(ii) Does there exist an R-set which is not a Riesz set?

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Every nuclear Fréchet space with a regular basis has the quasi-equivalence property

by

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Abstract. The following theorem is proved: *If X is a nuclear Fréchet space with a regular basis (x_n) and if (y_n) is another basis for X, then the bases (x_n) and (y_n) are quasi-equivalent.*

M. M. Dragilev has shown in [3] that nuclear Fréchet spaces in the classes (d_1) and (d_2) have the quasi-equivalence property. His results and techniques were reformulated and extended by C. Bessaga in [1]. B. S. Mitiagin has shown in [4] that nuclear centers of Hilbert scales have the quasi-equivalence property, and V. P. Zaharjuta extended this in [7] by replacing the hypothesis of nuclearity with the Schwartz condition, and finally Mitiagin [9] established this property for the centers of arbitrary Hilbert scales. Also Zaharjuta recently obtained the quasi-equivalence property for spaces which are products of a (d_1) and (d_2) space in [8]. However, the general problem of quasi-equivalence for nuclear Fréchet spaces remains.

In this paper we prove that any nuclear Köthe space with a regular basis has the quasi-equivalence property. The essential idea of the proof is that the diametral dimension $\delta(E)$ (as defined in [2]) distinguishes regular bases.

1. Definitions. For two sequences a and b, $a\cdot b$ will denote the sequence $(a_n b_n)$, and if B is a collection of sequences, $a\cdot B = \{a\cdot b\colon b\in B\}$. A *Köthe space* is the Fréchet space of sequences

$$\lambda = \bigcap \frac{1}{a^k}\cdot l_1 = \Big\{t\colon \forall k,\ \|t\|_k = \sum_{n=1}^{\infty} |t_n|\,a_n^k < +\infty\Big\},$$

with the topology generated by the norms $\|\cdot\|_k$, $k = 1, 2, \ldots$ We assume that for all k, n, $0 < a_n^k \leqslant a_n^{k+1}$. It is known that λ is nuclear if and only if for all k there exists m such that $\sum_n (a_n^k/a_n^m) < +\infty$, and that λ is a Schwartz space if and only if for all k there exists m such that $a_n^k/a_n^m \to 0$. If λ is a Köthe space, the sequences e^n, with $e_i^n = \delta_{ni}$, form an absolute basis for λ. On the other hand, every Fréchet space E with a continuous norm and an absolute basis (x^n) has a natural identification with a Köthe space. In fact, let $a_n^k = \|x^n\|_k$, where $(\|\ \|_k)$ is an increasing sequence of norms defining the topology on E. We also say that (x^n) is *represented* by the (a_n^k) (cf. [1], [3], or [5] for more information on the above topics).

If E is a Fréchet space with a continuous norm and an absolute basis (x^n) we say that (x^n) is *regular* if (x^n) is represented by a matrix (a_n^k) such that for each k and n,

$$\frac{a_n^k}{a_n^{k+1}} \geqslant \frac{a_{n+1}^k}{a_{n+1}^{k+1}}.$$

This concept was first introduced by Dragilev in [3].

If (x^n) and (y^n) are bases for the locally convex spaces (l.c.s.) E and F, respectively, we say that (x^n) and (y^n) are *equivalent* if $\sum t_n x^n$ converges if and only if $\sum t_n y^n$ converges. The bases (x^n) and (y^n) are *semi-equivalent* if there exists a sequence (a_n) of non-zero scalars such that (x^n) is equivalent to $(a_n y^n)$. (x^n) and (y^n) are *quasi-equivalent* if there exists a permutation Π of the natural numbers N such that (x^n) is semi-equivalent to $(y^{\Pi(n)})$. If E is a l.c.s. with a basis in which all bases are quasi-equivalent, we say that E has the *quasi-equivalence property*. (Cf. [1] or [3] for more details on quasi-equivalence.)

2. Results

LEMMA 1. *For each $p = 1, 2, \ldots$, let a^p and b^p be sequences of positive numbers such that for all p and q, $a^p \cdot b^q \in l_\infty$. Then there is a sequence, d, of positive numbers such that $a^p \cdot d \in l_\infty$ and $b^p/d \in l_\infty$ for all p.*

Proof. We define new collections of sequences $\{A^p\}_{p=1}^\infty$ and $\{B^p\}_{p=1}^\infty$ by induction as follows:

Let $A^1 = a^1$ and $B^1 = c_1 b^1$ where c_1 is a positive number chosen so that $A_n^1 B_n^1 \leqslant 1$, $\forall n$. Suppose that A^i and B^i have been defined for $i = 1, 2, \ldots, p-1$. Let $A^p = c_p' a^p$ where c_p' is a positive number chosen so that $A_n^p B_n^i \leqslant 1$ $\forall n$ and $i = 1, 2, \ldots, p-1$. Let $B^p = c_p b^p$ where c_p is a positive number chosen so that $A_n^i B_n^p \leqslant 1$ $\forall n$ and $i = 1, 2, \ldots, p$. The collections $\{A^p\}_{p=1}^\infty$ and $\{B^p\}_{p=1}^\infty$ satisfy the condition $A_n^p B_n^q \leqslant 1$ for all n, p and q.

The desired sequence, d, of positive numbers is defined by

$$d_n \equiv \sup_q B_n^q \leqslant \inf_p \frac{1}{A_n^p}, \quad \forall n.$$

LEMMA 2. *Let*

$$\lambda = \bigcap_p \frac{1}{a^p} l_1 \quad \textit{and} \quad \mu = \bigcap_p \frac{1}{b^p} l_1$$

be Köthe spaces and suppose

$$\bigcup_p \bigcap_q \frac{a^p}{a^q} c_0 = \bigcup_r \bigcap_s \frac{b^r}{b^s} c_0.$$

Then there is a sequence d of positive numbers such that $\lambda = d \cdot u$.

Proof. By assumption,

$$\forall p \quad \bigcap_q \frac{a^p}{a^q} c_0 \subset \bigcup_r \bigcap_s \frac{b^r}{b^s} c_0.$$

By [6], problem 33, p. 206,

$$\forall p \quad \exists r(p) \ni \bigcap_q \frac{a^p}{a^q} c_0 \subset \bigcap_s \frac{b^{r(p)}}{b^s} c_0, \quad \text{or} \quad \bigcup_s \frac{b^s}{b^{r(p)}} l_1 \subset \bigcup_q \frac{a^q}{a^p} l_1.$$

Thus

$$\forall p \exists r(p) \forall s \quad \frac{b^s}{b^{r(p)}} l_1 \subset \bigcup_q \frac{a^q}{a^p} l_1.$$

Again using the result from [6] we have

$$\forall p \exists r(p) \forall s \exists q(p, s) \ni \frac{b^s}{b^{r(p)}} l_1 \subset \frac{a^{q(p,s)}}{a^p} l_1$$

$$\text{or} \quad \frac{a^p}{b^{r(p)}} \frac{b^s}{a^{q(p,s)}} \in l_\infty.$$

Similarly one can show

$$\forall p \exists r'(p) \forall s \exists q'(p, s) \ni \frac{b^p}{a^{r'(p)}} \frac{a^s}{b^{q'(p,s)}} \in l_\infty.$$

For each p set

$$R(p) = \max\{r(p),\ q'(1, p),\ q'(2, p), \ldots, q'(p, p)\},$$

$$R'(p) = \max\{r'(p),\ q(1, p),\ q(2, p), \ldots, q(p, p)\}.$$

Then for all p and s,

$$\frac{a^p}{b^{R(p)}} \frac{b^s}{a^{R'(s)}} \in l_\infty.$$

[If $p \geqslant s$,

$$\frac{a^p}{b^{R(p)}} \frac{b^s}{a^{R'(s)}} \leqslant \frac{a^p}{b^{q'(s,p)}} \frac{b^s}{a^{r'(s)}} \in l_\infty;$$

if $s \geqslant p$,

$$\frac{a^p}{b^{R(p)}} \frac{b^s}{a^{R'(s)}} \leqslant \frac{a^p}{b^{r(p)}} \frac{b^s}{a^{q(p,s)}} \in l_\infty.]$$

By Lemma 1 there is a sequence d of positive numbers such that

$$\frac{b^s}{a^{R'(s)}}\Big/ d \in l_\infty \ \forall s \quad \text{and} \quad \frac{a^p}{b^{R(p)}}\cdot d \in l_\infty \ \forall p.$$

This is equivalent to the statement $\lambda = d\cdot\mu$.

THEOREM. *If E is a nuclear Köthe space with a regular basis, then all bases are quasi-equivalent.*

Proof. Let E be a nuclear Köthe space with the regular basis $\{x^n\}$ and let $\{y^n\}$ be an arbitrary basis for E. It is sufficient to show that $\{y^n\}$ is quasi-equival nt to $\{x^n\}$. By [3], Theorem 1, there is a permutation π of the positive integers such that $\{y^{\pi(n)}\}$ is a regular basis. Let (a_n^p) and (b_n^p) be regular representations of $\{x^n\}$ and $\{y^n\}$, respectively. Let

$$\mu = \bigcap_p \frac{1}{a^p} l_1 \quad \text{and} \quad \lambda = \bigcap_p \frac{1}{b^p} l_1.$$

To complete the proof of the theorem, we shall show that there is a sequence d of positive numbers such that $\lambda = d\cdot\mu$.

Bessaga, Pełczyński and Rolewicz ([2]) introduced a topological invariant δ, defined as follows: A sequence t is in δ if there is a neighborhood of zero, U, such that for all zero neighborhoods V $t_n/d_{n-1}(V, U)\in c_0$. It follows from (1.10) of [1] that δ has the two representations:

$$\bigcup_p \bigcap_q \frac{a^p}{a^q} c_0 = \delta = \bigcup_r \bigcap_s \frac{b^r}{b^s} c_0.$$

By Lemma 2, there is a sequence d of positive numbers such that $\lambda = d\cdot\mu$. Thus $\{x^n\}$ is equivalent to $\{d_n y^{\pi(n)}\}$.

Remark. The proof given shows that in a Schwartz Köthe space, all absolute regular bases are semi-equivalent.

The following corollary of Lemma 2, which solves Bessaga's conjecture ([1]) for stable spaces, was pointed out to us by Ed Dubinsky.

COROLLARY 1. *Let E and F be nuclear Fréchet spaces with continuous norms and regular bases, such that E is isomorphic to $E\times E$ and F is isomorphic to a complemented subspace of E. Let (x^n) be a regular basis for F. Then there exists a sequence (j_n) of integers with $j_1 < j_2 < \ldots$ such that $[x^{j_n}]$ is isomorphic to F.*

Proof. Applying Proposition 1 of [2] we see that $\delta(E) \subseteq \delta(E\times F) \subseteq \delta(E\times E) = \delta(E)$, so that $\delta(E) = \delta(E\times F)$. Let (y^n) be a basis for F. By Theorem 2.2 of [1] there exists a regular basis (z^n) for $E\times F$ such that for each n either $z^n = x^m$, for some m, or $z^n = y^i$ for some i. Applying Lemma 2 as in the proof of the theorem above, we obtain numbers (d_n) such that (z^n) is equivalent to $(d_n x^n)$. Then (y^n) is quasi-equivalent to a subsequence of (x^n).

The following Corollary solves, for Köthe spaces with a regular basis, a problem discussed in [8].

COROLLARY 2. *Let E be a Köthe space with a regular basis. Let $E^{(s)}$ be a closed subspace of E of codimension s, $s = 1, 2, \ldots$ Then either $E \cong E^{(1)}$ or E non $\cong E^{(s)}$ for any $s = 1, 2, \ldots$*

Proof. By Proposition 1 of [2], we see that $\delta(E^{(s)}) \subseteq \delta(E^{(1)}) \subseteq \delta(E)$. However, $E^{(s)}$ has a regular basis, so that by Lemma 2, $E^{(s)} \cong E$ if and only if $\delta(E^{(s)}) = \delta(E)$. This yields the result.

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