EVERY PLANAR GRAPH WITH NINE POINTS HAS A NONPLANAR COMPLEMENT

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In a *complete graph* every two points are joined by a line (are *adjacent*). A complete graph with n points is denoted by K_n. Let G be a graph with n points considered as a subgraph of K_n. The complement $\overline{G}$ of G is the graph obtained by removing all lines of G from K_n. The following problem was stated by Harary [2]: What is the least integer n such that every graph G with n points or its complement $\overline{G}$ is nonplanar? Harary [3] observed that $n \leqq 11$. It is readily seen that $n > 8$. In this note we shall outline the proof that $n = 9$, verifying a conjecture of J. L. Selfridge.

THEOREM. *If G is a graph with nine points, then G or its complement $\overline{G}$ is nonplanar.*

Let $p(G)$ be the number of points, $q(G)$ the number of lines, and $k(G)$ the number of components of graph G. Let $K_{m,n}$ be a graph with $m+n$ points, m points of one color and n points of another, in which two points are adjacent if and only if their colors are different. Kuratowski [5] proved the classic theorem that a graph is nonplanar if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.

PROPOSITION 1. *In each of the following cases, a graph G is nonplanar.*

(i) $p(G) \geqq 6$ *and* $k(\overline{G}) \geqq 4$.

(ii) $p(G) \geqq 7$, $k(\overline{G}) \geqq 3$ *and $\overline{G}$ has at most one isolated point.*

(iii) $p(G) \geqq 7$, $k(\overline{G}) = 2$ *and each component of $\overline{G}$ contains at least three points.*

(iv) $p(G) \geqq 9$ *and* $k(\overline{G}) \geqq 3$.

In each of these cases, it is easy to see that G contains $K_{3,3}$ as a subgraph. Thus G is nonplanar by Kuratowski's theorem.

PROPOSITION 2. *If $p(G) \geqq 9$, $k(\overline{G}) = 2$ and G is planar, then $\overline{G}$ is nonplanar.*

By Propositions 1 and 2, it is sufficient to prove the theorem under the hypothesis that $\overline{G}$ is connected.

Let G be a planar graph with $p(G) \geqq 4$. Imbed G into a 2-sphere S. By Fáry's theorem [1], there exists a triangulation T of S whose

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1-skeleton T_1 contains G as a subcomplex (subgraph) and whose 0-skeleton T_0 is the set of points of G.

PROPOSITION 3. *If G and $\bar{G}$ are both planar, then T_1 and $\bar{T}_1$ are both planar.*

Thus it is enough to prove the theorem when G is the 1-skeleton of a triangulation with nine points. Let $v_1, v_2, \cdots, v_9$ be the points of G and let d_i be the *degree* of the point v_i (the number of points with which v_i is adjacent). The vector $\pi(G)=(d_1, \cdots, d_9)$ is called *the partition* of G. It is convenient to write the degrees in $\pi(G)$ in nonincreasing order. Since $\bar{G}$ is connected and G is a triangulation, it follows easily that $3 \leqq d_i \leqq 7$ for each i.

PROPOSITION 4. *If there exists i such that $d_i = d_{i+1} = 3$, then $\bar{G}$ is nonplanar.*

PROPOSITION 5. *If there exists i such that $d_i = d_{i+1} = 4$, then $\bar{G}$ is nonplanar.*

There are several cases to discuss in order to establish Propositions 4 and 5. In each case, we can prove that $\bar{G}$ contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

Since G is a triangulation of a sphere, we see that $q(G)=21$. Therefore, $\sum_{i=1}^{9} d_i = 42$. Thus, if we omit the cases in Propositions 4 and 5, there is a unique possible partition of 42 into 9 summands d_i, $3 \leqq d_i \leqq 7$, namely:

$$\pi_0 = (5, 5, 5, 5, 5, 5, 5, 4, 3).$$

PROPOSITION 6. *There exists no triangulation of a sphere whose 1-skeleton has the partition π_0.*

Propositions 1–6 complete the proof of the theorem.

For any graph G with p points v_i and respective degrees d_i, $\sum_1^p d_i = 2q$. A *graphic partition* of an even number is one whose summands are the degrees of some graph G. Havel [4] has provided a characterization of graphic partitions. Call a graphic partition *simple* if it belongs to exactly one graph. A characterization of simple graphic partitions is an open problem. The particularly exhausting part of our proof stems from the fact that a graphic partition belonging to the 1-skeleton of a triangulation of a sphere need not be simple.

A graph G is called *biplanar* if there exists in G a planar subgraph whose complement in G is biplanar. From the proof of our theorem it is known that every proper subgraph of K_9 is decomposable. The following problem suggests itself: Characterize biplanar graphs.

Added in proof. Since this manuscript was submitted, we learned of two other subsequent proofs of the theorem. One is due to John R. Ball of the Carnegie Institute of Technology and is somewhat similar to our proof. The other was found by W. T. Tutte of the University of Waterloo who actually constructed every triangulation of the sphere having 9 vertices and verified for each that its complement is nonplanar!

References

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