Evolution and End Point of the Black String Instability: Large D Solution

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(Received 29 June 2015; published 28 August 2015)

We derive a simple set of nonlinear, $(1+1)$-dimensional partial differential equations that describe the dynamical evolution of black strings and branes to leading order in the expansion in the inverse of the number of dimensions D. These equations are easily solved numerically. Their solution shows that thin enough black strings are unstable to developing inhomogeneities along their length, and at late times they asymptote to stable nonuniform black strings. This proves an earlier conjecture about the end point of the instability of black strings in a large enough number of dimensions. If the initial black string is very thin, the final configuration is highly nonuniform and resembles a periodic array of localized black holes joined by short necks. We also present the equations that describe the nonlinear dynamics of anti–de Sitter black branes at large D.

DOI: 10.1103/PhysRevLett.115.091102 PACS numbers: 04.50.Gh, 04.25.-g, 04.60.Cf, 04.70.Bw

The instability of black strings and black branes discovered in [1] is a phenomenon with wide implications for the physics of higher-dimensional black holes and their applications to string theory and gauge-gravity duality [2]. Black strings behave similarly to tubular soap films and are prone, when thin enough, to rippling, i.e., developing nonuniformities along their direction. This instability is well established in perturbation theory, but its growth beyond the linearized approximation and its end point at asymptotically late times are notoriously hard problems. The numerical evolution of a perturbed five-dimensional black string in [3] is a landmark result: it gives strong evidence that the classical evolution does not stop at any stable configuration but proceeds in a self-similar cascade to smaller scales.

Important as the result of [3] is, it still leaves open many questions about the fate of unstable black strings and black branes. For instance, it would be very convenient to have a better, possibly analytic, understanding of the late-time evolution. Moreover, [3] presents a single calculation of a single system. Do all unstable black strings behave in the same manner? This is indeed unlikely: there has long been evidence that, by modifying the parameters of the system, the end point may be different. In the simplest instance, the parameter is the number of spacetime dimensions where the black string lives. Reference [4] found a critical dimension, $D=D_* \simeq 13.5$, above which weakly nonuniform static black strings (NUBS) have larger horizon area, for fixed string length and mass, than the uniform solutions. It is then possible that the classical evolution of the system in $D>D_*$ ends at a stable NUBS, as proposed (independently of D) in [5]. However, given the almost prohibitive cost of the numerical simulations (100000 CPU hours for [3]), the investigation of this possibility and the systematic study of the evolution of related systems has not been undertaken to date. This is strong motivation to search for simpler methods capable of capturing at least the main qualitative features of the phenomenon.

In this Letter we present an approach, based on an expansion in $1/D$, that allows us to address some of these issues. In particular, we show that for large enough D the end point of the instability is generically a stable NUBS. The simplification of the problem is dramatic: accurate numerical evolutions can be obtained in seconds (or less) in a conventional computer running a one-line NDSolve of Mathematica. This encourages further investigation of this and similar problems by these means.

The large D approach to black hole physics, initiated in [6–8], has been recently developed to deal with fully nonlinear deformations of horizons [9–12]. While the formalism of [10] incorporates time evolution, it is unclear to us whether it allows us, as it is, to study the system at hand, which involves horizon length scales $\sim 1/\sqrt{D}$. Therefore, we solve the problem *ab initio* in a formulation adapted to the dynamics of black branes.

Using ingoing Eddington-Finkelstein coordinates, the metric of a uniform black p-brane in $D=n+p+3$ dimensions, boosted along its world volume with velocity vector u_μ, $\eta^{\mu\nu}u_\mu u_\nu=-1$, $\mu,\nu=0,\ldots,p$, is

$$ds^2=\left[\eta_{\mu\nu}+\left(\frac{r_0}{r}\right)^n u_\mu u_\nu\right]d\sigma^\mu d\sigma^\nu-2u_\mu d\sigma^\mu dr+r^2 d\Omega_{n+1}. \quad (1)$$

The world volume directions are

$$\sigma^\mu = (t, \sigma^a), \qquad a = 1, \ldots, p. \tag{2}$$

We take p to be finite, so $D \to \infty$ is $n \to \infty$, and we will study deformations of the black brane that depend on σ^μ.

Since the wavelength of unstable fluctuations is known to be $\sim 1/\sqrt{n}$ [6,7], we rescale $\sigma^a \to \sigma^a/\sqrt{n}$. Furthermore, we consider small velocities, $u_\mu = (-1 + \mathcal{O}(n^{-1}), v_a/\sqrt{n})$, so, denoting $m = r_0^n$, the metric, Eq. (1), becomes

$$ds^2 = -\left(1 - \frac{m}{r^n}\right)dt^2 + 2\left(dt - \frac{v_a}{n}d\sigma^a\right)dr - \frac{2mv_a}{nr^n}d\sigma^a dt + \frac{1}{n}\left(\delta_{ab} + \frac{mv_a v_b}{nr^n}\right)d\sigma^a d\sigma^b + r^2 d\Omega_{n+1}. \tag{3}$$

We seek solutions that can be regarded as having m and v_a not constant but varying with the coordinates σ^μ. In order to find them, we take a metric ansatz of the form

$$ds^2 = -Adt^2 - 2(u_t dt + u_a d\sigma^a)dr - 2C_a d\sigma^a dt + G_{ab}d\sigma^a d\sigma^b + r^2 d\Omega_{n+1}, \tag{4}$$

with

$$A = \sum_{k\geq 0}\frac{A^{(k)}(\sigma^\mu, r)}{n^k}, \qquad u_t = \sum_{k\geq 0}\frac{u_t^{(k)}(\sigma^\mu, r)}{n^k},$$
$$u_a = \sum_{k\geq 0}\frac{u_a^{(k)}(\sigma^\mu, r)}{n^{k+1}}, \tag{5}$$

$$C_a = \sum_{k\geq 0}\frac{C_a^{(k)}(\sigma^\mu, r)}{n^{k+1}},$$
$$G_{ab} = \frac{1}{n}\left[\delta_{ab} + \sum_{k\geq 0}\frac{G_{ab}^{(k)}(\sigma^\mu, r)}{n^{k+1}}\right]. \tag{6}$$

The different scalings with n conform to Eq. (3). We have chosen a Bondi-type gauge where $g_{rr} = 0$ and r is the area radius of S^{n+1} to all orders in $1/n$. This leaves a gauge freedom in the choice of u_a, which is a shift vector on surfaces at constant r and is only restricted by boundary conditions. We have partially gauge fixed it to be independent of r.

We introduce the radial coordinate

$$\mathsf{R} = r^n, \tag{7}$$

such that when $D \to \infty$ keeping R finite, we focus on the near-horizon region, which we can always locate near $r = 1$, without loss of generality, by an appropriate choice of scales. The boundary conditions at large R (for near-horizon decoupled geometries [13]) are

$$A = 1 + \mathcal{O}(\mathsf{R}^{-1}), \qquad C_a = \mathcal{O}(\mathsf{R}^{-1}),$$
$$G_{ab} = \frac{1}{n}[\delta_{ab} + \mathcal{O}(n^{-1}, \mathsf{R}^{-1})]. \tag{8}$$

We also require regularity at the horizon, which is where $g_{\mu\nu}u^\mu u^\nu = 0$.

It is now straightforward to solve the Einstein equations perturbatively in $1/n$. To leading order, the solution is

$$A^{(0)} = 1 - \frac{m(\sigma^\mu)}{\mathsf{R}}, \quad C_a^{(0)} = \frac{p_a(\sigma^\mu)}{\mathsf{R}}, \quad G_{ab}^{(0)} = \frac{p_a(\sigma^\mu)p_b(\sigma^\mu)}{m(\sigma^\mu)\mathsf{R}}, \tag{9}$$

$$u_t^{(0)} = -1, \qquad u_a^{(0)} = \text{const}. \tag{10}$$

We have gauge fixed the σ^μ dependence of u_a in a manner consistent with boundary conditions. Solving similarly at the next order, the equations $R_{tt} = 0$ and $R_{ta} = 0$ are R-independent constraints that require, respectively,

$$\partial_t m - \partial_b \partial^b m = -\partial_b p^b, \tag{11}$$

and

$$\partial_t p_a - \partial_b \partial^b p_a = \partial_a m - \partial_b\left(\frac{p_a p^b}{m}\right) \tag{12}$$

(a, b indices are raised with the flat metric δ^{ab}). The remaining Einstein equations are readily solved yielding R-dependent metric components that preserve horizon regularity, but we will not need their explicit form.

Equations (11) and (12) are one of our main results. They are the effective equations for the collective variables, $m(\sigma^\mu)$ and $p_a(\sigma^\mu)$, that describe nonlinear fluctuations of the black brane. These variables give the energy and momentum densities of the black brane on sections at constant σ^a at a given time. The horizon of the solution is at $\mathsf{R} = m(\sigma^\mu)$, so the function $m(\sigma^\mu)$ is also interpreted as the area density, i.e., the area of the S^{n+1} (up to a factor of the unit-sphere area) at the horizon at a fixed value of t and σ^a, to leading order in $1/n$. For solutions that are spatially periodic in σ^a, the quantities

$$M = \int d\sigma^1 \ldots d\sigma^p m(t, \sigma^b), \qquad P_a = \int d\sigma^1 \ldots d\sigma^p p_a(t, \sigma^b), \tag{13}$$

are conserved in time. Up to normalization factors, they are the total mass and momentum of the black brane. Up to similar factors and to leading order, M is also the total horizon area, which therefore does not vary as the system evolves. If we define the surface gravity as the nonaffinity of the horizon generator ∂_t, it also remains constant.

The equations are invariant under Galilean boosts along σ^a with constant velocity v_a,

$$\sigma^a \to \sigma^a - v_a t, \qquad p_a \to p_a + m v_a, \tag{14}$$

which allows us to fix the rest frame of the black brane, in which $P_a = 0$, and set $u_a^{(0)} = 0$.

This effective formulation of black brane dynamics passes two important checks. (i) For small, linearized perturbations around the uniform black brane, with momentum k aligned with $\sigma^1 \equiv z$,

$$m(t,z) = 1 + \delta m e^{-i\omega t + ikz}, \quad p_a(t,z) = \delta p_a e^{-i\omega t + ikz}, \tag{15}$$

the solution frequencies are

$$\omega_\pm = i(\pm k - k^2), \qquad \omega = -ik^2, \tag{16}$$

which reproduce the frequencies of the sound and shear modes of the black brane to leading order at large D [7]. In particular, the frequency ω_+ for $0 < k < 1$ corresponds to the Gregory-Laflamme unstable mode. (ii) If we consider static, shear-free deformations, $m = m(z)$, $p_1 = p(z)$, and $p_{a\neq 1} = 0$, the resulting equation (with $p = m'$ in the rest frame),

$$m'' + m - \frac{(m')^2}{m} = \text{const}, \tag{17}$$

is equivalent to the equation for static black strings derived in [9,12] (using different gauges), with the variables being related by $m(z) = \exp[2\mathcal{P}(z)]$.

Observe that the sound deformations m, p_1 along a direction σ^1, are not affected by the shear $p_{a\neq 1}$. In the following, we set $p_{a\neq 1} = 0$ and consider black strings with $p_1(t, \sigma^1 = z) \equiv p(t,z)$. The direction z is compactified, $z \in [-L/2, L/2]$. We parametrize the periodicity L in terms of a wave number k_L as

$$k_L = \frac{2\pi}{L}. \tag{18}$$

Since we are fixing the string thickness $r_0 = 1$, the uniform strings are characterized by the value of k_L. Smaller values of k_L correspond to thinner black strings.

We solve numerically the black string equations

$$\partial_t m(t,z) - \partial_z^2 m(t,z) = -\partial_z p(t,z), \tag{19}$$

and

$$\partial_t p(t,z) - \partial_z^2 p(t,z) = \partial_z \left[m(t,z) - \frac{p(t,z)^2}{m(t,z)} \right]. \tag{20}$$

The Mathematica function NDSolve handles them without difficulty. We fix a value of k_L and introduce a small perturbation of the static uniform black string, $m(0,z) = 1 + \delta m_0(z)$, $p(0,z) = \delta p_0(z)$, such that the momentum P vanishes. We find that for $k_L > 1$, the perturbation quickly dissipates and the black string becomes uniform, in agreement with the absence in Eq. (16) of unstable linear modes with wave number $k > 1$.

For thinner black strings, with $k_L < 1$, after brief initial transients the deformation grows (at approximately the exponential rate of the linearized solution). Eventually, the system settles down at a stable configuration that solves the static equation (17) to a precision that can easily reach six digits. M and P in Eq. (13) remain constant throughout the evolution to very good accuracy (easily better than 10^{-9} for M and 10^{-6} for P).

In Figs. 1 and 2 we show two sample simulations. We plot $m(t,z)$, which gives the area of the S^{n+1} at a given z. This is different than the area radius of these spheres, which is

$$\mathcal{R}(t,z) = m(z)^{[1/(n+1)]} \simeq 1 + \frac{\ln m(t,z)}{n}. \tag{21}$$

For large deformations, $\ln m(t,z)$ can reveal structure that in $m(t,z)$ is exponentially suppressed (insets in Fig. 2).

Figure 1 is the evolution of a not-too-thin black string with $k_L = 0.98$. Since the perturbative unstable rate is

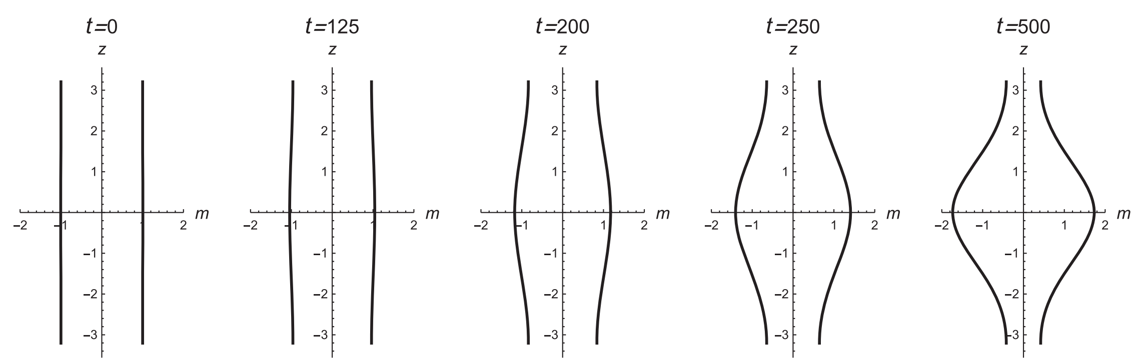

FIG. 1. Dynamical evolution of a perturbed string with $k_L = 0.98$. The horizontal axis is the S^{n+1}-area function, $m(t,z)$. For this simulation, the final state was reached before $t = 500$. The time it takes depends on the size and shape of the initial perturbation, but the final configuration is independent of them.

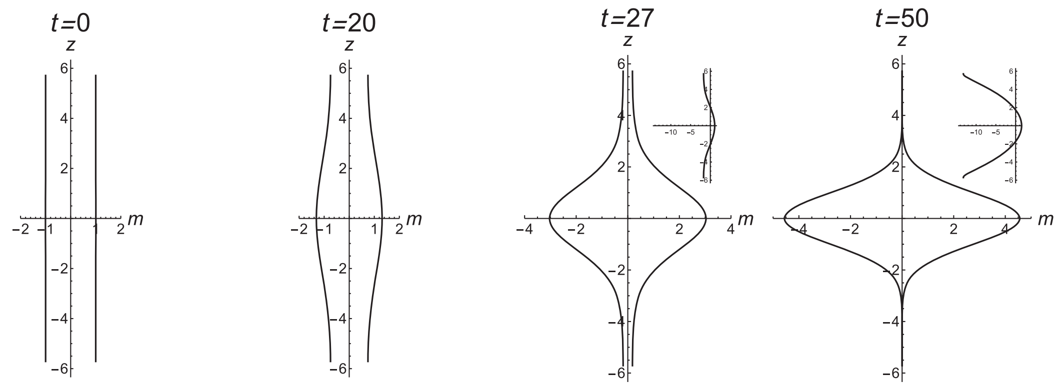

FIG. 2. Dynamical evolution of a perturbed string with $k_L = 0.55$. For the last two plots we display $\ln m(t,z)$ in insets, which show that the blobs extend to almost fill the compact direction. For this simulation, the final state was reached around $t = 40$.

small, the evolution is slow. The final profile is approximately sinusoidal, as expected for a small deformation.

Figure 2 shows the evolution for a thinner black string with $k_L = 0.55$, which evolves much faster and develops a large blob in its final state. As shown in [9,12], the profile of large static blobs is very approximately Gaussian

$$m(z) \simeq \frac{L}{\sqrt{2\pi}} e^{-z^2/2} \tag{22}$$

(the amplitude is fixed to have the same M as a uniform black string of length L). The radius $\mathcal{R}(z)$ for Eq. (22) decays only like $-z^2/2$ away from the center, as seen in the insets in Fig. 2, and the blob fills almost all the compact direction. Indeed, Eq. (22) gives an excellent approximation to the profile of $\ln m(z)$ for the final solution at all z (better than 1% for $k_L \lesssim 0.6$) except in a very short "neck" of length $(\Delta z)_{\text{neck}} \simeq (4/L)\ln L$ near $|z| = L/2$ [9]. As shown in [12], these blobs approximate very well the shape of a spherical black hole within a region of width $\sim 1/\sqrt{n}$ around its equator, as well as its total area. Bear in mind, however, that while the height of the blobs is $\mathcal{R} = \mathcal{O}(1)$, their proper length along z is $L/\sqrt{n}$; i.e., they are much thinner than a sphere.

When $1/2 < k_L < 1$ the NUBS at the end point of the evolution is unique for each value of k_L, independently of the shape and amplitude of the initial perturbation. When $0 < k_L < 1/2$ there are (at least) two different unstable modes that can be excited, with wave numbers k_L and $2k_L$ (or a higher multiple), and two possible final static NUBS. Since their mass and area are the same to leading order in $1/n$, the evolution depends on the relative growth rates of the unstable modes and on the shape of the initial perturbation. When the solution develops two or more blobs, it may be difficult to determine if these will remain in the asymptotic final state, or are instead only part of a very long-lived transient phase: since m is exponentially small in between the blobs, the interaction among them may be lost in numerical error. These situations, however, may fall outside the range of applicability of this formulation.

The large D approximation is valid when $|\ln m|$, $|\partial_{t,z} \ln m|$, $|\partial_{t,z} \ln p| \ll n$. In the solutions we consider, the most stringent condition is $|\ln m| \ll n$, which at the neck becomes $L \ll 2\sqrt{2n}$, i.e., $nk_L^2 \gg \pi^2/2$. Thus, our results are quantitatively accurate in a given dimension only for sufficiently large k_L, or conversely, for a given k_L only in sufficiently large n. Observe that near the limit of validity, at $L \sim \sqrt{n}$, the blob has proper size $\mathcal{O}(1)$ in all directions [9]. This suggests that this final state can be regarded as an array of roughly spherical black holes joined by thin, short necks. Additionally, when the neck becomes thin in Planck units, the black string may break up into separate black holes due to quantum gravity effects.

Within this range of validity, we conclude that our results are very strong evidence that the end point of the black string instability at large enough D is generically a stable NUBS.

There are several possible extensions of our study. With little extra effort we can obtain the equations that describe the leading large D nonlinear dynamics of anti–de Sitter (AdS) black branes. Either by directly solving the equations as in the previous case, or by applying the AdS/Ricci-flat correspondence of [14], we find

$$\partial_t m - \partial_b \partial^b m = -\partial_b p^b, \tag{23}$$

and

$$\partial_t p_a - \partial_b \partial^b p_a = -\partial_a m - \partial_b \left(\frac{p_a p^b}{m} \right). \tag{24}$$

It is easy to prove that these equations do not have any nonuniform static solutions. Small dynamical perturbations of the uniform configuration, Eq. (15), give as solutions the sound and shear quasinormal frequencies of the AdS black brane computed in [15], namely,

$$\omega_\pm = \pm k - ik^2, \qquad \omega = -ik^2. \tag{25}$$

Other quasinormal modes of the black brane have frequencies $\mathcal{O}(D)$ and do not appear in the near-horizon decoupling limit. Allowing variation in a number $p \sim D$ of horizon directions may involve qualitative changes.

The inclusion of $1/D$ corrections to Eqs. (11) and (12) is potentially very interesting. It has been shown in [12] that $1/D$ corrections allow us to accurately identify the critical dimension D_* below which static NUBS of a given area have higher mass than uniform black strings, so they cannot be the end points of the instability [16,17]. This suggests that the $1/D$ expansion may be able to reproduce, when $D < D_*$, an evolution qualitatively similar to the one observed in [3]. We hope to report on this in the future.

R. E. is supported by FPA2013-46570-C2-2-P, AGAUR 2009-SGR-168, and CPAN CSD2007-00042 Consolider-Ingenio 2010. K. T. was supported by JSPS Grant-in-Aid for Scientific Research No. 26-3387.

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