EVERY DIRECTED GRAPH HAS A SEMI-KERNEL

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In a directed graph, the distance $d(u,v)$ from a vertex u to a vertex v is the number of edges in the shortest directed path from u to v. It is well-known that every tournament has a vertex u such that $d(u,v) \leq 2$ for all v ; in fact, any vertex of largest outdegree is such a vertex [2] . This generalizes as follows :

Theorem : In a directed graph G, there is always a set S of vertices such that

(i) $d(u,v) \geq 2$ whenever $u,v \in S$ and $u \neq v$,

(ii) given any $v \notin S$ there is an $u \in S$ with $d(u,v) \leq 2$.

Proof. By induction on the number of vertices of G. Let w be a vertex of G ; let G' be the subgraph of G induced by $\{u:d(w,u)\geq 2\}$. By the induction hypothesis, there is a set S' which works for G'. If $d(u,w) \leq 1$ for some $u \in S'$, we set $S = S'$; otherwise we set $S = S' \cup \{w\}$. Obviously, S has the required properties.

REMARK. A set S satisfying (i) and such that

(iii) given any $v \notin S$ there is an $u \in S$ with $d(v,u) \leq 1$

is called a kernel (cf. [1]). Not every directed graph has a kernel.

REFERENCES

1. C. Berge, Graphs and Hypergraphs, North Holland, Amsterdam 1973, Chapter 14. Kernels and Grundy functions.

2. H.G. Landau, On dominance relations and the structure of animal societies,III ; the condition for a score structure, Bull. Math. Biophys. 15 (1955), 143-148.