. Then, UGAS of $\mathcal{A}$ is derived in view of Theorem 2.

Corollary 5: Consider system $\mathcal{H}_C$. Then, set $\mathcal{A}$ defined by (24) is UGAS. □

Remark 5: The stability property established in Corollary 5 can also be viewed as practical stability of the origin, which guarantees that trajectories converge to an adjustable attractor including the origin. Here, the attractor is $\mathcal{A}$ and it can be made as small as desired by adjusting parameter d. □

Remark 6: As in previous sections, the proposed small-gain approach provides enough flexibility to redesign—in this case—the fixed threshold triggering strategy. Particularly, we could consider the triggering condition $\gamma(|e|) \geq d + \delta\theta(|e|)$, for $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, instead of $\gamma(|e|) \geq d$, so that the small-gain condition in Theorem 1 holds with non-zero χ_1, χ_2. By doing so, we would change the attractor $\mathcal{A}_2$ in (24), which could make this redesign redundant as we can already change the attractor $\mathcal{A}_2$ by adjusting d. However, we note that the proposed redesigns in this paper are not necessarily exhaustive and many others can be proposed depending on the chosen objective, by following a similar philosophy. □

D. Decreasing threshold on network-induced error

An extra level of flexibility can be added to the fixed threshold strategy from Section V-C. Particularly, transmissions can be triggered when $|e|$ crosses a certain decreasing threshold. These techniques can be found in e.g., [13], [27]–[29].

1) Model: Similar to the dynamic triggering strategy in Section V-B, the triggering condition for these strategies can be written by using the scalar auxiliary variable η, i.e., $n_\eta = 1$. Particularly, the hybrid model is given by (7) with η–dynamics

$$\begin{aligned} \dot{\eta} &= -\beta(\eta), & q \in C, \\ \eta^+ &= \eta, & q \in D, \end{aligned} \tag{25}$$

where $\beta \in \mathcal{K}_\infty$, and the flow and jump sets are defined as

$$\begin{aligned} C &:= \{q : \gamma(|e|) \leq \sigma\beta(\eta) + d,\ \eta \geq 0,\ d > 0\}, \\ D &:= \{q : \gamma(|e|) \geq \sigma\beta(\eta) + d,\ \eta \geq 0,\ d > 0\}, \end{aligned} \tag{26}$$

for some $\sigma \in (0, 1)$. We use $\mathcal{H}_D$ to denote this hybrid system. Note that jumps occur whenever the norm of the network-induced error is greater or equal than a decreasing threshold on η plus a constant d. For $\eta(0, 0) = 0$ this rule reduces to the fixed threshold strategy.

2) Analysis: Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ with

$$\begin{aligned} \mathcal{A}_1 &:= \big\{(x, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} : \\ &\qquad V(x) \leq \overline{\alpha}_V \circ \alpha_V^{-1}(2d),\ \eta = 0\big\}, \\ \mathcal{A}_2 &:= \{e \in \mathbb{R}^{n_e} : \gamma(|e|) \leq 2d\}. \end{aligned} \tag{27}$$

As usual, we want to show that $\mathcal{A}$ is UGAS via Theorem 2. First, from Proposition 7 in the Appendix, we know that for this strategy all conditions in Theorem 1 are verified with

$$\chi_1(s) = 0,\ \ \chi_2(s) = \left[\gamma^{-1} \circ 2(1 + \varepsilon)\sigma\beta(s)\right]^2, \tag{28}$$

for all $s \geq 0$, any $\sigma \in (0, 1)$, $\varepsilon > 0$, and all $s \geq 0$. Completeness of maximal solutions and the dwell time condition follow similarly to the fixed threshold strategy as in Corollary 5. We can thus state the following result.

Corollary 6: Consider system $\mathcal{H}_D$. Then, set $\mathcal{A}$ defined by (27) is UGAS. □

3) Redesign: Once again we have that the conditions of Theorem 1 holds with $\chi_1 = 0$ for this strategy, and thus we present a redesign that aims to ensure UGAS with non-zero χ_1 and thus enlarges the flow set C with hopes of obtaining larger inter-event times. To that end, we modify the η–dynamics to

$$\begin{aligned} \dot{\eta} &= -\beta(\eta) + \delta(\gamma(|e|_{\mathcal{A}_2})), & q \in C, \\ \eta^+ &= \eta, & q \in D. \end{aligned} \tag{29}$$

Note that the redesigned dynamics (29) are different to (25), in the sense that (29) is no longer open-loop, but it now involves the network-induced error e. The goal here is to design $\delta \in \mathcal{K}_\infty$ such that the conditions of Theorem 1 hold. It follows from Proposition 8 that for $\delta(s) < \min\Big\{\frac{\nu}{2}\alpha_V \circ \overline{\alpha}_V^{-1} \circ \frac{1}{2}\beta^{-1}\left(\frac{s}{2(1+\varepsilon)\sigma}\right), \nu(1-\sigma)\beta \circ \frac{1}{2}\beta^{-1}\left(\frac{s}{2(1+\varepsilon)\sigma}\right)\Big\}$, with some $\nu \in (0, 1)$ and $\varepsilon > 0$, then the conditions of Theorem 1 are satisfied with

$$\begin{aligned} \chi_1(s) &= \max\Big\{2\overline{\alpha}_V \circ \alpha_V^{-1} \circ \frac{2}{\nu}\delta(\gamma(\sqrt{s})), \\ &\qquad 2\beta^{-1} \circ \frac{1}{\nu(1-\sigma)}\delta(\gamma(\sqrt{s}))\Big\}, \\ \chi_2(s) &= \left[\gamma^{-1} \circ 2(1 + \varepsilon)\sigma\beta(s)\right]^2, \end{aligned} \tag{30}$$

for all $s \geq 0$.

As we discussed at the end of Section V-B, we note that, for any given initial condition, the first jump generated by $\mathcal{H}_D^r$ occurs later than $\mathcal{H}_D$. This argument is used to show the semiglobal dwell-time condition. Lastly, completeness of maximal solutions follows similarly to the proof of Lemma 1 in the Appendix. Therefore, we can state the following.

Corollary 7: Consider system $\mathcal{H}_D^r$. Then, set $\mathcal{A}$ defined by (27) is UGAS. □

E. Decreasing threshold on V

Different from all the triggering rules studied above, where the triggering conditions are built upon the network-induced error, we now study the approach in [14], where the objective is to keep the value of the Lyapunov function V in SA1 below a time-varying designed threshold that decreases to the origin. This technique has also been applied for the self-triggered implementation of linear controllers in [44].

1) Model: Transmissions in [14] are triggered at the violation of $V(x(t)) \leq -\mu V(x(t_j))(t - t_j) + V(x(t_j))$, $t \in [t_j, t_{j+1})$, $j \in \mathbb{Z}_{\geq 0}$, $\mu > 0$, and t_j denotes the j-th transmission instant. We first note that SA1 implies

$$\langle \nabla V(x), f(x, e)\rangle \leq -\tilde{\alpha}_V(V(x)) + \gamma(|e|) \tag{31}$$

holds with $\tilde{\alpha}_V(s) := \alpha_V \circ \overline{\alpha}_V^{-1}(s)$, $s \geq 0$, for all $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$. The results of this section will exploit (31), as opposed to SA1, just to be consistent with [14]. This approach can be captured by the hybrid model (7) with η–dynamics

$$\left.\begin{aligned} \dot{\eta}_1 &= -\beta(\eta_1, \eta_2) \\ \dot{\eta}_2 &= 0 \end{aligned}\right\}\ q \in C, \qquad \left.\begin{aligned} \eta_1^+ &= V(x) \\ \eta_2^+ &= V(x) \end{aligned}\right\}\ q \in D, \tag{32}$$

with flow and jump sets defined as

$$C = \{q : V(x) \le \eta_1, \eta_2 \ge \eta_1\}, \tag{33}$$
$$D = \{q : V(x) \ge \eta_1, \tfrac{\partial V}{\partial x}(x) f(x,e) \ge -\sigma_1 \tilde{\alpha}_V(V(x)), \eta_1 \ge 0\}, \tag{34}$$

where $q := (x, e, \eta_1, \eta_2)$, β is assumed to satisfy $\beta(\cdot, r) \in \mathcal{K}_\infty$ for each $r > 0$, and $\beta(s, \cdot) \in \mathcal{K}_\infty$ for each $s > 0$, $\sigma_1 \in (0,1)$ is a parameter, and functions V and α_V come from SA1. We highlight that, in order to apply the small-gain view, the technique from [14] is first embedded into a novel hybrid model (32). In fact, note that we now require a vector $\eta := (\eta_1, \eta_2)$ to model the condition from [14], as opposed to scalar η–dynamics as we used for previous methods. We denote this hybrid system by $\mathcal{H}_E$, i.e., system (7) with η–dynamics (32) and flow/jump sets (38). We have considered a more general case than [14], since the strategy in [14] is recovered for $\beta(\eta_1, \eta_2) = \mu\eta_2$, $\mu > 0$. In general, β is designed such that

$$\beta(s_1, s_2) \ge \sigma\tilde{\alpha}_V(s_2), \quad s_1, s_2 \ge 0, \tag{35}$$

where $\sigma \in (0, \sigma_1)$, for $\sigma_1 \in (0,1)$ as per (39). With this choice of β, we note that $V(x)$ decreases faster than η_1 in view of (31). The condition $\frac{\partial V}{\partial x}(x) f(x,e) \ge -\sigma_1\tilde{\alpha}_V(V(x))$ in the definition of the set D is used to exclude the Zeno phenomenon. Particularly, after a jump, we have that $\eta_1 = V(x)$ and it is not necessary to jump again since the derivative of V needs to be greater or equal to $-\sigma_1\tilde{\alpha}_V(V(x))$ before jumping again.

2) Analysis: Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ with

$$\begin{aligned}\mathcal{A}_1 &:= \{(x, \eta_1, \eta_2) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} : \\ &\qquad (x, \eta_1, \eta_2) = (0,0,0)\}, \\ \mathcal{A}_2 &:= \mathbb{R}^{n_e}.\end{aligned} \tag{36}$$

We verify the conditions of Theorem 1 in Proposition 9 in the Appendix, and show that they hold with $\chi_1 = \chi_2 = 0$.

By following similar lines to [13, Theorem 6], we can show that system $\mathcal{H}_E$ admits a uniform semi-global dwell time outside $\mathcal{A}$. Also, its maximal solutions are complete by following similar arguments as in Lemma 1. We thus derive the following stability result with the help of Theorem 2.

Corollary 8: Consider system $\mathcal{H}_E$ and suppose Assumption 1 holds. Then, set $\mathcal{A}$ defined by (36) is UGAS. □

3) Redesign: We present two redesigns, which we parameterise by some constant $\xi \in \{0,1\}$. That is, we use $\xi = 0$ to define Redesign I, and $\xi = 1$ for Redesign II. We first modify η_1 in (32) such that

$$\begin{aligned}\dot{\eta}_1 &= -\beta(\eta_1, \eta_2) + (1-\xi)\delta(|x|) + \xi\delta_1(|e|), & q \in C, \\ \eta_1^+ &= V(x), & q \in D,\end{aligned} \tag{37}$$

and flow/jump sets as

$$C = \{q : V(x) + \xi\delta_2(|e|) \le \eta_1, \eta_2 \ge \eta_1\}, \tag{38}$$
$$D = \{q : V(x) + \xi\delta_2(|e|) \ge \eta_1, \tfrac{\partial V}{\partial x}(x) f(x,e) \ge -\sigma_1\tilde{\alpha}_V(V(x)), \eta_1 \ge 0\}, \tag{39}$$

where $\delta, \delta_1, \delta_2 \in \mathcal{K}_\infty$ are to be designed. We denote the redesigned version of $\mathcal{H}_E$ by $\mathcal{H}_E^r$. Note that δ relates to Redesign I ($\xi = 0$), and δ_1, δ_2 to Redesign II ($\xi = 1$). Consider the attractor sets

$$\begin{aligned}\mathcal{A}_1 &:= \{(x, \eta_1, \eta_2) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} : \\ &\qquad (x, \eta_1, \eta_2) = (0,0,0)\}, \\ \mathcal{A}_2 &:= \{e \in \mathbb{R}^{n_e} : \xi e = 0\}.\end{aligned} \tag{40}$$

Note that $\mathcal{A}_2 = \mathbb{R}^{n_e}$ for $\xi = 0$ and $\mathcal{A}_2 = \{0\}$ for $\xi = 1$. Hence, when $\xi = 0$, the attractor set $\mathcal{A}_1 \times \mathcal{A}_2$ is not compact, while it is compact when $\xi = 1$. For Redesign I ($\xi = 0$), we select $\delta \in \mathcal{K}_\infty$ so that

$$\delta(s) \le \sigma(\tfrac{1}{2} - \nu)\tilde{\alpha}_V \circ \underline{\alpha}_1(s), \tag{41}$$

holds for all $s \ge 0$, where $\sigma > 0$ comes from (35), $\nu \in (0, 1/2)$ is a design parameter, and $\underline{\alpha}_1(s) = \min\{(1/2)\underline{\alpha}_V(s/2), s/4\}$. Note that it is always possible to select a function $\delta \in \mathcal{K}_\infty$ that satisfies (41), take for instance $\delta = \sigma(\frac{1}{2} - \nu)\alpha_V \circ \underline{\alpha}_1$. Proposition 10 in Section V-E ensures the conditions of Theorem 1 hold with $\chi_1 = \chi_2 = 0$. Following the similar arguments surrounding Corollary 7, we can then state the following.

Corollary 9: Consider system $\mathcal{H}_E^r$ with $\xi = 0$ and suppose Assumption 1 holds. Then, set $\mathcal{A}$ defined by (40) is UGAS. □

Note that the Redesign I ($\xi = 0$) does not derive a non-zero χ_1 as in the previous sections. This is due to the fact that $\mathcal{A}_2$ in (40) is not compact for $\xi = 0$. However, the inter-transmission times may still increase since we are enlarging the flow set, as we illustrate with an example in Section VI.

On the other hand, for Redesign II ($\xi = 1$), given any $\varepsilon > 0$ and $\delta_2 \in \mathcal{K}_\infty$, we select $\delta_1 \in \mathcal{K}_\infty$ so that $\delta_1(s) < \sigma(1/2 - \nu)\tilde{\alpha}_V \circ \delta_2\left(\frac{1}{1+\varepsilon}s\right)$ for all $s > 0$. Then, from Proposition 11, the conditions of Theorem 1 are satisfied with non-zero gains

$$\begin{aligned}\chi_1(s) &= \tilde{\alpha}_V^{-1} \circ \frac{1}{(1/2 - \nu)\sigma}\delta_1(\sqrt{s}), \\ \chi_2(s) &= [(1+\varepsilon)\delta_2^{-1}(s)]^2,\end{aligned} \tag{42}$$

for all $s \ge 0$. Moreover, Redesign II allows for stronger stability results (for a compact attractor) compared to both [14] and Redesign I. We formalise it below.

Corollary 10: Consider system $\mathcal{H}_E^r$ with $\xi = 1$ and suppose Assumption 1 holds. Then, the origin is UGAS. □

We will see on an example in Section VI that Redesign II may provide larger inter-event times compared to [14] and Redesign I.

F. Combined triggering strategy

We have already highlighted the flexibility of the proposed approach in the sense that it not only covers many existing approaches in the literature, but it also allows for simple redesign. In this section, we aim to highlight such flexibility even further by studying a combination of existing ETC strategies, which we call *combined triggering strategy*. We only show one example of such combined strategies, but that many more can be considered. The combined strategy we consider below is the combination of the relative threshold in Section V-A and the decreasing threshold on $|e|$ from Section V-D.

1) Model: The resulting model for this strategy is (7) with η–dynamics as per (25), and flow and jump sets defined as

$$\begin{aligned} C &:= \{q : \gamma(|e|) \leq \sigma \max\{\beta(\eta), \alpha_V(|x|)\}, \eta \geq 0\}, \\ D &:= \{q : \gamma(|e|) \geq \sigma \max\{\beta(\eta), \alpha_V(|x|)\}, \eta \geq 0\}, \end{aligned} \tag{43}$$

where $q := (x, e, \eta)$ and $\sigma \in (0, 1)$. We denote this model by $\mathcal{H}_F$. Note that the combination of the ETC strategies is reflected in C and D. Also, for simplicity, we have considered only the purely decreasing part of the strategy from Section V-D, i.e., $d = 0$ in (26). This strategy has also been studied in [13, Section V.A] with a slightly different triggering condition.

2) Analysis: Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ with

$$\begin{aligned} \mathcal{A}_1 &:= \{(x, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} : (x, \eta) = (0, 0)\}, \\ \mathcal{A}_2 &:= \{e \in \mathbb{R}^{n_e} : e = 0\}. \end{aligned} \tag{44}$$

The conditions of Theorem 1 are verified in Proposition 12 in the Appendix. Particularly, we show that for this combined triggering strategy such conditions hold with gains

$$\begin{aligned} \chi_1(s) &= 0, \\ \chi_2(s) &= \left[\gamma^{-1} \circ (1 + \varepsilon)\sigma(\alpha_V \circ \underline{\alpha}_V^{-1}(s) + \beta(s))\right]^2, \end{aligned} \tag{45}$$

for all $s \geq 0$ and $\varepsilon > 0$.

The existence of a uniform semi-global dwell time outside $\mathcal{A}$ for system $\mathcal{H}_F$ is guaranteed by following similar lines as the proof in [13, Theorem 2]. This with the completeness of maximal solutions, as discussed in Corollary 3, implies the next result.

Corollary 11: Consider system $\mathcal{H}_F$ and suppose Assumption 1 holds. Then, set $\mathcal{A}$ defined by (44) is UGAS. □

3) Redesign: For this redesign, we modify the original η–dynamics in (25) as

$$\begin{aligned} \dot{\eta} &= -\beta(\eta) + \delta(\gamma(|e|)), & q \in C, \\ \eta^+ &= \eta, & q \in D, \end{aligned} \tag{46}$$

where functions $\beta, \delta \in \mathcal{K}_\infty$ are to be designed. We denote the redesign of $\mathcal{H}_F$ by $\mathcal{H}_F^r$, i.e., system (7) with η–dynamics (46) and flow/jump sets (43). It follows from Proposition 13 in the Appendix that for $\delta(s) < \nu\tilde{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ \hat{\alpha}_1^{-1} \circ \frac{1}{(1+\varepsilon)\sigma} s$, for some $\nu \in (0, 1)$, where $\tilde{\alpha}_1(s) := (1 - \sigma)\min\{\alpha_V(s/2), \beta(s/2)\}$, $\overline{\alpha}_1(s) = \overline{\alpha}_V(s) + s$, and $\hat{\alpha}_1(s) := \alpha_V \circ \underline{\alpha}_V^{-1}(s) + \beta(s)$, then the conditions of Theorem 1 are satisfied with gains

$$\begin{aligned} \chi_1(s) &= \overline{\alpha}_1 \circ \tilde{\alpha}_1^{-1} \circ \frac{1}{\nu}\delta\big(\gamma(s^{1/2})\big), \\ \chi_2(s) &= \left[\gamma^{-1} \circ (1 + \varepsilon)\sigma(\alpha_V \circ \underline{\alpha}_V^{-1}(s) + \beta(s))\right]^2, \end{aligned} \tag{47}$$

for all $s \geq 0$. The existence of a uniform semi-global dwell time outside $\mathcal{A}$ for system $\mathcal{H}_F^r$ is guaranteed by using similar arguments surrounding Corollary 4. This with the completeness of maximal solutions, as discussed in Corollary 3, implies the next result.

Corollary 12: Consider system $\mathcal{H}_F^r$ and suppose Assumption 1 holds. Then, set $\mathcal{A}$ defined by (44) is UGAS. □

Remark 7: An important advantage in studying combined triggering strategies is that they may generate less transmissions compared to each individual method. Moreover, the proposed small-gain view in this paper provides a systematic and simple approach to combine different existing ETC strategies and tackle their stability. □

VI. Numerical example

We now illustrate how the redesign flexibility that the small-gain perspective provides can be used to potentially reduce the number of transmissions. To that end, we study the redesigns from Sections V-A.3, V-B.3, V-D.3, V-E.3, and V-F.3, and we compare them to their original versions. Consider for this purpose the nonlinear dynamics of a single-link robot arm $\dot{x}_\mathrm{p} = Ax_\mathrm{p} + Bu - \phi(x_\mathrm{p})$, which is stabilised with a state-feedback controller of the form $u = Kx_\mathrm{p} + B^\top\phi(x_\mathrm{p})$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$, and $\phi(x_\mathrm{p}) = \begin{bmatrix} 0 & \sin(x_{\mathrm{p},1}) \end{bmatrix}^\top$. The gain K is designed such that the eigenvalues of $A+BK$ are -1 and -2. We consider that only the state x_p is sent over the network, and thus $e = \hat{x}_\mathrm{p} - x_\mathrm{p}$. Note that $x = x_\mathrm{p}$ in (6) since the controller is static and thus $n_\mathrm{c} = 0$ in (3). Under this setting, we can show that SA1 is verified with $V(x) = x^\top Px$, $P = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$, $\underline{\alpha}_V(s) = \lambda_{\min}(P)s^2$, $\overline{\alpha}_V(s) = \lambda_{\max}(P)s^2$, $\alpha_V(s) = (1/2)s^2$ and $\gamma(s) = 2(|P|+|PBK|)^2 s^2$ for any $s \geq 0$. Moreover, Assumption 2 is satisfied with $c = \lambda_{\max}(P)/\lambda_{\min}(P)$. We have chosen $\sigma = 0.1$, $\theta = 1$, and $\beta(\eta) = \eta$ for any $\eta \geq 0$. For the strategy in Section V-E, we have chosen $\beta(\eta_1, \eta_2)$ as per (35) with $\sigma_1 = 0.99$. Lastly, the redesign parameter δ for each triggering strategy has been chosen as per Sections V-A.3, V-B.3, V-D.3, V-E.3, and V-F.3, with $\nu = 0.99$ and $\varepsilon = 10^{-4}$, where applicable. For Redesign II in Section V-E, δ_1 is chosen to satisfy the condition above (42) with $\delta_2(s) = s, s \geq 0$.

We have run simulations for each triggering strategy for 10 different initial conditions $x(0, 0)$ uniformly distributed on the circle centered at the origin of radius 20. In all cases, we have selected $e(0, 0) = (0, 0)$ and $\eta(0, 0) = x(0, 0)^\top Px(0, 0)$, when relevant. The obtained average, minimum, and maximum inter-transmissions times over the 10 simulations are summarised in Table I. We can see that, in every triggering strategy, the redesigned technique generates larger average inter-event times than the corresponding original one. This translates in fewer transmissions, while still guaranteeing desirable stability conditions. A similar improvement is seen for the minimum inter-event times, with the exception of the decreasing threshold on $|e|$ strategy in Section V-D, and the combined strategy technique in Section V-F, which give the same value as the original technique. With regards to maximum inter-event times, it can be seen that the considered redesigns also provide an improvement, with the exception of the dynamic triggering strategy in Section V-B. We note that the redesigns proposed in the paper are not necessarily exhaustive: they open the door for potentially more extensive redesigns by following the same small-gain philosophy.

Lastly, we would like to highlight that the purpose of this section is to show the possible improvements brought by the redesigned techniques and *not* to compare the different triggering conditions with each other; this type of comparisons have already been done in [11] and [13], for instance.

VII. Conclusion

We proposed a unifying perspective to cover various commonly used ETC techniques in the literature under one um-

ETC strategy		τ_{avg}	$\tau_{\min}$	$\tau_{\max}$
Section V-A	Original	0.0587	0.0158	0.1080
	Redesign	**0.0888**	**0.0236**	**0.1611**
Section V-B	Original	0.4342	0.0794	**1.4226**
	Redesign	**0.4377**	**0.0798**	1.3237
Section V-D	Original	0.1790	0.0167	2.4225
	Redesign	**0.2152**	**0.0167**	**2.8364**
Section V-E	Original	0.6204	0.6103	0.6955
	Redesign I	**0.6209**	**0.6108**	**0.6968**
	Redesign II	**0.6749**	**0.6591**	**0.7950**
Section V-F	Original	0.2045	0.0173	1.2252
	Redesign	**0.2163**	**0.0173**	**1.3314**

TABLE I: Average (τ_{avg}), minimum ($\tau_{\min}$), and maximum ($\tau_{\max}$) inter-transmission times over 10 different initial conditions and an interval of 10 (continuous) time units.

brella. The design consists of choosing the dynamics of auxiliary variables and the construction of the jump and flow sets in an appropriate hybrid system description such that a hybrid small-gain theorem hold. We provided clear viewpoints on the essential differences and similarities of these ETC strategies. We also demonstrated the flexibility of the small-gain view as it easily provides redesigns for each of these schemes, which leads to the same stability guarantees as the original event generator; however, typically using less transmissions, as demonstrated by simulations.

We note that this viewpoint was not directly applicable to some triggering rules proposed in the literature, such as those based on time-regularisation [45], the dynamic rule in [13, Section V.B], and the event-holding strategy in [46]. It seems that for these strategies a dissipativity approach might be more appropriate, which will be investigated in future work. Another interesting direction is to study co-design of controller and ETC rules under a small-gain view. In this context, redesign of controllers for improved robustness is of interest, together with the inclusion of plant external disturbances. We believe that this work can serve as a foundation for these generalisations.

Appendix

Here we provide the detailed statements and proofs surrounding the main results in Section V.

A. Relative threshold strategy in Section V-A

The next proposition shows that the relative threshold strategy satisfies the conditions of Theorem 1.

Proposition 1: Consider system $\mathcal{H}_A$. Then, the conditions of Theorem 1 are satisfied with $\mathcal{A}_1$, $\mathcal{A}_2$ in (10), $V_1(x) = V(x)$, $V_2(e) = |e|^2$ for any $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$, $\underline{\alpha}_1(s) = \underline{\alpha}_V(s)$, $\overline{\alpha}_1(s) = \overline{\alpha}_V(s)$, $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$, χ_1, χ_2 in (11), $\alpha_1(s) = (1-\sigma)\alpha_V(s)$, any $\alpha_2 \in \mathcal{K}_\infty$, with γ, α_V coming from Assumption 1, for any $s \geq 0$. □

Proof. We first consider the x–system with $V_1 = V$ and show that the conditions of Theorem 1 are satisfied. V_1 satisfies item (i) of Theorem 1 in view of item (i) of SA1. Let $q \in C$. The definition of set C in (9) implies that $\gamma(|e|) \leq \sigma\alpha_V(|x|)$, which with SA1, leads to $\langle \nabla V_1(x), f(x,e)\rangle \leq -\alpha_V(|x|) + \gamma(|e|) \leq -(1-\sigma)\alpha_V(|x|) =: -\alpha_1(|x|)$. Hence, item (ii) of Theorem 1 holds with $\chi_1(s) = 0$ for any $s \geq 0$. Let $q \in D$. Since x does not change at jumps, we have that[2] $V_1(x^+) = V_1(x)$ in view of (7). Then, item (iii) of Theorem 1 holds.

We now consider the e–system with Lyapunov function $V_2(e) = |e|^2$. V_2 satisfies item (i) of Theorem 1 with $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$ for any $s \geq 0$. Let $q \in C$. It follows from (9) and item (i) of SA1 that $\gamma(|e|) \leq \sigma\alpha_V(|x|) \leq \sigma\alpha_V \circ \underline{\alpha}_V^{-1}(V_1(x))$, and thus $V_2(e) = |e|^2 \leq \left(\gamma^{-1} \circ \left(\sigma\alpha_V \circ \underline{\alpha}_V^{-1}(V_1(x))\right)\right)^2 =: \overline{\chi}_2(V_1(x))$. Hence, for any $\varepsilon > 0$, $V_2(e) > (1+\varepsilon)\overline{\chi}_2(V_1(x)) =: \chi_2(V_1(x))$ contradicts the fact that $q \in C$. Consequently, for any $\alpha_2 \in \mathcal{K}_\infty$, $V_2(e) > \chi_2(V_1(x)) \Rightarrow \langle \nabla V_2(e), g(x,e)\rangle \leq -\alpha_2(|e|)$ vacuously holds. On the other hand, when $V_2(e) = \chi_2(V_1(x))$, noting that $V_2(e) \leq \overline{\chi}_2(V_1(x)) = \frac{1}{1+\varepsilon}\chi_2(V_1(x))$ on C, we necessarily have that $x = 0$ and thus $e = 0$, and thus $\langle \nabla V_2(e), g(x,e)\rangle = -\alpha_2(|e|) = 0$ holds in this case, for any $\alpha_2 \in \mathcal{K}_\infty$. Therefore, item (ii) of Theorem 1 is verified. Let $q \in D$. We have that $V_2(e^+) = 0$ in view of (7).

Lastly, the small-gain condition in item (iv) holds as $\chi_1(s) = 0$ for any $s \geq 0$, concluding the proof. ■

The next proposition applies to the redesigned relative threshold strategy presented in Section V-A.3.

Proposition 2: Consider system $\mathcal{H}_A^r$ and suppose Assumptions 1 and 2 hold. Then, the conditions of Theorem 1 are satisfied with $\mathcal{A}_1$, $\mathcal{A}_2$, V_1, V_2, $\underline{\alpha}_1, \overline{\alpha}_1, \underline{\alpha}_2, \overline{\alpha}_2$ in Proposition 1, χ_1, χ_2 in (13), $\alpha_1 = (1-\nu)(1-\sigma)\alpha_V$ for some $\nu \in (0,1)$, and any $\alpha_2 \in \mathcal{K}_\infty$. □

Proof. We only need to prove that items (ii) and (iv) of Theorem 1 are verified in view of Proposition 1.

Let $q \in C$ and consider the x–system with $V_1 = V$. In view of item (ii) of SA1 and (12), $\langle \nabla V_1(x), f(x,e)\rangle \leq -(1-\sigma)\alpha_V(|x|) + \delta\gamma(|e|)$. Hence, $\delta\gamma(|e|) \leq \nu(1-\sigma)\alpha_V(|x|)$, with ν as above, implies $\langle \nabla V_1(x), f(x,e)\rangle \leq -(1-\nu)(1-\sigma)\alpha_V(|x|) =: -\alpha_1(|x|)$. In view of item (i) of SA1 and using $V_2(e) = |e|^2$, $\delta\gamma(\sqrt{V_2(e)}) \leq \nu(1-\sigma)\alpha_V \circ \overline{\alpha}_V^{-1}(V_1(x))$ implies $\delta\gamma(|e|) \leq \nu(1-\sigma)\alpha_V(|x|)$ and thus item (ii) of Proposition 1 holds with χ_1 in (13).

Consider the e-system now and $q \in C$. In view of (12) and item (i) of SA1, $\gamma(|e|) \leq \dfrac{\sigma}{1-\delta}\alpha_V \circ \underline{\alpha}_V^{-1}(V_1(x))$ and, since $V_2(e) = |e|^2$, $V_2(e) \leq \left[\gamma^{-1} \circ \left(\frac{\sigma}{1-\delta}\alpha_V \circ \underline{\alpha}_V^{-1}(V_1(x))\right)\right]^2$. As a result, in view of the definition of χ_2 in (13) and since $\gamma^{-1} \in \mathcal{K}_\infty$, $V_2(e) > \chi_2(V_1(x)) = \left[\gamma^{-1} \circ (1+\varepsilon)\frac{\sigma}{1-\delta}\alpha_V \circ \underline{\alpha}_V^{-1}(V_1(x))\right]^2$, which contradicts the fact that $q \in C$. We then follow the same reasoning as in the proof of Proposition 1.

The last property we need to check is the small-gain condition. Since $\delta \in \left(0, \frac{1-\sigma}{1-\sigma+\sigma c}\right)$, there exist $\nu \in (0,1)$ sufficiently close to 1 and $\varepsilon > 0$ sufficiently small such that

$$\frac{\delta}{1-\delta}(1+\varepsilon)\frac{\sigma}{\nu(1-\sigma)} < \frac{1}{c}, \tag{48}$$

[2] In all the proofs of the appendix, when we write (with some abuse of notation) x^+, e^+, η^+ for vectors x, e, η, we mean $x, 0, \ell_2(q)$ in view of (8), respectively.

where c comes from Assumption 2. From (13), for any $s \geq 0$,

$$\chi_1 \circ \chi_2(s) = \overline{\alpha}_V \circ \alpha_V^{-1} \left(\frac{\delta(1+\varepsilon)}{\nu(1-\sigma)} \frac{\sigma}{1-\delta} \alpha_V \circ \underline{\alpha}_V^{-1}(s) \right).$$

Let $s > 0$. From (48), and since the involved functions are strictly increasing, we derive $\chi_1 \circ \chi_2(s) < \overline{\alpha}_V \circ \alpha_V^{-1}\left(\frac{1}{c}\alpha_V \circ \underline{\alpha}_V^{-1}(s)\right)$. Invoking Assumption 2, we derive $\chi_1 \circ \chi_2(s) < \overline{\alpha}_V \circ \alpha_V^{-1}\left(\alpha_V \circ \overline{\alpha}_V^{-1}(s)\right) = s$. Hence, the small-gain condition is verified, which concludes the proof. ■

B. Dynamic triggering strategy in Section V-B

The next proposition shows that the dynamic triggering strategy satisfies the conditions of Theorem 1 with $\theta > 0$.

Proposition 3: Consider system $\mathcal{H}_B$ with $\theta > 0$. Then, the conditions of Theorem 1 are satisfied with $\mathcal{A}_1$ and $\mathcal{A}_2$ in (17), $V_1(x,\eta) = V(x) + \eta$ for any $(x,\eta) \in \mathbb{R}^{n_x} \times \mathbb{R}$, $V_2(e) = |e|^2$ for any $e \in \mathbb{R}^{n_e}$, $\underline{\alpha}_1(s) = \min\{\underline{\alpha}_V(s/2), s/2\}$, $\overline{\alpha}_1(s) = \overline{\alpha}_V(s) + s$, $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$, $\alpha_1(s) = \min\{(1-\sigma)\alpha_V\left(\frac{s}{2}\right), \beta\left(\frac{s}{2}\right)\}$, any $\alpha_2 \in \mathcal{K}_\infty$, and χ_1, χ_2 in (18), for any $s \geq 0$. □

Proof. We first study the (x,η)–system with the Lyapunov function $V_1(x,\eta) = V(x) + \eta$. Item (i) of Theorem 1 is satisfied in view of item (i) of SA1 and [47, Remark 2.3]. Let $q \in C$. In view of item (ii) of SA1,

$$\begin{aligned}
&\langle \nabla V_1(x,\eta), \mathcal{F}_1(x,e,\eta)\rangle \\
&\qquad \leq -(1-\sigma)\alpha_V(|x|) - \beta(\eta) \\
&\qquad \leq -\min\{(1-\sigma)\alpha_V(|(x,\eta)|/2), \beta(|(x,\eta)|/2)\} \\
&\qquad =: -\alpha_1(|(x,\eta)|),
\end{aligned}$$

where the last inequality follows from [47, Remark 2.3]. Consequently, item (ii) of Theorem 1 holds with $\chi_1(s) = 0$ for any $s \geq 0$. Let $q \in D$. We have that $V_1(x^+,\eta^+) = V_1(x,\eta)$ in view of (7) and (14). We now consider the e-system with $V_2(e) = |e|^2$. Items (i) and (iii) of Theorem 1 hold in view of Proposition 1. We now verify item (ii) of Theorem 1. Let $q \in C$. From (i) of SA1, (15) and the definition of V_1, $\gamma(|e|) \leq \frac{1}{\theta}\eta + \sigma\alpha_V(|x|) \leq \overline{\chi}_2(V_1(x,\eta))$, with $\overline{\chi}_2$ defined after (18). This implies, by definition of V_2, that $V_2(e) \leq \left[\gamma^{-1}\left(\overline{\chi}_2(V_1(x,\eta))\right)\right]^2$. Hence, $V_2(e) > \left[\gamma^{-1}\left((1+\varepsilon)\overline{\chi}_2(V_1(x,\eta))\right)\right]^2 =: \chi_2(V_1(x,\eta))$, which contradicts the fact that $q \in C$. Thus, item (ii) of Theorem 1 holds by following the same reasoning as in the proof of Proposition 1. Lastly, the small-gain condition holds trivially as $\chi_1 = 0$. ■

The next proposition shows that the dynamic triggering strategy satisfies the conditions of Theorem 1 with $\theta = 0$.

Proposition 4: Consider system $\mathcal{H}_B$ with $\theta = 0$. Then, the conditions of Theorem 1 are satisfied with $\mathcal{A}_1$ and $\mathcal{A}_2$ in (17), $V_1, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1$ as per Proposition 3, and $V_2(e) = |e|^2_{\mathcal{A}_2}$ for any $e \in \mathbb{R}^{n_e}$, any $\underline{\alpha}_2(s), \overline{\alpha}_2(s) \in \mathcal{K}_\infty$ satisfying $\underline{\alpha}_2(s) \leq \overline{\alpha}_2(s)$ for $s \geq 0$, any $\alpha_2 \in \mathcal{K}_\infty$, and $\chi_1(s) = \chi_2(s) = 0$, $s \geq 0$. □

Proof: Consider the (x,η)–system. Items (i), (ii), and (iii) in Theorem 1 for the (x,η)–system follow exactly as in the proof of Proposition 3. Now consider the e–system with $V_2(e) = |e|^2_{\mathcal{A}_2}$. Since $\theta = 0$ implies $\mathcal{A}_2 = \mathbb{R}^{n_e}$, then $V_2(e) = |e|^2_{\mathcal{A}_2} = 0$ for all $e \in \mathbb{R}^{n_e}$. Therefore, item (i) follows for any $\underline{\alpha}_2(s), \overline{\alpha}_2(s) \in \mathcal{K}_\infty$ satisfying $\underline{\alpha}_2(s) \leq \overline{\alpha}_2(s)$, for any $s \geq 0$. Moreover, items (ii) and (iii) trivially follow. Lastly, the small-gain condition in item (iv) follows straightforwardly given $\chi_1 = \chi_2 = 0$, concluding the proof. ■.

The below lemma shows that all maximal solutions to system $\mathcal{H}_B$ are complete.

Lemma 1: Each maximal solution to $\mathcal{H}_B$ is complete. □

Proof. Let $q = (x,e,\eta) \in C \setminus D$. If $\eta > 0$, then there exists a neighbourhood U of q such that for any $q' = (x',e',\eta') \in U$, $\eta' > 0$. Then, for any $q' \in U$, $\mathcal{F}(q') \cap T_C(q') \neq \emptyset$, since $T_C(q') = \mathbb{R}^{n_q}$ and $\mathcal{F}(q') \in T_C(q')$, where $\mathcal{F}(q') = (f(x',e'), g(x',e'), -\beta(\eta') + \sigma\alpha_V(|x'|) - \gamma(|e'|))$. On the other hand, if $\eta = 0$, necessarily $\sigma\alpha_V(|x|) - \gamma(|e|) > 0$, as $q \in C \setminus D$. Hence, there exists a neighbourhood U of q such that for any $q' = (x',e',\eta') \in U$, $\sigma\alpha_V(|x'|) - \gamma(|e'|) > 0$. Then, $\mathcal{F}(q') \cap T_C(q') \neq \emptyset$ for any $q' \in U$, since $T_C(q') = \mathbb{R}^{n_x+n_e} \times \mathbb{R}_{\geq 0}$ and $\mathcal{F}(q') = (f(x',e'), g(x',e'), 0) \in T_C(q')$. Moreover, note that $\mathcal{G}(D) \subset C \cup D$, where $\mathcal{G}(q) = (x, 0, \eta)$. Hence, any maximal solution is complete in view of [22, Proposition 6.10]. ■

The next proposition applies to the redesigned strategy proposed in Section V-B.3.

Proposition 5: Consider system $\mathcal{H}_B^r$. Then, the conditions of Theorem 1 are satisfied with $\mathcal{A}_1$, $\mathcal{A}_2$, V_1, V_2, $\underline{\alpha}_1, \overline{\alpha}_1, \underline{\alpha}_2, \overline{\alpha}_2$ from Proposition 3, χ_1, χ_2 in (22), $\alpha_1 = (1-\nu)\tilde{\alpha}_1(s)$ for some $\nu \in (0,1)$ with $\tilde{\alpha}_1(s) := \min\{(1-\sigma)\alpha_V(s/2), \beta(s/2)\}$, and any $\alpha_2 \in \mathcal{K}_\infty$. □

Proof. We only need to prove items (ii) and (iv) of Theorem 1; the proofs of items (i) and (iii) follow as per Proposition 3.

Let $q \in C$ and consider the (x,η)–system with V_1 as given in Proposition 3. In view of SA1 and (19),

$$\begin{aligned}
&\langle \nabla V_1(x,\eta), \mathcal{F}_1(x,e,\eta)\rangle \\
&\qquad \leq -(1-\sigma)\alpha_V(|x|) - \beta(\eta) + \delta(\gamma(|e|)) \\
&\qquad \leq -\tilde{\alpha}_1(|(x,\eta)|) + \delta(\gamma(|e|)).
\end{aligned}$$

Note that if $V_1(x,\eta) \geq \chi_1(V_2(e))$ with χ_1 in (22), then $\delta(\gamma(|e|)) \leq \nu\tilde{\alpha}_1(|(x,\eta)|)$, with $\nu \in (0,1)$, and thus $\langle \nabla V_1(x,\eta), \mathcal{F}_1(x,e,\eta)\rangle \leq -\alpha_1(|(x,\eta)|)$, as desired.

Consider the e–system and let $q \in C$. We have that $(\mathbb{I} - \delta) \circ \gamma(|e|) \leq \frac{\eta}{\theta} + \sigma\alpha_V(|x|) \leq a_2(V_1(x,\eta))$ with $a_2(s) := \frac{s}{\theta} + \sigma\alpha_V(\underline{\alpha}_V^{-1}(s))$. This implies that $V_2(e) \leq \left[\gamma^{-1} \circ (\mathbb{I} - \delta)^{-1} \circ a_2(V_1(x,\eta))\right]^2$ for $q \in C$. Now, from the definition of χ_2 in (22), if $V_2(e) > \chi_2(V_1(x,\eta))$, then $q \in C$ is contradicted, so the right-hand side of item (ii) of Theorem 1 vacuously holds in this case for any $\alpha_2 \in \mathcal{K}_\infty$. The case where $V_2(e) = \chi_2(V_1(x,\eta))$ can only occur at the origin, like in the proof of Proposition 1, which also leads to the satisfaction of the right-hand side of item (ii) of Theorem 1.

We are left with proving that the small-gain condition holds. With the definition of χ_1, χ_2 in (22), we have that $\chi_1 \circ \chi_2(s) = \overline{\alpha}_1 \circ \tilde{\alpha}_1^{-1}\left(\frac{1}{\nu}\delta\left((1+\varepsilon)(\mathbb{I} - \delta)^{-1} \circ a_2(s)\right)\right)$. Therefore, we need $\delta \in \mathcal{K}_\infty$ to satisfy

$$\delta(s) < \nu\tilde{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ a_2^{-1} \circ (\mathbb{I} - \delta) \circ \frac{1}{1+\varepsilon}s, \qquad (49)$$

so that $\chi_1 \circ \chi_2(s) < s$ for all $s > 0$. Let $\varrho := \tilde{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ a_2^{-1}$. Then, (49) implies $\varrho^{-1}((1/\nu)\delta(s)) + \delta(s/(1+\varepsilon)) < s/(1+\varepsilon)$, which holds if we have $\varrho^{-1}((1/\nu)\delta(s)) < \frac{s}{2(1+\varepsilon)}$ and $\delta(s/(1+\varepsilon)) < \frac{s}{2(1+\varepsilon)}$. In other words, $\delta(s) <$

$\min\left\{\nu\varrho\left(\frac{s}{2(1+\varepsilon)}\right), s/2\right\}$, for any $s \geq 0$, ensures the small condition holds, and the proof is thus complete. ■

C. Fixed threshold strategy in Section V-C

The next proposition shows the fixed threshold strategy in Section V-C.2 satisfies the conditions of Theorem 1.

Proposition 6: Consider system $\mathcal{H}_C$. Then, the conditions of Theorem 1 are satisfied with $\mathcal{A}_1$ and $\mathcal{A}_2$ as in (24), $V_1(x) = \max\{V(x) - \overline{\alpha}_V \circ \alpha_V^{-1}(2d), 0\}$, $V_2(e) = |e|^2_{\mathcal{A}_2}$ for any $x \in \mathbb{R}^{n_x}$ and $e \in \mathbb{R}^{n_e}$, some $\underline{\alpha}_1, \overline{\alpha}_1 \in \mathcal{K}_\infty$, $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$, $\chi_1 = \chi_2 = 0$, $\alpha_1(s) = \frac{1}{2}\alpha_V \circ \overline{\alpha}_V^{-1} \circ \underline{\alpha}_1(s)$, for any $s \geq 0$, any $\alpha_2 \in \mathcal{K}_\infty$, with $\alpha_V, \overline{\alpha}_V, \underline{\alpha}_V$ from Assumption 1. □

Proof. We consider the x–system and show that the conditions of Theorem 1 hold. Item (i) of Theorem 1 holds by [22, p.54]. Let $q \in C$. The definition of set C in (23) implies that $\gamma(|e|) \leq d$, i.e., $e \in \mathcal{A}_2$, which with item (ii) of SA1, leads to $\langle \nabla V_1(x), f(x, e)\rangle \leq -\alpha_V(|x|) + d$. When $q \in C \setminus \mathcal{A}$, necessarily $x \notin \mathcal{A}_1$ as $e \in \mathcal{A}_2$. Thus, $V_1(x) = V(x) - \overline{\alpha}_V \circ \alpha_V^{-1}(2d)$. On the other hand, $\frac{1}{2}\alpha_V \circ \overline{\alpha}_V^{-1}(V_1(x)) + d = \frac{1}{2}\Big(\alpha_V \circ \overline{\alpha}_V^{-1}(V_1(x)) + \alpha_V \circ \overline{\alpha}_V^{-1} \circ \overline{\alpha}_V \circ \alpha_V^{-1}(2d)\Big) \leq \alpha_V \circ \overline{\alpha}_V^{-1}(V_1(x) + \overline{\alpha}_V \circ \alpha_V^{-1}(2d)) = \alpha_V \circ \overline{\alpha}_V^{-1}(V(x))$. Hence, in view of item (i) of SA1,

$$
\begin{aligned}
\langle \nabla V_1(x), f(x, e)\rangle &\leq -\alpha_V \circ \overline{\alpha}_V^{-1}(V(x)) + d \\
&\leq -\frac{1}{2}\alpha_V \circ \overline{\alpha}_V^{-1}(V_1(x)) \\
&\leq -\frac{1}{2}\alpha_V \circ \overline{\alpha}_V^{-1} \circ \underline{\alpha}_1(|x|_{\mathcal{A}_1}) \\
&= -\alpha_1(|x|_{\mathcal{A}_1}). \qquad (50)
\end{aligned}
$$

When $q \in \mathcal{A}$, (50) also holds since $\langle \nabla V_1(x), f(x, e)\rangle = 0 = -\alpha_1(|x|_{\mathcal{A}_1})$ in this case. Then, item (ii) of Theorem 1 holds with $\chi_1 = 0$. Item (iii) of Theorem 1 holds as x does not change at jumps.

We now consider the e–system and $V_2(e) = |e|^2_{\mathcal{A}_2}$. V_2 satisfies item (i) of Theorem 1 with $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$ for any $s \geq 0$. Let $q \in C$. Note that V_2 is locally Lipschitz, see [48, Lemma 1.2]. We distinguish two cases: (i) $\gamma(|e|) < d$ and (ii) $\gamma(|e|) = d$. For case (i), for any $\alpha_2 \in \mathcal{K}_\infty$, $V_2^\circ(e; g(x, e)) = 0 = -\alpha_2(|e|_{\mathcal{A}_2})$ holds as $|e|_{\mathcal{A}_2} = 0$. Consider case (ii), which means that $q \in C \cap D$, and consider $\mathcal{F}(q) \in T_C(q)$, $q := (x, e)$. [13, Lemma 4] shows that $0 = \big\langle \nabla d, f(x, e)\big\rangle \geq \big\langle \nabla\gamma(|e|), g(x, e)\big\rangle$. Then, we have that item (ii) of Theorem 1 holds with $\chi_2(s) = 0$ for any $s \geq 0$, in view of the definition of V_2 and γ is of class-$\mathcal{K}_\infty$. Item (iii) holds as per the other proofs in the Appendix. Lastly, item (iv) holds as $\chi_1 = \chi_2 = 0$, completing the proof. ■

D. Decreasing threshold on network-induced error in Section V-D

The next proposition shows the decreasing threshold strategy on the network-induced error satisfies conditions of Theorem 1.

Proposition 7: Consider system $\mathcal{H}_D$. Then, all conditions of Theorem 1 are satisfied with $\mathcal{A}_1, \mathcal{A}_2$ as in (27), $V_1(x, \eta) = \max\{V(x) - \overline{\alpha}_V \circ \alpha_V^{-1}(2d), 0\} + \eta$ for any $(x, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0}$, $V_2(e) = |e|^2_{\mathcal{A}_2}$ for any $e \in \mathbb{R}^{n_e}$, some $\underline{\alpha}_1, \overline{\alpha}_1 \in \mathcal{K}_\infty$, $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$, $\alpha_1(s) := \max\big\{\min\{(1-\sigma)\beta(s/2), (1/2)\alpha_V \circ \overline{\alpha}_V^{-1}(\underline{\alpha}_1(s/2)/2)\}, \beta(\underline{\alpha}_1(s))\big\}$, any $\alpha_2 \in \mathcal{K}_\infty$, and χ_1, χ_2 as per (28). □

Proof: Consider the (x, η)–system with $V_1(x, \eta) = \max\{V(x) - \overline{\alpha}_V \circ \alpha_V^{-1}(2d), 0\} + \eta$. Since V_1 is continuous, zero only on $\mathcal{A}_1$, and radially unbounded with respect to $\mathcal{A}_1$, then item (i) of Theorem 1 holds by [22, p.54]. Let $q \in C$. We consider two cases: (i) $(x, \eta) \notin \mathcal{A}_1$ and (ii) $(x, \eta) \in \mathcal{A}_1$. Consider case (i), which holds either for (ia) $V(x) > \overline{\alpha}_V \circ \alpha_V^{-1}(2d)$ and $\eta \geq 0$, or (ib) $V(x) \leq \overline{\alpha}_V \circ \alpha_V^{-1}(2d)$ and $\eta > 0$. For (ia), we have that, from SA1, and since $q \in C$,

$$
\begin{aligned}
&\langle \nabla V_1(x, \eta), \mathcal{F}_1(x, e, \eta)\rangle \\
&\quad \leq -\alpha_V(|x|) + \gamma(|e|) - \beta(\eta) \\
&\quad \leq -(1-\sigma)\beta(\eta) - (\alpha_V \circ \overline{\alpha}_V^{-1}(V(x)) - d) \\
&\quad < -(1-\sigma)\beta(\eta) \\
&\qquad\quad - \tfrac{1}{2}\alpha_V \circ \overline{\alpha}_V^{-1}\left(V(x) - \overline{\alpha}_V \circ \alpha_V^{-1}(2d)\right) \\
&\quad \leq -(1-\sigma)\beta(\eta) - \tfrac{1}{2}\alpha_V \circ \overline{\alpha}_V^{-1}\left(V_1(x, \eta) - \eta\right),
\end{aligned}
$$

where the latter follows from $V(x) > \overline{\alpha}_V \circ \alpha_V^{-1}(2d)$ and the definition of V_1. By writing $V_1 = V_1/2 + V_1/2$, we have from the above that, if $\eta \leq V_1(x, \eta)/2$, then $\langle \nabla V_1(x, \eta), \mathcal{F}_1(x, e, \eta)\rangle \leq -(1-\sigma)\beta(\eta) - \frac{1}{2}\alpha_V \circ \overline{\alpha}_V^{-1}(V_1(x, \eta)/2) \leq -\min\{(1-\sigma)\beta(|(x, \eta)|_{\mathcal{A}_1}/2), (1/2)\alpha_V \circ \overline{\alpha}_V^{-1}(\underline{\alpha}_1(|(x, \eta)|_{\mathcal{A}_1}/2)/2)\}$, where we used [47, Remark 2.3]. On the other hand, if $\eta > V_1(x, \eta)/2$, then $\langle \nabla V_1(x, \eta), \mathcal{F}_1(x, e, \eta)\rangle < -(1-\sigma)\beta(V_1(x, \eta)/2) \leq -(1-\sigma)\beta(\underline{\alpha}_1(|(x, \eta)|_{\mathcal{A}_1})/2)$ since $\sigma \in (0, 1)$ and β is non-decreasing. Therefore, for case (ia), $\langle \nabla V_1(x, \eta), \mathcal{F}_1(x, e, \eta)\rangle \leq -\max\{\min\{(1-\sigma)\beta(|(x, \eta)|_{\mathcal{A}_1}/2), (1/2)\alpha_V \circ \overline{\alpha}_V^{-1}(\underline{\alpha}_1(|(x, \eta)|_{\mathcal{A}_1}/2)/2)\}, (1-\sigma)\beta(\underline{\alpha}_1(|(x, \eta)|_{\mathcal{A}_1}/2))\}$. Now consider case (ib), where $V(x) \leq \overline{\alpha}_V \circ \alpha_V^{-1}(2d)$ and $\eta > 0$. Then, $V_1(x, \eta) = \eta$ and thus $\langle \nabla V_1(x, \eta), \mathcal{F}_1(x, e, \eta)\rangle = -\beta(\eta) = -\beta(V_1(x, \eta)) \leq -\beta(\underline{\alpha}_1(|(x, \eta)|_{\mathcal{A}_1}))$. Now consider case (ii), i.e., $(x, \eta) \in \mathcal{A}_1$. In this case we have that $V_1(x, \eta) = 0$, and thus $\langle \nabla V_1(x, \eta), \mathcal{F}_1(x, e, \eta)\rangle = 0$. Consequently, from all the above cases, item (ii) of Theorem 1 holds with $\chi_1 = 0$, and $\alpha_1(s) := -\max\big\{\min\{(1-\sigma)\beta(s/2), (1/2)\alpha_V \circ \overline{\alpha}_V^{-1}(\underline{\alpha}_1(s/2)/2)\}, \beta(\underline{\alpha}_1(s))\big\}$. Item (iii) holds in view of (7) and (25), i.e., $V_1(x^+, \eta^+) \leq V_1(x, \eta)$.

Now consider the e–system with $V_2(e) = |e|^2_{\mathcal{A}_2}$. V_2 satisfies item (i) of Theorem 1 with $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$, for any $s \geq 0$. Let $q \in C$. We distinguish two cases: (i) $e \notin \mathcal{A}_2$, and case (ii) $e \in \mathcal{A}_2$. Consider case (i), i.e., $\gamma(|e|) > 2d$. Since $q \in C$, we have that $\gamma(|e|) \leq \sigma\beta(\eta) + d \leq \sigma\beta(V_1(x, \eta)) + d < \sigma\beta(V_1(x, \eta)) + \frac{1}{2}\gamma(|e|) < 2\sigma\beta(V_1(x, \eta))$. Then, when $V_2(e) = |e|^2_{\mathcal{A}_2} \geq \chi_2(V_1(x, \eta))$, we have that, since $\varepsilon > 0$, $\gamma(|e|) \geq \gamma(|e|_{\mathcal{A}_2}) > 2\sigma\beta(V_1(x, \eta))$, which contradicts $q \in C$. Therefore, for any $\alpha_2 \in \mathcal{K}_\infty$ and χ_2 in (28), $V_2(e) \geq \chi_2(V_1(x)) \Rightarrow V_2^\circ(e; g(x, e)) \leq -\alpha_2(|e|_{\mathcal{A}_2})$ vacuously holds. Now consider case (ii). Since here $e \in \mathcal{A}_2$, we have $|e|_{\mathcal{A}_2} = 0$, and thus $V_2^\circ(e; g(x, e)) = -\alpha_2(|e|_{\mathcal{A}_2}) = 0$ holds for any $\alpha_2 \in \mathcal{K}_\infty$. Hence, item (ii) of Theorem 1 holds. Item (iii) holds since $V_2(e^+) = 0 \leq V_2(e)$. Lastly, the small-gain condition in item (iv) trivially holds since $\chi_1 = 0$, concluding the proof. ■

The next proposition relates to Section V-D.3.

Proposition 8: Consider system $\mathcal{H}_D^r$. Then, all conditions of Theorem 1 are satisfied with $\mathcal{A}_1$ and $\mathcal{A}_2$ as in (27), $V_1(x,\eta) = \max\{V(x) - \overline{\alpha}_V \circ \alpha_V^{-1}(2d), 0\} + \eta$ for any $(x,\eta) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0}$, $V_2(e) = |e|_{\mathcal{A}_2}^2$ for any $e \in \mathbb{R}^{n_e}$, some $\underline{\alpha}_1, \overline{\alpha}_1 \in \mathcal{K}_\infty$, $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$, $\alpha_1(s) := \max\big\{\min\{(1-\sigma)\beta(s/2), ((1-\nu)/2)\alpha_V \circ \overline{\alpha}_V^{-1}(\underline{\alpha}_1(s/2)/2)\}, (1-\nu)\beta(\underline{\alpha}_1(s))\big\}$, any $\alpha_2 \in \mathcal{K}_\infty$, and χ_1, χ_2 as per (30). □

Proof: We only analyse the (x,η)–system since the η–dynamics are the only change in the redesign, i.e., χ_2 remains unchanged. In addition, we only verify item (ii) in Theorem 1 since items (i) and (iii) follow exactly as in Proposition 7. Let $q \in C$ and we proceed using the same cases as in Proposition 7, i.e., for case (ia), if $\eta \leq V_1(x,\eta)/2$, then $\langle \nabla V_1(x,\eta), \mathcal{F}_1(x,e,\eta)\rangle \leq -(1-\sigma)\beta(\eta) - \frac{1}{2}\alpha_V \circ \overline{\alpha}_V^{-1}(V_1(x,\eta)/2) + \delta(\gamma(|e|_{\mathcal{A}_2}))$. Then, if $V_1 \geq 2\overline{\alpha}_V \circ \alpha_V^{-1} \circ \frac{2}{\nu}\delta(\gamma(\sqrt{V_2}))$, for some $\nu \in (0,1)$, then $\delta(\gamma(|e|_{\mathcal{A}_2})) \leq \frac{\nu}{2}\alpha_V \circ \overline{\alpha}_V^{-1}(V_1(x,\eta)/2)$, and $\langle \nabla V_1(x,\eta), \mathcal{F}_1(x,e,\eta)\rangle \leq -(1-\sigma)\beta(\eta) - \frac{(1-\nu)}{2}\alpha_V \circ \overline{\alpha}_V^{-1}(V_1(x,\eta)/2)$. Next, case (ib) when $\eta > V_1(x,\eta)/2$ implies that $\langle \nabla V_1(x,\eta), \mathcal{F}_1(x,e,\eta)\rangle \leq -(1-\sigma)\beta(V_1(x,\eta)/2) + \delta(\gamma(|e|_{\mathcal{A}_2}))$. Then, when $V_1 \geq 2\beta^{-1} \circ \frac{1}{\nu(1-\sigma)}\delta(\gamma(\sqrt{V_2}))$, we have $\delta(\gamma(|e|_{\mathcal{A}_2})) \leq \nu(1-\sigma)\beta(V_1(x,\eta)/2)$, and thus $\langle \nabla V_1(x,\eta), \mathcal{F}_1(x,e,\eta)\rangle \leq -(1-\nu)(1-\sigma)\beta(V_1(x,\eta)/2) \leq -(1-\nu)(1-\sigma)\beta(\underline{\alpha}_1(|(x,\eta)|_{\mathcal{A}_1})/2)$. Next, for case (ib), we get $\langle \nabla V_1(x,\eta), \mathcal{F}_1(x,e,\eta)\rangle = -\beta(V_1(x,\eta)) + \delta(\gamma(|e|_{\mathcal{A}_2}))$. Then, if $V_1 \geq \beta^{-1} \circ \frac{1}{\nu}\delta(\gamma(\sqrt{V_2}))$, we get $\delta(\gamma(|e|_{\mathcal{A}_2})) \leq \nu\beta(V_1(x,\eta))$ and $\langle \nabla V_1(x,\eta), \mathcal{F}_1(x,e,\eta)\rangle \leq -(1-\nu)\beta(V_1(x,\eta)) \leq -(1-\nu)\beta(\underline{\alpha}_1(|(x,\eta)|_{\mathcal{A}_1}))$. Lastly, case (ii) follows exactly as in Proposition 7. From all the above, we note that item (ii) of Theorem 1 is verified with χ_1 in (30).

Lastly, we verify the small-gain condition. Using (28) we have that $\chi_1 \circ \chi_2(s) = \max\Big\{2\overline{\alpha}_V \circ \alpha_V^{-1} \circ \frac{2}{\nu}\delta(2(1+\varepsilon)\sigma\beta(s)), 2\beta^{-1} \circ \frac{1}{\nu(1-\sigma)}\delta(2(1+\varepsilon)\sigma\beta(s))\Big\}$. Then, as $\delta(s) < \min\Big\{\frac{\nu}{2}\alpha_V \circ \overline{\alpha}_V^{-1} \circ \frac{1}{2}\beta^{-1}\left(\frac{s}{2(1+\varepsilon)\sigma}\right), \nu(1-\sigma)\beta \circ \frac{1}{2}\beta^{-1}\left(\frac{s}{2(1+\varepsilon)\sigma}\right)\Big\}$, $\chi_1 \circ \chi_2(s) < s$ for any $s > 0$. ■

E. Decreasing threshold on V strategy in Section V-E

The following proposition is for the triggering rule of decreasing threshold on V in Section V-E.2.

Proposition 9: Consider system $\mathcal{H}_E$. Then, the conditions of Theorem 1 hold with $\mathcal{A}_1$, $\mathcal{A}_2$ in (36), $V_1(x,\eta_1,\eta_2) = \max\{V(x), \frac{1}{2}(\eta_1 + \eta_2)\}$, $V_2(e) = |e|_{\mathcal{A}_2}^2$ for any $(x,\eta_1,\eta_2) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and $e \in \mathbb{R}^{n_e}$, $\underline{\alpha}_1(s) = \min\{(1/2)\underline{\alpha}_V(s/2), s/4\}$, $\overline{\alpha}_1(s) = \overline{\alpha}_V(s) + s$, any $\underline{\alpha}_2, \overline{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$ satisfying $\underline{\alpha}_2(s) \leq \overline{\alpha}_2(s)$, $\alpha_1(s) = \frac{1}{2}\sigma\tilde{\alpha}_V(\underline{\alpha}_1(s))$, $s \geq 0$, $\tilde{\alpha}_V$ as per (31), and $\chi_1 = \chi_2 = 0$. □

Proof: Consider the (x,η_1,η_2)–system. V_1 satisfies item (i) of Theorem 1 in view of item (i) of SA1, and from [47, Remarks 2.2–2.3]. Let $q \in C$ and note that here $\eta_2 \geq \eta_1$. Therefore, $\eta_2 \geq \frac{1}{2}(\eta_2 + \eta_1)$, and from (35) we can write $\beta(\eta_1,\eta_2) \geq \sigma\tilde{\alpha}_V(\eta_2) \geq \sigma\tilde{\alpha}_V(\frac{1}{2}(\eta_2 + \eta_1))$. Moreover, since $\eta_2 \geq \eta_1 \geq V(x)$, then $V_1(x,\eta_1,\eta_2) = \frac{1}{2}(\eta_1 + \eta_2)$. Consequently, from all the above we get $\langle \nabla V_1(x,\eta_1,\eta_2), \mathcal{F}_1(q)\rangle = -\frac{1}{2}\beta(\eta_1,\eta_2) \leq -(1/2)\sigma\tilde{\alpha}_V(V_1(x,\eta_1,\eta_2)) \leq -(1/2)\sigma\tilde{\alpha}_V(\underline{\alpha}_1(|(x,\eta_1,\eta_2)|))$. Then, item (ii) of Theorem 1 holds with $\chi_1 = 0$.

Let $q \in D$. $V_1(x^+,\eta_1^+,\eta_2^+) = \max\{V(x^+), \frac{1}{2}\eta_1^+ + \frac{1}{2}\eta_2^+\} = V(x) \leq V_1(x,\eta_1,\eta_2)$, and thus item (iii) of Theorem 1 holds.

Consider the e–system now. It is straightforward that item (i) of Theorem 1 holds with any $\underline{\alpha}_2, \overline{\alpha}_2 \in \mathcal{K}_\infty$ satisfying $\underline{\alpha}_2(s) \leq \overline{\alpha}_2(s)$. Since $\mathcal{A}_2 = \mathbb{R}^{n_e}$ when $q \in C$, then $|e|_{\mathcal{A}_2} = 0$ and $\langle \nabla V_2(e), \mathcal{F}_2(q)\rangle = 0 = -\alpha_2(|e|_{\mathcal{A}_2})$ for any $\alpha_2 \in \mathcal{K}_\infty$, and hence, item (ii) of Theorem 1 holds. Item (iii) holds as $V_2(e^+) = 0$. Lastly, the small-gain condition holds as $\chi_1 = \chi_2 = 0$, concluding the proof. ■

Proposition 10: Consider system $\mathcal{H}_E^r$ with $\xi = 0$. Then, the conditions of Theorem 1 hold with $\mathcal{A}_1, \mathcal{A}_2, V_1, V_2, \underline{\alpha}_1, \overline{\alpha}_1, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \chi_1, \chi_2$ from Proposition 9, and $\alpha_1(s) = \nu\sigma\tilde{\alpha}_V(\underline{\alpha}_1(s))$, $s \geq 0$, $\tilde{\alpha}_V$ as per (31). □

Proof. We only verify item (ii) of Theorem 1 for the (x,η_1,η_2)–system since the other items follow as per Proposition 9. Particularly, here we have $\langle \nabla V_1(x,\eta_1,\eta_2), \mathcal{F}_1(q)\rangle = -\frac{1}{2}\beta(\eta_1,\eta_2) + \delta(|x|) \leq -\frac{1}{2}\sigma\tilde{\alpha}_V(\underline{\alpha}_1(|(x,\eta_1,\eta_2)|)) + \delta(|(x,\eta_1,\eta_2)|)$. The proof is complete by using the choice of δ in (41). ■

Proposition 11: Consider system $\mathcal{H}_E^r$ with $\xi = 1$. Then, the conditions of Theorem 1 hold with $\mathcal{A}_1, \mathcal{A}_2$ as per (40), $V_1, \underline{\alpha}_1, \overline{\alpha}_1$ from Proposition 9, $V_2(e) = |e|^2$, $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$, $\alpha_1(s) = \nu\sigma\tilde{\alpha}_V(\underline{\alpha}_1(s))$, any $\alpha_2 \in \mathcal{K}_\infty$, $s \geq 0$, and χ_1, χ_2 as per (42). □

Proof. We only verify items (ii) and (iv) of Theorem 1, since the other items follow similar to Proposition 9. Consider the (x,η_1,η_2)–system. Let $q \in C$, then, proceeding similarly to Proposition 9, we can write $\langle \nabla V_1(x,\eta_1,\eta_2), \mathcal{F}_1(q)\rangle = -\frac{1}{2}\beta(\eta_1,\eta_2) + \delta_1(|e|) \leq -\frac{1}{2}\sigma\tilde{\alpha}_V(V_1(x,\eta_1,\eta_2)) + \delta_1(|e|)$. Therefore, if $V_1(x,\eta_1,\eta_2) \geq \tilde{\alpha}_V^{-1} \circ \frac{1}{(1/2-\nu)\sigma}\delta_1(\sqrt{V_2(e)}) =: \chi_1(V_2(e))$, we have $\delta_1(|e|) \leq \sigma(1/2-\nu)\tilde{\alpha}_V(V_1(x,\eta_1,\eta_2))$, which in turn implies $\langle \nabla V_1(x,\eta_1,\eta_2), \mathcal{F}_1(q)\rangle \leq -\nu\sigma\tilde{\alpha}_V(V_1(x,\eta_1,\eta_2)) \leq -\nu\sigma\tilde{\alpha}_V(\underline{\alpha}_1(|(x,\eta_1,\eta_2)|))$.

Consider the e–system now. Let $q \in C$ and note that $\delta_2(|e|) \leq V(x) + \delta_2(|e|) \leq \eta_1 \leq V_1(x,\eta_1,\eta_2)$. Hence, in C, we have $V_2(e) = |e|^2 \leq [\delta_2^{-1}(V_1(x,\eta_1,\eta_2))]^2$. Hence, if $V_2(e) > \chi_2(V_1(x,\eta_1,\eta_2))$, with χ_2 in (42), then $q \in C$ is contradicted, so the right-hand side of item (ii) of Theorem 1 vacuously holds in this case for any $\alpha_2 \in \mathcal{K}_\infty$. The case where $V_2(e) = \chi_2(V_1(x,\eta_1,\eta_2))$ can only occur at the origin, like in the proof of Proposition 1, which leads to the satisfaction of item (ii) of Theorem 1.

Lastly, we design δ_1 so that the small-gain condition in item (iv) of Theorem 1 holds. By definition of χ_1 and χ_2 in (42), we have $\chi_1 \circ \chi_2(s) = \tilde{\alpha}_V^{-1} \circ \frac{1}{(1/2-\nu)\sigma}\delta_1((1+\varepsilon)\delta_2^{-1}(s))$. Then, as $\delta_1(s) < \sigma(1/2-\nu)\tilde{\alpha}_V \circ \delta_2\left(\frac{1}{1+\varepsilon}s\right)$, for some $\varepsilon > 0$ and $\delta_2 \in \mathcal{K}_\infty$, we have $\chi_1 \circ \chi_2(s) < s$ for any $s > 0$. ■

F. Combined strategy I in Section V-F

The next proposition applies to the combined triggering strategy from Section V-F.

Proposition 12: Consider system $\mathcal{H}_F$. Then, the conditions of Theorem 1 are satisfied with $\mathcal{A}_1$ and $\mathcal{A}_2$ in (44), $V_1(x,\eta) = V(x) + \eta$, $V_2(e) = |e|^2$ for any $x \in \mathbb{R}^{n_x}$, $\eta \in \mathbb{R}_{\geq 0}$ and $e \in \mathbb{R}^{n_e}$, $\underline{\alpha}_1(s) = \min\{\underline{\alpha}_V(s/2), s/2\}$, $\overline{\alpha}_1(s) = \overline{\alpha}_V(s) + s$,

$\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$, $\alpha_1(s) = (1-\sigma)\min\{\alpha_V(s/2), \beta(s/2)\}$, any $\alpha_2 \in \mathcal{K}_\infty$, and χ_1, χ_2 as per (45). □

Proof: Consider the (x, η)–system with $V_1(x, \eta) = V(x) + \eta$. Item (i) of Theorem 1 is satisfied as per Proposition 3. Let $q \in C$. Then, from SA1 and the definition of C, $\langle \nabla V_1(x, \eta), \mathcal{F}_1(x, e, \eta)\rangle \leq -(1-\sigma)(\alpha_V(|x|) + \beta(\eta)) \leq -\alpha_1(|(x, \eta)|)$, which implies item (ii) in Theorem 1 holds with $\chi_1 = 0$. Item (iii) is satisfied as x and η do not change at jumps. Consider the e–system with $V_2(e) = |e|^2$. Item (i) of Theorem 1 is satisfied with $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$ for all $s \geq 0$. Let $q \in C$, then $\gamma(|e|) \leq \sigma\alpha_V(|x|) + \sigma\beta(\eta) \leq \sigma[\alpha_V \circ \underline{\alpha}_V^{-1}(V_1(x, \eta)) + \beta(V_1(x, \eta))]$. Hence, if $V_2(e) = |e|^2 \geq \chi_2(V_1(x, \eta))$, then $\gamma(|e|) \geq (1+\varepsilon)\sigma[\alpha_V \circ \underline{\alpha}_V^{-1}(V_1(x, \eta)) + \beta(V_1(x, \eta))]$ which contradicts $q \in C$ since $\varepsilon > 0$, and thus item (ii) of Theorem 1 is satisfied using the arguments in the proof of Proposition 1 for any $\alpha_2 \in \mathcal{K}_\infty$. Lastly, item (iv) follows immediately since $\chi_1 = 0$, completing the proof. ■

Proposition 13: Consider system $\mathcal{H}_F^r$. Then, the conditions of Theorem 1 are satisfied with $\mathcal{A}_1$ and $\mathcal{A}_2$ in (44), $V_1(x, \eta) = V(x) + \eta$, $V_2(e) = |e|^2$ for any $x \in \mathbb{R}^{n_x}$, $\eta \in \mathbb{R}_{\geq 0}$ and $e \in \mathbb{R}^{n_e}$, $\underline{\alpha}_1(s) = \min\{\underline{\alpha}_V(s/2), s/2\}$, $\overline{\alpha}_1(s) = \overline{\alpha}_V(s) + s$, $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s^2$, $\alpha_1(s) = (1-\nu)(1-\sigma)\min\{\alpha_V(s/2), \beta(s/2)\}$, $\nu \in (0, 1)$, any $\alpha_2 \in \mathcal{K}_\infty$, and χ_1, χ_2 as per (47). □

Proof: We only focus on proving items (ii) and (iv) of Theorem 1 for the x–system, since the proof of other items and the e–system follow exactly as in Proposition 12. Let $q \in C$, then, from the proof of Proposition 12 we have that $\langle \nabla V_1(x, \eta), \mathcal{F}_1(x, e, \eta)\rangle \leq -\tilde{\alpha}_1(|(x, \eta)|) + \delta(\gamma(|e|))$, where $\tilde{\alpha}_1(s) := (1-\sigma)\min\{\alpha_V(s/2), \beta(s/2)\}$. Therefore, if $V_1(x, \eta) \geq \overline{\alpha}_1 \circ \tilde{\alpha}_1^{-1} \circ \frac{1}{\nu}\delta(\gamma(|e|))$ for some $\nu \in (0, 1)$, then $\delta(\gamma(|e|)) \leq \nu\tilde{\alpha}_1(|(x, \eta)|)$. Hence, $\langle \nabla V_1(x, \eta), \mathcal{F}_1(x, e, \eta)\rangle \leq -\alpha_1(|(x, \eta)|)$. Lastly, we design δ so that the small-gain condition in item (iv) is satisfied. We have that $\chi_1 \circ \chi_2 = \overline{\alpha}_1 \circ \tilde{\alpha}_1^{-1} \circ \frac{1}{\nu}\delta\left((1+\varepsilon)\sigma(\alpha_V \circ \underline{\alpha}_V^{-1} + \beta)\right)$. Then, since we design δ such that $\delta(s) < \nu\tilde{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ \hat{\alpha}_1^{-1} \circ \frac{1}{(1+\varepsilon)\sigma}s$, where $\hat{\alpha}_1(s) := \alpha_V \circ \underline{\alpha}_V^{-1}(s) + \beta(s)$, then $\chi_1 \circ \chi_2(s) < s$ for any $s > 0$, completing the proof. ■

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