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Every separable Banach space has a bounded strong norming biorthogonal sequence which is also a Steinitz basis

by

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Abstract. Every separable, infinite-dimensional Banach space X has a biorthogonal sequence $\{z_n, z_n^*\}$, with $\operatorname{span}\{z_n^*\}$ norming on X and $\{\|z_n\|+\|z_n^*\|\}$ bounded, so that, for every x in X and x^* in X^*, there exists a permutation $\{\pi(n)\}$ of $\{n\}$ so that

$$x \in \overline{\operatorname{conv}}\Big\{\text{finite subseries of } \sum_{n=1}^{\infty} z_n^*(x) z_n\Big\} \quad \text{and} \quad x_n^*(x) = \sum_{n=1}^{\infty} z_{\pi(n)}^*(x) x^*(z_{\pi(n)}).$$

Introduction. This note concerns the search for the best sequence capable of representing the elements of a separable Banach space X.

A sequence $\{x_n\}$ in X is said to be *complete* or *fundamental* if $\overline{\operatorname{span}}\{x_n\} = X$. If $\{x_n^*\} \subset X^*$ (the dual space) then $\{x_n, x_n^*\}$ is said to be *biorthogonal* if $x_m^*(x_n) = \delta_{mn}$ (Kronecker symbol).

A biorthogonal sequence $\{x_n, x_n^*\}$ is said to be

- *complete* if $\{x_n\}$ is complete;
- *total* if $[\operatorname{span}\{x_n^*\}]^{\perp}$ $(= \{x \in X : x_n^*(x) = 0 \text{ for each } n\}) = \{0\}$;
- *norming* if there exists a number H such that, for each x in X, $\|x\| \le H \sup\{|x^*(x)|/\|x^*\| : x^* \in \operatorname{span}\{x_n^*\}\}$;
- *strong* if for each decomposition $\{n\} = \{n_k\} \cup \{n'_k\}$, $\{n_k\} \cap \{n'_k\} = \emptyset$, of the positive integers, $\overline{\operatorname{span}}\{x_n\}_{n \in \{n_k\}} = [\overline{\operatorname{span}}\{x_n^*\}_{n\in\{n'_k\}}]^{\perp}$.

If a complete biorthogonal sequence $\{x_n, x_n^*\}$ is total (resp. norming, strong) then $\{x_n\}$ is said to be an *M-basis* (resp. a *norming M-basis*, *strong M-basis*).

$\{x_n, x_n^*\}$ is said to be *bounded* (and $\{x_n\}$ *uniformly minimal*) if $\{x_n\}$ and $\{x_n^*\}$ are both bounded.

Moreover, in this note we say that $\{x_n, x_n^*\}$ is *convex strong* if, for each x in X, $x \in \overline{\operatorname{conv}}\{\text{finite subseries of } \sum_{n=1}^{\infty} x_n^*(x) x_n\}$.

We recall three characterizations of strong biorthogonal sequences:

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$\{x_n, x_n^*\}$ *is a strong biorthogonal sequence*

$\Leftrightarrow$ *for each decomposition* $\{n\} = \{n_k\} \cup \{n'_k\}, \{n_k\} \cap \{n'_k\} = \emptyset$, *of the positive integers, setting* $X_0 = \overline{\text{span}}\{x_{n'_k}\}$, *there exists* $\{F_k\} \subset (X/X_0)^*$ *such that* $\{x_{n_k} + X_0, F_k\}$ *is an M-basis of* X/X_0 ([20], p. 243)

$\Leftrightarrow$ *for each couple of infinite subsequences* $\{n_k\}$ *and* $\{n'_k\}$ *of* $\{n\}$, $\overline{\text{span}}\{x_{n_k}\} \cap \overline{\text{span}}\{x_{n'_k}\} = \overline{\text{span}}\{x_k\}_{k\in\{n_k\}\cap\{n'_k\}}$ ([17])

$\Leftrightarrow$ *for each* x *in* X, $x \in \overline{\text{span}}\{x_n^*(x)x_n\}$ ([20], p. 762).

Hence "convex strong" implies "strong".

Finally, if $\{x_n, x_n^*\}$ is biorthogonal then $\{x_n\}$ is said to be

- a *Steinitz basis* if, for each x in X and x^* in X^*, there exists a permutation $\{\pi(n)\}$ so that

$$x^*(x) = \sum_{n=1}^{\infty} x^*_{\pi(n)}(x)x^*(x_{\pi(n)});$$

- a *basis* if, for each x in X,

$$x = \sum_{n=1}^{\infty} x_n^*(x)x_n.$$

From [7] we recall the following characterization

I*. *A bounded biorthogonal sequence is convex strong if and only if it is a Steinitz basis.*

The search for a best complete sequence originates already in Banach's book [1] (1932) with the famous problems of existence of a basis and of a complete bounded biorthogonal sequence; the problem of existence of a strong biorthogonal sequence originates in a paper of Ruckle ([18], 1970) (see also [19] and [3]).

The story of this research goes through a number of intermediate results on existence of an M-basis (Markushevich [13], 1943), existence of a complete norming biorthogonal sequence (Mackey [11], 1946) and other improvements (Davis–Johnson [2], 1973).

Finally, the basis problem was given a negative answer by Enflo [5] (1973); while Ovsepian and Pełczyński proved the existence of a complete bounded biorthogonal sequence ([15], 1975; refined by Pełczyński [16], 1976).

For a long period of time we can see refinements of the negative answer of Enflo (for example, in these last years, Szarek [21] (1987) and Mankiewicz and Nielsen [12] (1989)); while the positive answer of Ovsepian and Pełczyński did not gain further improvements.

The aim of this note is to present the following positive answer:

THEOREM. *Every separable Banach space has a bounded norming convex strong biorthogonal sequence.*

That is, *every separable Banach space has a uniformly minimal norming convex strong M-basis which* (by I*) *is also a Steinitz basis.*

Remark 1. We showed in [24] that the concepts of norming M-basis, uniformly minimal M-basis and strong M-basis are quite independent.

Remark 2. Actually, the proof of §2 gives the following property: Every separable Banach space X has a uniformly minimal norming M-basis $\{\widetilde{z}_n\}$, with $\{\widetilde{z}_n, \widetilde{z}_n^*\}$ biorthogonal, such that there is an increasing sequence $\{q_m\}$ of positive integers so that for every $\overline{x}$ in X and for each $\varepsilon > 0$ there exists an integer m_ε so that, for every $m \ge m_\varepsilon$,

$$\Big\|\overline{x} - \Big\{\sum_{n=1}^{\overline{q}_m} \widetilde{z}_n^*(\overline{x})\widetilde{z}_n + \overline{\varepsilon}_m \sum_{k=1}^{N(m)} \widetilde{z}^*_{n(k,m)}(\overline{x})\widetilde{z}_{n(k,m)}\Big\}\Big\| < \varepsilon$$

for some $0 \le \overline{\varepsilon}_m < 1$ and some

$$q_m \le \overline{q}_m < n(1,m) < \ldots < n(N(m),m) \le q_{m+1}$$

with

$$\overline{\varepsilon}_m \Big\|\sum_{n=1}^{\overline{q}_m} \widetilde{z}_n^*(\overline{x})\widetilde{z}_n\Big\| < \varepsilon.$$

Hence we also have

$$\Big\|\overline{x} - \Big\{(1-\overline{\varepsilon}_m)\sum_{n=1}^{\overline{q}_m} \widetilde{z}_n^*(\overline{x})\widetilde{z}_n + \overline{\varepsilon}_m \sum_{k=1}^{N(m)} \widetilde{z}^*_{n(k,m)}(\overline{x})\widetilde{z}_{n(k,m)}\Big\}\Big\| < 2\varepsilon.$$

Remark 3. We recall that, if $\{x_n, x_n^*\}$ is a complete bounded biorthogonal sequence, then $\lim_{n\to\infty} x_n^*(x)x_n = 0$ for each x in X.

Acknowledgments are due to the referee for improving the presentation of this note.

1. Main tools of the proof.

The main tool in §2 is the following property ([22]; recall also [9] and [10]).

II*. *If* $\{x_n, x_n^*\}$ *is a complete norming biorthogonal sequence in* X *then there exists an increasing sequence* $\{r_m\}$ *of positive integers so that, for every* $\widehat{x}$ *in* X,

$$\widehat{x} = \lim_{m\to\infty}\Big[\sum_{n=1}^{r_m} x_n^*(\widehat{x})x_n + \sum_{n=r_m+1}^{r_{m+1}} \widehat{a}_n x_n\Big]$$

where $\{\widehat{a}_n\}$ *depends on* $\widehat{x}$ *while* $\{r_m\}$ *does not; moreover, if there exists an infinite subsequence* $\{m_k\}$ *of* $\{m\}$ *so that* $x_n^*(\widehat{x}) = 0$ *for* $r_{m_k}+1 \le n \le r_{m_k+1}$

for every k, then setting $r_{m_0}=0$ *we have*

$$\widehat{x}=\sum_{k=0}^{\infty}\sum_{n=r_{m_k}+1}^{r_{m_{k+1}}} x_n^*(\widehat{x})x_n.$$

The first statement follows from Theorem I of [22]; for the second, following the proof of Corollary 2 of [22] and setting

$$Y=\operatorname{span}\Big\{\{x_n\}_{n=1}^{r_{m_1}}\cup\Big\{\bigcup_{k=1}^{\infty}\{x_n\}_{n=r_{m_{k+1}}+1}^{r_{m_{k+1}}}\Big\}\Big\}$$

we have

$$x+Y=\sum_{k=1}^{\infty}\sum_{n=r_{m_k}+1}^{r_{m_k+1}}(x_n^*(x)x_n+Y)$$

for every x in X, and

$$x=\sum_{n=1}^{r_{m_1}}x_n^*(x)x_n+\sum_{k=1}^{\infty}\sum_{n=r_{m_{k+1}}+1}^{r_{m_{k+1}}}x_n^*(x)x_n$$

for every x in Y; thus in our case $\widehat{x}\in Y$ by the hypothesis and by the first of these two relations; then the assertion follows from the second relation since $x_n^*(\widehat{x})=0$ for $r_{m_k}+1\le n\le r_{m_{k+1}}$ for every k.

We point out that an M-basis has property II* if and only if it is norming [6].

The next main tool in §2 is the following property, which appears in [15] (see also [20], p. 248) and which is a modification of a lemma of Olevskiĭ [14]:

III*. *Let* $\{x_n,x_n^*\}_{n=1}^{2^Q}$ *be a biorthogonal sequence in* X. *Then there exists another biorthogonal sequence* $\{y_n,y_n^*\}_{n=1}^{2^Q}$ *with* $\operatorname{span}\{y_n\}_{n=1}^{2^Q}=\operatorname{span}\{x_n\}_{n=1}^{2^Q}$ *and* $\operatorname{span}\{y_n^*\}_{n=1}^{2^Q}=\operatorname{span}\{x_n^*\}_{n=1}^{2^Q}$ *and such that for every* n *with* $1\le n\le 2^Q$,

$$\|y_n\|<\|x_1\|/2^{Q/2}+(1+2^{1/2})\max\{\|x_k\|:2\le k\le 2^Q\},$$
$$\|y_n^*\|<\|x_1^*\|/2^{Q/2}+(1+2^{1/2})\max\{\|x_k^*\|:2\le k\le 2^Q\}.$$

More precisely,

$$y_n=\sum_{j=1}^{2^Q}\beta_{Qnj}x_j\quad\text{and}\quad y_n^*=\sum_{j=1}^{2^Q}\beta_{Qnj}x_j^*$$

where $\beta_{Qn1}=1/2^{Q/2}$ *for* $1\le n\le 2^Q$, *and moreover, for every* k *with* $0\le k\le Q-1$ *and every* j *with* $1\le j\le 2^k$, *we have*

$$\beta_{Q,i,2^k+j}=\begin{cases}1/2^{(Q-k)/2} & \text{for } (2j-2)2^{Q-k-1}+1\le i\le(2j-1)2^{Q-k-1},\\ -1/2^{(Q-k)/2} & \text{for } (2j-1)2^{Q-k-1}+1\le i\le 2j\cdot2^{Q-k-1},\\ 0 & \text{for } 1\le i\le(2j-2)2^{Q-k-1}\\ & \quad\text{and for } 2j\cdot2^{Q-k-1}+1\le i\le 2^Q.\end{cases}$$

We also use in §2 the following property ([23], see in particular (f) of the introduction):

IV*. *If* $\{x_n,f_n\}$ *is a norming (on* $\operatorname{span}\{x_n\}$*) bounded biorthogonal sequence in* X *then there exist* $\{y_n\}$ *in* X *and* $\{x_n^*\}\cup\{y_n^*\}$ *in* X^* *so that* $\{x_n,x_n^*\}\cup\{y_n,y_n^*\}$ *is a complete norming bounded biorthogonal sequence in* X.

Another main tool in §2 is the following property, which comes from the Dvoretzky theorem [4] that l^2 is finitely represented in every infinite-dimensional Banach space.

V*. *There exists in* X *a norming bounded biorthogonal sequence* $\{x_n,x_n^*\}$ *with* $\{x_n\}=\bigcup_{m=1}^{\infty}\{x_{mn}\}_{n=1}^{m}$ *such that, for every* m *and for every sequence* $\{a_n\}_{n=1}^{m}$ *of numbers*,

$$(1-2^{-m})\Big(\sum_{n=1}^{m}|a_n|^2\Big)^{1/2}\le\Big\|\sum_{n=1}^{m}a_nx_{mn}\Big\|\le(1+2^{-m})\Big(\sum_{n=1}^{m}|a_n|^2\Big)^{1/2}.$$

Indeed, let $\{y_n\}$ be a basic sequence of X, with a basis constant K. By [4] there exists an increasing sequence $\{r_m\}$ of positive integers so that, for every m, $\operatorname{span}\{y_n\}_{n=r_m+1}^{r_{m+1}}$ contains a sequence $\{x_{mn}\}_{n=1}^{m}$ with the property of the assertion. It is sufficient to prove that, for every fixed $p>1$ and for every k with $1\le k\le p$,

$$\Big\|\sum_{m=1}^{p-1}\sum_{n=1}^{m}a_{mn}x_{mn}+\sum_{n=1}^{k}a_{pn}x_{pn}\Big\|\le 8K\Big\|\sum_{m=1}^{p}\sum_{n=1}^{m}a_{mn}x_{mn}\Big\|$$

for every sequence $\{\{a_{mn}\}_{n=1}^{m}\}_{m=1}^{p}$ of numbers (indeed, it will then follow that $\{x_n\}=\bigcup_{m=1}^{\infty}\{x_{mn}\}_{n=1}^{m}$ is basic, with basis constant $\le 8K$, therefore norming and bounded too, where we use the intrinsic characterization (f) of [23] for norming sequences). Set

$$u=\sum_{m=1}^{p-1}\sum_{n=1}^{m}a_{mn}x_{pn},\quad v=\sum_{n=1}^{k}a_{pn}x_{pn},\quad w=\sum_{n=k+1}^{p}a_{pn}x_{pn}.$$

We know that $\|u\|\le K\|u+v+w\|$ since K is the basis constant of $\{y_n\}$; moreover, $\|v\|\le 2\|v+w\|$ since $\{x_{pn}\}_{n=1}^{p}$ has the property of the assertion. Then if $\|u\|\ge\|u+v\|/4$ we have

$$\|u+v\|\le 4\|u\|\le 4K\|u+v+w\|;$$

while if $\|u\| < \|u+v\|/4$, that is, $\|u\| < \|v\|/3$, it follows that

$$\begin{aligned}\|u+v\| &< (4/3)\|v\| = 8(\|v\|/2 - \|v\|/3) < 8(\|v\|/2 - \|u\|)\\ &\le 8(\|v+w\| - \|u\|) \le 8\|u+v+w\|.\end{aligned}$$

This completes the proof.

2. Proof of Theorem. By IV* and V*, together with the techniques of [23], there exists in X a norming M-basis $\{x_n\}$, with $\{x_n, x_n^*\}$ biorthogonal, such that $\|x_n\| = 1$ and $\|x_n^*\| < M$ for every n and $\{x_n\} = \{x_{n''}\} \cup \{x_{n'}\}$ with $\{x_{n'}\} = \bigcup_{m=1}^{\infty}\{x_{mn}\}_{n=1}^{m}$, where, for every m and for every sequence $\{a_n\}_{n=1}^{m}$ of numbers,

$$(1)\qquad (1-2^{-m})\Big(\sum_{n=1}^{m} a_n^2\Big)^{1/2} \le \Big\|\sum_{n=1}^{m} a_n x_{mn}\Big\| \le (1+2^{-m})\Big(\sum_{n=1}^{m} a_n^2\Big)^{1/2}.$$

We shall construct two biorthogonal sequences $\{y_n, y_n^*\}$ and $\{z_n, z_n^*\}$ by means of a suitable block perturbation of $\{x_n, x_n^*\}$, that is, there will be an increasing sequence $\{q_m\}$ of positive integers such that, for every m,

$$(2)\qquad \begin{aligned}\operatorname{span}\{y_n\}_{n=q_m+1}^{q_{m+1}} &= \operatorname{span}\{z_n\}_{n=q_m+1}^{q_{m+1}} = \operatorname{span}\{x_n\}_{n=q_m+1}^{q_{m+1}},\\ \operatorname{span}\{y_n^*\}_{n=q_m+1}^{q_{m+1}} &= \operatorname{span}\{z_n^*\}_{n=q_m+1}^{q_{m+1}} = \operatorname{span}\{x_n^*\}_{n=q_m+1}^{q_{m+1}}.\end{aligned}$$

We shall define $\{q_m\}$ by means of the sequence $\{r_m\}$ of II*, that is, we shall find an increasing sequence $\{t(m)\}$ of positive integers such that $q_m = r_{t(m)}$ for every m.

We start with

$$\{y_n, y_n^*\}_{n=1}^{q_1} = \{z_n, z_n^*\}_{n=1}^{q_1} = \{x_n, x_n^*\}_{n=1}^{r_1}$$

and we proceed by induction. Suppose we have defined $\{y_n, y_n^*\}_{n=1}^{q_m}$ and $\{z_n, z_n^*\}_{n=1}^{q_m}$ for some $m \ge 1$. We now construct $\{y_n, y_n^*\}_{n=q_m+1}^{q_{m+1}}$ and $\{z_n, z_n^*\}_{n=q_m+1}^{q_{m+1}}$.

First, we set

$$S_{m1} = 2^{m+2}Mr_{t(m)+1}, \quad Q_{m1} = 2(S_{m1}+m)M, \quad N_{m1} = 4^{m+2Q_{m1}+MS_{m1}}.$$

Now we choose a sequence $\{v_{m1n}\}_{n=1}^{L_{m1}}$ which is $(1/S_{m1})$-dense in the ball of radius $2S_{m1}$ in $\operatorname{span}\{x_n\}_{n=r_{t(m)+1}+1}^{r_{t(m)+2}}$. Next we set, by means of the sequences of (1),

$$\begin{aligned}&s'(m,1) = L_{m1}2^{Q_{m1}}N_{m1},\\ &s(m,1) = \text{the first integer} \ge s'(m,1)\\ &\qquad\qquad \text{such that } \{x_{s(m,1),n}\}_{n=1}^{s(m,1)} \subset \{x_n\}_{n>r_{t(m)+2}}.\end{aligned}$$

We arrange the first $s'(m,1)$ vectors of the sequence $\{x_{s(m,1),n}\}_{n=1}^{s(m,1)}$ in the following way:

$$\{x_{s(m,1),n}\}_{n=1}^{s'(m,1)} = \{\{\{x_{m1nkj}\}_{j=1}^{N_{m1}}\}_{k=1}^{2^{Q_{m1}}}\}_{n=1}^{L_{m1}}.$$

Now, we set

$$y_{q_m+1} = x_{q_m+1}/S_{m1} - \sum_{n=1}^{L_{m1}}\sum_{j=1}^{N_{m1}} x_{m1n1j} \quad\text{and}\quad y_{q_m+1}^* = S_{m1}x_{q_m+1}^*;$$

moreover, for every n and j with $1 \le n \le L_{m1}$ and $1 \le j \le N_{m1}$, we set

$$y_{m1n1j} = x_{m1n1j} + v_{m1n} \quad\text{and}\quad y_{m1n1j}^* = x_{m1n1j}^* + S_{m1}x_{q_m+1}^*,$$

while, for $2 \le k \le 2^{Q_{m1}}$, we set $y_{m1nkj} = x_{m1nkj}$ and $y_{m1nkj}^* = x_{m1nkj}^*$. Then there exists

$$\{y_n^*\}_{n=r_{t(m)+1}+1}^{r_{t(m)+2}} \subset \operatorname{span}\{x_{q_m+1}^* \cup \{x_n^*\}_{n=r_{t(m)+1}+1}^{r_{t(m)+2}} \cup \{x_{s(m,1),n}^*\}_{n=1}^{s'(m,1)}\}$$

such that, on setting $y_n = x_n$ for $r_{t(m)+1}+1 \le n \le r_{t(m)+2}$, the sequence

$$\{y_{q_m+1}, y_{q_m+1}^*\} \cup \{y_n, y_n^*\}_{n=r_{t(m)+1}+1}^{r_{t(m)+2}} \cup \{y_{s(m,1),n}, y_{s(m,1),n}^*\}_{n=1}^{s'(m,1)}$$

is biorthogonal; namely, if

$$v_{m1n} = \sum_{l=r_{t(m)+1}+1}^{r_{t(m)+2}} b_{m1nl}x_l \quad\text{for } 1 \le n \le L_{m1}$$

then

$$y_l^* = x_l^* - \sum_{n=1}^{L_{m1}}\sum_{j=1}^{N_{m1}} b_{m1nl}y_{m1n1j}^* \quad\text{for } r_{t(m)+1}+1 \le l \le r_{t(m)+2}.$$

At this point, by III* of §1 and by (1), there exists a sufficiently large positive integer $t(m,1)$ such that, on setting

$$\{x_n\}_{n=r_{t(m)+2}+1}^{r_{t(m,1)}} = \{x_{s(m,1),n}\}_{n=1}^{s'(m,1)} \cup \{x_{m1n}\}_{n=1}^{T_{m1}}$$

and $y_{m1n} = x_{m1n}$ and $y_{m1n}^* = x_{m1n}^*$ for $1 \le n \le T_{m1}$, there exists a block perturbation

$$\{z_{q_m+1}, z_{q_m+1}^*\} \cup \{z_n, z_n^*\}_{n=r_{t(m)+1}+1}^{r_{t(m)+2}} \cup \{z_{m1n}, z_{m1n}^*\}_{n=1}^{T_{m1}}$$

of

$$\{y_{q_m+1}, y_{q_m+1}^*\} \cup \{y_n, y_n^*\}_{n=r_{t(m)+1}+1}^{r_{t(m)+2}} \cup \{y_{m1n}, y_{m1n}^*\}_{n=1}^{T_{m1}}$$

such that $\max\{\|z_{q_m+1}\|, \|z_{q_m+1}^*\|/M; \|z_n\|, \|z_n^*\|/M$ for $r_{t(m)+1}+1 \le n \le r_{t(m)+2}; \|z_{m1n}\|, \|z_{m1n}^*\|/M$ for $1 \le n \le T_{m1}\} < 3$.

On the other hand, since by the above $2^{Q_{m1}/2} > 2^m MS_{m1}$, by III* and (1), for every n and j with $1 \le n \le L_{m1}$ and $1 \le j \le N_{m1}$, there exists a block perturbation

$$\{z_{m1nkj}, z_{m1nkj}^*\}_{k=1}^{2^{Q_{m1}}} \quad\text{of}\quad \{y_{m1nkj}, y_{m1nkj}^*\}_{k=1}^{2^{Q_{m1}}}$$

such that

$$\{\|z_{m1nkj}\|, \|z^*_{m1nkj}\|/M : 1 \le k \le 2^{Q_{m1}}\} < 3.$$

We now pass to the definition in the general case: that is, we fix an integer i with $1 < i \le r_{t(m)+1} - r_{t(m)}$ and we suppose to have defined

$$\{y_{q_m+l}, y^*_{q_m+l}\}_{l=1}^{i-1} \cup \{y_n, y^*_n\}_{n=r_{t(m)+1}+1}^{r_{t(m,i-1)}}$$

and

$$\{z_{q_m+l}, z^*_{q_m+l}\}_{l=1}^{i-1} \cup \{z_n, z^*_n\}_{n=r_{t(m)+1}+1}^{r_{t(m,i-1)}};$$

then we are going to define

$$\{y_{q_m+i}, y^*_{q_m+i}\} \cup \{y_n, y^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}$$

and

$$\{z_{q_m+i}, z^*_{q_m+i}\} \cup \{z_n, z^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}.$$

First, we set

$$(3)\qquad \begin{aligned} S_{mi} &= 2^{m+2} M r_{t(m,i-1)},\\ Q_{mi} &= 2(S_{mi}+m)M, \qquad N_{mi} = 4^{m+2Q_{mi}+MS_{mi}}. \end{aligned}$$

Again we choose a sequence $v = \{v_{min}\}_{n=1}^{L_{mi}}$ such that

(4) v is $(1/S_{mi})$-dense in the ball of radius $2S_{mi}$ in $\mathrm{span}\{x_n\}_{r_{t(m,i-1)}+1}^{n=r_{t(m,i-1)+1}}$

(then, on setting $t(m,0) = t(m)+1$, the definition of v_{m1n} agrees with this general definition).

Next, we set, by means of the sequences of (1),

$$(5)\qquad \begin{aligned} &s'(m,i) = L_{mi} 2^{Q_{mi}} N_{mi},\\ &s(m,i) = \text{the first integer } \ge s'(m,i) \text{ such that}\\ &\qquad \{x_{s(m,i),n}\}_{n=1}^{s(m,i)} \subset \{x_n\}_{n > r_{t(m,i-1)+1}}, \end{aligned}$$

We arrange the first $s'(m,i)$ vectors of $\{x_{s(m,i),n}\}_{n=1}^{s(m,i)}$ in the following way:

$$\{x_{s(m,i),n}\}_{n=1}^{s'(m,i)} = \{\{\{x_{minkj}\}_{j=1}^{N_{mi}}\}_{k=1}^{2^{Q_{mi}}}\}_{n=1}^{L_{mi}}.$$

Now, we set

$$y_{q_m+i} = x_{q_m+i}/S_{mi} - \sum_{n=1}^{L_{mi}}\sum_{j=1}^{N_{mi}} x_{min1j} \quad\text{and}\quad y^*_{q_m+i} = S_{mi} x^*_{q_m+i};$$

moreover, for every n and j with $1 \le n \le L_{mi}$ and $1 \le j \le N_{mi}$, we set

$$(6)\qquad y_{min1j} = x_{min1j} + v_{min} \quad\text{and}\quad y^*_{min1j} = x^*_{min1j} + S_{mi} x^*_{q_m+i},$$

while, for $2 \le k \le 2^{Q_{mi}}$, we set $y_{minkj} = x_{minkj}$ and $y^*_{minkj} = x^*_{minkj}$. Again as for $i = 1$ there exists

$$\{y^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}} \subset \mathrm{span}\{x^*_{q_m+i} \cup \{x^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}} \cup \{x^*_{s(m,i),n}\}_{n=1}^{s'(m,i)}\}$$

such that, on setting $y_n = x_n$ for $r_{t(m,i-1)} + 1 \le n \le r_{t(m,i-1)+1}$, the sequence

$$\{y_{q_m+i}, y^*_{q_m+i}\} \cup \{y_n, y^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}} \cup \{y_{s(m,i),n}, y^*_{s(m,i),n}\}_{n=1}^{s'(m,i)}$$

is biorthogonal.

Now, by III* and by (1) and (6), there exists a sufficiently large positive integer $t(m,i)$ such that, on setting

$$\{x_n\}_{n=r_{t(m,i-1)+1}+1}^{r_{t(m,i)}} = \{x_{s(m,i),n}\}_{n=1}^{s'(m,i)} \cup \{x_{min}\}_{n=1}^{T_{mi}}$$

and $y_{min} = x_{min}$ and $y^*_{min} = x^*_{min}$ for $1 \le n \le T_{mi}$, there exists a block perturbation

$$\{z_{q_m+i}, z^*_{q_m+i}\} \cup \{z_n, z^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}} \cup \{z_{min}, z^*_{min}\}_{n=1}^{T_{mi}}$$

of

$$\{y_{q_m+i}, y^*_{q_m+i}\} \cup \{y_n, y^*_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}} \cup \{y_{min}, y^*_{min}\}_{n=1}^{T_{mi}}$$

such that

$$(7)\qquad \begin{aligned} &\max\{\|z_{q_m+i}\|, \|z^*_{q_m+i}\|/M; \|z_n\|, \|z^*_n\|/M \text{ for}\\ &\quad r_{t(m,i-1)} + 1 \le n \le r_{t(m,i-1)+1}; \|z_{min}\|, \|z^*_{min}\|/M, 1 \le n \le T_{mi}\} < 3. \end{aligned}$$

Again by III* and (1), (3), (5) and (6) for every n and j with $1 \le n \le L_{mi}$ and $1 \le j \le N_{mi}$, there exists a block perturbation

$$\{z_{minkj}, z^*_{minkj}\}_{k=1}^{2^{Q_{mi}}} \quad\text{of}\quad \{y_{minkj}, y^*_{minkj}\}_{k=1}^{2^{Q_{mi}}}$$

such that, for every k with $1 \le k \le 2^{Q_{mi}}$,

$$(8)\qquad \|z_{minkj}\| < 3, \qquad \|z^*_{minkj}\|/M < 3.$$

We proceed in this way till y_{q_m+i} and z_{q_m+i} for $i = r_{t(m)+1} - r_{t(m)}$; then we set $q_{m+1} = r_{t(m,i)}$ for $i = r_{t(m)+1} - r_{t(m)}$; it follows that (2) is satisfied, and moreover, $\{z_n\}$ is uniformly minimal.

Now we consider the following permutation of $\{z_n\}_{n=q_m+1}^{q_{m+1}}$: By (6), (7) and (8) we have

$$\{z_n\}_{n=q_m+1}^{q_{m+1}} = \{z_{q_m+i}\}_{i=1}^{r_{t(m)+1}-r_{t(m)}} \cup \{\{z_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}}\}_{i=1}^{r_{t(m)+1}-r_{t(m)}}\}$$

where, for every i with $1 \le i \le r_{t(m)+1} - r_{t(m)}$,

$$\begin{aligned} \{z_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}} &= \{z_n\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i-1)+1}} \cup \{z_{min}\}_{n=1}^{T_{mi}}\\ &\quad \cup \{\{\{z_{miknj}\}_{j=1}^{N_{mi}}\}_{k=1}^{2^{Q_{mi}}}\}_{n=1}^{L_{mi}}. \end{aligned}$$

Now we take a biorthogonal sequence $\{\widetilde{z}_n, \widetilde{z}_n^*\}_{n=q_m+1}^{q_{m+1}}$ which is a permutation of $\{z_n, z_n^*\}_{n=q_m+1}^{q_{m+1}}$ where

$$
\begin{aligned}
&\{\widetilde{z}_n, \widetilde{z}_n^*\}_{n=q_m+1}^{q_m+r_{t(m,1)}-r_{t(m)+1}+1} = \{z_{q_m+1}, z_{q_m+1}^*\} \cup \{z_n, z_n^*\}_{n=r_{t(m)+1}+1}^{r_{t(m,1)}},\\
&\{\widetilde{z}_n, \widetilde{z}_n^*\}_{n=q_m+r_{t(m,i-1)}-r_{t(m,0)}+(i-1)+1}^{q_m+r_{t(m,i)}-r_{t(m,0)}+i}\\
&= \{z_{q_m+i}, z_{q_m+i}^*\} \cup \{z_n, z_n^*\}_{n=r_{t(m,i-1)}+1}^{r_{t(m,i)}} \quad \text{for } 1 < i \le r_{t(m)+1} - r_{t(m)}.
\end{aligned} \tag{9}
$$

Let us check that the assertion is satisfied. Let $\overline{x} \in X$ with $\|\overline{x}\| = 1$; there are two possibilities:

(A) There exists an integer m_0 such that, for each integer $m \ge m_0$, there exists another integer $i(m)$ so that

$$
\begin{aligned}
&1 \le i(m) \le r_{t(m)+1} - r_{t(m)},\\
&|x_{q_m+i(m)}^*(\overline{x})| > M/S_{m,i(m)},\\
&|x_{q_m+j}^*(\overline{x})| \le M/S_{m,j} \quad \text{for } i(m)+1 \le j \le r_{t(m)+1} - r_{t(m)}.
\end{aligned} \tag{10}
$$

We fix $\varepsilon > 0$. Since $\{x_n\}$ is uniformly minimal, by (1) and by Remark 3 of the introduction there exists an integer $m'(\varepsilon)$ such that

$$
\begin{aligned}
&1/2^{m'(\varepsilon)} < \varepsilon/2,\\
&|x_n^*(\overline{x})| < \varepsilon/2^3 \quad \text{for each } n > r_{t(m)} \text{ and } m \ge m'(\varepsilon),\\
&\Big\| \sum_{j=i(m)}^{r_{t(m)+1}-r_{t(m)}} x_{q_m+j}^*(\overline{x}) x_{q_m+j} \Big\| < \varepsilon/2^2
\end{aligned} \tag{11}
$$

(where the third inequality follows from the second and from the third inequality of (10)). By II* of §1 there exists another integer $m(\varepsilon) \ge m'(\varepsilon)$ so that, for each $m \ge m(\varepsilon)$, there exists v_m so that

$$
\begin{aligned}
&\Big\| \overline{x} - \Big\{ \sum_{n=1}^{r_{t(m,i(m)-1)}} x_n^*(\overline{x}) x_n + v_m \Big\} \Big\| < \varepsilon/2^2,\\
&v_m \in \operatorname{span}\{x_n\}_{n=r_{t(m,i(m)-1)}+1}^{r_{t(m,i(m)-1)+1}},\\
&\|v_m\| < 2M r_{t(m,i(m)-1)}.
\end{aligned} \tag{12}
$$

Indeed, by (1) and the first inequality we have

$$
\begin{aligned}
\|v_m\| &< \Big\| \overline{x} - \sum_{n=1}^{r_{t(m,i(m)-1)}} x_n^*(\overline{x}) x_n \Big\| + \varepsilon/2^2\\
&< 1 + M r_{t(m,i(m)-1)} + \varepsilon/2^2.
\end{aligned}
$$

On the other hand, by hypothesis and by (6)–(8) we have

$$
\begin{aligned}
\sum_{n=1}^{q_m+i(m)-1} x_n^*(\overline{x}) x_n + \sum_{n=r_{t(m)+1}+1}^{r_{t(m,i(m)-1)}} x_n^*(\overline{x}) x_n
&= \sum_{n=1}^{q_m+i(m)-1} y_n^*(\overline{x}) y_n + \sum_{n=r_{t(m)+1}+1}^{r_{t(m,i(m)-1)}} y_n^*(\overline{x}) y_n\\
&= \sum_{n=1}^{q_m+i(m)-1} z_n^*(\overline{x}) z_n + \sum_{n=r_{t(m)+1}+1}^{r_{t(m,i(m)-1)}} z_n^*(\overline{x}) z_n
\end{aligned}
$$

(where the indices n with $r_{t(m)+1} + 1 \le n \le r_{t(m,i(m)-1)}$ do not appear if $i(m) = 1$). Therefore, since

$$
\sum_{j=i(m)}^{r_{t(m)+1}-r_{t(m)}} x_{q_m+j}^*(\overline{x}) x_{q_m+j} = \sum_{n=q_m+i(m)}^{r_{t(m)+1}} x_n^*(\overline{x}) x_n,
$$

by (11) and (12) we obtain

$$
\begin{aligned}
&\Big\| \overline{x} - \Big\{ \sum_{n=1}^{q_m+i(m)-1} x_n^*(\overline{x}) x_n + \sum_{n=r_{t(m)+1}+1}^{r_{t(m,i(m)-1)}} x_n^*(\overline{x}) x_n + v_m \Big\} \Big\|\\
&= \Big\| \overline{x} - \Big\{ \sum_{n=1}^{q_m+i(m)-1} z_n^*(\overline{x}) z_n + \sum_{n=r_{t(m)+1}+1}^{r_{t(m,i(m)-1)}} z_n^*(\overline{x}) z_n + v_m \Big\} \Big\| < \varepsilon/2.
\end{aligned} \tag{13}
$$

By (10) we have

$$
S_{m,i(m)} |x_{q_m+i(m)}^*(\overline{x})| > M. \tag{14}
$$

Hence by (4), (11) and (12), for every j with $1 \le j \le N_{m,i(m)}$, there exists an integer $n(j,m)$, with $1 \le n(j,m) \le L_{m,i(m)}$, such that

$$
\Big\| \frac{x_{q_m+i(m)}^*(\overline{x})}{|x_{q_m+i(m)}^*(\overline{x})|} v_{m,i(m),n(1,m)} - v_m \Big\| < \frac{1}{S_{m,i(m)}};
$$

and for $2 \le j \le N_{m,i(m)}$,

$$
\begin{aligned}
&\Big\| \frac{x_{q_m+i(m)}^*(\overline{x})}{|x_{q_m+i(m)}^*(\overline{x})|} v_{m,i(m),n(j,m)}\\
&\quad + \frac{x_{m,i(m),n(j-1,m),1,j-1}^*(\overline{x})}{S_{m,i(m)} |x_{q_m+i(m)}^*(\overline{x})|} v_{m,i(m),n(j-1,m)} - v_m \Big\| < \frac{1}{S_{m,i(m)}}
\end{aligned} \tag{15}
$$

(since $v_{m,i(m),n(j-1,m)}$ and v_m belong to $\operatorname{span}\{x_n\}_{n=r_{t(m,i(m)-1)}+1}^{r_{t(m,i(m)-1)+1}}$, and moreover, by (11) and (14), $|x^*_{m,i(m),n(j-1,m),1,j-1}(\overline{x})/(S_{m,i(m)}x^*_{q_m+i(m)}(\overline{x}))| < \varepsilon/(2^3M)$).

Now set

$$\overline{q}_m = q_m + i(m) - 1 + r_{t(m,i(m)-1)} - r_{t(m)+1},$$

$$\overline{\varepsilon}_m = \frac{1}{N_{m,i(m)}S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|},$$

(16) $$\{\widetilde{z}_{n(k,m)}\}_{k=1}^{N(m)} = \{\{z_{m,i(m),n(j,m),k,j}\}_{k=1}^{2^{Q_{m,i(m)}}}\}_{j=1}^{N_{m,i(m)}},$$

$$A = \Big\|\overline{x} - \Big\{(1-\overline{\varepsilon}_m)\sum_{n=1}^{\overline{q}_m}\widetilde{z}^*_n(\overline{x})\widetilde{z}_n + \overline{\varepsilon}_m\sum_{k=1}^{N(m)}\widetilde{z}^*_{n(k,m)}(\overline{x})\widetilde{z}_{n(k,m)}\Big\}\Big\|.$$

Since by (9),

$$q_m \le \overline{q}_m < n(1,m) < \ldots < n(k,m) < \ldots < n(N(m),m) \le q_m + 1,$$

it is sufficient to prove that

(17) $$A < \varepsilon.$$

By (8), (9), (13) and (16) we have

$$\begin{aligned} A = \Big\|\overline{x} - \Big\{&\Big(1 - \frac{1}{N_{m,i(m)}S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|}\Big) \\ &\times \Big(\sum_{n=1}^{q_m+i(m)-1} z^*_n(\overline{x})z_n + \sum_{n=r_{t(m)+1}+1}^{r_{t(m,i(m)-1)}} z^*_n(\overline{x})z_n\Big) \\ &+ \frac{1}{N_{m,i(m)}S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} \\ &\times \sum_{j=1}^{N_{m,i(m)}}\sum_{k=1}^{2^{Q_{m,i(m)}}} z^*_{m,i(m),n(j,m),k,j}(\overline{x})z_{m,i(m),n(j,m),k,j}\Big\}\Big\| \\ < \varepsilon/2 + A_1 + A_{1,0}& \end{aligned}$$

with

$$A_1 = \frac{1}{N_{m,i(m)}S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} \times \Big\|\sum_{n=1}^{q_m+i(m)-1} x^*_n(\overline{x})x_n + \sum_{n=r_{t(m)+1}+1}^{r_{t(m,i(m)-1)}} x^*_n(\overline{x})x_n\Big\|,$$

$$A_{1,0} = \Big\|\frac{1}{N_{m,i(m)}S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} \times \sum_{j=1}^{N_{m,i(m)}}\sum_{k=1}^{2^{Q_{m,i(m)}}} y^*_{m,i(m),n(j,m),k,j}(\overline{x})y_{m,i(m),n(j,m),k,j} - v_m\Big\|.$$

By (1) and (3)–(14) we have

$$\begin{aligned} A_1 &< \frac{r_{t(m,i(m)-1)}M}{N_{m,i(m)}S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} < \frac{r_{t(m,i(m)-1)}}{N_{m,i(m)}} \\ &< 1/4^{m+2Q_{m,i(m)}} < 1/2^{m+3}. \end{aligned}$$

By (6) and (8) we have $A_{1,0} \le A_2 + A_{2,0}$ with

$$A_2 = \Big\|\frac{1}{N_{m,i(m)}S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} \times \sum_{j=1}^{N_{m,i(m)}}\sum_{k=2}^{2^{Q_{m,i(m)}}} x^*_{m,i(m),n(j,m),k,j}(\overline{x})x_{m,i(m),n(j,m),k,j}\Big\|,$$

$$A_{2,0} = \frac{1}{N_{m,i(m)}}\Big\|\sum_{j=1}^{N_{m,i(m)}}\Big(\frac{1}{(S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} \times y^*_{m,i(m),n(j,m),1,j}(\overline{x})y_{m,i(m),n(j,m),1,j} - v_m\Big)\Big\|.$$

By (1), (3), (5), (11) and (14) we obtain

$$\begin{aligned} A_2 &< \frac{2}{N_{m,i(m)}S_{m,i,(m)}|x^*_{q_m+i(m)}(\overline{x})|}(N_{m,i(m)}2^{Q_{m,i(m)}})^{1/2} \\ &< \frac{2^{Q_{m,i(m)}/2+1}}{(N_{m,i(m)})^{1/2}M} < 2/4^{m+MS_{m,i(m)}} < 1/2^{m+3}. \end{aligned}$$

By (6) and (8) we see that

$$\begin{aligned} A_{2,0} = \frac{1}{N_{m,i(m)}}\Big\|\sum_{j=1}^{N_{m,i(m)}}\Big\{&\frac{1}{S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} \\ &\times (x^*_{(m,i(m),n(j,m),1,j}(\overline{x}) + S_{m,i(m)}x^*_{q_m+i(m)}(\overline{x})) \\ &\times (x_{(m,i(m),n(j,m),1,j} + v_{m,i(m),n(j,m)}) - v_m\Big\}\Big\| \le A_3 + A_{3,0} \end{aligned}$$

with

$$A_3 = \frac{1}{N_{m,i(m)}} \times \Big\| \sum_{j=1}^{N_{m,i(m)}} \Big(\frac{x^*_{m,i(m),n(j,m),1,j}(\overline{x})}{S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} + \frac{x^*_{q_m+i(m)}(\overline{x})}{|x^*_{q_m+i(m)}(\overline{x})|} \Big) x_{m,i(m),n(j,m),1,j} \Big\|,$$

$$A_{3,0} = \frac{1}{N_{m,i(m)}} \Big\| \sum_{j=1}^{N_{m,i(m)}} \Big\{ \Big(\frac{x^*_{m,i(m),n(j,m),1,j}(\overline{x})}{S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} v_{m,i(m),n(j,m)} + \frac{x^*_{q_m+i(m)}(\overline{x})}{|x^*_{q_m+i(m)}(\overline{x})|} v_{m,i(m),n(j,m)} - v_m \Big\} \Big\|.$$

By (1), (3), (5), (11) and (14) we have

$$A_3 < \frac{2}{N_{m,i(m)}} \{N_{m,i(m)}(\varepsilon/(M\cdot 2^3)+1)\}^{1/2} < 4/(N_{m,i(m)})^{1/2} < 4/2^{5m+5S_{m,i(m)}} < 1/2^{m+3}.$$

On the other hand, $A_{3,0} \le A_4 + A_5$ with

$$A_4 = \frac{1}{N_{m,i(m)}} \Big\| \Big\{ \frac{x^*_{q_m+i(m)}(\overline{x})}{|x^*_{q_m+i(m)}(\overline{x})|} v_{m,i(m),n(1,m)} - v_m \Big\} + \sum_{j=2}^{N_{m,i(m)}} \Big\{ \frac{x^*_{m,i(m),n(j-1,m),1,j-1}(\overline{x})}{S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} v_{m,i(m),n(j-1,m)} + \frac{x^*_{q_m+i(m)}(\overline{x})}{|x^*_{q_m+i(m)}(\overline{x})|} v_{m,i(m),n(j,m)} - v_m \Big\} \Big\|,$$

$$A_5 = \frac{1}{N_{m,i(m)}} \Big\| \frac{x^*_{m,i(m),n(N_{m,i(m)},m),1,N_{m,i(m)}}(\overline{x})}{S_{m,i(m)}|x^*_{q_m+i(m)}(\overline{x})|} v_{m,i(m),n(N_{m,i(m)},m)} \Big\|.$$

By (3) and (15) we have $A_4 < 1/S_{m,i(m)} < 1/2^{m+2}$, while by (3), (4), (11) and (14),

$$A_5 < \frac{2\varepsilon S_{m,i(m)}}{M\cdot 2^3 N_{m,i(m)}} < \frac{2\varepsilon}{2^3 4^{m+2Q_{m,i(m)}}} < \frac{1}{2^{m+3}}.$$

Consequently,

$$A < \varepsilon/2 + A_1 + A_2 + A_3 + A_4 + A_5 < \varepsilon/2 + 1/2^m < \varepsilon.$$

That is, (17) is proved.

(B) If (A) does not occur then there exists a subsequence $\{m(k)\}$ of $\{m\}$ such that, for every k,

$$|x^*_{q_{m(k)}+i}(\overline{x})| \le M/S_{m(k),i} \quad \text{for } 1 \le i \le r_{t(m(k))+1} - r_{t(m(k))}.$$

Hence, by (1) and (3), for every k we have

$$\Big\| \sum_{n=r_{t(m(k))}+1}^{r_{t(m(k))+1}} x^*_n(\overline{x}) x_n \Big\| < \sum_{n=r_{t(m(k))}+1}^{r_{t(m(k))+1}} |x^*_n(\overline{x})| < 1/2^{m(k)}.$$

If we set

$$\overline{x} = x' + x'' \quad \text{with} \quad x'' = \sum_{k=1}^{\infty} \sum_{n=r_{t(m(k))}+1}^{r_{t(m(k))+1}} x^*_n(\overline{x}) x_n,$$

it follows that $x^*_n(x') = 0$ for $r_{t(m(k))}+1 \le n \le r_{t(m(k))+1}$ for every k; hence by the second part of II* of §1 we have

$$x' = \sum_{n=1}^{r_{t(m(1))}} x^*_n(x') x_n + \sum_{k=1}^{\infty} \sum_{n=r_{t(m(k))}+1}^{r_{t(m(k+1))}} x^*_n(x') x_n = \sum_{n=1}^{r_{t(m(1))}} x^*_n(\overline{x}) x_n + \sum_{k=1}^{\infty} \sum_{n=r_{t(m(k))+1}+1}^{r_{t(m(k+1))}} x^*_n(\overline{x}) x_n;$$

therefore, setting $q_{m(k)} = r_{t(m(k))}$ for every k and $q_{m(0)} = 0$, we have

$$\overline{x} = \sum_{k=0}^{\infty} \sum_{n=q_{m(k)}+1}^{q_{m(k+1)}} x^*_n(\overline{x}) x_n = \sum_{k=0}^{\infty} \sum_{n=q_{m(k)}+1}^{q_{m(k+1)}} y^*_n(\overline{x}) y_n = \sum_{k=0}^{\infty} \sum_{n=q_{m(k)}+1}^{q_{m(k+1)}} \widetilde{y}^*_n(\overline{x}) \widetilde{y}_n = \sum_{k=0}^{\infty} \sum_{n=q_{m(k)}+1}^{q_{m(k+1)}} z^*_n(\overline{x}) z_n.$$

This completes the proof of the Theorem.

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Operators in finite distributive subspace lattices II

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Abstract. In a previous paper we gave an example of a finite distributive subspace lattice $\mathcal{L}$ on a Hilbert space and a rank two operator of $\operatorname{Alg}\mathcal{L}$ that cannot be written as a finite sum of rank one operators from $\operatorname{Alg}\mathcal{L}$. The lattice $\mathcal{L}$ was a specific realization of the free distributive lattice on three generators. In the present paper, which is a sequel to the aforementioned one, we study $\operatorname{Alg}\mathcal{L}$ for the general free distributive lattice with three generators (on a normed space). Necessary and sufficient conditions are given for 1) a finite rank operator of $\operatorname{Alg}\mathcal{L}$ to be written as a finite sum of rank ones from $\operatorname{Alg}\mathcal{L}$, and 2) a realization of $\mathcal{L}$ to contain a finite rank operator of $\operatorname{Alg}\mathcal{L}$ with the preceding property. These results are then used to show the curiosity that the product of two finite rank operators of $\operatorname{Alg}\mathcal{L}$ always has the above property.

1. Introduction. This paper is a continuation of [7], of which we shall assume familiarity and whose notation we follow.

Briefly, if $\mathcal{L}$ is a subspace lattice on a normed space $\mathcal{X}$, a general question is whether every finite rank operator of $\operatorname{Alg}\mathcal{L}$ has the FRP, i.e. whether it can be written as a finite sum of rank one operators from $\operatorname{Alg}\mathcal{L}$. The question is more natural in the case of completely distributive $\mathcal{L}$, as $\operatorname{Alg}\mathcal{L}$ then has a large supply of rank one operators [4]. Indeed, in the special case of a nest $\mathcal{L}$ the answer is affirmative [1, 6] and so is the case when $\mathcal{L}$ is a complete atomic Boolean subspace lattice [5, 3]. (In some of these results $\mathcal{X}$ was assumed a Hilbert space.) For general completely distributive lattices the answer was again shown to be affirmative if the underlying space was finite-dimensional [5] but the question was finally settled negatively by Hopenwasser and Moore [2] in infinite dimensions. In the same paper they give an affirmative answer if $\mathcal{L}$ is a finite width (see [2] for the definition) commutative subspace lattice. Their example of a completely distributive subspace lattice $\mathcal{L}$ for which $\operatorname{Alg}\mathcal{L}$ fails the FRP has an infinite number of elements. This then left open the case of finite distributive subspace lattices $\mathcal{L}$, which was settled negatively in [7]. There, a specific realization of the free distributive lattice $\mathcal{L}_3$ was

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