

# Event-Triggered Distributed State Estimation with Randomly Occurring Uncertainties and Nonlinearities over Sensor Networks: A Delay-Fractioning Approach

Jun Hu<sup>a,b,c</sup>, Zidong Wang<sup>d,e</sup>, Jinling Liang<sup>a,e,\*</sup> and Hongli Dong<sup>f</sup>

## Abstract

This paper is concerned with the problem of event-triggered distributed state estimation for a class of discrete nonlinear stochastic systems with time-varying delays, randomly occurring uncertainties and randomly occurring nonlinearities. Both the uncertainties and nonlinearities enter into the system in a random way characterized by random variables obeying the Bernoulli distribution. An event-triggered scheme is introduced to reduce the number of excessive executions of the signal transmissions. The aim of this paper is to design a distributed state estimator such that the estimation error dynamics is asymptotically mean-square stable. By constructing a Lyapunov-Krasovskii functional and employing the delay-fractioning approach, sufficient conditions are established to guarantee the desired performance requirements and then the explicit form of the distributed estimator gains is parameterized. An illustrative example is finally provided to demonstrate the effectiveness of the developed distributed state estimation scheme with the event-triggered communication mechanism.

## Keywords

Sensor networks, distributed state estimation, randomly occurring uncertainties, randomly occurring nonlinearities, time-varying delay.

## I. INTRODUCTION

A sensor network is composed of a group of sensor nodes equipped with the communication infrastructure monitoring and collecting information at diverse locations [26]. During the past decade, the sensor networks have received an increasing research interest due to their successful applications in a variety of domains such as industrial automation, traffic and environmental monitoring, medical device monitoring, and wireless networks etc [6, 24, 31]. In terms of theoretical research, considerable effort has been devoted to the synchronization,

This work was supported in part by the National Natural Science Foundation of China under Grants 61329301, 61134009, 11301118, 61174136 and 61374127, the China Postdoctoral Science Foundation under Grant 2014M560376, Jiangsu Planned Projects for Postdoctoral Research Funds under Grant 1402004A, and the Alexander von Humboldt Foundation of Germany.

<sup>a</sup> Department of Mathematics, Southeast University, Nanjing 210096, China.

<sup>b</sup> Department of Applied Mathematics, Harbin University of Science and Technology, Harbin 150080, China.

<sup>c</sup> Department of Electrical and Computer Engineering, University of Kaiserslautern, Kaiserslautern 67663, Germany.

<sup>d</sup> Department of Computer Science, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom.

<sup>e</sup> Communication Systems and Networks (CSN) Research Group, Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia.

<sup>f</sup> College of Electrical and Information Engineering, Northeast Petroleum University, Daqing 163318, China.

\* Corresponding author. E-mail: jinlliang@gmail.com

estimation and filtering problems over sensor networks [7, 32, 34]. To be specific, the design of the centralized time-varying Kalman filtering has been studied in [30] over a wireless sensor network with correlated fading channels, where the stochastic stability of the developed filtering algorithm has been discussed. A stochastic sampled-data approach has been proposed in [35] to design the distributed  $H_\infty$  filtering in sensor networks, and a sufficient condition has been given to ensure the exponential mean-square stability of the resulting filtering error dynamics as well as the  $H_\infty$  performance requirement. In [38–40], the distributed synchronization problems have been addressed for dynamical networks with nonlinear disturbances and stochastic coupling.

As is well known, time delays are inevitable in various practical engineering systems including networked control systems and process control systems [9, 10, 15, 41]. The occurrence of time delays causes lasting impact on the desirable system performance and may even yield the instability of the controlled systems [17, 19, 44]. So far, a number of approaches have been developed to deal with the analysis and synthesis problems for time-delay systems. It should be pointed out that, so far, much effort has been made to design the state estimators for time-delay systems over sensor networks [18, 20, 21]. To mention a few, in [18], the Gaussian mixture Kalman particle filtering algorithm has been developed to cope with the Gaussian or non-Gaussian nature of the random network delays. The distributed state estimation problems have been investigated in [20, 21] for discrete nonlinear systems with time-delays and nonlinear disturbances.

On the other hand, note that the measured signals collected by the sensors are commonly transmitted over a shared communication channel in the networked systems [33, 43]. Since a network is typically of limited communication capacity, it would be significant yet challenging to avoid unnecessary waste of the communication and computation resources. In this regard, the event-triggering mechanism has recently gained particular research focus because of its capability of decreasing the executions of the signal transmissions while maintaining the admissible performance [8, 16, 22, 29, 37]. To be specific, the event-triggered filtering and fault detection problems have been investigated in [16, 22] for networked systems with communication delays. In [37], the optimal event-triggered approach to state estimation has been developed for linear discrete time systems where the multiple sensors provide the measurement updates according to the individual event-triggering conditions. It is worth mentioning that, however, little research attention has been paid on the distributed state estimation problem for time-delay systems with the event-triggered communication mechanism, and this constitutes one of the motivations of this paper.

Nonlinearities and uncertainties are arguably two of the most important kinds of complexities that have received considerable research attention in the past few decades [1, 4, 14, 23]. With the rapid development of the network technologies, it has now well recognized that both the nonlinearities and uncertainties may occur in a probabilistic way with certain types/intensity due primarily to the random changes of the network circumstances. As such, the randomly occurring uncertainties (ROUs) and the randomly occurring nonlinearities (RONs) should be properly taken care of when designing the control systems. In [45], sufficient conditions have been established to guarantee the asymptotic synchronization in the mean-square sense for discrete stochastic complex networks subject to RONs, multiple stochastic disturbances and mixed time delays. Very recently, in [11], a sliding mode control scheme has been developed for a class of discrete networked nonlinear systems with both ROUs and RONs. Up to now, despite its practical significance, the event-based distributed state estimation problem has not yet been tackled for discrete stochastic systems in the simultaneous presence of

ROUs and RONs, not to mention the case where the communication delays are also involved. It is, therefore, the purpose of this paper to shorten such a gap by initiating a study on the distributed state estimation problem with both ROUs and RONs by using the delay-fractioning approach [12,28] with hope to reduce the possible conservatism.

Motivated by the above discussions, in this paper, we aim to discuss the distributed state estimation problem for a class of discrete nonlinear stochastic systems with ROUs, RONs and time-varying delay over sensor networks. By constructing a Lyapunov-Krasovskii functional based on the delay-fractioning approach, sufficient conditions are given such that the resulting estimation error systems are asymptotically stable in the mean-square sense. Moreover, the derived explicit form of the distributed estimator gains can be easily solved by using the semi-definite programme approach. Finally, an illustrative example is provided to show the usefulness of the distributed estimation scheme. The main contributions of this paper lie in the following two aspects: (i) the delay-fractioning approach is, for the first time, introduced to deal with the event-based distributed state estimation problem; and (ii) intensive stochastic analysis is conducted to handle the ROUs, RONs and time-varying delay in a unified framework under the event-triggered communication mechanism.

The rest of this paper is organized as follows. In Section II, the problem addressed is formulated and some preliminaries are briefly introduced. By constructing the Lyapunov-Krasovskii functional based on the delay-fractioning approach, sufficient conditions are presented in Section III to ensure the asymptotical mean-square stability of the estimation error systems. Subsequently, the explicit form of the distributed estimator gain are derived that can be easily solved by using the semi-definite programme method. In Section IV, a numerical example is provided to illustrate the feasibility and effectiveness of the proposed estimation method. Finally, conclusions are drawn in Section V.

*Notations.* The notations used throughout the paper are standard except where otherwise stated. The superscript “ $T$ ” stands for matrix transposition.  $\mathbb{R}^n$  ( $\mathbb{R}^{n \times m}$ ) denote the  $n$ -dimensional Euclidean space and the set of all  $n \times m$  matrices, respectively. The notation  $P > 0$  ( $P \geq 0$ ) means that matrix  $P$  is real symmetric and positive definite (positive semi-definite).  $(\Omega, \mathcal{F}, \text{Prob})$  is a probability space, where  $\text{Prob}$ , the probability measure, has a total mass 1.  $\mathbb{E}\{x\}$  stands for the expectation of  $x$ .  $\otimes$  represents the Kronecker product.  $\mathbf{1}_N$  is the  $N$ -dimensional vector with all elements being 1.  $I$  and  $0$  denote the identity matrix and a zero matrix with appropriate dimensions, respectively.  $\text{diag}\{X_1, X_2, \dots, X_n\}$  stands for a block-diagonal matrix with matrices  $X_1, X_2, \dots, X_n$  on the diagonal.  $\|\cdot\|$  denotes the Euclidean norm of a vector or its induced norm of a matrix. In symmetric block matrices or long matrix expressions, we use a star ( $*$ ) to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## II. PROBLEM FORMULATION AND PRELIMINARIES

Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$  be a directed graph of order  $N$  with the set of nodes  $\mathcal{V} = \{1, 2, \dots, N\}$ , the set of edges  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ , and the weighted adjacency matrix  $\mathcal{L} = [l_{ij}]$  with nonnegative adjacency element  $l_{ij}$ . Here, the matrix  $\mathcal{L}$  characterizes the interconnection topology of the nodes and the ordered pair  $(i, j)$  denotes an edge of  $\mathcal{G}$ . The adjacency elements associated with the edges of the graph are positive, i.e.,  $l_{ij} > 0 \iff (i, j) \in \mathcal{E}$  which represents that there exists the information transmission from sensor  $j$  to sensor  $i$ . Also, assume that  $l_{ii} = 1$

for all  $i \in \mathcal{V}$ . The set of the neighbors of node  $i \in \mathcal{V}$  plus the node itself is denoted by  $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$ .

In this paper, we consider the following class of discrete nonlinear stochastic systems:

$$x_{k+1} = (A + \alpha_k \Delta A)x_k + A_d x_{k-d_k} + \beta_k B f(x_k) + D x_k \omega_k \quad (1)$$

where  $x_k \in \mathbb{R}^n$  is the state vector,  $f(x_k)$  is the known nonlinear function, and  $\omega_k$  is a one-dimensional, zero-mean Gaussian white noise sequence on a probability space  $(\Omega, \mathcal{F}, \text{Prob})$  with  $\mathbb{E}\{\omega_k^2\} = 1$ . The positive integer  $d_k$  describes the discrete time-varying delay satisfying  $d_m \leq d_k \leq d_M$ , where  $d_m$  and  $d_M$  are known positive integers representing the lower and upper bounds of  $d_k$ , respectively. The lower bound of delay  $d_m$  can always be described by  $d_m = \tau m$  with  $\tau$  and  $m$  being positive integers.  $x_s = \varphi_s$  ( $s = -d_M, -d_M + 1, \dots, 0$ ) is the initial condition.  $A$ ,  $A_d$ ,  $B$  and  $D$  are known real matrices with appropriate dimensions.

The real-valued matrix  $\Delta A$  stands for the norm-bounded parameter uncertainty:

$$\Delta A = HFM \quad (2)$$

where  $H$  and  $M$  are known real constant matrices, and  $F \in \mathbb{R}^{n_1 \times n_2}$  is an unknown matrix satisfying  $F^T F \leq I$ . The nonlinearity  $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies

$$\|f(x) - f(y)\| \leq \|\Phi(x - y)\|, \quad \forall x, y \in \mathbb{R}^n \quad (3)$$

with  $\Phi$  being a known matrix.

The random variables  $\alpha_k \in \mathbb{R}$  and  $\beta_k \in \mathbb{R}$  are Bernoulli distributed white noise sequences taking the values 0 or 1 by

$$\begin{aligned} \text{Prob}\{\alpha_k = 1\} &= \mathbb{E}\{\alpha_k\} = \bar{\alpha}, \quad \text{Prob}\{\alpha_k = 0\} = 1 - \bar{\alpha} \\ \text{Prob}\{\beta_k = 1\} &= \mathbb{E}\{\beta_k\} = \bar{\beta}, \quad \text{Prob}\{\beta_k = 0\} = 1 - \bar{\beta} \end{aligned} \quad (4)$$

with  $\bar{\alpha} \in [0, 1]$  and  $\bar{\beta} \in [0, 1]$  being known scalars. Here, we assume that  $\alpha_k$ ,  $\beta_k$  and  $\omega_k$  are mutually independent.

Assume that the sensors are distributed in the space according to certain rules. The states of the target plant are to be estimated based on the measurement outputs collected by sensors. For the  $i$ -th sensor, the measurement output is

$$y_{k,i} = C_i x_k, \quad (5)$$

where  $y_{k,i} \in \mathbb{R}^p$  is the output measured by sensor  $i$  from the target plant,  $C_i$  ( $i = 1, 2, \dots, N$ ) are known real matrices with appropriate dimensions.

To reduce the data transmission frequency, the following event generator functions  $\psi_i(\cdot, \cdot)$  ( $i = 1, 2, \dots, N$ ) are introduced

$$\psi_i(\phi_{k,i}, \delta_i) = \phi_{k,i}^T \Omega_i \phi_{k,i} - \delta_i r_{k,i}^T \Omega_i r_{k,i} \quad (6)$$

where  $r_{k,i} = y_{k,i} - C_i \hat{x}_{k,i}$  is the innovation sequence exchanged via the network with  $\hat{x}_{k,i} \in \mathbb{R}^n$  being the state estimation of the target plant in the  $i$ -th sensor node,  $\Omega_i > 0$ ,  $\delta_i \in [0, 1)$ , and  $\phi_{k,i} = r_{k',i} - r_{k,i}$  with  $r_{k',i}$  being the broadcast innovation at the latest event time.

The execution is triggered if

$$\psi_i(\phi_{k,i}, \delta_i) > 0 \quad (7)$$

holds. Hence, the sequence of the event-triggered instants  $0 = k_{0,i} \leq k_{1,i} \leq \dots \leq k_{k',i} \leq \dots$  is determined iteratively by

$$k_{k'+1,i} = \inf\{k \in \mathbb{N} | k > k_{k',i}, \psi_i(\phi_{k,i}, \delta_i) > 0\}, \quad (8)$$

where  $k_{0,i} = 0$  is the initial event-triggered instant.

*Remark 1:* In the extreme case, when  $\delta_i = 0$  ( $i = 1, 2, \dots, N$ ), we can see that  $\{k_{0,i}, k_{1,i}, k_{2,i}, \dots\} = \{0, 1, 2, \dots\}$ . It means that all measurements are transmitted to the side of the state estimator at each sampling instant. Then, the addressed event-triggered state estimation problem reduces to the traditional one.

In this paper, we construct the following event-based distributed state estimator for sensor node  $i$ :

$$\hat{x}_{k+1,i} = A\hat{x}_{k,i} + \bar{\beta}Bf(\hat{x}_{k,i}) + \sum_{j \in \mathcal{N}_i} l_{ij}K_{ij}r_{k_{k'},j} \quad (9)$$

where  $\hat{x}_{0,i} = 0$  and  $K_{ij}$  ( $j \in \mathcal{N}_i$ ) are the estimator gains on the sensor node  $i$  to be designed.

By letting  $\tilde{x}_{k,i} = x_k - \hat{x}_{k,i}$ , we have

$$\begin{aligned} \tilde{x}_{k+1,i} &= A\tilde{x}_{k,i} + \alpha_k \Delta A x_k + A_d x_{k-d_k} + \bar{\beta}B\tilde{f}(\tilde{x}_{k,i}) \\ &\quad + (\beta_k - \bar{\beta})Bf(x_k) + D x_k \omega_k - \sum_{j \in \mathcal{N}_i} l_{ij}K_{ij}(\phi_{k,j} + C_j \tilde{x}_{k,j}), \end{aligned} \quad (10)$$

where  $\tilde{f}(\tilde{x}_{k,i}) = f(x_k) - f(\hat{x}_{k,i})$ . For convenience of later developments, set

$$\begin{aligned} \tilde{x}_k &= \begin{bmatrix} \tilde{x}_{k,1}^T & \tilde{x}_{k,2}^T & \cdots & \tilde{x}_{k,N}^T \end{bmatrix}^T, \quad \tilde{A} = I_N \otimes A, \quad \Delta \tilde{A} = 1_N \otimes \Delta A, \\ \tilde{A}_d &= 1_N \otimes A_d, \quad \tilde{B}_1 = I_N \otimes B, \quad \tilde{B}_2 = 1_N \otimes B, \\ \tilde{C} &= \text{diag}\{C_1, C_2, \dots, C_N\}, \quad \tilde{D} = 1_N \otimes D, \\ \tilde{f}(\tilde{x}_k) &= \begin{bmatrix} \tilde{f}^T(\tilde{x}_{k,1}) & \tilde{f}^T(\tilde{x}_{k,2}) & \cdots & \tilde{f}^T(\tilde{x}_{k,N}) \end{bmatrix}^T, \\ \tilde{\phi}_k &= \begin{bmatrix} \phi_{k,1}^T & \phi_{k,2}^T & \cdots & \phi_{k,N}^T \end{bmatrix}^T, \end{aligned} \quad (11)$$

and

$$\tilde{K} = \begin{bmatrix} \tilde{K}_{ij} \end{bmatrix}_{N \times N} \quad \text{with} \quad \tilde{K}_{ij} = \begin{cases} l_{ij}K_{ij}, & i = 1, 2, \dots, N, \quad j \in \mathcal{N}_i \\ 0, & i = 1, 2, \dots, N, \quad j \notin \mathcal{N}_i \end{cases} \quad (12)$$

Noting that  $l_{ij} = 0$  if  $j \notin \mathcal{N}_i$ , it is easy to see that  $\tilde{K}$  is a sparse matrix in  $\mathcal{W}_{n \times p}$  with

$$\mathcal{W}_{n \times p} = \left\{ \tilde{U} = [U_{ij}] \in \mathbb{R}^{Nn \times Np} | U_{ij} \in \mathbb{R}^{n \times p}, U_{ij} = 0 \text{ if } j \notin \mathcal{N}_i \right\}. \quad (13)$$

Then, (10) can be rewritten by

$$\begin{aligned} \tilde{x}_{k+1} &= (\tilde{A} - \tilde{K}\tilde{C})\tilde{x}_k + \alpha_k \Delta \tilde{A} x_k + \tilde{A}_d x_{k-d_k} + \bar{\beta}\tilde{B}_1\tilde{f}(\tilde{x}_k) \\ &\quad + (\beta_k - \bar{\beta})\tilde{B}_2f(x_k) + \tilde{D}x_k\omega_k - \tilde{K}\tilde{\phi}_k. \end{aligned} \quad (14)$$

It follows from (3) and the definition of  $\tilde{f}(\tilde{x}_k)$  that

$$\|\tilde{f}(\tilde{x}_k)\| \leq \|\tilde{\Phi}\tilde{x}_k\| \quad (15)$$

with  $\tilde{\Phi} = I_N \otimes \Phi$ .

By denoting  $\eta_k = \begin{bmatrix} x_k^T & \tilde{x}_k^T \end{bmatrix}^T$ , we have

$$\begin{aligned} \eta_{k+1} &= (\mathcal{A} + \bar{\alpha}\Delta\mathcal{A})\eta_k + (\alpha_k - \bar{\alpha})\Delta\mathcal{A}\eta_k + \mathcal{A}_d\eta_{k-d_k} + \bar{\beta}\mathcal{B}_1F(\eta_k) \\ &\quad + (\beta_k - \bar{\beta})\mathcal{B}_2F(\eta_k) + \mathcal{D}\eta_k\omega_k - \bar{K}\tilde{\phi}_k, \end{aligned} \quad (16)$$

with

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} A & 0 \\ 0 & \tilde{A} - \tilde{K}\tilde{C} \end{bmatrix}, \quad \Delta\mathcal{A} = \begin{bmatrix} \Delta A & 0 \\ \Delta\tilde{A} & 0 \end{bmatrix}, \quad \mathcal{A}_d = \begin{bmatrix} A_d & 0 \\ \tilde{A}_d & 0 \end{bmatrix}, \\ \mathcal{B}_1 &= \begin{bmatrix} B & 0 \\ 0 & \tilde{B}_1 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} B & 0 \\ \tilde{B}_2 & 0 \end{bmatrix}, \quad F(\eta_k) = \begin{bmatrix} f(x_k) \\ \tilde{f}(\tilde{x}_k) \end{bmatrix}, \\ \mathcal{D} &= \begin{bmatrix} D & 0 \\ \tilde{D} & 0 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} 0 \\ \tilde{K} \end{bmatrix}. \end{aligned} \quad (17)$$

It is not difficult to test that

$$\|F(\eta_k)\| \leq \|\bar{\Phi}\eta_k\| \quad (18)$$

where  $\bar{\Phi} = \text{diag}\{\Phi, \tilde{\Phi}\}$  and  $\tilde{\Phi}$  is defined in (15).

To proceed, we introduce the following lemmas that will be used for further developments.

*Lemma 1:* [5] Given constant matrices  $\mathcal{S}_1$ ,  $\mathcal{S}_2$  and  $\mathcal{S}_3$  where  $\mathcal{S}_1 = \mathcal{S}_1^T$  and  $\mathcal{S}_2 = \mathcal{S}_2^T > 0$ . Then  $\mathcal{S}_1 + \mathcal{S}_3^T\mathcal{S}_2^{-1}\mathcal{S}_3 < 0$  if and only if

$$\begin{bmatrix} \mathcal{S}_1 & \mathcal{S}_3^T \\ * & -\mathcal{S}_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\mathcal{S}_2 & \mathcal{S}_3 \\ * & \mathcal{S}_1 \end{bmatrix} < 0.$$

*Lemma 2:* [42] Let  $Q = Q^T$ ,  $N$  and  $H$  be real matrices of appropriate dimensions. Then, for any  $F$  satisfying  $F^TF \leq I$ ,  $Q + NFH + H^TF^TN^T < 0$  if and only if there exists a scalar  $\varepsilon > 0$  such that  $Q + \varepsilon^{-1}NN^T + \varepsilon H^TH < 0$ , or equivalently,

$$\begin{bmatrix} Q & N & \varepsilon H^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0.$$

*Lemma 3:* [34] Let  $P = \text{diag}\{P_1, P_2, \dots, P_N\}$  with  $P_i \in \mathbb{R}^{n \times n}$  ( $i = 1, 2, \dots, N$ ) being invertible matrices. If  $X = PU$  for  $U \in \mathbb{R}^{nN \times pN}$ , then we have  $U \in \mathcal{W}_{n \times p} \iff X \in \mathcal{W}_{n \times p}$ .

The purpose of this paper is to design the event-based distributed state estimator of form (9) on each sensor node  $i$  such that the resulting estimation error system (16) is mean-square asymptotically stable irrespective of the time-varying delays, ROUs and RONs. To be specific, by using the delay-fractioning approach, we aim to design the estimator gains  $K_{ij}$  so as to ensure the asymptotical stability of the estimation error system in the mean-square sense.

## III. MAIN RESULTS

In this section, by constructing a Lyapunov-Krasovskii functional and using the delay-fractioning approach, a sufficient condition is established to ensure the mean-square asymptotical stability of the estimation error system. Subsequently, the explicit form of the distributed state estimator gains is provided.

*Theorem 1:* For given  $\bar{K}$  and a scalar  $\gamma \in (0, 1)$ , assume that there exist matrices  $\mathcal{P} = \text{diag}\{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{N+1}\}$  with  $\mathcal{P}_i > 0$  ( $i = 1, 2, \dots, N+1$ ),  $\mathcal{Q} > 0$ ,  $\mathcal{R} > 0$ ,  $\mathcal{S} > 0$ ,  $\mathcal{X}_1 \geq 0$ ,  $\mathcal{X}_2 \geq 0$ , matrices  $\mathcal{Y}_i$  ( $i = 1, 2, 3$ ), and scalars  $\lambda_1 > 0$ ,  $\lambda_2 > 0$  satisfying

$$\Pi_1 + \Pi_2 + \Pi_2^T + \Pi_3 < 0, \quad (19)$$

$$\Lambda_1 = \begin{bmatrix} \mathcal{X}_1 & \mathcal{Y}_1 \\ * & \gamma\mathcal{P} \end{bmatrix} \geq 0, \quad \Lambda_2 = \begin{bmatrix} \mathcal{X}_2 & \mathcal{Y}_2 \\ * & \gamma\mathcal{P} \end{bmatrix} \geq 0, \quad \Lambda_3 = \begin{bmatrix} \mathcal{X}_2 & \mathcal{Y}_3 \\ * & \gamma\mathcal{P} \end{bmatrix} \geq 0, \quad (20)$$

where

$$\begin{aligned} \Pi_1 &= \tau\mathcal{X}_1 + (d_M - d_m)\mathcal{X}_2, \\ \Pi_2 &= \begin{bmatrix} \mathcal{Y}_1 & \mathcal{Y}_2 & \mathcal{Y}_3 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \frac{I_{(N+1)n \times (N+1)n} - I_{(N+1)n \times (N+1)n} \quad 0_{(N+1)n \times [(N+1)(m+2)n + Np]} \\ 0_{(N+1)n \times (N+1)nm} \quad I_{(N+1)n \times (N+1)n} \quad -I_{(N+1)n \times (N+1)n} \quad 0_{(N+1)n \times [2(N+1)n + Np]} \\ 0_{(N+1)n \times (N+1)(m+1)n} \quad I_{(N+1)n \times (N+1)n} \quad -I_{(N+1)n \times (N+1)n} \quad 0_{(N+1)n \times [(N+1)n + Np]} \end{bmatrix}, \\ \Pi_3 &= (1 + 2\gamma\hbar)\Xi_1^T \mathcal{P} \Xi_1 + \tilde{\alpha}(1 + 2\gamma\hbar)\Xi_2^T \mathcal{P} \Xi_2 + \tilde{\beta}(1 + 2\gamma\hbar)\Xi_5^T \mathcal{B}_2^T \mathcal{P} \mathcal{B}_2 \Xi_5 \\ &\quad + \Xi_3^T [(1 + 2\gamma\hbar)\mathcal{D}^T \mathcal{P} \mathcal{D} + (2\gamma\hbar - 1)\mathcal{P} + (d_M - d_m + 1)\mathcal{Q} + \mathcal{S} + \lambda_1 \bar{\Phi}^T \bar{\Phi} + \lambda_2 \mathcal{C}^T \Psi \Omega \mathcal{C}] \Xi_3 \\ &\quad - \Xi_4^T \mathcal{Q} \Xi_4 - \Xi_5^T \mathcal{S} \Xi_5 - \lambda_1 \Xi_6^T \Xi_6 - \lambda_2 \Xi_7^T \Omega \Xi_7 + \Xi_{\bar{\mathcal{R}}}^T \bar{\mathcal{R}} \Xi_{\bar{\mathcal{R}}}, \\ \Xi_1 &= \begin{bmatrix} \mathcal{A} + \bar{\alpha}\Delta\mathcal{A} & 0_{(N+1)n \times (N+1)mn} & \mathcal{A}_d & 0_{(N+1)n \times (N+1)n} & \bar{\beta}\mathcal{B}_1 & -\bar{K} \end{bmatrix}, \\ \Xi_2 &= \begin{bmatrix} \Delta\mathcal{A} & 0_{(N+1)n \times [(N+1)(m+3)n + Np]} \end{bmatrix}, \\ \Xi_3 &= \begin{bmatrix} I_{(N+1)n \times (N+1)n} & 0_{(N+1)n \times [(N+1)(m+3)n + Np]} \end{bmatrix}, \\ \Xi_4 &= \begin{bmatrix} 0_{(N+1)n \times (N+1)(m+1)n} & I_{(N+1)n \times (N+1)n} & 0_{(N+1)n \times [2(N+1)n + Np]} \end{bmatrix}, \\ \Xi_5 &= \begin{bmatrix} 0_{(N+1)n \times (N+1)(m+2)n} & I_{(N+1)n \times (N+1)n} & 0_{(N+1)n \times [(N+1)n + Np]} \end{bmatrix}, \\ \Xi_6 &= \begin{bmatrix} 0_{(N+1)n \times (N+1)(m+3)n} & I_{(N+1)n \times (N+1)n} & 0_{(N+1)n \times Np} \end{bmatrix}, \\ \Xi_7 &= \begin{bmatrix} 0_{Np \times (N+1)(m+4)n} & I_{Np \times Np} \end{bmatrix}, \\ \hbar &= d_M - \tau m + \tau, \quad \tilde{\alpha} = \bar{\alpha}(1 - \bar{\alpha}), \quad \tilde{\beta} = \bar{\beta}(1 - \bar{\beta}), \\ \Xi_{\bar{\mathcal{R}}} &= \begin{bmatrix} \frac{I_{(N+1)mn \times (N+1)mn} \quad 0_{(N+1)mn \times [4(N+1)n + Np]} \\ 0_{(N+1)mn \times (N+1)n} \quad I_{(N+1)mn \times (N+1)mn} \quad 0_{(N+1)mn \times [3(N+1)n + Np]} \end{bmatrix}, \\ \bar{\mathcal{R}} &= \text{diag}\{\mathcal{R}, -\mathcal{R}\}, \quad \Omega = \text{diag}\{\Omega_1, \Omega_2, \dots, \Omega_N\}, \\ \Psi &= \text{diag}\{\delta_1 I, \delta_2 I, \dots, \delta_N I\}, \quad \mathcal{C} = \begin{bmatrix} 0 & \tilde{C} \end{bmatrix}. \end{aligned} \quad (21)$$

Then, the estimation error system (16) is mean-square asymptotically stable.

*Proof:* Based on the delay-fractioning idea, let us construct the following Lyapunov-Krasovskii functional for (16):

$$V(\eta_k) = \sum_{l=1}^4 V_l(\eta_k) \quad (22)$$

where

$$\begin{aligned} V_1(\eta_k) &= \eta_k^T \mathcal{P} \eta_k, \\ V_2(\eta_k) &= \sum_{l=k-d_k}^{k-1} \eta_l^T \mathcal{Q} \eta_l + \sum_{j=-d_M+1}^{-\tau m} \sum_{l=k+j}^{k-1} \eta_l^T \mathcal{Q} \eta_l, \\ V_3(\eta_k) &= \sum_{l=k-\tau}^{k-1} \Gamma_l^T \mathcal{R} \Gamma_l + \sum_{l=k-d_M}^{k-1} \eta_l^T \mathcal{S} \eta_l, \\ V_4(\eta_k) &= \sum_{j=-\tau+1}^0 \sum_{l=k+j-1}^{k-1} \zeta_l^T \gamma \mathcal{P} \zeta_l + \sum_{j=-d_M+1}^{\tau m} \sum_{l=k+j-1}^{k-1} \zeta_l^T \gamma \mathcal{P} \zeta_l \\ \Gamma_l &= \left[ \eta_l^T \quad \eta_{l-\tau}^T \quad \cdots \quad \eta_{l-(m-1)\tau}^T \right]^T, \quad \zeta_l = \eta_{l+1} - \eta_l \end{aligned}$$

with  $\mathcal{P} > 0$ ,  $\mathcal{Q} > 0$ ,  $\mathcal{R} > 0$  and  $\mathcal{S} > 0$  to be determined. Along the state trajectory of the system (16), one has

$$\begin{aligned} \mathbb{E}\{\Delta V_1(\eta_k)\} &= \eta_k^T (\mathcal{A} + \bar{\alpha} \Delta \mathcal{A})^T \mathcal{P} (\mathcal{A} + \bar{\alpha} \Delta \mathcal{A}) \eta_k + 2\eta_k^T (\mathcal{A} + \bar{\alpha} \Delta \mathcal{A})^T \mathcal{P} \mathcal{A}_d \eta_{k-d_k} \\ &\quad + 2\bar{\beta} \eta_k^T (\mathcal{A} + \bar{\alpha} \Delta \mathcal{A})^T \mathcal{P} \mathcal{B}_1 F(\eta_k) - 2\eta_k^T (\mathcal{A} + \bar{\alpha} \Delta \mathcal{A})^T \mathcal{P} \bar{K} \tilde{\phi}_k \\ &\quad + \tilde{\alpha} \eta_k^T \Delta \mathcal{A}^T \mathcal{P} \Delta \mathcal{A} \eta_k + \eta_{k-d_k}^T \mathcal{A}_d^T \mathcal{P} \mathcal{A}_d \eta_{k-d_k} \\ &\quad + 2\bar{\beta} \eta_{k-d_k}^T \mathcal{A}_d^T \mathcal{P} \mathcal{B}_1 F(\eta_k) - 2\eta_{k-d_k}^T \mathcal{A}_d^T \mathcal{P} \bar{K} \tilde{\phi}_k \\ &\quad + \bar{\beta}^2 F^T(\eta_k) \mathcal{B}_1^T \mathcal{P} \mathcal{B}_1 F(\eta_k) - 2\bar{\beta} F^T(\eta_k) \mathcal{B}_1^T \mathcal{P} \bar{K} \tilde{\phi}_k \\ &\quad + \tilde{\beta} F^T(\eta_k) \mathcal{B}_2^T \mathcal{P} \mathcal{B}_2 F(\eta_k) + \eta_k^T \mathcal{D}^T \mathcal{P} \mathcal{D} \eta_k \\ &\quad + \tilde{\phi}_k^T \bar{K}^T \mathcal{P} \bar{K} \tilde{\phi}_k - \eta_k^T \mathcal{P} \eta_k, \end{aligned} \quad (23)$$

where  $\tilde{\alpha}$  and  $\tilde{\beta}$  are defined in (21). Similarly, through straightforward algebraic manipulations, we have

$$\begin{aligned} \mathbb{E}\{\Delta V_2(\eta_k)\} &= \eta_k^T \mathcal{Q} \eta_k - \eta_{k-d_k}^T \mathcal{Q} \eta_{k-d_k} + \sum_{l=k+1-d_{k+1}}^{k-1} \eta_l^T \mathcal{Q} \eta_l - \sum_{l=k+1-d_k}^{k-1} \eta_l^T \mathcal{Q} \eta_l \\ &\quad + (d_M - \tau m) \eta_k^T \mathcal{Q} \eta_k - \sum_{l=k-d_M+1}^{k-\tau m} \eta_l^T \mathcal{Q} \eta_l \\ &\leq (d_M - \tau m + 1) \eta_k^T \mathcal{Q} \eta_k - \eta_{k-d_k}^T \mathcal{Q} \eta_{k-d_k}, \end{aligned} \quad (24)$$

$$\mathbb{E}\{\Delta V_3(\eta_k)\} = \Gamma_k^T \mathcal{R} \Gamma_k - \Gamma_{k-\tau}^T \mathcal{R} \Gamma_{k-\tau} + \eta_k^T \mathcal{S} \eta_k - \eta_{k-d_M}^T \mathcal{S} \eta_{k-d_M}, \quad (25)$$



$$\begin{aligned}
\mathbb{E}\{\Delta V_4(\eta_k)\} &= \sum_{j=-\tau+1}^0 \left( \sum_{l=k+j}^k \zeta_l^T \gamma \mathcal{P} \zeta_l - \sum_{l=k-1+j}^{k-1} \zeta_l^T \gamma \mathcal{P} \zeta_l \right) \\
&\quad + \sum_{j=-d_M+1}^{\tau m} \left( \sum_{l=k+j}^k \zeta_l^T \gamma \mathcal{P} \zeta_l - \sum_{l=k-1+j}^{k-1} \zeta_l^T \gamma \mathcal{P} \zeta_l \right) \\
&= \bar{h} \zeta_k^T \gamma \mathcal{P} \zeta_k - \sum_{l=k-\tau}^{k-1} \zeta_l^T \gamma \mathcal{P} \zeta_l - \sum_{l=k-d_k}^{k-\tau m-1} \zeta_l^T \gamma \mathcal{P} \zeta_l - \sum_{l=k-d_M}^{k-d_k-1} \zeta_l^T \gamma \mathcal{P} \zeta_l \\
&\leq 2\gamma \bar{h} \eta_{k+1}^T \mathcal{P} \eta_{k+1} + 2\gamma \bar{h} \eta_k^T \mathcal{P} \eta_k - \sum_{l=k-\tau}^{k-1} \zeta_l^T \gamma \mathcal{P} \zeta_l \\
&\quad - \sum_{l=k-d_k}^{k-\tau m-1} \zeta_l^T \gamma \mathcal{P} \zeta_l - \sum_{l=k-d_M}^{k-d_k-1} \zeta_l^T \gamma \mathcal{P} \zeta_l, \tag{26}
\end{aligned}$$

where  $\bar{h}$  is defined in (21).

Subsequently, for any appropriately dimensioned matrices  $\mathcal{X}_1 \geq 0$  and  $\mathcal{X}_2 \geq 0$ , we have

$$0 = \tau \xi_k^T \mathcal{X}_1 \xi_k - \sum_{l=k-\tau}^{k-1} \xi_k^T \mathcal{X}_1 \xi_k, \tag{27}$$

$$0 = (d_M - d_m) \xi_k^T \mathcal{X}_2 \xi_k - \sum_{l=k-d_k}^{k-d_m-1} \xi_k^T \mathcal{X}_2 \xi_k - \sum_{l=k-d_M}^{k-d_k-1} \xi_k^T \mathcal{X}_2 \xi_k, \tag{28}$$

where  $\xi_k = \left[ \Gamma_k^T \quad \eta_{k-d_m}^T \quad \eta_{k-d_k}^T \quad \eta_{k-d_M}^T \quad F^T(\eta_k) \quad \tilde{\phi}_k^T \right]^T$ . Noting  $\zeta_l = \eta_{l+1} - \eta_l$ , for any matrices  $\mathcal{Y}_i$  ( $i = 1, 2, 3$ ), the following equations are true:

$$0 = 2\xi_k^T \mathcal{Y}_1 \left[ \eta_k - \eta_{k-\tau} - \sum_{l=k-\tau}^{k-1} \zeta_l \right], \tag{29}$$

$$0 = 2\xi_k^T \mathcal{Y}_2 \left[ \eta_{k-d_m} - \eta_{k-d_k} - \sum_{l=k-d_k}^{k-d_m-1} \zeta_l \right], \tag{30}$$

$$0 = 2\xi_k^T \mathcal{Y}_3 \left[ \eta_{k-d_k} - \eta_{k-d_M} - \sum_{l=k-d_M}^{k-d_k-1} \zeta_l \right]. \tag{31}$$

In addition, according to the event-triggering condition (7), we obtain

$$\tilde{\phi}_k^T \Omega \tilde{\phi}_k - \eta_k^T \mathcal{C}^T \Psi \Omega \mathcal{C} \eta_k \leq 0, \tag{32}$$

where  $\Omega$ ,  $\mathcal{C}$  and  $\Psi$  are defined in (21). Then, together with (18), (20), (23)-(32), for two scalars  $\lambda_1 > 0$  and  $\lambda_2 > 0$ , we have

$$\begin{aligned}
\mathbb{E}\{\Delta V(\eta_k)\} &\leq \eta_k^T (\mathcal{A} + \bar{\alpha}\Delta\mathcal{A})^T \mathcal{P} (\mathcal{A} + \bar{\alpha}\Delta\mathcal{A}) \eta_k + 2\eta_k^T (\mathcal{A} + \bar{\alpha}\Delta\mathcal{A})^T \mathcal{P} \mathcal{A}_d \eta_{k-d_k} \\
&\quad + 2\bar{\beta}\eta_k^T (\mathcal{A} + \bar{\alpha}\Delta\mathcal{A})^T \mathcal{P} \mathcal{B}_1 F(\eta_k) - 2\eta_k^T (\mathcal{A} + \bar{\alpha}\Delta\mathcal{A})^T \mathcal{P} \bar{K} \tilde{\phi}_k \\
&\quad + \tilde{\alpha}\eta_k^T \Delta\mathcal{A}^T \mathcal{P} \Delta\mathcal{A} \eta_k + \eta_{k-d_k}^T \mathcal{A}_d^T \mathcal{P} \mathcal{A}_d \eta_{k-d_k} \\
&\quad + 2\bar{\beta}\eta_{k-d_k}^T \mathcal{A}_d^T \mathcal{P} \mathcal{B}_1 F(\eta_k) - 2\eta_{k-d_k}^T \mathcal{A}_d^T \mathcal{P} \bar{K} \tilde{\phi}_k - \eta_k^T \mathcal{P} \eta_k \\
&\quad + \bar{\beta}^2 F^T(\eta_k) \mathcal{B}_1^T \mathcal{P} \mathcal{B}_1 F(\eta_k) - 2\bar{\beta} F^T(\eta_k) \mathcal{B}_1^T \mathcal{P} \bar{K} \tilde{\phi}_k \\
&\quad + \tilde{\beta} F^T(\eta_k) \mathcal{B}_2^T \mathcal{P} \mathcal{B}_2 F(\eta_k) + \eta_k^T \mathcal{D}^T \mathcal{P} \mathcal{D} \eta_k + \tilde{\phi}_k^T \bar{K}^T \mathcal{P} \bar{K} \tilde{\phi}_k \\
&\quad + (d_M - \tau m + 1) \eta_k^T \mathcal{Q} \eta_k - \eta_{k-d_k}^T \mathcal{Q} \eta_{k-d_k} + \Gamma_k^T \mathcal{R} \Gamma_k \\
&\quad - \Gamma_{k-\tau}^T \mathcal{R} \Gamma_{k-\tau} + \eta_k^T \mathcal{S} \eta_k - \eta_{k-d_M}^T \mathcal{S} \eta_{k-d_M} \\
&\quad + 2\gamma \hbar \eta_{k+1}^T \mathcal{P} \eta_{k+1} + 2\gamma \hbar \eta_k^T \mathcal{P} \eta_k + \xi_k^T [\tau \mathcal{X}_1 + (d_M - d_m) \mathcal{X}_2] \xi_k \\
&\quad + 2\xi_k^T \mathcal{Y}_1(\eta_k - \eta_{k-\tau}) + 2\xi_k^T \mathcal{Y}_2(\eta_{k-d_m} - \eta_{k-d_k}) + 2\xi_k^T \mathcal{Y}_3(\eta_{k-d_k} - \eta_{k-d_M}) \\
&\quad - \sum_{l=k-\tau}^{k-1} \varsigma_{k,l}^T \Lambda_1 \varsigma_{k,l} - \sum_{l=k-d_k}^{k-\tau m-1} \varsigma_{k,l}^T \Lambda_2 \varsigma_{k,l} - \sum_{l=k-d_M}^{k-d_k-1} \varsigma_{k,l}^T \Lambda_3 \varsigma_{k,l} \\
&\quad + \lambda_1 \eta_k^T \bar{\Phi}^T \bar{\Phi} \eta_k - \lambda_1 F^T(\eta_k) F(\eta_k) + \lambda_2 \eta_k^T \mathcal{C}^T \Psi \Omega \mathcal{C} \eta_k - \lambda_2 \tilde{\phi}_k^T \Omega \tilde{\phi}_k \\
&\leq \xi_k^T (\Pi_1 + \Pi_2 + \Pi_2^T + \Pi_3) \xi_k - \sum_{l=k-\tau}^{k-1} \varsigma_{k,l}^T \Lambda_1 \varsigma_{k,l} \\
&\quad - \sum_{l=k-d_k}^{k-\tau m-1} \varsigma_{k,l}^T \Lambda_2 \varsigma_{k,l} - \sum_{l=k-d_M}^{k-d_k-1} \varsigma_{k,l}^T \Lambda_3 \varsigma_{k,l} \tag{33}
\end{aligned}$$

where  $\varsigma_{k,l} = \begin{bmatrix} \xi_k^T & \zeta_l^T \end{bmatrix}^T$ ,  $\Pi_1$ ,  $\Pi_2$ ,  $\Pi_3$ ,  $\Lambda_1$ ,  $\Lambda_2$  and  $\Lambda_3$  are defined in (21). According to (19)-(20), we can conclude that the estimation error system (16) is mean-square asymptotically stable.  $\blacksquare$

Now, we are in a position to deal with the uncertainties in (19) and then derive the explicit form of the distributed estimator gains  $K_{ij}$ .

*Theorem 2:* For a given scalar  $\gamma \in (0, 1)$ , assume that there exist matrices  $\mathcal{P} = \text{diag}\{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{N+1}\}$  with  $\mathcal{P}_i > 0$  ( $i = 1, 2, \dots, N+1$ ),  $\mathcal{Q} > 0$ ,  $\mathcal{R} > 0$ ,  $\mathcal{S} > 0$ ,  $\mathcal{X}_1 \geq 0$ ,  $\mathcal{X}_2 \geq 0$ , matrices  $\mathcal{Y}_i$  ( $i = 1, 2, 3$ ), matrix  $\mathcal{K} \in \mathcal{W}_{n \times p}$ , and scalars  $\lambda_1 > 0$ ,  $\lambda_2 > 0$ ,  $\varepsilon > 0$  satisfying (20) and

$$\begin{bmatrix}
\Pi_1 + \Pi_2 + \Pi_2^T + \bar{\Pi}_3 & \sqrt{1 + 2\gamma\hbar}(\bar{\Xi}_1^T \mathcal{P} + \hat{\Xi}_1^T) & 0 & 0 & \varepsilon \bar{\mathcal{M}}^T \\
* & -\mathcal{P} & 0 & \bar{\alpha} \sqrt{1 + 2\gamma\hbar} \mathcal{P} \bar{H} & 0 \\
* & * & -\mathcal{P} & \sqrt{\bar{\alpha}(1 + 2\gamma\hbar)} \mathcal{P} \bar{H} & 0 \\
* & * & * & -\varepsilon I & 0 \\
* & * & * & * & -\varepsilon I
\end{bmatrix} < 0, \tag{34}$$

with

$$\begin{aligned}
\bar{\Xi}_1 &= \begin{bmatrix} \mathcal{A}_0 & 0_{(N+1)n \times (N+1)mn} & \mathcal{A}_d & 0_{(N+1)n \times (N+1)n} & \bar{\beta}\mathcal{B}_1 & 0_{(N+1)n \times Np} \end{bmatrix}, \\
\hat{\Xi}_1 &= \begin{bmatrix} -\bar{\mathcal{K}}\mathcal{C} & 0_{(N+1)n \times (N+1)(m+3)n} & -\bar{\mathcal{K}} \end{bmatrix}, \quad \bar{\mathcal{K}} = \begin{bmatrix} 0 & \mathcal{K}^T \end{bmatrix}^T, \\
\mathcal{A}_0 &= \text{diag}\{A, \tilde{A}\}, \quad \bar{H} = 1_{N+1} \otimes H, \\
\bar{\Pi}_3 &= \tilde{\beta}(1+2\gamma\hbar)\Xi_5^T \mathcal{B}_2^T \mathcal{P} \mathcal{B}_2 \Xi_5 + \Xi_3^T [(1+2\gamma\hbar)\mathcal{D}^T \mathcal{P} \mathcal{D} + (2\gamma\hbar-1)\mathcal{P} \\
&\quad + (d_M - d_m + 1)\mathcal{Q} + \mathcal{S} + \lambda_1 \bar{\Phi}^T \bar{\Phi} + \lambda_2 \mathcal{C}^T \Psi \Omega \mathcal{C}] \Xi_3 - \Xi_4^T \mathcal{Q} \Xi_4 \\
&\quad - \Xi_5^T \mathcal{S} \Xi_5 - \lambda_1 \Xi_6^T \Xi_6 - \lambda_2 \Xi_7^T \Omega \Xi_7 + \Xi_{\bar{\mathcal{R}}}^T \bar{\mathcal{R}} \Xi_{\bar{\mathcal{R}}}, \\
\bar{\mathcal{M}} &= \begin{bmatrix} M & 0_{n_2 \times [(N+1)nm+4Nn+3n+Np]} \end{bmatrix}, \tag{35}
\end{aligned}$$

then the estimation error system (16) is mean-square asymptotically stable. If the above linear matrix inequalities are feasible, the matrix  $\tilde{K}$  can be given by

$$\tilde{K} = (V^T \mathcal{P} V)^{-1} \mathcal{K} \tag{36}$$

with  $V = \begin{bmatrix} 0_{Nn \times n} & I_{Nn \times Nn} \end{bmatrix}^T$ . Accordingly, the distributed state estimator gains  $K_{ij}$  ( $i = 1, 2, \dots, n, j \in \mathcal{N}_i$ ) can be obtained by (12).

*Proof:* By using Lemma 1,  $\Pi_1 + \Pi_2 + \Pi_2^T + \Pi_3 < 0$  is equivalent to

$$\begin{bmatrix} \Pi_1 + \Pi_2 + \Pi_2^T + \bar{\Pi}_3 & \sqrt{1+2\gamma\hbar}\Xi_1^T \mathcal{P} & \sqrt{\tilde{\alpha}(1+2\gamma\hbar)}\Xi_2^T \mathcal{P} \\ * & -\mathcal{P} & 0 \\ * & * & -\mathcal{P} \end{bmatrix} < 0 \tag{37}$$

where  $\bar{\Pi}_3$  is defined in (35). Then, we can rewrite (37) into the following form:

$$\Theta + \mathcal{H} F \mathcal{M} + \mathcal{M}^T F^T \mathcal{H}^T < 0, \tag{38}$$

where

$$\begin{aligned}
\Theta &= \begin{bmatrix} \Pi_1 + \Pi_2 + \Pi_2^T + \bar{\Pi}_3 & \sqrt{1+2\gamma\hbar}(\Xi_1^T + \tilde{\Xi}_1^T)\mathcal{P} & 0 \\ * & -\mathcal{P} & 0 \\ * & * & -\mathcal{P} \end{bmatrix}, \\
\tilde{\Xi}_1 &= \begin{bmatrix} -\bar{\mathcal{K}}\mathcal{C} & 0_{(N+1)n \times (N+1)(m+3)n} & -\bar{\mathcal{K}} \end{bmatrix}, \\
\mathcal{H} &= \begin{bmatrix} 0_{n_1 \times [(N+1)(m+4)+p]n} & \bar{\alpha}\sqrt{1+2\gamma\hbar}\bar{H}^T \mathcal{P} & \sqrt{\tilde{\alpha}(1+2\gamma\hbar)}\bar{H}^T \mathcal{P} \end{bmatrix}^T, \\
\mathcal{M} &= \begin{bmatrix} M & 0_{n_2 \times [(N+1)nm+6Nn+5n+Np]} \end{bmatrix}.
\end{aligned}$$

By denoting  $\mathcal{P}\bar{K} = \bar{\mathcal{K}}$  and using Lemma 2, it is not difficult to verify that (38) holds if (34) is true. Moreover, according to Lemma 3, it is easy to show that  $\tilde{K} \in \mathcal{W}_{n \times p}$ . Hence, the proof of this theorem is complete. ■

*Remark 2:* Up to now, the distributed state estimation problem is studied for a class of discrete nonlinear stochastic systems with time-varying delay, ROUs and RONs. Based on the event-triggered mechanism, a new distributed state estimator is constructed where the available neighbor information of the sensor nodes is employed. It is shown that the explicit form of the estimator gains can be easily solved by using the semi-definite programme approach.

## IV. AN ILLUSTRATIVE EXAMPLE

In this section, we provide a numerical example to demonstrate the feasibility and effectiveness of the distributed state estimation scheme.

The sensor network is represented by a directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$  with the set of nodes  $\mathcal{V} = \{1, 2, 3, 4\}$ , the set of edges  $\mathcal{E} = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 3), (4, 4)\}$ , and the following adjacency matrix

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The system parameters are given as follows:

$$\begin{aligned} A &= \begin{bmatrix} -0.24 & -0.32 \\ 0.4 & -0.08 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.8 & 0.8 \\ -1.2 & -0.4 \end{bmatrix}, \\ B &= \begin{bmatrix} -0.6 & -0.02 \\ 0.02 & -0.7 \end{bmatrix}, \quad D = \begin{bmatrix} -0.4 & 0 \\ -0.3 & -0.2 \end{bmatrix}, \\ H &= \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^T, \quad M = \begin{bmatrix} 0.05 & 0.1 \end{bmatrix}, \quad F = \sin(0.5k) \\ C_1 &= \begin{bmatrix} 0.8 & 1.6 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 5.6 & 2.4 \end{bmatrix}, \\ C_3 &= \begin{bmatrix} 0.8 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 2.4 & 3.2 \end{bmatrix}. \end{aligned}$$

Assume that the time-varying delay  $d_k$  satisfies  $2 \leq d_k \leq 5$ ,  $\Phi = \text{diag}\{0.2, 0.3\}$ ,  $\bar{\alpha} = 0.9$ ,  $\bar{\beta} = 0.85$ ,  $\Omega_i = 1$  and  $\delta_i = 0.5$  ( $i = 1, 2, 3, 4$ ). Set  $m = 1$  and  $\gamma = 0.01$ . By solving (20) and (34) in Theorem 2, we have

$$\begin{aligned} K_{11} &= \begin{bmatrix} 0.0097 & -0.0984 \end{bmatrix}^T, \quad K_{21} = \begin{bmatrix} -0.0701 & 0.0079 \end{bmatrix}^T, \\ K_{22} &= \begin{bmatrix} -0.1277 & 0.0652 \end{bmatrix}^T, \quad K_{31} = \begin{bmatrix} -0.0064 & 0.0518 \end{bmatrix}^T, \\ K_{32} &= \begin{bmatrix} 0.0177 & -0.3597 \end{bmatrix}^T, \quad K_{33} = \begin{bmatrix} 0.0151 & 0.0401 \end{bmatrix}^T, \\ K_{43} &= \begin{bmatrix} -0.0378 & -0.0236 \end{bmatrix}^T, \quad K_{44} = \begin{bmatrix} 0.0178 & 0.0086 \end{bmatrix}^T, \end{aligned}$$

which confirm the feasibility of the proposed state estimation approach.

## V. CONCLUSIONS

In this paper, the event-based distributed state estimation problem has been investigated for a class of nonlinear stochastic systems with time-varying delay, ROUs and RONS. The phenomena of the ROUs and RONS have been depicted by introducing two random variables obeying the Bernoulli distribution. By considering the event-triggered mechanism and the characteristic of the sensor works, a new distributed state estimator has been constructed. Based on the delay-fractioning idea, a sufficient condition has been proposed to guarantee the mean-square asymptotical stability of the estimation error system and the explicit form of the estimator gains has also been given. Finally, a numerical example has been provided to demonstrate the usefulness of the developed estimation scheme. Further research topics include the extension of the proposed

results to more complex systems with event-triggering communication mechanism such as nonlinear polynomial systems [2, 3], time-varying systems [13, 43], mechanical systems [25] and multi-machine power systems [27].

## REFERENCES

- [1] M. V. Basin and D. Calderon-Alvarez, Optimal filtering over linear observations with unknown parameters, *Journal of the Franklin Institute*, vol. 347, no. 6, pp. 988–1000, 2010.
- [2] M. V. Basin, P. Shi, and P. Soto, Central suboptimal  $H_\infty$  filtering for nonlinear polynomial systems with multiplicative noise, *Journal of the Franklin Institute*, vol. 347, no. 9, pp. 1740–1754, 2010.
- [3] M. V. Basin and P. C. Rodriguez-Ramirez, Sliding mode controller design for stochastic polynomial systems with unmeasured states, *IEEE Transactions on Industrial Electronics*, vol. 61, no. 1, pp. 387–396, 2014.
- [4] K. Bouibed, L. Seddiki, K. Guelton, and H. Akdag, Actuator and sensor fault detection and isolation of an actuated seat via nonlinear multi-observers, *Systems Science and Control Engineering: An Open Access Journal*, vol. 2, no. 1, pp. 150–160, 2014.
- [5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadelphia: SIAM, 1994.
- [6] V. K. Chaurasiya, N. Jain, and G. C. Nandi, A novel distance estimation approach for 3D localization in wireless sensor network using multi dimensional scaling, *Information Fusion*, vol. 15, pp. 5–18, 2014.
- [7] H. Dong, Z. Wang, and H. Gao, Distributed  $H_\infty$  filtering for a class of Markovian jump nonlinear time-delay systems over lossy sensor networks, *IEEE Transactions on Industrial Electronics*, vol. 60, no. 10, pp. 4665–4672, 2013.
- [8] M. C. F. Donkers and W. P. M. H. Heemels, Output-based event-triggered control with guaranteed  $L_\infty$ -gain and improved and decentralized event-triggering, *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1362–1376, 2012.
- [9] H. Gao, Z. Fei, J. Lam, and B. Du, Further results on exponential estimates of Markovian jump systems with mode-dependent time-varying delays, *IEEE Transactions on Automatic Control*, vol. 56, no. 1, pp. 223–229, 2011.
- [10] H. Gao and X. Li,  $H_\infty$  filtering for discrete-time state-delayed systems with finite frequency specifications, *IEEE Transactions on Automatic Control*, vol. 56, no. 12, pp. 2932–2939, 2011.
- [11] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, Robust sliding mode control for discrete stochastic systems with mixed time delays, randomly occurring uncertainties, and randomly occurring nonlinearities, *IEEE Transactions on Industrial Electronics*, vol. 59, no. 7, pp. 3008–3015, 2012.
- [12] J. Hu, Z. Wang, Y. Niu, and L. K. Stergioulas,  $H_\infty$  sliding mode observer design for a class of nonlinear discrete time-delay systems: a delay-fractioning approach, *International Journal of Robust and Nonlinear Control*, vol. 22, no. 16, pp. 1806–1826, 2012.
- [13] J. Hu, Z. Wang, B. Shen, and H. Gao, Quantized recursive filtering for a class of nonlinear systems with multiplicative noises and missing measurements, *International Journal of Control*, vol. 86, no. 4, pp. 650–663, 2013.
- [14] J. Hu, Z. Wang, B. Shen, and H. Gao, Gain-constrained recursive filtering with stochastic nonlinearities and probabilistic sensor delays, *IEEE Transactions on Signal Processing*, vol. 61, no. 5, pp. 1230–1238, 2013.
- [15] J. Hu, D. Chen, and J. Du, State estimation for a class of discrete nonlinear systems with randomly occurring uncertainties and distributed sensor delays, *International Journal of General Systems*, vol. 43, no. 3-4, pp. 387–401, 2014.
- [16] S. Hu and D. Yue, Event-based  $H_\infty$  filtering for networked systems with communication delay, *Signal Processing*, vol. 92, no. 9, pp. 2029–2039, 2012.
- [17] M. Kermani and A. Sakly, Stability analysis for a class of switched nonlinear time-delay systems, *Systems Science and Control Engineering: An Open Access Journal*, vol. 2, no. 1, pp. 80–89, 2014.
- [18] J. Kim, J. Lee, E. Serpedin, and K. Qaraqe, A robust estimation scheme for clock phase offsets in wireless sensor networks in the presence of non-Gaussian random delays, *Signal Processing*, vol. 89, no. 6, pp. 1155–1161, 2009.
- [19] A. Khandekar and B. Patre, Discrete sliding mode control for robust tracking of time-delay systems, *Systems Science and Control Engineering: An Open Access Journal*, vol. 2, no. 1, pp. 457–464, 2014.
- [20] J. Liang, Z. Wang, and X. Liu, Distributed state estimation for discrete-time sensor networks with randomly varying nonlinearities and missing measurements, *IEEE Transactions on Neural Networks*, vol. 22, no. 3, pp. 486–496, 2011.

- [21] J. Liang and W. Zheng, Distributed state estimation for uncertain sensor networks with mixed time delays and sensor saturations, *International Journal of General Systems*, vol. 43, no. 3-4, pp. 332–345, 2014.
- [22] J. Liu and D. Yue, Event-based fault detection for networked systems with communication delay and nonlinear perturbation, *Journal of the Franklin Institute*, vol. 350, no. 9, pp. 2791–2807, 2013.
- [23] Y. Luo, G. Wei, X. Ding and Y. Liu, Controller design for 2-D stochastic nonlinear Roesser model: a probability-dependent gain-scheduling approach, *Journal of the Franklin Institute*, vol. 351, no. 11, pp. 5182–5203, 2014.
- [24] M. S. Mahmoud, H. M. Khalid, and M. Sabih, Improved distributed estimation method for environmental physical variables in static sensor networks, *IET Wireless Sensor Systems*, vol. 3, no. 3, pp. 216–232, 2013.
- [25] M. Mohamed, K. Nahla, and B. Safya, Design of a nonlinear observer for mechanical systems with unknown inputs, *Systems Science and Control Engineering: An Open Access Journal*, vol. 1, no. 1, pp. 105–112, 2013.
- [26] S. Movaghati and M. Ardakani, Optimum bit-sensor assignment for distributed estimation in inhomogeneous sensor networks, *IEEE Communications Letters*, vol. 18, no. 4, pp. 668–671, 2014.
- [27] M. Ouassaid, M. Maaroufi, and M. Cherkaoui, A real-time nonlinear decentralized control of multimachine power systems, *Systems Science and Control Engineering: An Open Access Journal*, vol. 2, no. 1, pp. 135–142, 2014.
- [28] D. Peaucelle, D. Arzelier, D. Henrion, and F. Gouaisbaut, Quadratic separation for feedback connection of an uncertain matrix and an implicit linear transformation, *Automatica*, vol. 43, no. 5, pp. 795–804, 2007.
- [29] C. Peng and T. Yang, Event-triggered communication and  $H_\infty$  control co-design for networked control systems, *Automatica*, vol. 49, no. 5, pp. 1326–1332, 2013.
- [30] D. E. Quevedo, A. Ahlen, and K. H. Johansson, State estimation over sensor networks with correlated wireless fading channels, *IEEE Transactions on Automatic Control*, vol. 58, no. 3, pp. 581–593, 2013.
- [31] J. M. Rabaey, M. J. Ammer, J. L. da Silva, D. Patel, and S. Roundy, Picoradio supports ad hoc ultra-low power wireless networking, *Computer*, vol. 33, no. 7, pp. 42–48, 2000.
- [32] W. Rukpakavong, L. Guan, and I. Phillips, Dynamic node lifetime estimation for wireless sensor networks, *IEEE Sensors Journal*, vol. 14, no. 5, pp. 1370–1379, 2014.
- [33] L. Schenato, Optimal estimation in networked control systems subject to random delay and packet drop, *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1311–1317, 2008.
- [34] B. Shen, Z. Wang, and Y. S. Hung, Distributed consensus  $H_\infty$  filtering in sensor networks with multiple missing measurements: the finite-horizon case, *Automatica*, vol. 46, no. 10, pp. 1682–1688, 2010.
- [35] B. Shen, Z. Wang, and X. Liu, A stochastic sampled-data approach to distributed  $H_\infty$  filtering in sensor networks, *IEEE Transactions on Circuits and Systems I-Regular Papers*, vol. 58, no. 9, pp. 2237–2246, 2011.
- [36] D. Shi, T. Chen, and L. Shi, Event-triggered maximum likelihood state estimation, *Automatica*, vol. 50, no. 1, pp. 247–254, 2014.
- [37] D. Shi, T. Chen, and L. Shi, An event-triggered approach to state estimation with multiple point- and set-valued measurements, *Automatica*, vol. 50, no. 6, pp. 1641–1648, 2014.
- [38] Y. Tang, H. Gao, W. Zou, and J. Kurths, Distributed synchronization in networks of agent systems with nonlinearities and random switchings, *IEEE Transactions on Cybernetics*, vol. 43, no. 1, pp. 358–370, 2013.
- [39] Y. Tang, H. Gao, and J. Kurths, Distributed robust synchronization of dynamical networks with stochastic coupling, *IEEE Transactions on Circuits and Systems I-Regular Papers*, vol. 61, no. 5, pp. 1508–1519, 2014.
- [40] Y. Tang and W. K. Wong, Distributed synchronization of coupled neural networks via randomly occurring control, *IEEE Transactions on Neural Networks and Learning Systems*, vol. 24, no. 3, pp. 435–447, 2013.
- [41] L. Wang, G. Wei, and H. Shu, State estimation for complex networks with randomly occurring coupling delays, *Neurocomputing*, vol. 122, pp. 513–520, 2013.
- [42] Y. Wang, L. Xie, and C. E. de Souza, Robust control of a class of uncertain nonlinear systems, *Systems & Control Letters*, vol. 19, no. 2, pp. 139–149, 1992.
- [43] Z. Wang, H. Dong, B. Shen, and H. Gao, Finite-horizon  $H_\infty$  filtering with missing measurements and quantization effects, *IEEE Transactions on Automatic Control*, vol. 58, no. 7, pp. 1707–1718, 2013.
- [44] Z. Wang, B. Shen, H. Shu, and G. Wei, Quantized  $H_\infty$  control for nonlinear stochastic time-delay systems with missing measurements, *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1431–1444, 2012.

- [45] Z. Wang, Y. Wang, and Y. Liu, Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and mixed time delays, *IEEE Transactions on Neural Networks*, vol. 21, no. 1, pp. 11–25, 2010.