Acta Math., 175 (1995), 197–226

Everywhere discontinuous harmonic maps into spheres

by

TRISTAN RIVIÈRE

Ecole Normale Supérieure de Cachan
Cachan, France

1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^n$, S^2 the unit sphere of $\mathbf{R}^3$ and (Σ, h) a surface homeomorphic to S^2 with a metric h. We may assume, using the Nash–Moser theorem, that Σ is isometrically imbedded in some Euclidean space $\mathbf{R}^k$. We consider the Sobolev space

$$H^1(\Omega,\Sigma)=\{u\in H^1(\Omega,\mathbf{R}^k): u(x)\in\Sigma,\ \text{a.e. } x\in\Omega\}.$$

Let $E(u)=\int_\Omega |\nabla u|^2$ be the Dirichlet energy for any u in $H^1(\Omega,\Sigma)$. For a sufficiently small neighborhood V of Σ in $\mathbf{R}^k$ the projection π of a point x of V is well defined. Weakly harmonic maps from Ω into Σ are critical points in $H^1(\Omega,\Sigma)$ of the Dirichlet energy in the following way:

$$u \text{ is weakly harmonic if } \forall\xi\in C_c^\infty(\Omega,\mathbf{R}^n),\ \frac{d}{dt}E(\pi(u+t\xi))\Big|_{t=0}=0. \tag{1}$$

This is equivalent to the fact that u verifies the Euler–Lagrange equation

$$-\Delta u = A(u)(\nabla u,\nabla u)\ \text{ in } \mathcal{D}'(\Omega,\mathbf{R}^m),\quad u\in\Sigma\ \text{ a.e.}, \tag{2}$$

where $A(u)$ is the second fundamental form of Σ and where we have used the notation

$$A(u)(\nabla u,\nabla u)=A(u)\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial x}\right)+A(u)\left(\frac{\partial u}{\partial y},\frac{\partial u}{\partial y}\right)+A(u)\left(\frac{\partial u}{\partial z},\frac{\partial u}{\partial z}\right).$$

In this paper we are interested in the singular set of such maps.

The system (2) is non-linear elliptic, and the non-linearity, produced by the fact that our map takes its values in a non-flat manifold, has a quadratic growth for the gradient:

$$-\Delta u = f(u,\nabla u)\ \text{ in } \mathcal{D}'(\Omega,\mathbf{R}^m)\quad \text{where } |f(x,p)|\leqslant C(|x|^s+|p|^2). \tag{3}$$

Systems of this kind are studied in [8] under the name of "systems verifying a natural growth condition". Under the only hypothesis that u belongs to $L^\infty \cup W^{1,2}$, there are no general results concerning regularity, or partial regularity, of weak solutions of (3). However, the problem of the regularity of weakly harmonic maps represents an almost independent theory.

Let u be in $H^1(\Omega, \Sigma)$. By $\operatorname{Sing} u$ we denote the complement of the largest open set where u is C^∞. In the case of weakly harmonic maps this set coincides with the set where u is discontinuous: this follows directly from a result of Hildebrandt, Kaul and Widman (see [13]) which asserts that any weakly harmonic map from any open set with values in a sufficiently small part of Σ is C^∞.

In a previous paper [15] we presented most of the results obtained concerning the size of $\operatorname{Sing} u$ for u belonging to different subclasses of the weakly harmonic maps. It was already well-known that, for $n=3$, contrary to dimension two (see the result of Helein [12]), we cannot expect $\operatorname{Sing} u$ to be the empty set in the general case: the map $x/|x|$ from B^3 into S^2 is a simple example of a non-regular weakly harmonic map. In the particular case where u is a weakly harmonic map which minimizes the Dirichlet energy, in dimension three for instance, a result of Schoen and Uhlenbeck [18] asserts that this set cannot be larger than isolated points. In [15] we gave examples of weakly harmonic maps from B^3 into S^2 whose singular set is exactly a segment. This proved in particular that $\mathcal{H}^1(\operatorname{Sing} u)$ (where $\mathcal{H}^1$ denotes the Hausdorff measure of dimension one) is not necessarily zero, which is valid for the subclass of the stationary weakly harmonic maps (see the results of Evans [7] and Bethuel [2]). We also conjectured that we could obtain, in dimension three, larger singular sets.

Our main result is the following (see Theorem 1): for any given non-constant boundary condition $\phi \in C^\infty(\partial B^3, \Sigma)$ there exists a weakly harmonic map from B^3 into Σ taking the value ϕ on the boundary and such that $\operatorname{Sing} u$ coincides with the closed ball $\overline{B^3}$.

It is natural to conjecture that any closed subset of B^3 can be the singular set of a weakly harmonic map with values in Σ. We can also ask ourself whether there exist non-constant weakly harmonic maps having a constant trace on ∂B^3. Since any regular weakly harmonic map having a constant trace on ∂B^3 is constant, this question is equivalent to the question of the existence of a weakly harmonic map having a constant boundary value and a non-empty singular set.

One can point out that our result clearly implies that, for any integers $n \geqslant 3$ and $p \geqslant 2$, there exists an everywhere discontinuous harmonic map U from B^n into S^p: on each intersection of B^n with the 3-dimensional subspaces having a fixed direction define our map U to be a fixed everywhere discontinuous harmonic map u from dimension three into S^2 (since Δu is perpendicular to S^2 in $\mathbf{R}^3$, Δu is still perpendicular to S^p in $\mathbf{R}^{p+1}$

and U is a harmonic map into S^p).

We may now ask the question whether harmonic maps can be so singular when the target surface is no more homeomorphic to the sphere. In [17] we prove that weakly harmonic maps with values in any 2-dimensional torus of revolution are necessarily regular. We conjecture that this remains true in the general case of target surfaces with a non-null genus: the lack of topological obstruction ($\pi_2(\Sigma)=0$) could imply regularity for the weakly harmonic maps. This conjecture can also be extended to target manifolds of dimension greater than two which do not contain harmonic spheres, i.e. target manifolds for which there do not exist non-constant regular harmonic maps from S^2.

The paper is organised as follows: In §2.1 we recall basic elements of the relaxed energy theory that we extend to any metric on the sphere. Most of the tools presented in this section are introduced in [6], [3], [9] and [10]. §2.2 deals with the problem of the insertion of point singularities into regular maps in dimension 3 taking their values in Σ. Chapter 3 is devoted to the construction of singular maps and to the proof of our main theorem whose precise statement is given at the beginning of §3.1. In the Appendix we prove the technical lemmas of Chapter 2.

2. Preliminaries

2.1. Minimal connections and relaxed energies

As from now $n=3$ and $\Omega=B^3$.

If Σ is diffeomophic to S^2, let g be the metric pullback of h on S^2 by this diffeomorphism. By definition this diffeomorphism is an isometry between (S^2,g) and (Σ,h), hence it is equivalent to take (S^2,g) or (Σ,h) as target surface. By S^2 we will denote the sphere with the induced metric in $\mathbf{R}^3$ and by Σ the same sphere but with the metric g.

Consider a point a in $\mathbf{R}^3$ and $u\in C^\infty(B_\varrho(a)\setminus\{a\},\Sigma)$ with $\varrho>0$. Let S be a sphere centered at a and included in $B_\varrho(a)$. Then u restricted to S has a topological degree in $\mathbf{Z}$. By continuity this number is independent of the choice of S: this is the degree of u at a.

Let $(P_i)_{1\leqslant i\leqslant n}$ be a sequence of points of B^3 (the same point can be repeated several times) and let $(N_i)_{1\leqslant i\leqslant n}$ be another sequence of points of B^3. The minimal connection between the $(P_i)_{1\leqslant i\leqslant n}$ and the $(N_i)_{1\leqslant i\leqslant n}$ is the number

$$L(P_i,N_i)=\min_{\sigma\in\mathcal{S}_n}\left\{\sum_{i=1}^{n}|P_i-N_{\sigma(i)}|\right\} \tag{4}$$

where $\mathcal{S}_n$ is the set of the permutations in $\mathbf{N}_n=\{1,...,n\}$.

Suppose now that $(P_i)_{1\leqslant i\leqslant p}$ and $(N_i)_{1\leqslant i\leqslant n}$ are two finite sequences of points of B^3 with $p\neq n$ (we may take $p<n$ here). In such a case the additional points of the largest sequence are connected to the boundary ∂B^3. Thus the minimal connection between those two sequences becomes

$$L(P_i,N_i)=\min_{\sigma\in\mathcal{I}_{p,n}}\left\{\sum_{i=1}^{n}|P_i-N_{\sigma(i)}|+\sum_{k\in\mathbf{N}_n\setminus\sigma(\mathbf{N}_p)}d(N_k,\partial B^3)\right\} \tag{5}$$

where $\mathcal{I}_{p,n}$ is the set of injections from $\mathbf{N}_p$ into $\mathbf{N}_n$. Let $\sigma_0\in\mathcal{I}_{p,n}$ realise the previous minimum. By minimal connection, we also call the following union of segments:

$$\bigcup_{i=1,\dots,p}[P_i,N_{\sigma_0(i)}]\cup\bigcup_{k\in\mathbf{N}_n\setminus\sigma_0(\mathbf{N}_p)}[N_k,\Pi_{\partial B^3}(N_k)] \tag{6}$$

where $\Pi_{\partial B^3}(N_k)$ is the projection of N_k onto the boundary of B^3. We observe that the sum of the lengths of the previous segments is equal to $L(P_i,N_i)$. In fact, we also call this number the length of the minimal connection. Since σ_0 is not necessarily unique, the minimal connection of two given sequences $(P_i)_{1\leqslant i\leqslant p}$ and $(N_i)_{1\leqslant i\leqslant n}$, seen as a union of segments, is also not necessarily unique (what is unique is the length $L(P_i,N_i)$), but, in this paper, couples of finite sequences of points of B^3 having a unique minimal connection will play an important role (see §2.3 and §3.3).

The maps of $H^1(B^3,\Sigma)$ having a finite number of singularities will be useful for two reasons: their singular set is very simple, moreover a result of Bethuel and Zheng (see [4]) asserts that this set is dense in $H^1(B^3,\Sigma)$. Let $R(B^3,\Sigma)$ be the subset of $H^1(B^3,\Sigma)$ which consists of the maps having a finite number of singularities with degree ±1. We also have the density result $\overline{R(B^3,\Sigma)}^{H^1}=H^1(B^3,\Sigma)$.

Let u be in $R(B^3,\Sigma)$. By $L(u)$ we denote the minimal connection between the singularities of u with degree $+1$ and the singularities of u with degree -1. Let ϕ be in $C^\infty(\partial B^3,\Sigma)$. By $R_\phi(B^3,\Sigma)$ we denote the set of maps in $R(B^3,\Sigma)$ with value ϕ on the boundary.

Let u and v be in $R_\phi(B^3,\Sigma)$. By $L(u,v)$ we denote the minimal connection between the union of the singularities of u of degree $+1$ and the singularities of v of degree -1 and the union of the singularities of u of degree -1 and the singularities of v of degree $+1$.

Remark. For u and v in $R_\phi(B^3,\Sigma)$, u and v have the same singularities with the same degree if and only if $L(u,v)=0$.

In [3], Bethuel, Brezis and Coron extend the notion of minimal connection to any map in $H^1(B^3,S^2)$ for ϕ of degree zero. $L(u)$ becomes:

$$L(u)=\frac{1}{4\pi}\sup_{\substack{\xi:B^3\to\mathbf{R}\\ \|\nabla\xi\|_{L^\infty}\leqslant1}}\left\{\int_{B^3}D(u)\cdot\nabla\xi-\int_{\partial B^3}D(u)\cdot n\,\xi\right\} \tag{7}$$

where $D(u)=(u\cdot u_y\wedge u_z, u\cdot u_z\wedge u_x, u\cdot u_x\wedge u_y)$ and n is the exterior normal of ∂B^3.

For u and v in $H^1_\phi(B^3,\Sigma)$, $L(u,v)$ becomes:

$$L(u,v)=\frac{1}{4\pi}\sup_{\substack{\xi:B^3\to\mathbf{R}\\ \|\nabla\xi\|_{L^\infty}\leqslant 1}}\left\{\int_{B^3}(D(u)-D(v))\cdot\nabla\xi\right\}. \tag{8}$$

For the convenience of the reader we give briefly, in the Appendix, the main arguments which show that (7) coincides with the definition of the minimal connection given below for maps having a finite number of singularities.

Remark. $D(u)$ is the adjoint vector field to the 2-form $u^{\#}\omega$, the pullback by u of the volume form on S^2 (for the standard metric): $\omega=x\,dy\wedge dz+y\,dz\wedge dx+z\,dx\wedge dy$. In fact, in the definitions of $L(u)$ and $L(u,v)$, $D(u)$ can be replaced by the adjoint vector field of the pullback by u of any non-exact two form ω' on S^2 such that $\int_{S^2}\omega'\neq 0$:

$$L(u)=\left(\int_{S^2}\omega'\right)^{-1}\sup_{\substack{\xi:B^3\to\mathbf{R}\\ \|\nabla\xi\|_{L^\infty}\leqslant 1}}\left\{\int_{B^3}*u^{\#}\omega'\cdot\nabla\xi-\int_{\partial B^3}*u^{\#}\omega'\cdot n\,\xi\,d\sigma\right\}. \tag{9}$$

This is a consequence of the fact that the integral definition of the degree does not depend on the non-exact 2-form that we choose on the sphere.

This remark is useful for proving lower semi-continuity in the weak H^1-topology of the relaxed energies, defined below, in the case of any metric on the sphere: we will replace ω by the volume form ω' associated to the metric g.

In [3] the authors also introduce the notion of relaxed energy which plays an essential role in our work. Let u be in $H^1(B^3,\Sigma)$. By $F(u)$ we denote the relaxed energy of u:

$$F(u)=E(u)+2\left(\int_{S^2}\omega'\right)L(u). \tag{10}$$

Let ϕ be in $C^\infty(\partial B^3,\Sigma)$ and u,v in $H^1_\phi(B^3,\Sigma)$. By $F_v(u)$ we denote the relaxed energy relative to v:

$$F_v(u)=E(u)+2\left(\int_{S^2}\omega'\right)L(u,v) \tag{11}$$

Let us denote by $A(\Sigma)=\int_{S^2}\omega'$ the volume of Σ.

We now present some properties of the minimal connections and the relaxed energies we will use in this paper.

(a) $L(u)$ and $L(u,v)$ are continuous in the strong H^1- and $H^1\times H^1$-topologies respectively.

(b) $F(u)$ and $F_v(u)$ (for a fixed v) are lower semi-continuous in the weak H^1-topology.

(c) The critical points of the relaxed energies $F(u)$ and $F_v(u)$ are weakly harmonic maps.

(d) If v is in $R_\phi(B^3,\Sigma)$ any minimizer u of $F_v(u)$ satisfies $\mathcal{H}^1(\operatorname{Sing} u)<+\infty$.

Properties (a), (b), (c) and (d) are proved in [3] and [9] in the case of S^2. (a) does not depend on the metric, (c) and (d) can be established exactly in the same way as in [3] and [9]. To prove (b) in the case of any metric it is easier to work with the pullback of the volume form associated to the metric than to work with $D(u)$. For the convenience of the reader we give the complete proof of (b) in the Appendix.

2.2. Construction of dipoles

The basic tool of this paper, which we present in this section, is the insertion of point singularities into regular maps. Most of the time the singularities we will add have the degree ± 1. For topological reasons such insertions must contain as many singularities of degree $+1$ as singularities of degree -1. Naturally the most elementary insertion is the insertion of a couple of singularities having degrees $+1$ and -1: such a singular configuration is called a "dipole". We will now deal with the problem of inserting a dipole into any regular map in dimension three taking its values in S^2 by using as little Dirichlet energy as possible.

The first time the word dipole appeared in that context was in [6] related to the fact that the infimum of $E(u)$ among the maps u in $H^1(\mathbf{R}^3,S^2)$, which have two fixed isolated singularities of degrees $+1$ and -1, is exactly 8π times the minimal connection of this singular configuration: that is, 8π times the length of the segment formed by the two singularities. This can be interpreted in the following way: if we want to insert a dipole into a constant map in dimension three, taking its values in S^2, it is necessary to spend strictly more than 8π times the length of the dipole. In the case where we want to insert a dipole into non-constant maps we can take advantage of the variations of those maps to make a construction which uses in all cases strictly less energy than 8π times the length of the dipole. This was suggested in [3], and proved in the axially symmetric case by Hardt, Lin and Poon (see [11]). Let us mention also that the idea of this strict inequality was first introduced, for the 2-dimensional case, by Brezis and Coron in [5]. As we will see in Chapter 3 that is the key point of our proof of the existence of totally discontinuous harmonic maps. This was also the key point of our proof of the existence of infinitely many weakly harmonic maps for a given non-constant boundary condition from a domain in dimension three, into Σ (see [16]). We give now the precise statement of this construction (Σ is still the notation for (S^2, g)):

LEMMA A1. *Consider a regular map u from a ball $B_r(x)$ of center x and radius r taking its values into Σ and such that $\nabla u(x) \neq 0$. For any $\varrho < r$ there exist a bipoint (P, N) of center x included in $B_\varrho(x)$ and a map $\tilde{u}$ in $H^1(B_r(x), \Sigma)$ such that*

$$\begin{cases} \operatorname{Sing} \tilde{u} = \{P, N\} \text{ and } \deg(\tilde{u}, P) = -\deg(\tilde{u}, N) = +1, \\ \tilde{u} = u \text{ in } B_r(x) \subset B_\varrho(x), \\ E(\tilde{u}) < E(u) + 2A(\Sigma)|P - N|. \end{cases} \tag{12}$$

The proof of this lemma is given in the Appendix.

We note that, in the previous construction, the dipole which we insert is not prescribed; we just know its existence. The following lemma will be useful.

LEMMA A2 [1]. *Let u be in $C^\infty(\bar{\Omega}, S^2)$ and P, N be two distinct points of Ω. For any $C > 2A(\Sigma)$ and for any open set $O \subset \Omega$ which contains the segment $[P, N]$, there exists $\tilde{u} \in H^1(\Omega, \Sigma) \cap C^\infty(\bar{\Omega} \setminus \{P, N\}, \Sigma)$ such that*

$$\begin{cases} \tilde{u} = u \text{ in } \Omega \setminus O, \\ \deg(\tilde{u}, P) = -\deg(\tilde{u}, N) = -1, \\ \int_\Omega |\nabla(u(x) - \tilde{u}(x))|^2 \, dx \leqslant C|P - N|. \end{cases} \tag{13}$$

We now have a construction of a prescribed dipole but the expense of energy is larger.

2.3. Maps having a unique minimal connection

Let $\mathcal{R}_\phi(B^3, \Sigma)$ be the maps of $R_\phi(B^3, \Sigma)$ having a unique minimal connection. For any u in $\mathcal{R}_\phi(B^3, \Sigma)$, each singularity is connected to another singularity having an opposite degree, or to a point of the boundary, in a unique way for realising $L(u)$. The couple of points connected in that way are the dipoles of u.

We are interested in the addition of point singularities to a map in $\mathcal{R}_\phi(B^3, \Sigma)$ in the way described in Lemmas A1 and A2, but we want to ensure that the new map is always in $\mathcal{R}_\phi(B^3, \Sigma)$. The following lemma, which will be proved in the Appendix, will be very useful.

LEMMA A3. *Let $(P_i)_{1 \leqslant i \leqslant p}$ and $(N_i)_{1 \leqslant i \leqslant n}$ be two finite sequences of points of B^3 such that there exists a unique minimal connection C between those two sequences. Let x be a point of $B^3 \setminus C$. Then there exists $r > 0$ such that, for any couple (P, N) of $B_r(x)^2$,* *there exists a unique minimal connection between* $(P_i)_{1\leqslant i\leqslant p}\cup\{P\}$ *and* $(N_i)_{1\leqslant i\leqslant n}\cup\{N\}$ *and this connection is equal to* $C\cup[P,N]$.

We now introduce a last definition needed in §3.2: Consider u in $R_\phi(B^3,\Sigma)$ and v in $\mathcal{R}_\phi(B^3,\Sigma)$. A mixed chain $u-v$ is a union of segments

$$C=[a,a_1]\cup\bigcup_{i=1,\dots,n-1}[a_{i+1},b_i]\cup[b_n,b] \tag{14}$$

where (a_i,b_i) are dipoles of v, and (a,b) are two singularities of u with the degrees

$$\deg(a,u)=\deg(a_i,v)=-\deg(b_i,v)=-\deg(b,u)=1. \tag{15}$$

By definition the length of such a chain is the number

$$L(C)=|a-a_1|+\sum_{i=1}^{n-1}|a_{i+1}-b_i|+|b_n-b|. \tag{16}$$

3. Existence of everywhere singular harmonic maps from B^3 into Σ

3.1. Reduction of the problem

Chapter 3 is devoted to the proof of the following theorem.

THEOREM 1. *Let* ϕ *be a non-constant map in* $C^\infty(\partial B^3,\Sigma)$. *There exists* v *in* $H^1_\phi(B^3,\Sigma)$ *such that any minimizer* u *of* $F_v(u)=E(u)+2A(\Sigma)L(u,v)$ *is singular everywhere, i.e.* $\operatorname{Sing} u=\overline{B^3}$.

Let x_n be a sequence of points in B^3 and $\varrho_n>0$ such that

$$\lim_{n\to+\infty}\varrho_n=0,\quad B_{\varrho_n}(x_n)\subset B^3,\quad (x_n) \text{ is dense in } \overline{B^3}. \tag{17}$$

$B_{\varrho_n}(x_n)$ is denoted B_n.

We show in §3.2 that Theorem 1 is a consequence of the following technical lemma.

LEMMA 1. *There exists a sequence of maps* v_n *in* $\mathcal{R}_\phi(B^3,\Sigma)$ *such that*:

(1) v_n *converges strongly to some* $v\in H^1_\phi(B^3,\Sigma)$,

(2) $\operatorname{Sing} v_{n+1}=\operatorname{Sing} v_n\cup\{P_{n+1},N_{n+1}\}$ *where* $P_{n+1},N_{n+1}\in B_{n+1}$ *and* $\{P_{n+1},N_{n+1}\}$ *is a dipole of* v_{n+1},

(3) *for any minimizer* u *of* F_v, *for any* $n\in\mathbf{N}$ *and for any dipole* $\delta=\{P,N\}$ *of* v_n *which is not a dipole of* v_0,

$$L(u,v_n)<L(u,v_n^\delta)+|P-N| \tag{18}$$

where v_n^δ *is any map of* $\mathcal{R}_\phi(B^3,\Sigma)$ *such that* $\operatorname{Sing} v_n^\delta=\operatorname{Sing} v_n\setminus\{P,N\}$.

Naturally, in the previous lemma, the strict inequality in (3) is the most difficult condition to establish. This strict inequality comes fundamentally from the strict inequality in the construction of the dipole presented above in Lemma A1.

3.2. Lemma 1 implies Theorem 1

Let v_n be a sequence given by Lemma 1 and u be a minimizer of F_v where v is the strong H^1-limit of v_n.

Assume that u is regular in a small ball $B_r(x)$ included in B^3. Our aim in this section is to derive a contradiction under this assumption.

From the density result presented in §2.1, we know that there exists a sequence u_q of maps in $R_\phi(B^3,\Sigma)$, regular in $B_r(x)$, such that u_q converges strongly to u in H^1.

Let $u_{q(n)}$ be a subsequence of u_q, denoted by u_n, chosen such that for any dipole $\delta=(P,N)$ of v_n which is not a dipole of v_0,

$$L(u_n,v_n)<L(u_n,v_n^\delta)+|P-N| \tag{19}$$

where v_n^δ is any map of $\mathcal{R}_\phi(B^3,\Sigma)$ having exactly the same singularities as v_n, without the dipole (P,N). We prove now that the strict inequality (19) implies that any singularity of v_n, in any minimal connection between u_n and v_n, is connected to other singularities, necessarily by a mixed chain u_n-v_n. More precisely we prove the following lemma.

LEMMA 2. *Let u_n and v_n be two sequences as above, in $R_\phi(B^3,\Sigma)$ and $\mathcal{R}_\phi(B^3,\Sigma)$ respectively. Let C be a minimal connection between u_n and v_n, and a be a singular point of v_n which is not a singular point of v_0. Then there exist a mixed chain u_n-v_n, C' and a union of segments C'' such that*

$$a\in C' \quad \text{and} \quad C=C'\cup C''. \tag{20}$$

Proof of Lemma 2. Let a be a singular point of v_n which is not a singular point of v_0. Then a is one of the singularities of a dipole (P,N) of v_n (we may suppose that $a=P$). Let C be a minimal connection between u_n and v_n. There are three possibilities to connect P in C:

(1) The dipole (P,N) is connected to itself. More precisely, there exists a union of segments C'' such that

$$C=[P,N]\cup C''. \tag{21}$$

(2) The singularities P and N are connected to other singularities of v_n in a closed chain which does not contain singularities of u. More precisely, there exist a sequence of

dipoles of v_n, $(P_{i_k}, N_{i_k})_{1\leqslant k\leqslant q}$, and a union of segments C'' such that

$$C=[P,N_{i_1}]\cup\bigcup_{k=1,\dots,q-1}[P_{i_k},N_{i_{k+1}}]\cup[P_{i_q},N]\cup C''. \tag{22}$$

(3) P and N are connected in a mixed chain: there exist a mixed chain u_n-v_n, C' and a union of segments C'' such that

$$P\in C' \quad\text{and}\quad C=C'\cup C''. \tag{23}$$

We now prove that cases (1) and (2) do not occur.

Suppose that (1) holds. C'' is a connection between the singularities of u_n and the singularities of v_n excluding P and N. This implies that

$$L(C'')\geqslant L(u_n,v_n^\delta) \tag{24}$$

where δ denotes the dipole (P,N). Thus, using (21) and (24), we obtain

$$L(u_n,v_n)=L(C)=L(C'')+|P-N|\geqslant L(u_n,v_n^\delta)+|P-N|. \tag{25}$$

This contradicts the strict inequality (19).

Suppose now that (2) holds. We then have

$$L(u_n,v_n)=L(C)=|P-N_{i_1}|+\sum_{k=1}^{q-1}|P_{i_k}-N_{i_{k+1}}|+|P_{i_q}-N|+L(C''). \tag{26}$$

Since v_n is in $\mathcal{R}_\phi(B^3,\Sigma)$, it admits a unique minimal connection which is the union of the segments realised by the dipoles. Thus the following strict inequality holds:

$$|P-N_{i_1}|+\sum_{k=1}^{q-1}|P_{i_k}-N_{i_{k+1}}|+|P_{i_q}-N|>|P-N|+\sum_{k=1}^{q}|P_{i_k}-N_{i_k}|. \tag{27}$$

Thus, if, in C, we replace the chain $[P,N_{i_1}]\cup\bigcup_{k=1}^{q-1}[P_{i_k},N_{i_{k+1}}]\cup[P_{i_q},N]$ by the original dipoles $[P,N]\cup\bigcup_{k=1}^{q}[P_{i_k},N_{i_k}]$, we strictly decrease the length of the connection C and this contradicts the fact that C is a minimal connection.

Thus, possibility (2) never holds and Lemma 2 is proved. □

Consider now the dipoles of v_n included in the ball $B_{r/2}(x)$. As we just proved, those singularities of v_n are indirectly connected to singularities of u_n, out of $B_r(x)$, by mixed chains u_n-v_n. We are going to prove that the minimal length among all the connections between the singularities of v_n in $B_{r/2}(x)$ and the singularities of u_n out of

$B_r(x)$ necessarily goes to infinity as $n \to +\infty$. This will imply that $L(u_n, v_n) \to +\infty$ and since u_n and v_n converge strongly in H^1 this contradicts the continuity of L in $H^1 \times H^1$ presented above.

Let C_n be a sequence of minimal connections between u_n and v_n. Let $\mathcal{E}_n$ be the union of mixed chains $u_n - v_n$ contained in C_n which connect singularities of v_n in the ball $B_{r/2}(x)$. Finally let α_n be the number of elements of $\mathcal{E}_n$.

First case: α_n is uniformly bounded by an integer N.

Let $C_{r/4}(x)$ be the cube $C_{r/4}(x) = \prod_{i=1}^{3} \left(x_i - \frac{1}{4}r, x_i + \frac{1}{4}r\right)$ where $x = (x_1, x_2, x_3)$. We clearly have $C_{r/4}(x) \subset B_{r/2}(x)$. Let p be an integer and divide $C_{r/4}(x)$ into p^3 small disjoint cubes $(C_p^i)_{i=1,\dots,p^3}$ of length $r/4p$:

$$C_{r/4}(x) = \bigcup_{i=1,\dots,p^3} C_p^i. \tag{28}$$

Consider C_p^i, one of those cubes. We denote by $\frac{1}{2}C_p^i$ the cube homothetic to C_p^i with rate $\frac{1}{2}$ having the same center as C_p^i. For n sufficiently large there exists a dipole of v_n included in $\frac{1}{2}C_p^i$. Since $\alpha_n < N$, there exists a mixed chain $u_n - v_n$ which connects p^3/N of those dipoles. Letting D_n be such a chain we easily verify that

$$L(D_n) \geqslant \frac{p^3}{N} \cdot \frac{r}{4p} - L(v_n). \tag{29}$$

This implies that

$$\forall p \in \mathbf{N}\ \exists n \in \mathbf{N}: \quad L(u_n, v_n) = L(C_n) \geqslant L(D_n) \geqslant \frac{r}{4N} p^2 - L(v_n), \tag{30}$$

which contradicts the fact that $L(u_n, v_n)$ and $L(v_n)$ are bounded.

Second case: $\forall N \in \mathbf{N}\ \exists n \in \mathbf{N}$ such that $\alpha_n > N$.

Up to a subsequence, we may suppose that $\alpha_n > n$. C_n has to "cross" the domain $B_r(x) \setminus B_{r/2}(x)$ from the complement of $B_r(x)$ into $B_{r/2}(x)$ at least $2n$ times. This implies that

$$L(u_n, v_n) \geqslant nr - L(v_n), \tag{31}$$

which also contradicts the fact that $L(u_n, v_n)$ and $L(v_n)$ are bounded.

In any case, the assumption that u is regular in a small ball $B_r(x)$ yields a contradiction.

Thus Lemma 1 implies Theorem 1. □

3.3. Proof of Lemma 1

Step 1: Construction of v_n.

We construct by induction several sequences simultaneously:

$$\begin{cases} P_n \text{ and } N_n \in B_n \text{ such that } \delta_n = |P_n - N_n| \leqslant \min(\delta_{n-1}^2, \mu_{n-1}^2), \\ v_n \in \mathcal{R}_\phi(B^3, \Sigma) \text{ such that } \operatorname{Sing} v_n = \operatorname{Sing} v_{n-1} \cup \{P_n, N_n\}, \\ w_n \text{ a minimizer of } F_{v_n}, \\ \widetilde{w}_n \text{ a perturbation of } w_n, \\ \mu_{n+1} = (E(w_n) - E(\widetilde{w}_n))/2A(\Sigma) + |P_{n+1} - N_{n+1}|. \end{cases} \tag{32}$$

Let v_0 be any map in $\mathcal{R}_\phi(B^3, \Sigma)$, w_0 any minimizer of F_{v_0}, and $\delta_0 = \mu_0 = \alpha$ where α will be chosen small enough later.

Suppose v_n, w_n, δ_n and μ_n to be constructed. Let us now add to v_n a dipole (P_{n+1}, N_{n+1}) and construct $\widetilde{w}_n$, v_{n+1}, δ_{n+1}, μ_{n+1} and w_{n+1}. (P_{n+1}, N_{n+1}) will be added in B_{n+1} but the exact insertion locus in B_{n+1} and the size of this new dipole will be determined from w_n in the following way: w_n is a minimizer of F_{v_n}, since v_n has only a finite number of singularities, $\mathcal{H}^1(\operatorname{Sing} w_n) < +\infty$ (see the preliminaries). Let y_{n+1} be a point in B_{n+1} such that $y_{n+1} \in B^3 \setminus \operatorname{Sing} w_n$ and y_{n+1} is not in the minimal connection of v_n. Let r_{n+1} be a small radius given by Lemma A3 which permits the assertion that for any couple of distinct points (P, N) in $B_{r_{n+1}}(y_{n+1})$ there exists a unique minimal connection between {the positive singularities of v_n}$\cup\{P\}$ and {the negative singularities of v_n}$\cup\{N\}$, and (P, N) realises a dipole of this minimal connection. r_{n+1} is also taken small enough such that

$$w_n \text{ is regular in } B_{r_{n+1}}(y_{n+1}), \quad r_{n+1} < \min(\delta_n^2, \mu_n^2), \quad B_{r_{n+1}}(y_{n+1}) \subset B_{n+1}. \tag{33}$$

Suppose w_n to be constant in $B_{r_{n+1}}(y_{n+1})$. Since w_n is regular and harmonic in the complement of $\operatorname{Sing} w_n$, from [14] we know that w_n is real analytic in this set, and since $\operatorname{Sing} w_n$ is closed and $\mathcal{H}^1(\operatorname{Sing} w_n) < +\infty$, the complement of $\operatorname{Sing} w_n$ is a connected dense open set, which implies that w_n is constant up to the boundary, and this contradicts $\phi \neq$ constant.

Thus there exists $z_{n+1} \in B_{r_{n+1}}(y_{n+1})$ such that $\nabla w_n(z_{n+1}) \neq 0$. We can apply Lemma A1 to w_n at the point z_{n+1} in $B_{\alpha_{n+1}}(z_{n+1})$ (where α_{n+1} is positive and such that $B_{\alpha_{n+1}}(z_{n+1}) \subset B_{r_{n+1}}(y_{n+1})$). Let $\widetilde{w}_n$ be the perturbation of w_n given by the lemma, and (P_{n+1}, N_{n+1}) the couple of singularities of $\widetilde{w}_n$ in $B_{\alpha_{n+1}}(z_{n+1})$. We have

$$E(\widetilde{w}_n) < E(w_n) + 2A(\Sigma)|P_{n+1} - N_{n+1}|. \tag{34}$$

Let

$$\mu_{n+1} = \frac{E(w_n)-E(\widetilde{w}_n)}{2A(\Sigma)} + |P_{n+1}-N_{n+1}|. \tag{35}$$

As is proved in [1] it is possible to remove the two singularities P_{n+1} and N_{n+1} of $\widetilde{w}_{n+1}$ by using a quantity of Dirichlet energy as close to $2A(\Sigma)|P_{n+1}-N_{n+1}|$ as we wish. Since such a transformation gives a map having the same singular set as w_n and since w_n minimizes F_{v_n}, we easily obtain that

$$\mu_{n+1} \leqslant 2|P_{n+1}-N_{n+1}| \leqslant 2\mu_n^2. \tag{36}$$

We now insert the two fixed singularities P_{n+1} and N_{n+1} into v_n as described in Lemma A2 for a constant $C>2A(\Sigma)$ independent of n. Let v_{n+1} be this perturbation of v_n. We have

$$\int_{B^3} |\nabla(v_{n+1}-v_n)|^2\,dx \leqslant C|P_{n+1}-N_{n+1}|. \tag{37}$$

As we explained above, r_{n+1} has been taken sufficiently small such that v_{n+1} is in $\mathcal{R}_\phi(B^3,\Sigma)$ and (P_{n+1},N_{n+1}) is a dipole of v_{n+1}: this follows from Lemma A3.

Finally let $\delta_{n+1}=|P_{n+1}-N_{n+1}|$.

Step 2: v_n satisfies the conditions of Lemma 1.

It is clear that (37) and the condition $\delta_{n+1}<\delta_n^2$, for $\alpha=\delta_0$ chosen sufficiently small, imply that v_n is a Cauchy sequence in H^1. Thus condition (1) of Lemma 1 is verified.

By construction, condition (2) of the lemma is also fullfilled.

We will prove now that condition (3) of the lemma is fullfilled by v_n. Let u be a minimizer of F_v, let $n>0$ and $\delta=(P,N)$ be a dipole of v_n which is not a dipole of v_0. Then there exists $1\leqslant q\leqslant n$ such that $(P,N)=(P_q,N_q)$. Since u is a minimizer of F_v,

$$E(u)+2A(\Sigma)L(u,v) \leqslant E(\widetilde{w}_{q-1})+2A(\Sigma)L(\widetilde{w}_{q-1},v). \tag{38}$$

Thus

$$E(u)+2A(\Sigma)L(u,v) \leqslant E(w_{q-1})+2A(\Sigma)L(\widetilde{w}_{q-1},v)+2A(\Sigma)|P_q-N_q|-2A(\Sigma)\mu_q. \tag{39}$$

We are going to use several times the following general inequality:

$$L(u_1,u_3) \leqslant L(u_1,u_2)+L(u_2,u_3) \quad \forall u_1,u_2,u_3 \in H^1_\phi(B^3,\Sigma). \tag{40}$$

This inequality follows directly from the definition of L.

By construction we have

$$L(\widetilde{w}_{q-1},v_q) = L(w_{q-1},v_{q-1}). \tag{41}$$

Applying (40) to $\widetilde{w}_{q-1}$, v_q and v, and using (41), (39) implies

$$\begin{aligned} E(u)+2A(\Sigma)L(u,v) \leqslant E(w_{q-1})+2A(\Sigma)L(w_{q-1},v_{q-1}) \\ +2A(\Sigma)|P-N|-2A(\Sigma)(\mu_q-L(v_q,v)). \end{aligned} \tag{42}$$

Since w_{q-1} minimizes $F_{v_{q-1}}$, (42) implies

$$\begin{aligned} E(u)+2A(\Sigma)L(u,v) \leqslant E(u)+2A(\Sigma)L(u,v_{q-1}) \\ +2A(\Sigma)|P-N|-2A(\Sigma)(\mu_q-L(v_q,v)). \end{aligned} \tag{43}$$

This is equivalent to

$$L(u,v) \leqslant L(u,v_{q-1})+|P-N|-(\mu_q-L(v_q,v)). \tag{44}$$

Applying (40) to u, v_n and v, we get

$$L(u,v) \geqslant L(u,v_n)-L(v_n,v). \tag{45}$$

Using (45), (44) implies

$$L(u,v_n) \leqslant L(u,v_{q-1})+|P-N|-(\mu_q-L(v_q,v)-L(v_n,v)). \tag{46}$$

Let v_n^δ be any map of $\mathcal{R}_\phi(B^3,\Sigma)$ having exactly the same singularities as v_n except the dipole $\delta=(P,N)$. Applying (40) to u, v_n^δ and v_{q-1}, we get

$$L(u,v_{q-1}) \leqslant L(u,v_n^\delta)+L(v_n^\delta,v_{q-1}) \leqslant L(u,v_n^\delta)+L(v_n,v_q). \tag{47}$$

Using (47), (46) implies

$$L(u,v_n) \leqslant L(u,v_n^\delta)+|P-N|-(\mu_q-L(v_q,v)-L(v_n,v)-L(v_n,v_q)) \tag{48}$$

but

$$\begin{aligned} L(v_q,v) &\leqslant \sum_{k=q+1}^{+\infty} \delta_k, \\ L(v_n,v) &\leqslant \sum_{k=n+1}^{+\infty} \delta_k \leqslant \sum_{k=q+1}^{+\infty} \delta_k, \\ L(v_q,v_n) &\leqslant \sum_{k=q+1}^{n} \delta_k \leqslant \sum_{k=q+1}^{+\infty} \delta_k. \end{aligned} \tag{49}$$

Since $\delta_{k+1} \leqslant \delta_k^2$ $\forall k\in\mathbf{N}$ and $\delta_{q+1} \leqslant \mu_q^2$, we get

$$\sum_{k=q+1}^{+\infty} \delta_k \leqslant \sum_{k=0}^{+\infty} \delta_{q+1}^{(2^k)} \leqslant \sum_{k=1}^{+\infty} \mu_q^{(2^k)}. \tag{50}$$

(36) implies easily that for $\delta_0=\alpha$ chosen small enough, μ_n is decreasing and $\sum_{k=1}^{+\infty}\mu_q^{(2^k)}\leqslant 2\mu_q^2$. Combining (49), (50) and the previous remark we obtain

$$\mu_q-L(v_q,v)-L(v_n,v)-L(v_q,v_n)\geqslant\mu_q-6\mu_q^2. \tag{51}$$

For $\alpha=\mu_0$ chosen small enough, $\mu_q-6{\mu_q}^2>0\ \forall q\geqslant 0$. (48), (51) and the previous remark imply

$$L(u,v_n)<L(u,v_n^\delta)+|P-N|. \tag{52}$$

This concludes the proof of Lemma 1. □

A. Appendix

A.1. Integral definition of the minimal connection

In this section we prove that, for any map u in $R_\phi(B^3,\Sigma)$ whose singular points are $(P_i)_{1\leqslant i\leqslant n}$ for the degree $+1$ and $(N_i)_{1\leqslant i\leqslant n}$ for the degree -1 (ϕ is supposed here to have degree zero),

$$L(P_i,N_i)=\min_{\sigma\in\mathcal{S}_n}\left\{\sum_{i=1}^n|P_i-N_{\sigma(i)}|\right\}, \tag{53}$$

where $\mathcal{S}_n$ is the set of the permutations in $\mathbf{N}_n=\{1,\dots,n\}$, is equal to

$$L(u)=\frac{1}{4\pi}\sup_{\substack{\xi:B^3\to\mathbf{R}\\ \|\nabla\xi\|_{L^\infty}\leqslant 1}}\left\{\int_{B^3}D(u)\cdot\nabla\xi-\int_{\partial B^3}D(u)\cdot n\,\xi\right\} \tag{54}$$

where $D(u)=(u\cdot u_y\wedge u_z,u\cdot u_z\wedge u_x,u\cdot u_x\wedge u_y)$ and n is the exterior normal of ∂B^3.

Remark. If we only suppose that u has a finite number of singularities having non-zero degrees, without requiring those degrees to be equal to $+1$ or -1, the previous result (i.e. $L(u)=L(P_i,N_i)$) remains the same modulo the following notation: each singular point of positive (or negative) degree is repeated, in the sequence $(P_i)_{1\leqslant i\leqslant n}$ (or $(N_i)_{1\leqslant i\leqslant n}$), according to the multiplicity of its degree.

Let ξ be a regular function on B^3 such that $\|\nabla\xi\|_{L^\infty}\leqslant 1$. Applying Federer's coarea formula we get

$$\int_{B^3}D(u)\cdot\nabla\xi=\int_{s\in S^2}\omega(s)\left(\int_{u^{-1}(s)}\frac{D(u)}{J_2u}\cdot\nabla\xi\right) \tag{55}$$

where ω is the volume form of S^2, immersed in $\mathbf{R}^3$, and J_2u is the square root of the sum of the squares of the minors of order 2 of ∇u. More precisely,

$$J_2u=\sqrt{(u_y\wedge u_z)^2+(u_z\wedge u_x)^2+(u_x\wedge u_y)^2}=|D(u)|. \tag{56}$$

Since $D(u)$ is the adjoint vector to the pullback by u of ω, for any regular image s by u on S^2 and for $x\in u^{-1}(\{s\})$, $(D(u)/|D(u)|)(x)$ is the tangent vector at x to the 1-dimensional manifold in $B^3\setminus\mathrm{Sing}\{u\}$ realised by the coimage of s. Moreover $u^{-1}(\{s\})$ is the union of curves of two kinds, oriented by $(D(u)/|D(u)|)(x)$:

(1) closed curves $\Gamma_1\cup\Gamma_2\cup...\cup\Gamma_k$,

(2) curves joining $\partial B^3\cup\{P_i\}\cup\{N_i\}$.

For topological reasons the algebraic number of times the second class of curves of $u^{-1}(\{s\})$, starting from P_i (or N_i), is the degree of u at P_i (or N_i). Thus,

$$\int_{u^{-1}(\{s\})}\frac{D(u)}{|D(u)|}\cdot\nabla\xi=-\sum_{i=1}^n\xi(P_i)+\sum_{i=1}^n\xi(N_i)+\sum_{y\in\partial B^3\cap u^{-1}(\{s\})}\xi(y)\,\mathrm{sign}\left(\frac{D(u)}{|D(u)|}\cdot n\right)\tag{57}$$

and

$$\int_{B^3}D(u)\cdot\nabla\xi=4\pi\left(-\sum_{i=1}^n\xi(P_i)+\sum_{i=1}^n\xi(N_i)\right)+\int_{s\in S^2}\omega(s)\sum_{y\in\partial B^3\cap u^{-1}(\{s\})}\xi(y)\,\mathrm{sign}(D(u)\cdot n).\tag{58}$$

On the other hand, if σ denotes the volume form of the boundary ∂B^3 of the domain B^3, we have

$$(n\cdot D(u))\sigma=(n\cdot *u^{\#}\omega)\sigma=\langle\sigma,u^{\#}\omega\rangle\sigma,\tag{59}$$

where the restriction of $u^{\#}\omega$ to ∂B^3 depends only on the restriction of u to ∂B^3. More precisely, this coincides with $\phi^{\#}\omega$ where $u=\phi$ on ∂B^3, yielding

$$\int_{\partial B^3}(n\cdot D(u))\xi\sigma=\int_{\partial B^3}\xi\phi^{\#}\omega.\tag{60}$$

Using one more time the coarea formula we have

$$\begin{aligned}\int_{\partial B^3}\xi\phi^{\#}\omega&=\int_{s\in S^2}\omega(s)\sum_{y\in\partial B^3\cap u^{-1}(\{s\})}\xi(y)\,\mathrm{sign}(\det\nabla\phi)\\&=\int_{s\in S^2}\omega(s)\sum_{y\in\partial B^3\cap u^{-1}(\{s\})}\xi(y)\,\mathrm{sign}(D(u)\cdot n).\end{aligned}\tag{61}$$

Combining (58) and (61) we obtain

$$L(u)=\sup_{\substack{\xi:B^3\to\mathbf{R}\\ \|\nabla\xi\|_{L^\infty}\leqslant 1}}\left\{-\sum_{i=1}^n\xi(P_i)+\sum_{i=1}^n\xi(N_i)\right\}.\tag{62}$$

It is proved in [6], using theorems of Birkhoff and Kantorovich, that (62) coincides with the definition (53) given for $L(P_i,N_i)$.

A.2. Proof of property (b) in §2.1

The outline of the proof is the same as the outline of the proof of Theorem 3 in [3]:

Since a supremum of sequentially lower semi-continuous functions is also sequentially lower semi-continuous, it is sufficient to prove that, for any fixed $\xi\colon B^3\to\mathbf{R}$ with $\|\nabla\xi\|_\infty\leqslant 1$, the function

$$H^1_\phi(B^3,\Sigma)\ni u\to F_\xi(u)=\int_{B^3}|\nabla u|^2+2\int *u^\#\omega'\cdot\nabla\xi \tag{63}$$

is lower semi-continuous in the weak H^1-topology.

Remark. Let v be in $H^1_\phi(B^3,\Sigma)$. We will often write $\int|\nabla u|^2$ and $\int *u^\#\omega'\cdot\nabla\xi$ in the form

$$\int g_{11}(v)|\nabla v^1|^2+g_{22}(v)|\nabla v^2|^2+2g_{12}(v)\nabla v^1\cdot\nabla v^2$$

and

$$\int\sqrt{g_{11}(v)g_{22}(v)-g_{12}^2(v)}\;\nabla v^1\times\nabla v^2\cdot\nabla\xi$$

respectively. Generally this makes no sense because the g_{ij} are metric coefficients defined for u in a subset of Σ, but it can be understood as a sum of such quantities multiplied by regular cut-off functions on Σ adapted to the atlas. The multiplication by such functions will have no influence on the nature of the convergence of those quantities presented below.

Let $u_n\rightharpoonup u$ weakly in $H^1_\phi(B^3,\Sigma)$. By $v[g]$ and $v[g]_n$ denote the following quantities:

$$v[g]=\sqrt{g_{11}g_{22}-g_{12}^2}\,(u)\quad\text{and}\quad v[g]_n=\sqrt{g_{11}g_{22}-g_{12}^2}\,(u_n).$$

We have

$$\begin{aligned}\int_{B^3}*u^\#\omega'\cdot\nabla\xi-\int_{B^3}*u_n^\#\omega'\cdot\nabla\xi\\ &=\int_{B^3}v[g]\nabla u^1\times\nabla u^2\cdot\nabla\xi-v[g]_n\nabla u_n^1\times\nabla u_n^2\cdot\nabla\xi\\ &=A_n+B_n+C_n+D_n\end{aligned}\tag{64}$$

where we have used the notation

$$\begin{aligned}A_n&=\int_{B^3}(v[g]-v[g]_n)\nabla u^1\wedge\nabla u^2\cdot\nabla\xi,\\ B_n&=-\int_{B^3}v[g]_n\nabla u^1\times(\nabla u_n^2-\nabla u^2)\cdot\nabla\xi,\\ C_n&=-\int_{B^3}v[g]_n(\nabla u_n^1-\nabla u^1)\times\nabla u^2\cdot\nabla\xi\,dx,\\ D_n&=-\int_{B^3}v[g]_n(\nabla u_n^1-\nabla u^1)\times(\nabla u_n^2-\nabla u^2)\cdot\nabla\xi.\end{aligned}\tag{65}$$

Since $u_n \rightharpoonup u$ weakly in $H^1_\phi(B^3,\Sigma)$,

$$v[g]_n \rightharpoonup v[g] \quad \text{in weak* } L^\infty$$

and the three first integrals A_n, B_n and C_n tend to zero. We cannot hope to have weak sequential continuity in H^1 of $\int_{B^3} *u^\# \omega' \cdot \nabla\xi\, dx$, because topology properties are generally not invariant by weak convergence in H^1. Thus, in order to establish the lower semi-continuity of F_ξ we have to compare the fourth integral D_n with the Dirichlet energies of u_n and u. We use the following elementary lemma.

LEMMA A4. *Let g_{11}, g_{22} and g_{12} be in $\mathbf{R}$ such that $g_{11}, g_{22}, g_{11}g_{22}-g_{12}^2>0$. If $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$ are in $\mathbf{R}^3$ we have*

$$\sqrt{g_{11}g_{22}-g_{12}^2}\,\|a\times b\| \leqslant \tfrac{1}{2}(g_{11}\|a\|^2+g_{22}\|b\|^2+2g_{12}a\cdot b). \tag{67}$$

Proof of Lemma A4. If $a'=\sqrt{g_{11}}\,a$ and $b'=\sqrt{g_{22}}\,b$ we have to prove the inequality

$$\sqrt{1-\left(\frac{g_{12}}{\sqrt{g_{11}g_{22}}}\right)^2}\,\|a'\times b'\| \leqslant \frac{1}{2}\left(\|a'\|^2+\|b'\|^2+2\frac{g_{12}}{\sqrt{g_{11}g_{22}}}a'\cdot b'\right). \tag{68}$$

Let ϕ be such that $\sin\phi=\|a'\wedge b'\|/\|a'\|\cdot\|b'\|$ and $\cos\phi=a'\cdot b'/\|a'\|\cdot\|b'\|$ and let $\alpha=g_{12}/\sqrt{g_{11}g_{22}}$. Using these notations (68) becomes

$$(\sqrt{1-\alpha^2}\sin\phi-\alpha\cos\phi)\|a'\|\cdot\|b'\| \leqslant \tfrac{1}{2}(\|a'\|^2+\|b'\|^2). \tag{69}$$

Let β be such that $\sin\beta=\alpha$ and $\cos\beta=\sqrt{1-\alpha^2}$. Using this notation (70) becomes

$$\sin(\phi-\beta)\|a'\|\cdot\|b'\| \leqslant \tfrac{1}{2}(\|a'\|^2+\|b'\|^2). \tag{70}$$

This is clearly true and Lemma A4 is proved. □

Using the previous lemma we may now compare the fourth integral D_n with the Dirichlet energies of u and u_n:

$$\begin{aligned}|D_n| &= \left|\int_{B^3} v[g]_n(\nabla u_n^1-\nabla u^1)\times(\nabla u_n^2-\nabla u^2)\cdot\nabla\xi\,dx\right|\\ &\leqslant \int_{B^3} v[g]_n\|(\nabla u_n^1-\nabla u^1)\times(\nabla u_n^2-\nabla u^2)\|\\ &\leqslant \int_{B^3} \tfrac{1}{2}(g_{11}^n\|\nabla(u_n^1-u^1)\|^2+g_{22}^n\|\nabla(u_n^2-u^2)\|^2\\ &\qquad +2g_{12}^n\nabla(u_n^1-u^1)\cdot\nabla(u_n^2-u^2))\end{aligned} \tag{71}$$

where we have used the notation g_{ij}^n and g_{ij} for $g_{ij}(u_n)$ and $g_{ij}(u)$. Let

$$E_n = \int \tfrac{1}{2}(g_{11}^n\|\nabla(u_n^1-u^1)\|^2+g_{22}^n\|\nabla(u_n^2-u^2)\|^2 \\ +2g_{12}^n\nabla(u_n^1-u^1)\cdot\nabla(u_n^2-u^2)). \tag{72}$$

We have

$$\begin{aligned} E_n &= \int_{B^3} \tfrac{1}{2}(g_{11}^n\|\nabla u_n^1\|^2+g_{22}^n\|\nabla u_n^2\|^2+2g_{12}^n\nabla u_n^1\cdot\nabla u_n^2) \\ &\quad -\int_{B^3} \tfrac{1}{2}(g_{11}\|\nabla u^1\|^2+g_{22}\|\nabla u^2\|^2+2g_{12}\nabla u^1\cdot\nabla u^2) \\ &\quad +F_n+G_n \end{aligned} \tag{73}$$

where F_n and G_n denote the quantities

$$\begin{aligned} F_n &= -\int_{B^3} \tfrac{1}{2}((g_{11}^n-g_{11})\|\nabla u^1\|^2+(g_{22}^n-g_{22})\|\nabla u_n^2\|^2 \\ &\quad +2(g_{12}^n-g_{12})\nabla u^1\cdot\nabla u^2), \\ G_n &= -\int_{B^3} g_{11}^n\nabla(u_n^1-u^1)\cdot\nabla u^1+g_{22}^n\nabla(u_n^2-u^2)\cdot\nabla u^2 \\ &\quad +g_{12}^n\nabla(u_n^1-u^1)\cdot\nabla u^2+g_{12}^n\nabla(u_n^2-u^2)\cdot\nabla u^1. \end{aligned}$$

As before, in view of (66), the two last integrals of (73), F_n and G_n, tend to zero.

Using the previous remark, (71) may be written in the form

$$|D_n| \leqslant \int_{B^3}|\nabla u_n|^2-\int_{B^3}|\nabla u|^2+\varepsilon(n) \tag{74}$$

where $\varepsilon(n)$ goes to zero at infinity. Finally combining (64) and (74) we obtain

$$\int_{B^3}|\nabla u|^2+2\int *u^{\#}\omega'\cdot\nabla\xi\,dx \leqslant \int_{B^3}|\nabla u_n|^2\,dx+2\int_{B^3} *u_n^{\#}\omega'\cdot\nabla\xi\,dx+\varepsilon'(n) \tag{75}$$

where $\varepsilon'(n)$ goes to zero when n tends to infinity.

Thus F_ξ is lower semi-continuous in the weak H^1-topology for any ξ and property (b) in §2.1 is proved. □

A.3. Proof of Lemma A1

A.3.1. Presentation

(a) *Notations.* We may always replace x_0 by $(0,0,0)$. Let ψ be a conformal diffeomorphism from S^2 into Σ. We denote by $\hat{u}$ the map

$$\hat{u}=\psi^{-1}\circ u. \tag{76}$$

For p in S^2 we denote by $\mu(p)$ the conformal coefficient of ψ at p:

$$|\nabla\psi_{(p)}\cdot v| = \mu(p)|v| \quad \forall v\in T_pS^2. \tag{77}$$

We set $\mu=\mu(\hat{u}(0,0,0))$.

Choose an orthonormal basis $(\vec{\imath},\vec{\jmath},\vec{k})$ such that

$$\nabla_{xy}\hat{u}(0,0,0)\neq 0 \quad (\text{i.e. } \nabla_{xy}u(0,0,0)\neq 0). \tag{78}$$

We may assume that $\hat{u}_x(0,0,0)\cdot\hat{u}_y(0,0,0)=0$. Indeed this can be obtained by a simple rotation $(\vec{\imath},\vec{\jmath})$ (see [5]). This implies in particular that $u_x(0,0,0)\cdot u_y(0,0,0)=0$. We may also suppose that $\hat{u}_x(0,0,0)\neq 0$ (i.e. $u_x(0,0,0)\neq 0$).

For δ sufficiently small and z in $[-\delta+\delta^2,\delta-\delta^2]$, let us consider the unit vector fields

$$I(z)=\frac{\hat{u}_x(0,0,z)}{||\hat{u}_x(0,0,z)||}, \quad K(z)=\hat{u}(0,0,z). \tag{79}$$

Since $\hat{u}$ takes its values in the unit sphere of $\mathbf{R}^3$, it is clear that $I(z)$ is in the tangent vector plane to the sphere at $\hat{u}(0,0,z)$ and that I and K are orthogonal.

Let $a=||\hat{u}_x(0,0,0)||$ and $b=||\hat{u}_y(0,0,0)||$.

If $b\neq 0$, let $J(z)$ be the unique regular vector field such that $(I(z),J(z),K(z))$ is an orthonormal basis having the same orientation as $(\hat{u}_x(0,0,0),\hat{u}_y(0,0,0),\hat{u}(0,0,0))$.

If $b=0$, let $J(z)$ be the unique regular vector field such that $(I(z),J(z),K(z))$ is a direct orthonormal basis. In this case the choice of the orientation is arbitrary but has to be the same for all z.

We easily verify that, in the two cases,

$$\begin{aligned}\hat{u}_x(0,0,z)&=(a+O(z))I(z),\\ \hat{u}_y(0,0,z)&=O(z)I(z)+(b+O(z))J(z).\end{aligned} \tag{80}$$

(b) *Sketch of the construction.* Letting δ be sufficiently small, we transform u in the cylinder C_δ centered at 0, with axis along the z-axis, of radius $2\delta^2$ and of length $2(\delta+\delta^2)$. We denote by u^δ the transformed map. $\tilde{u}$ will be equal to u^δ for a δ chosen sufficiently small at the end of the proof.

(1) $\tilde{u}=u$ outside C^δ.

(2) At each z between $\delta-\delta^2$ and $-\delta+\delta^2$ in C^δ (i.e. in the subcylinder of C^δ denoted c^δ), we linearly interpolate u outside c^δ and a conformal map which maps the horizontal disk centered at the point $(0,0,z)$ and of radius δ^2 onto a "big part" of Σ exactly as it is made for $\Sigma=S^2$ by H. Brezis and J.-M. Coron in [5, Lemma 2].

(3) Let $p=(0,0,\delta)$ and $n=(0,0,-\delta)$, in the small cylinder c^p_δ (or c^n_δ), centered at p (or n), with axis along the z-axis, of radius $2\delta^2$ and of length $2\delta^2$, if we denote by π^+ (or π^-) the radial projection centered at p (or n) onto the boundary of c^p_δ (or c^n_δ), the transformed map u^δ is the composition of π^+ (or π^-) and the value of u^δ on this boundary.

A.3.2. The construction of $\tilde{u}$

(a) *Construction of u^δ for z in $[-\delta+\delta^2, \delta-\delta^2]$ and estimates for $E(u^\delta)$ in c^δ.* Let (r,θ) be the polar coordinates corresponding to (x,y). For any z in $[-\delta+\delta^2, \delta-\delta^2]$ we construct u^δ as follows:

(1) If $r>2\delta^2$:

$$u^\delta = u. \tag{81}$$

(2) If $r<\delta^2$:

$$u^\delta = \psi\left(\frac{2\lambda}{\lambda^2+r^2}(xI(z)+yJ(z)-\lambda K(z))+K(z)\right) \tag{82}$$

where $\lambda=c\delta^4$, c will be fixed later.

(3) If $\delta^2\leqslant r\leqslant 2\delta^2$:

$$u^\delta = \psi\big((A_1r+B_1)I(z)+(A_2r+B_2)J(z)+\sqrt{1-(A_1r+B_1)^2-(A_2r+B_2)^2}\,K(z)\big), \tag{83}$$

where A_i and B_i only depend on θ, δ and z as follows:

$$\begin{aligned} &2\delta^2A_i+B_i = \hat{u}_i(2\delta^2\cos\theta, 2\delta^2\sin\theta, z) \quad \text{for } i=1,2,\\ &\delta^2A_1+B_1 = \frac{2\lambda\delta^2}{\lambda^2+\delta^4}\cos\theta,\\ &\delta^2A_2+B_2 = \frac{2\lambda\delta^2}{\lambda^2+\delta^4}\sin\theta. \end{aligned} \tag{84}$$

($\hat{u}_i$ is the ith coordinate of $\hat{u}$ in $(I(z), J(z), K(z))$. Thus,

$$2\delta^2A_i+B_i = \hat{u}_i(0,0,z)+2\delta^2\cos\theta\frac{\partial\hat{u}_i}{\partial x}(0,0,z)+2\delta^2\sin\theta\frac{\partial\hat{u}_i}{\partial y}(0,0,z)+O(\delta^4). \tag{85}$$

But $\hat{u}_i(0,0,z)=0$ for $i=1,2$. Moreover,

$$\begin{aligned} &\frac{\partial\hat{u}_1}{\partial x}(0,0,z)=a+O(z), \quad \frac{\partial\hat{u}_2}{\partial x}(0,0,z)=0,\\ &\frac{\partial\hat{u}_1}{\partial y}(0,0,z)=O(z), \qquad \frac{\partial\hat{u}_2}{\partial y}(0,0,z)=b+O(z). \end{aligned} \tag{86}$$

Thus,

$$\begin{aligned} 2\delta^2A_1+B_1 &= 2a\delta^2\cos\theta+O(\delta^3),\\ 2\delta^2A_2+B_2 &= 2b\delta^2\sin\theta+O(\delta^3). \end{aligned} \tag{87}$$

Similarly the last two equations of (84) give

$$\begin{aligned} \delta^2A_1+B_1 &= 2c\delta^2\cos\theta+O(\delta^3),\\ \delta^2A_2+B_2 &= 2c\delta^2\sin\theta+O(\delta^3). \end{aligned} \tag{88}$$

Finally we obtain

$$\begin{aligned} A_1 &= 2(a-c)\cos\theta + O(\delta),\\ B_1 &= 2(2c-a)\delta^2\cos\theta + O(\delta^3),\\ A_2 &= 2(b-c)\sin\theta + O(\delta),\\ B_2 &= 2(2c-b)\delta^2\sin\theta + O(\delta^3). \end{aligned} \tag{89}$$

Exactly in the same way we verify

$$\begin{aligned} \frac{\partial A_1}{\partial\theta} &= -2(a-c)\sin\theta + O(\delta),\\ \frac{\partial B_1}{\partial\theta} &= -2(2c-a)\delta^2\sin\theta + O(\delta^3),\\ \frac{\partial A_2}{\partial\theta} &= 2(b-c)\cos\theta + O(\delta),\\ \frac{\partial B_2}{\partial\theta} &= 2(2c-b)\delta^2\cos\theta + O(\delta^3). \end{aligned} \tag{90}$$

For $\delta^2 \leqslant r \leqslant 2\delta^2$ we have

$$\hat{u}_1^\delta = O(\delta^2), \quad \hat{u}_2^\delta = O(\delta^2). \tag{91}$$

Since $(\hat{u}_3^\delta)^2 = 1-(\hat{u}_1^\delta)^2-(\hat{u}_2^\delta)^2$ we have

$$\hat{u}_3^\delta = 1 + O(\delta^4) \quad \text{for } \delta^2 \leqslant r \leqslant 2\delta^2, \tag{92}$$

and using (84), (89) and (90) we obtain

$$\frac{\partial \hat{u}_3^\delta}{\partial r} = O(\delta^2) \quad \text{and} \quad \frac{1}{r}\cdot\frac{\partial \hat{u}_3^\delta}{\partial\theta} = O(\delta^2). \tag{93}$$

We can now compute the energy of u^δ in c^δ for $\delta^2 \leqslant r \leqslant 2\delta^2$:

$$\begin{aligned} &\int_{\delta^2-\delta}^{\delta-\delta^2}\int_{\delta^2\leqslant r\leqslant 2\delta^2} |\nabla_{xy}u^\delta|^2\,dx\,dy\,dz\\ &= \int_{\delta^2-\delta}^{\delta-\delta^2}\int_{\delta^2\leqslant r\leqslant 2\delta^2}\left[\left|\frac{\partial u^\delta}{\partial x}\right|^2 + \left|\frac{\partial u^\delta}{\partial y}\right|^2\right] dx\,dy\,dz\\ &= \int_{\delta^2-\delta}^{\delta-\delta^2}\int_{\delta^2\leqslant r\leqslant 2\delta^2}\left[\left|\nabla\psi_{(\hat{u}^\delta(x,y,z))}\cdot\frac{\partial \hat{u}^\delta}{\partial x}\right|^2 + \left|\nabla\psi_{(\hat{u}^\delta(x,y,z))}\cdot\frac{\partial \hat{u}^\delta}{\partial y}\right|^2\right] dx\,dy\,dz\\ &= \int_{\delta^2-\delta}^{\delta-\delta^2}\int_{\delta^2\leqslant r\leqslant 2\delta^2}\left[\mu^2(\hat{u}^\delta(x,y,z))\left(\left|\frac{\partial \hat{u}^\delta}{\partial x}\right|^2 + \left|\frac{\partial \hat{u}^\delta}{\partial y}\right|^2\right)\right] dx\,dy\,dz\\ &= \int_{\delta^2-\delta}^{\delta-\delta^2}\int_{\delta^2\leqslant r\leqslant 2\delta^2}\left[\mu^2\left(\left|\frac{\partial \hat{u}^\delta}{\partial x}\right|^2 + \left|\frac{\partial \hat{u}^\delta}{\partial y}\right|^2\right) + O(\delta)\right] dx\,dy\,dz. \end{aligned} \tag{94}$$

After some computations we find

$$\int_{\delta^2-\delta}^{\delta-\delta^2}\int_{\delta^2\leqslant r\leqslant 2\delta^2}|\nabla_{xy}u^\delta|^2\,dx\,dy\,dz=8\pi\delta^5\mu^2(a^2+b^2-2c^2 \\ +(a^2+b^2+8c^2-4ac-4bc)\ln 2)+O(\delta^6). \tag{95}$$

Concerning the third coordinate:

$$2\delta^2\frac{\partial A_i}{\partial z}+\frac{\partial B_i}{\partial z}=\frac{\partial \hat{u}_i}{\partial z}(2\delta^2\cos\theta,2\delta^2\sin\theta,z)=\frac{\partial\hat{u}_i}{\partial z}(0,0,z)+O(\delta^2). \tag{96}$$

On the other hand, $\hat{u}_i(0,0,z)=0$ for $i=1,2$ and $\forall z\in[-\delta+\delta^2,\delta-\delta^2]$, and hence

$$2\delta^2\frac{\partial A_i}{\partial z}+\frac{\partial B_i}{\partial z}=O(\delta^2).$$

Thus we easily obtain that

$$\frac{\partial\hat{u}_1^\delta}{\partial z}=O(\delta^2)\quad\text{and}\quad\frac{\partial\hat{u}_2^\delta}{\partial z}=O(\delta^2)\quad\text{for } \delta^2\leqslant r\leqslant 2\delta^2. \tag{97}$$

Once again, since $(\hat{u}_3^\delta)^2=1-(\hat{u}_1^\delta)^2-(\hat{u}_2^\delta)^2$ we deduce that

$$\frac{\partial\hat{u}_3^\delta}{\partial z}=O(\delta^4)\quad\text{for } \delta^2\leqslant r\leqslant 2\delta^2. \tag{98}$$

Using the estimates above we estimate the gradient of u^δ along the z-axis using (91), (92), (97) and (98), and obtain

$$\begin{aligned}\left|\frac{\partial u^\delta}{\partial z}(x,y,z)\right|^2&=\mu^2(\hat{u}^\delta(x,y,z))\left|\frac{\partial\hat{u}^\delta}{\partial z}(x,y,z)\right|^2\\&=\mu^2(\hat{u}^\delta(x,y,z))\left|\frac{\partial\hat{u}_1^\delta}{\partial z}I(z)+\frac{\partial\hat{u}_2^\delta}{\partial z}J(z)+\frac{\partial\hat{u}_3^\delta}{\partial z}K(z)\right.\\&\qquad\left.+\hat{u}_1^\delta\frac{dI}{dz}+\hat{u}_2^\delta\frac{dI}{dz}+\hat{u}_3^\delta\frac{dK}{dz}\right|^2.\end{aligned} \tag{99}$$

This implies

$$\left|\frac{\partial u^\delta}{\partial z}(x,y,z)\right|^2=\mu^2(\hat{u}^\delta(x,y,z))\left|\frac{dK}{dz}\right|^2+O(\delta^2). \tag{100}$$

Moreover,

$$\hat{u}^\delta(x,y,z)=\hat{u}^\delta(0,0,z)+O(\delta^2)=K(z)+O(\delta^2)\quad\text{for } \delta^2\leqslant r\leqslant 2\delta^2. \tag{101}$$

Thus,

$$\mu^2(\hat{u}^\delta(x,y,z))=\mu^2(\hat{u}^\delta(0,0,z))+O(\delta^2)\quad\text{for } \delta^2\leqslant r\leqslant 2\delta^2, \tag{102}$$

and finally, combining (99) and (102) we obtain

$$\begin{aligned}\left|\frac{\partial u^\delta}{\partial z}(x,y,z)\right|^2 &= \mu^2(\hat{u}^\delta(0,0,z))\left|\frac{dK}{dz}\right|^2 + O(\delta^2)\\ &= \mu^2(\hat{u}^\delta(0,0,z))\left|\frac{\partial \hat{u}}{\partial z}(0,0,z)\right|^2 + O(\delta^2)\\ &= \left|\frac{\partial u}{\partial z}(0,0,z)\right|^2 + O(\delta^2).\end{aligned} \tag{103}$$

Integrating this part of the energy in c^δ for $\delta^2 \leqslant r \leqslant 2\delta^2$ we are led to

$$\begin{aligned}\int_{\delta^2-\delta}^{\delta-\delta^2}\int_{\delta^2\leqslant r\leqslant 2\delta^2}\left|\frac{\partial u^\delta}{\partial z}\right|^2 dx\,dy\,dz\\ = \pi((2\delta^2)^2-(\delta^2)^2)\int_{\delta^2-\delta}^{\delta-\delta^2}\left|\frac{\partial u}{\partial z}(0,0,z)\right|^2 dz + O(\delta^7).\end{aligned} \tag{104}$$

From now on, in this section, we evaluate the energy of u^δ in c^δ for $r \leqslant \delta^2$.

For $r \leqslant \delta^2$ and for a fixed z, u^δ is a conformal diffeomorphism from the horizontal disk $B^2((0,0,z),\delta^2)$ into Σ. Thus,

$$\int_{r\leqslant\delta^2} |\nabla_{xy} u^\delta(x,y,z)|\,dx\,dy = 2\,\mathrm{Area}(u^\delta(B^2((0,0,z),\delta^2),z)). \tag{105}$$

The image by $\hat{u}^\delta(\cdot\,,z)$ of $B^2((0,0,z),\delta^2)$ is all of the sphere S^2 without the geodesic ball $B_{S^2}(\hat{u}(0,0,z),\gamma)$ in S^2 of center $\hat{u}(0,0,z)$ and of radius $\gamma = 2c\delta^2 + O(\delta^4)$. Thus,

$$\begin{aligned}\mathrm{Area}(u^\delta(B^2((0,0,z),\delta^2),z)) &= \mathrm{Area}(\psi(S^2 \setminus B_{S^2}(\hat{u}(0,0,z),\gamma)))\\ &= A(\Sigma) - \mathrm{Area}(\psi(B_{S^2}(\hat{u}(0,0,z),\gamma)))\\ &= A(\Sigma) - \tfrac{1}{2}\textstyle\int_{B_{S^2}(\hat{u}(0,0,z),\gamma)} |\nabla\psi(p)|^2\,d\sigma\\ &= A(\Sigma) - 4\pi\mu^2(\hat{u}(0,0,z))c^2\delta^4 + O(\delta^6)\\ &= A(\Sigma) - 4\pi\mu^2 c^2\delta^4 + O(\delta^5).\end{aligned} \tag{106}$$

For any z in $[-\delta+\delta^2, \delta-\delta^2]$ we have

$$\int_{r\leqslant\delta^2} |\nabla_{xy} u^\delta(x,y,z)|\,dx\,dy = 2A(\Sigma) - 8\pi\mu^2 c^2\delta^4 + O(\delta^5), \tag{107}$$

and finally

$$\int_{\delta^2-\delta}^{\delta-\delta^2}\int_{r\leqslant\delta^2} |\nabla_{xy} u^\delta(x,y,z)|\,dx\,dy\,dz = 2A(\Sigma)2(\delta-\delta^2) - 16\pi\mu^2 c^2\delta^5 + O(\delta^6). \tag{108}$$

Concerning the third derivative, for any z in $[-\delta+\delta^2, \delta-\delta^2]$, we distinguish the case where $r \leqslant \delta^3$ and where $\delta^3 \leqslant r \leqslant \delta^2$:

$$\begin{aligned}\int_{r\leqslant\delta^3}\left|\frac{\partial u^\delta}{\partial z}(x,y,z)\right|^2 dx\,dy &= \int_{r\leqslant\delta^3}\mu^2(\hat{u}^\delta(x,y,z))\left|\frac{\partial \hat{u}^\delta}{\partial z}(x,y,z)\right|^2 dx\,dy \\ &\leqslant \sup_{S^2}\mu\int_{r\leqslant\delta^3}\left|\frac{2\lambda}{\lambda^2+r^2}\left(x\frac{dI}{dz}+y\frac{dJ}{dz}-\lambda\frac{dK}{dz}\right)+\frac{dK}{dz}\right|^2 dx\,dy.\end{aligned} \tag{109}$$

This gives, in particular,

$$\int_{r\leqslant\delta^3}\left|\frac{\partial u^\delta}{\partial z}(x,y,z)\right|^2 dx\,dy = O(\delta^6), \tag{110}$$

which implies

$$\int_{-\delta+\delta^2}^{\delta-\delta^2}\int_{r\leqslant\delta^3}\left|\frac{\partial u^\delta}{\partial z}(x,y,z)\right|^2 dx\,dy\,dz = O(\delta^7). \tag{111}$$

In the other part, for $\delta^3 \leqslant r \leqslant \delta^2$, we have

$$|\hat{u}_3^\delta(x,y,z)-1| = \left|\frac{2\lambda^2}{\lambda^2+r^2}\right| = \left|\frac{2c^2\delta^8}{c^2\delta^8+r^2}\right| = O(\delta^2). \tag{112}$$

Thus

$$|\hat{u}^\delta(x,y,z)-\hat{u}(0,0,z)| = O(\delta) \quad \text{for } \delta^3 \leqslant r \leqslant \delta^2. \tag{113}$$

Let us now estimate the energy coming from the derivation in z for $\delta^3 \leqslant r \leqslant \delta^2$.

$$\begin{aligned}&\int_{-\delta+\delta^2}^{\delta-\delta^2}\int_{\delta^3\leqslant r\leqslant\delta^2}\left|\frac{\partial u^\delta}{\partial z}(x,y,z)\right|^2 dx\,dy\,dz \\ &\quad= \int_{-\delta+\delta^2}^{\delta-\delta^2}\int_{\delta^3\leqslant r\leqslant\delta^2}\mu^2(\hat{u}^\delta(x,y,z))\left|\frac{\partial \hat{u}^\delta}{\partial z}(x,y,z)\right|^2 dx\,dy\,dz \\ &\quad= \int_{-\delta+\delta^2}^{\delta-\delta^2}\mu^2(\hat{u}^\delta(0,0,z))\left(\int_{\delta^3\leqslant r\leqslant\delta^2}\left|\frac{\partial \hat{u}^\delta}{\partial z}(x,y,z)\right|^2 dx\,dy\right)dz + O(\delta^6).\end{aligned} \tag{114}$$

By definition

$$\left|\frac{\partial \hat{u}^\delta}{\partial z}(x,y,z)\right|^2 = \left|\frac{2\lambda}{\lambda^2+r^2}\left(x\frac{dI}{dz}+y\frac{dJ}{dz}\right)+\frac{r^2-\lambda^2}{r^2+\lambda^2}\cdot\frac{dK}{dz}\right|^2.$$

We notice that

$$\left|\frac{2\lambda}{\lambda^2+r^2}\left(x\frac{dI}{dz}+y\frac{dJ}{dz}\right)\right| \leqslant C\frac{\lambda}{r} = O(\delta) \quad \text{for } \delta^3 \leqslant r \leqslant \delta^2. \tag{115}$$

Using this estimate, (114) becomes

$$\begin{aligned}&\int_{-\delta+\delta^2}^{\delta-\delta^2}\int_{\delta^3\leqslant r\leqslant\delta^2}\left|\frac{\partial u^\delta}{\partial z}(x,y,z)\right|^2 dx\,dy\,dz\\ &=2\pi\int_{\delta^3}^{\delta^2}\left(\frac{r^2-\lambda^2}{r^2+\lambda^2}\right)r\,dr\int_{-\delta+\delta^2}^{\delta-\delta^2}\mu^2(\hat{u}^\delta(0,0,z))\left|\frac{\partial\hat{u}}{\partial z}(0,0,z)\right|^2 dz+O(\delta^6)\\ &=\pi\delta^4\int_{-\delta+\delta^2}^{\delta-\delta^2}\left|\frac{\partial u}{\partial z}(0,0,z)\right|^2 dz+O(\delta^6).\end{aligned}\tag{116}$$

(b) *Estimates for* $E(u^\delta)$ *in* c_p^δ *and* c_n^δ. As we have said in §A.3.1 (b), u^δ is, in c_p^δ (or c_n^δ), the composition of the radial projection π^+ (or π^-) of center p (or n) and the value of u^δ on ∂c_p^δ (or ∂c_n^δ). Precisely, π^+ (or π^-) sends any point x (different from p (or n)) in c_p^δ (or c_n^δ) to the point $\pi^+(x)$ (or $\pi^-(x)$) of ∂c_p^δ (or ∂c_n^δ) such that the segment $[\pi^+(x),p]$ (or $[\pi^-(x),n]$) contains the point x.

We divide c_p^δ into two parts:

- Let G be $(\pi^+)^{-1}(\partial c_p^\delta\cap\partial c^\delta)$, a small cone of vertex p.
- Let H be the complement of G in c_p^δ, i.e. $H=(\pi^+)^{-1}(\partial c_p^\delta\setminus\partial c^\delta)$.

On $\partial H\setminus(\partial G\cap\partial H)$, $u^\delta=u$. Since u is regular we easily conclude that

$$\int_H|\nabla u^\delta|^2\,dx\,dy\,dz=\int_H|\nabla(u\circ\pi^+)|^2\,dx\,dy\,dz=O(\delta^6).\tag{117}$$

$\partial G\setminus(\partial G\cap\partial H)$ is the horizontal disk $D_{2\delta^2}$ of centre $(0,0,\delta-\delta^2)$ and of radius $2\delta^2$. On this disk, u^δ is the "linear interpolation" on S^2 between u at the boundary $\partial D_{2\delta^2}$ and a conformal map on the concentric disk D_{δ^2} (see part (a) of this section). All of this implies:

- On $D_{2\delta^2}\setminus D_{\delta^2}$, $|\nabla_{xy}u^\delta(x,y,\delta-\delta^2)|$ is bounded by a constant independent of δ. Thus,

$$\int_{(\pi^+)^{-1}(D_{2\delta^2}\setminus D_{\delta^2})}|\nabla u^\delta|^2\,dx\,dy\,dz=O(\delta^6).\tag{118}$$

- On D_{δ^2},

$$|\nabla_{xy}u^\delta(x,y,\delta-\delta^2)|^2=\mu^2(u^\delta(x,y,\delta-\delta^2))\leqslant\sup_{S^2}\mu|\nabla_{xy}\hat{u}^\delta(x,y,\delta-\delta^2)|^2.$$

Using the majoration in [5, p. 207] we obtain

$$|\nabla_{xy}u^\delta|^2\leqslant C\frac{\delta^8}{(\delta^8+r^2)^2}\quad\text{on } D_{\delta^2}.\tag{119}$$

Let G' be the intersection of $(\pi^+)^{-1}(D_{\delta^2})$ and the ball centered at p and of radius δ^2. By homogeneity we have

$$\int_{G'} |\nabla u^\delta|^2\,dx\,dy\,dz = \delta^2 \int_{\partial G'\cap G} |\nabla_T u^\delta|^2\,d\sigma. \tag{120}$$

Moreover, π^+ is a conformal map from $\partial G'$ into D_{δ^2}. Thus,

$$\int_{\partial G'\cap G} |\nabla_T u^\delta|^2\,d\sigma = \int_{D_{\delta^2}} |\nabla_{xy} u^\delta|^2\,dx\,dy. \tag{121}$$

Using (120) and (107) we obtain

$$\int_{G'} |\nabla u^\delta|^2\,dx\,dy\,dz = \delta^2(2A(\Sigma)+O(\delta^4)) = 2A(\Sigma)\delta^2+O(\delta^6). \tag{122}$$

Finally we estimate $\int_{G\setminus G'} |\nabla u^\delta|^2\,dx\,dy\,dz$: using Fubini and (119) we have

$$\begin{aligned}\int_{G\setminus G'} |\nabla u^\delta|^2\,dx\,dy\,dz &\leqslant C\int_0^{\delta^2} \big(\sqrt{\delta^4+r^2}-\delta^2\big)\frac{\delta^8}{(\delta^8+r^2)^2}\,r\,dr \\ &\leqslant C\int_0^{\delta^2} \delta^6 \frac{r^3}{(\delta^8+r^2)^2}\,dr = O(\delta^6\ln(1/\delta)).\end{aligned} \tag{123}$$

Combining (117), (118), (122) and (123) we obtain

$$\int_{c_p^\delta} |\nabla u^\delta|^2\,dx\,dy\,dz = 2A(\Sigma)\delta^2+O(\delta^6\ln(1/\delta)).$$

(c) *Choice of c and δ.* Combining (95), (104), (108), (116) and (124) we obtain

$$\begin{aligned}&\int_{C^\delta} |\nabla u^\delta|^2\,dx\,dy\,dz \\ &= 2A(\Sigma)2\delta - 8\pi\mu^2\delta^5(4c^2-a^2-b^2-(a^2+b^2+8c^2-4ac-4bc)\ln 2) \\ &\quad + 4\pi\delta^4\int_{-\delta+\delta^2}^{\delta-\delta^2} \left|\frac{\partial u}{\partial z}(0,0,z)\right|^2 dz + O(\delta^6\ln(1/\delta)).\end{aligned} \tag{125}$$

Moreover, since u is regular,

$$\begin{aligned}\int_{C^\delta}\left(\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial u}{\partial y}\right|^2\right)dx\,dy\,dz &= \int_{C^\delta}\mu^2(a^2+b^2)\,dx\,dy\,dz+O(\delta^6) \\ &= 8\pi\delta^5\mu^2(a^2+b^2)+O(\delta^6).\end{aligned} \tag{126}$$

Concerning the third coordinate, we have

$$\int_{C^\delta}\left|\frac{\partial u}{\partial z}\right|^2 dx\,dy\,dz = 4\pi\delta^4\int_{-\delta+\delta^2}^{\delta-\delta^2}\left|\frac{\partial u}{\partial z}(0,0,z)\right|^2 dz+O(\delta^6). \tag{127}$$

Finally, since we have

$$\int_{\Omega\setminus C^\delta}|\nabla u^\delta|^2=E(u)-\int_{C^\delta}\left(\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial u}{\partial y}\right|^2+\left|\frac{\partial u}{\partial z}\right|^2\right)dx\,dy\,dz. \tag{128}$$

Using (125), (126), (127) and (128) we get

$$\begin{aligned}\int_\Omega|\nabla u^\delta|^2=E(u)+2A(\Sigma)2\delta-8\pi\mu^2\delta^5(4c^2-(a^2+b^2+8c^2-4ac-4bc)\ln 2)\\ +O(\delta^6\ln(1/\delta)).\end{aligned} \tag{129}$$

Exactly as in [5], we can choose c such that

$$4c^2-(a^2+b^2+8c^2-4ac-4bc)\ln 2>0, \tag{130}$$

for instance $c=\max\{\frac12 a,\frac12 b\}$.

For δ sufficiently small we have the desired strict inequality and u^δ is a solution to the problem. □

A.4. Proof of Lemma A3

We may suppose that $n\geqslant p$. Let $k=n-p$. Let L be the length of the unique minimal connection C between the P_i and the N_i, and L' the minimum of the lengths of the other connections.

In C, the indexation is chosen such that, for $i\leqslant p$, N_i is connected to P_i and for $i>p$, N_i is connected to ∂B^3. For $i>p$ let us denote by $P_i=\Pi_{\partial B^3}(N_i)$ the projection of N_i on ∂B^3. (It is necessarily unique (i.e. $N_i\neq 0$) because of the uniqueness of the minimal connection.) Thus we have $C=\bigcup_{i=1}^n[P_i,N_i]$.

Let $x\in B^3\setminus C$ and $r>0$ such that

$$r<\min\left(\tfrac14(L'-L);\tfrac14\min_{i=1,\dots,n}\{|P_i-x|+|N_i-x|-|P_i-N_i|\}\right). \tag{131}$$

Let P and N be in $B_r(x)$ with $P\neq N$.

Suppose that there exists a minimal connection C' between $(P_i)_{1\leqslant i\leqslant p}\cup\{P\}$ and $(N_i)_{1\leqslant i\leqslant n}\cup\{N\}$ which does not contain the segment $[P,N]$. Thus, C' contains a union $[P,N_k]\cup[P',N]$ where P' is either a point P_l, with $l\leqslant p$, or the projection of N on ∂B^3. Let $P''=P_l$ in the first case and $P''=\Pi_{\partial B^3}(N_k)$ in the second case. We remark that $|N_k-P''|\leqslant|N_k-P'|$.

Let C'' be the following union of segments:

$$C''=(C'\cup[N_k,P''])\setminus([P,N_k]\cup[P',N]). \tag{132}$$

This is a connection between $(P_i)_{1\leqslant i\leqslant p}$ and $(N_i)_{1\leqslant i\leqslant n}$. Of course we have

$$\begin{aligned} L(C'') &= L(C')+|N_k-P''|-(|P-N_k|-|P'-N|) \\ &\leqslant L(C')+|N_k-P'|-(|P-N_k|+|P'-N|). \end{aligned} \tag{133}$$

Since $C\cup[P,N]$ is a connection between $(P_i)_{1\leqslant i\leqslant p}\cup\{P\}$ and $(N_i)_{1\leqslant i\leqslant n}\cup\{N\}$ we have

$$L(C') \leqslant L(C)+|P-N|. \tag{134}$$

Combining (133) and (134) we obtain

$$\begin{aligned} L(C'') &\leqslant L(C')+|N_k-P'|-(|P-N_k|+|P'-N|) \\ &\leqslant L(C)+|N-P|+|N_k-P'|-(|P-N_k|+|P'-N|). \end{aligned} \tag{135}$$

If $P''=P_k$ then $P'=P_k$ and (135) becomes

$$|P-N_k|+|P_k-N| \leqslant |N-P|+|N_k-P_k|. \tag{136}$$

But r has been chosen sufficiently small such that

$$\begin{aligned} |P-N|+2r &< |P_k-x|+|N_k-x|-|P_k-N_k| \\ &\leqslant |P_k-P|+|N_k-N|-|P_k-N_k|+|P-x|+|N-x| \\ &\leqslant |P_k-P|+|N_k-N|-|P_k-N_k|+2r. \end{aligned} \tag{137}$$

(137) implies

$$|P_N|+|P_k-N_k| < |P_k-P|+|N_k-N|$$

and this contradicts (136). Thus $P''\neq P$, but in this case C'' is necessarily different from C and we have $L(C'')>L+4r$. Using (135) we obtain

$$4r < |N-P|+|N_k-P'|-(|P-N_k|+|P'-N|). \tag{138}$$

This implies

$$|P'-N|+|N-P|+|P-N_k| < |P'-N_k|, \tag{139}$$

which is a contradiction.

We conclude that any minimal connection between

$$(P_i)_{1\leqslant i\leqslant p}\cup\{P\} \quad \text{and} \quad (N_i)_{1\leqslant i\leqslant n}\cup\{N\}$$

necessarily contains the segment $[P,N]$. The other part of the connection is between $(P_i)_{1\leqslant i\leqslant p}$ and $(N_i)_{1\leqslant i\leqslant n}$: it has to be minimal, thus it is C. The lemma is proved. □

Acknowledgements. The author is grateful to Fabrice Bethuel for having drawn his attention to this problem and for useful discussions. He would also like to thank Haïm Brezis and Jean-Michel Coron for their attention to his work and for their constant support.

References

[1] BETHUEL, F., A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7 (1990), 269–286.

[2] — On the singular set of stationary harmonic maps. *Manuscripta Math.*, 78 (1993), 417–443.

[3] BETHUEL, F., BREZIS, H. & CORON, J.-M., Relaxed energies for harmonic maps, in *Variational Methods* (H. Berestycki, J.-M. Coron & I. Ekeland, eds.). Birkhäuser, Boston, MA, 1990.

[4] BETHUEL, F. & ZHENG, X., Density of smooth functions between two manifolds in Sobolev spaces. *J. Funct. Anal.*, 80 (1988), 60–75.

[5] BREZIS, H. & CORON, J.-M., Large solutions for harmonic maps in two dimensions. *Comm. Math. Phys.*, 92 (1983), 203–215.

[6] BREZIS, H., CORON, J.-M. & LIEB, E., Harmonic maps with defects. *Comm. Math. Phys.*, 107 (1986), 649–705.

[7] EVANS, L. C., Partial regularity for stationary harmonic maps into the sphere. *Arch. Rational Mech. Anal.*, 116 (1991), 101–113.

[8] GIAQUINTA, M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.* Princeton Univ. Press, Princeton, NJ, 1983.

[9] GIAQUINTA, M., MODICA G. & SOUČEK, J., The Dirichlet energy of mappings with values into the sphere. *Manuscripta Math.*, 65 (1989), 489–507.

[10] — Cartesian currents and variational problems for mappings into spheres. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (*4*), 16 (1990), 393–485.

[11] HARDT, R., LIN F. H. & POON, C., Axially symmetric harmonic maps minimizing a relaxed energy. *Comm. Pure Appl. Math.*, 45 (1992), 417–459.

[12] HELEIN, F., Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne. *C. R. Acad. Sci. Paris Sér I Math.*, 312 (1991), 591–596.

[13] HILDEBRANDT, S., KAUL, H. & WIDMAN, K. O., An existence theorem for harmonic mappings of Riemannian manifolds. *Acta Math.*, 138 (1977), 1–15.

[14] MORREY, C. B., *Multiple Integrals in the Calculus of Variations.* Springer-Verlag, New York, 1966.

[15] RIVIÈRE, T., Harmonic maps from B^3 into S^2 having a line of singularities. *C. R. Acad. Sci. Paris Sér. I Math.*, 101 (1991), 583–587.

[16] — Construction d'un dipôle. Preprint 9303, Ecole Normale Supérieure de Cachan, 1993.

[17] — Regularité des applications harmoniques à valeur dans les tores de révolution. Ph.D. Thesis, Université de Paris VI, 1993.

[18] SCHOEN R. & UHLENBECK, K., A regularity theory for harmonic maps. *J. Differential Geom.*, 17 (1982), 307–335.

TRISTAN RIVIÈRE
Centre de Mathématiques et de leurs Applications
Ecole Normale Supérieure de Cachan
Unité associée au CNRS URA-1611
61, avenue du Président Wilson
94235 Cachan Cedex
France
riviere@cmla9.ens-cachan.fr

Received December 14, 1993