Evolution equations and scales of Banach spaces

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发展方程与巴拿赫空间链

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and
SCALES OF BANACH SPACES

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PROEFSCHRIFT

Ter verkrijging van de graad van doctor aan de technische universiteit Eindhoven, op gezag van de Rector Magnificus, Prof. ir. M. Tels voor een commissie aangewezen door het College van Dekanen in het openbaar te verdedigen op woensdag 28 juni 1989 te 16.00 uur.

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发展方程与巴拿赫空间链

学位论文

一九八九年六月廿八日（星期三）十六时整在以德国侯温大学校长合尔斯教授为首的委员会主持下进行公开博士论文答辩.

刘 贵 忠

（出生于中国陕西志丹）

献给我的父母和祖先

Dedicated to my parents and ancestors

Contents

0. Preliminaries and Introduction

Many problems in mathematical physics and other sciences reduce to an equation of the form

$$\frac{du}{dt} + Au = 0. \tag{1}$$

Here $u = u(t,x)$ (t a time variable and $x \in \Omega$ a space variable) is the unknown function with $u(t,\cdot)$ in a certain function space X for each t. Further, $A : D(A) \subset X \to X$ is a linear operator. In many cases a mathematical or physical consideration suggests that a certain Banach or Hilbert space X would be appropriate. In this setting we naturally have the following definitions.

Definitions 0.1. The initial value problem

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad t > 0 \ (\text{in } X) \\ u(0) = u_0 \in X \end{cases} \tag{2}$$

is said to be well posed if there exists a dense subspace $D \subset D(A)$ of X such that

(i) For any given $u_0 \in D$, there exists a continuously differentiable function $u : [0,\infty) \to X$ such that (2) is satisfied.

(ii) For any sequence of continuously differentiable functions $u_n : [0,\infty) \to X$ which satisfy equation (1), $u_n(0) \to 0$ implies that $u_n(t) \to 0$ uniformly on compacta of $t \in [0,\infty)$. □

Definition 0.2. A family $\{T(t) \mid t \geq 0\}$ of continuous mappings on a Banach space $(X, \|\cdot\|)$ is said to be a strongly continuous semigroup on X, or a C_0 semigroup on X, if the conditions below are satisfied:

(i) $T(0) = I$ the identity mapping on X;

(ii) $T(t_1)\,T(t_2) = T(t_1 + t_2)$ for all $t_1, t_2 \geq 0$ (the semigroup property);

(iii) For each $u \in X$, the mapping $T(\cdot)\,u : [0,\infty) \to X$ is continuous. □

Definition 0.3. Let $\{T(t) \mid t \geq 0\}$ be a C_0 semigroup on X. Its infinitesimal generator is a linear operator $-A$ in X such that

$$D(-A) = \{u \in X \mid \lim_{t \to 0+} \frac{T(t)\,u - u}{t} \ \textit{exists} \text{ in } X\}$$

$$-A\,u = \lim_{t \to 0+} \frac{T(t)\,u - u}{t} \quad \text{for } u \in D(-A).$$

□

Theorem 0.4. The initial value problem (2) is well posed iff $-A$ is closable and its closure $-\bar{A}$ is the infinitesimal operator of a C_0 semigroup $\{T(t) \mid t \geq 0\}$ on X. □

This theorem deserves a little more detailed explanation. In fact, if the initial value problem (2) is well posed, then A and hence $\overline{A}$ has D as a core and the semigroup $\{T(t) \mid t \geq 0\}$ is obtained by a continuation of the solution operator $u(t)(\cdot) : D \rightarrow X$ $(t \geq 0)$. Conversely, if $-\overline{A}$ is the infinitesimal generator of a C_0 semigroup $\{T(t) \mid t \geq 0\}$ then D can be any core of $-A$. For convenience of the readers we mention that a subspace D of X is said to be a core of A if $D \subset D(A)$ and $\overline{(A \restriction_D)} = \overline{A}$.

From the above readily follows

Theorem 0.5. A dense subspace $D \subset D(-A)$ is a core of the generator $-A$ of a C_0 semigroup $\{T(t) \mid t \geq 0\}$ on X if $T(t)D \subset D$ for all $t \geq 0$. □

The following well known theorem of Hille and Yosida in 1948 provides a characterization of an infinitesimal generator $-A$ of a C_0 semigroup $\{T(t) \mid t \geq 0\}$ in terms of the spectral property of the operator A.

Theorem 0.6. (Hille-Yosida) A linear operator $-A : D(-A) \subset X \rightarrow X$ in a Banach space $(X, \|\cdot\|)$ is the infinitesimal generator of a C_0 semigroup $\{T(t) \mid t \geq 0\}$ on X iff the following conditions hold:

(i) A is densely defined and closed.

(ii) There exists $\omega \in \mathbb{R}$ and $M \geq 0$ such that $(\omega, \infty) \subset \rho(A)$ and

$$\|(\lambda I + A)^{-k}\| \leq M(\lambda - \omega)^{-k}, \quad \forall k \in \mathbb{N}_0, \quad \forall \lambda > \omega. \tag{3}$$

□

The proof of the above theorem can be reduced to that for the so called contractive semigroup by a renorming procedure. A C_0 semigroup $\{T(t) \mid t \geq 0\}$ is called contractive if $\|T(t)\| \leq 1$ for all $t \geq 0$. For such a class of C_0 semigroup the above theorem reads as follows.

Theorem 0.6'. A linear operator $-A : D(-A) \subset X \rightarrow X$ is the infinitesimal generator of a contractive semigroup on X iff

(i) A is densely defined and closed.

(ii) $(0, \infty) \subset \rho(A)$ and

$$\|(\lambda I + A)^{-1}\| \leq \lambda^{-1}, \quad \forall \lambda > 0. \tag{4}$$

□

Obviously condition (4) is equivalent to

$$\| u + \mu A u \| \geq \| u \|, \ \forall \mu > 0, \ \forall u \in D(A). \tag{5}$$

This is in turn equivalent to that

$$\mathrm{Re}(Au, u) \geq 0, \ \forall u \in D(A) \tag{6}$$

where

$$\mathrm{Re}(v, u) = \lim_{t \to 0+} \frac{\| u + tv \| - \| u \|}{t} = \sup_{w \in J(u)} < v, w >. \tag{7}$$

Here $J : X \to X^*$ is the duality mapping (possibly multivalued). Such an operator A is said to be accretive. If for some $\omega \in \mathbb{R}$ the operator $\omega I + A$ is accretive, then A is called quasi-accretive.

As it happens very often, information on the dual operator A^* of A is useful for the study of A itself. For instance we have

Theorem 0.7. Let A be a densely defined closed operator on a Banach space X. Then
(i) $R(A) = X$ iff $\exists M > 0 \ \forall u \in D(A^*) \ [\| A^* u \| \geq M \| u \|]$
(ii) $R(A^*) = X^*$ iff $\exists M > 0 \ \forall u \in D(A) \ [\| A u \| \geq M \| u \|]$ □

Theorem 0.6' with condition (4) replaced by (6) constitutes the Philips-Lumer characterization of infinitesimal generators of contractive semigroups. In particular, a sufficient condition for $-A$ to generate a contractive semigroup on X is

$$\left.\begin{array}{l} \mathrm{Re}(Au, u) \geq 0, \ \forall u \in D(A) \\ \mathrm{Re}(A^* v, v) \geq 0, \ \forall v \in D(A^*). \end{array}\right\} \tag{8}$$

From Theorem 0.6 follows easily

Theorem 0.8. If $A : D(A) \subset X \to X$ is a generator of a C_0 semigroup $\{T(t) \mid t \geq 0\}$ on a reflexive Banach space X, then A^* is the generator of the C_0 semigroup $\{[T(t)]^* \mid t \geq 0\}$. □

Note that for a non-reflexive Banach space X, $[T(\cdot)]^* u : [0, \infty) \to X^*$ is not necessarily continuous, so it is in general not a C_0 semigroup.

Theorem 0.6 and its variants give perfect necessary and sufficient conditions for an operator $-A$ to generate a C_0 semigroup on a Banach space X. Unfortunately these conditions are usually hard to check in concrete situations. The following two results, one by Rellich, Kato, Gustafson and Chernoff in terms of perturbations, one by De Graaf in terms of the so called auxiliary operators, provide useful sufficient conditions which are easier to verify in some concrete cases.

Theorem 0.9. (Rellich-Kato-Gustafson-Chernoff) Suppose that $A : D(A) \subset X \to X$ be an infinitesimal generator of a C_0 semigroup on a Banach space $(X, \|\cdot\|)$. Let P be another operator in X such that $D(P) \supset D(A)$ and

$$\|Pu\| \leq M\|u\| + \alpha\|Au\|, \quad \forall u \in D(A) \tag{9}$$

where M and α are nonnegative constants with $\alpha < 1$. Then the operator $A+P$ as defined on $D(A)$ generates a C_0 semigroup on X.

If instead of (9) above a weaker condition

$$\|Pu\| \leq M\|u\| + \|Au\|, \quad \forall u \in D(A) \tag{10}$$

is satisfied, then the operator $A+P$ as defined in $D(A)$ is closable and its closure $\overline{A+P}$ generates a C_0 semigroup on X. □

Theorem 0.10. (De Graaf) Let A be a closable densely defined operator in a Hilbert space $(H, (\cdot,\cdot))$. Assume that there exists a strictly positive self-adjoint operator Q such that $D(Q) \subset D(A)$, $R(Q) = H$ and

$$\left.\begin{array}{l} \mathrm{Re}(u, Au) \leq \omega(u,u) \\ \mathrm{Re}(Qu, Au) \leq \omega(Qu,u) \end{array}\right\} \forall u \in D(Q). \tag{11}$$

Then $D(Q)$ is a core for the operator A and $\bar{A}$ generates a quasi-contractive C_0 semigroup on X (i.e. a C_0 semigroup $\{T(t) \mid t \geq 0\}$ such that $\{e^{-\omega t}T(t) \mid t \geq 0\}$ is contractive for some $\omega \geq 0$). □

Theorem 0.10 was given in [Gr1]. All the remaining results are quite standard and can be found in standard books on functional analysis or monographs on operator semigroups. Cf. e.g. [Fa], [Go], [Pa], [R-S], [Ta] and [Yo].

Now we give a brief description of the structure of this thesis. For more details we refer to the introductions of the respective chapters.

In Chapter I we are concerned with the construction of the so called regular spaces and hyper-spaces as well as linear operators therein. For a Banach space $(X, \|\cdot\|)$ and an invertible operator B of positive type in X a scale of Banach spaces $\{X_B^\sigma \mid \sigma \in \mathbb{R}\}$ is defined. More precisely, for $\sigma \geq 0$, $X_B^\sigma = (D(B^\sigma), \|\cdot\|_\sigma)$ with $\|u\|_\sigma = \|B^\sigma u\|$ for $u \in D(B^\sigma)$, and $X_B^{-\sigma}$ = completion of X with respect to the norm $\|u\|_{-\sigma} = \|B^{-\sigma}u\|$ for $u \in X$. From these Banach spaces we form their natural inductive limits $X_B^{\sigma+} = \bigcup_{\tau>\sigma} X_B^\tau$ $(\sigma \in [-\infty,+\infty))$ and projective limits $X_B^{\sigma-} = \bigcap_{\tau<\sigma} X_B^\tau$ $(\sigma \in (-\infty,+\infty])$. Thus we have the scheme

$$X_B^{(-\sigma)-} \supset X_B^{-\sigma} \supset X_B^{(-\sigma)+} \supset X \supset X_B^{\sigma-} \supset X_B^\sigma \supset X_B^{\sigma+}.$$

In the above diagram the spaces to the right of X are called regular spaces and those to the left of X hyper-spaces. We are thus led to the study of the inductive limits and projective limits of a sequence of Banach spaces in general, as well as linear operators therein. In doing so, the topological structures of the regular spaces and hyper-spaces are clarified and several types of continuous linear mappings on them are characterized. By choosing different space X and different operator B various classical spaces of functions or generalized functions are realized functional-analytically as regular spaces or hyper-spaces in the above sense.

In the second chapter we discuss the regularity and extendibility of a C_0 semigroup e^{-tA} on X with respect to a scale of Banach spaces $\{X_B^\sigma \mid \sigma \in \mathbb{R}\}$ constructed in Chapter I. More precisely, conditions between the operators A and B^σ (or A^* and $(B^*)^\sigma$) are given so that the semigroup e^{-tA} on X restricts to a C_0 semigroup on X_B^σ (or extends to a C_0 semigroup on $X_B^{-\sigma}$). Applications to matrix operators in l^2 and to second order partial differential operators on $L^2(\mathbb{R}^n)$ are presented. We also set two criteria for an infinite matrix (a_{jk}) to generate a C_0 semigroup on l^2.

In Chapter III we formulate and prove a Hille-Yosida type theorem for locally equi-continuous semigroups on the inductive limit space of a sequence of Banach spaces. We emphasize that instead of semi-norms of the inductive limit, which are hard to find and to deal with, we use the norms of the constituents of the inductive limit. The result together with that of Ouchi applies readily to the spaces $X_B^{\sigma\pm}$ defined in Chapter I.

In the last chapter weighted L^2 spaces of harmonic functions on $\mathbb{R}^q (q \geq 2)$ and several naturally arising linear operators in them are studied. The central idea is an identification of a weighted L^2 space of harmonic functions on $\mathbb{R}^q$ with the domain of a suitable positive self-adjoint operator in $L^2(S^{q-1})$ (S^{q-1} the unit sphere in $\mathbb{R}^q$); the identification is the natural restriction-extension procedure. In particular, we have natural weighted L^2 spaces of harmonic functions on $\mathbb{R}^q$ wherein the differentiation operators are continuous or even compact. Also, working in the opposite direction we arrive at a complete characterization of the ranges of the propagator of the fractional spherical diffusion equation $\frac{\partial u}{\partial t} = -(-\Delta_{LB})^{\nu/2} u$, where Δ_{LB} is the Laplace-Beltrami operator on S^{q-1}.

I. Regular Spaces and Hyper-spaces

This chapter is devoted to the construction of regular spaces, hyper-spaces and their linear operators.

In the first section we discuss topological properties of the inductive limit E^+ of an increasing sequence of Banach spaces $\{(E_n, \|\cdot\|_{E_n}) \mid n \in \mathbb{N}_0\}$ and of the projective limit F^- of a decreasing sequence of Banach spaces $\{(F_n, \|\cdot\|_{F_n}) \mid n \in \mathbb{N}_0\}$, as well as linear operators in them. One point is the idea of using the so called interpolation type inequalities in the study of the topological properties of inductive limits. Four types of continuous mappings T, i.e. $E^+ \to \tilde{E}^+$, $E^+ \to F^-$, $F^- \to \tilde{F}^-$ and $F^- \to E^+$ ($\tilde{E}^+$ and $\tilde{F}^+$ are spaces of type E^+ and F^- respectively), are characterized in terms of, in particular, estimates of the form

$$\| T u \|_k \leq M_{n,k} \| u \|_n .$$

The essential trick here is a "two dimensional" and a "three dimensional" diagonal argument.

In the second section so called regular spaces and hyperspaces are defined and studied. Given a linear operator $B : D(B) \subset X \to X$ of positive type in a reflexive Banach space $(X, \|\cdot\|)$ we define a scale of Banach spaces $\{X_B^\sigma \mid \sigma \in \mathbb{R}\}$ as follows: For $\sigma \geq 0, X_B^\sigma = (D(B^\sigma), \|\cdot\|_\sigma)$ with $\| u \|_\sigma = \| B^\sigma u \|$; for $\sigma < 0$ X_B^σ is the completion of $(X, \|\cdot\|_{-\sigma})$ with $\| u \|_{-\sigma} = \| B^{-\sigma} u \|$ for $u \in X$. In terms of this family of Banach spaces $\{X_B^\sigma \mid \sigma \in \mathbb{R}\}$ we define $X_B^{\sigma_0 +} = \bigcup_{\sigma > \sigma_0} X_B^\sigma$ with inductive limit topology for $\sigma_0 \in (\infty, -\infty]$ and $X_B^{\sigma_0 -} = \bigcap_{\sigma < \sigma_0} X_B^\sigma$ with projective limit topology for $\sigma_0 \in [\infty, \infty)$. By the properties of fractional powers of linear operators and by the results in the first section we are able to clarify the relations among the spaces defined above and their topological properties as well as to give characterizations of continuous mappings thereupon. Together with B the dual operator B^* is an operator of positive type in the dual space X^*; thus we have the spaces $(X^*)_{B^*}^\sigma$, $(X^*)_{B^*}^{\sigma+}$ and $(X^*)_{B^*}^{\sigma-}$. There hold the natural duality relations $(X_B^\sigma)^* = (X^*)_{B^*}^{-\sigma}$, $[(X^*)_{B^*}^{-\sigma}]^* = X_B^\sigma$ and similarly for the inductive and projective limits. We have the diagram

$$X_B^{-\infty} \supset X_B^{(\sigma)-} \supset X_B^{-\sigma} \supset X_B^{(-\sigma)+} \supset X_B^{0-} \supset X \supset X_B^{0+} \supset X_B^{\sigma-} \supset X_B^{\sigma} \supset X_B^{\sigma+} \supset X_B^{\infty}$$

$$(X^*)_{B^*}^{-\infty} \supset (X^*)_{B^*}^{(-\sigma)+} \supset (X^*)_{B^*}^{-\sigma} \supset (X^*)_{B^*}^{(-\sigma)-} \supset (X^*)_{B^*}^{0-} \supset X^*$$

$$X^* \supset (X^*)_{B^*}^{0+} \supset (X^*)_{B^*}^{\sigma-} \supset (X^*)_{B^*}^{\sigma} \supset (X^*)_{B^*}^{\sigma+} \supset (X^*)_{B^*}^{\infty} .$$

The spaces to the right of X and X^* are called regular spaces and the spaces to the left hyperspaces. Conditions are given for a densely defined operator $A : D(A) \subset X \to X$ to be extendible to a continuous operator acting between a pair of hyperspaces, in terms of its dual operator $A^* : D(A^*) \subset X^* \to X^*$.

Finally, in the third section we present various examples of regular spaces and hyperspaces by choosing different Banach spaces X and operators B. In this way a number of classical function spaces and generalized function spaces are realized as regular spaces and hyperspaces respectively. More precisely we have the following illustrative table.

	X	B	X_B^σ comments
1)	l^p	diag $\{\lambda_k\}$	$l^{p,\sigma}\{\lambda_k\}$ weighted l^p spaces
2)	L^p	$u(x) \mapsto \Lambda(x)u(x)$	$L^{p,\sigma}\{\Lambda(x)\}$ weighted L^p spaces
3)	$L^p(\mathbb{R}^n)$	$I-\Delta$(Δ Laplacian)	$W^{p,2\sigma}$ Sobolev spaces on $\mathbb{R}^n$
4)	$L^2([0,2\pi]^n)$	$I-\Delta$	$W_{\text{per}}^{2\sigma}$ Periodic Sobolev spaces on $\mathbb{R}^n$ W_{per}^{∞} periodic test space $W_{\text{per}}^{-\infty}$ periodic generalized space
5)	$L^2(\mathbb{R}^n)$	$x^2-\Delta$	$(LW)^{2,2\sigma}$ Modified Sobolov spaces $(LW)^{2,\infty}$ Schwartz test space $(LW)^{2,-\infty}$ Schwartz tempered
6)	$L^p(\mathbb{R})$	$e^{-\frac{d^2}{dx^2}}$	$\approx A^{p,\frac{1}{4\sigma}}$ Ranges of heat-diffusion equation
7)	$L^2(\mathbb{R})$	$e^{x^2-\frac{d^2}{dx^2}}$	$X_{\tanh(\sigma/2)}$ Spaces of Van Eijndhoven-Meyers $X_B^{0+} = S_{1/2}^{1/2}$ Gelfand-Shilov
8)	$L^2(S^{q-1})$	$e^{(-\Delta_{LB})^{1/2}}$	$HA_{\Delta_{LB}}^q(\nu,\sigma)$ Weighted spaces of harmonic functions on $\mathbb{R}^q$
	(S^{q-1} unit sphere in $\mathbb{R}^q$)	(Δ_{LB} Laplace-Beltrami)	$HA^q(\mu)$

Linear operators are discussed in these spaces.

I.1. Inductive Limits and Projective Limits of Sequences of Banach Spaces

Let there be given a sequence of Banach spaces $\{(E_n, \|\cdot\|_n) \mid n \in \mathbb{N}_0\}$ such that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}_0$ and the embeddings are all continuous, i.e.,

$$\|u\|_{n+1} \leq M_n \|u\|_n \quad \forall n \in \mathbb{N}_0, \forall u \in E_n \tag{1}$$

where the M_n's are positive constants. In this case there always exists a corresponding sequence of new norms $|\cdot|_n$ on E_n which are each equivalent to $\|\cdot\|_n$ and which are monotone decreasing, i.e.

$$| u |_{n+1} \leq | u |_n \quad \forall n \in \mathbb{N}_0, \forall u \in E_n. \tag{2}$$

Indeed it suffices to set

$$\begin{cases} | \cdot |_0 = \| \cdot \|_0 \\ | \cdot |_{n+1} = (M_1 M_2 \cdots M_n)^{-1} \| \cdot \|_{n+1} \quad n \in \mathbb{N}_0. \end{cases} \tag{3}$$

Therefore in the sequel of this section we always assume the monotonicity of the sequence of norms $\| \cdot \|_n$.

On the vector space $E^+ = \bigcup_{n \in \mathbb{N}_0} E_n$ the locally convex inductive limit topology σ_{ind} is imposed; a balanced convex set u in E^+ is a neighbourhood of zero in $(E^+, \sigma_{\text{ind}})$ iff $u \cap E_n$ is a neighbourhood of zero in E_n for all $n \in \mathbb{N}_0$. In general, topological properties of a subset G in E^+ cannot simply be reduced to the corresponding properties of the sets $G \cap E_n$ in $E_n (n \in \mathbb{N}_0)$. For instance, Makarov has given various examples of inductive limits in which bounded sets in $(E^+, \sigma_{\text{ind}})$ are not bounded in any of the spaces $(E_n, \| \cdot \|_n)$, or even not situated in any of them. However, there do exist conditions which ensure that a set G is bounded in $(E^+, \sigma_{\text{ind}})$ iff it is bounded in some E_{n_0} (n_0 depends on G); such a sequence $\{E_n \mid n \in \mathbb{N}_0\}$ or its inductive limit E^+ is said to be regular. We cite the following result of Floret [Fl2].

Theorem I.1.1. Let K_n be the closed unit ball in E_n. If for all sequences $\{\varepsilon_m \mid m \in \mathbb{N}_0\}$ of positive numbers and all $n \in \mathbb{N}_0$, $\sum_{m=0}^{n} \varepsilon_m K_m$ is closed in E_{n+1}, then the inductive limit $E^+ = \operatorname{ind}_{n \to \infty} E_n$ is regular. □

As consequences of the above theorem we have the following corollaries.

Corollary I.1.1. ([Fl2]) If there exists a semireflexive locally convex space E and an injective continuous operator $T : E^+ = \operatorname{ind}_{n \to \infty} E_n \to E$ such that $T K_n$ is closed for all n, then E^+ is regular. □

Corollary I.1.2. ([Fl2]) The inductive limit of a sequence of dual Banach space with the inclusion mappings being dual mappings is regular. □

As pointed out by Floret, the familiar fact that inductive limits of sequences of locally convex spaces with (weakly) compact linking mappings are regular follows from Corollary I.1.2 in view of the following lemma ([Gro]).

Lemma I.1.3. Let K be an absolutely convex, weakly compact subset of a locally convex space V. Then there is a Banach space E such that $[\![K]\!] = E^*$ isometrically and $[\![K]\!] = E^* \hookrightarrow V$ is $\sigma(E^*, E) - \sigma(V, V^*)$ continuous. Here $[\![K]\!]$ denotes the linear hull of K equipped with the Minkowski norm m_K. □

If the inductive limit $E^+ = \operatorname{ind}_{n\to\infty} E_n$ is regular, then it readily follows that E^+ is bornological and barreled. If, in addition, an interpolation-type inequality is satisfied, then we can also characterize converging nets or sequences, Cauchy nets or sequences and compact sets in E^+ and consequently obtain the completeness of E^+.

Theorem I.1.4. Suppose that the inductive limit E^+ is regular and that E^+ is continuously embedded in some Banach space E. Assume for each $n \in \mathbb{N}_0$ there is $k(>n)$ and a function $\phi_{n,k} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that the interpolation inequality

$$\| u \|_k \leq \phi_{n,k}(\| u \|_n , \| u \|), \quad \forall u \in E_n \tag{4}$$

holds. Here the function $\phi_{n,k}(t,s)$ is monotone in each of its variables and such that $\phi_{n,k}(t,s) \to 0$ as $s \to 0$ for each fixed t. Then we have

(i) A bounded set $\{u_\alpha \mid \alpha \in I\}$ in E^+ converges (to zero) in E^+ iff it converges (to zero) in some E_n.

(ii) A bounded set $\{u_\alpha \mid \alpha \in I\}$ in E^+ is a Cauchy net in E^+ iff it is a Cauchy net in some E_n.

(iii) A subset G in E^+ is compact iff it is compact in some E_n. (The same is true for relative compactness.)

(iv) E^+ is complete.

Proof.

(i) Assume that a bounded set $\{u_\alpha \mid \alpha \in I\}$ converges to zero in E^+. The regularity of the inductive limit E^+ implies the existence of some $n \in \mathbb{N}_0$ such that $\{u_\alpha \mid \alpha \in I\} \subset E_n$ and $\| u_\alpha \| \leq M_n$ for all $\alpha \in I$ where M_n is a positive constant. On the other hand the continuity of the embedding from E^+ into E ensures that the net converges to zero in E, i.e., $\| u_\alpha \| \to 0$. Then, by assumption, there exists some $k > n$ and a function $\phi_{n,k}$ such that (4) is satisfied. In particular

$$\begin{aligned} \| u_\alpha \|_k &\leq \phi_{n,k}(\| u_\alpha \|_n , \| u_\alpha \|) \\ &\leq \phi_{n,k}(M_n , \| u_\alpha \|) \end{aligned}$$

from which follows that $\| u_\alpha \|_k \to 0$. The converse is trivial.

(ii) The proof is entirely similar to that of (i) and is omitted.

(iii) Let the subset G be compact in E^+. In particular it is bounded in E^+ and the regularity of the inductive limit implies the existence of some $n \in \mathbb{N}_0$ such that $G \subset E_n$ and $\| u \|_n \leq M_n$ for all $u \in G$, where M_n is a positive constant. By our assumption we can choose some $k > n$ such that (4) is valid. Then, for a given sequence $\{u_m \mid m \in \mathbb{N}_0\} \subset G$, the compactness of G in E^+ implies the existence of a subsequence $\{u_{m'} \to v \in G$ in E^+ and hence in E, i.e. $\| u_m - v \| \to 0$ as $m' \to \infty$. From

$$\| u_{m'} - v \|_k \leq \phi_{n,k}(\| u_{m'} - v \|_n , \| u_{m'} - v \|)$$
$$\leq \phi_{n,k}(2M_n , \| u_{m'} - v \|)$$

follows readily $\{u_{m'}\} \to v$ in E_k. This shows the sequential compactness of G in E_k, which is however equivalent to compactness in the Banach space E_k. The converse is trivial.

(iv) follows directly from (ii) and from the equivalence of bounded completeness and completeness (Satz 4.3 in [Fl1]). □

In analysis we meet projective limits as well as inductive limits of Banach spaces. However, the theory for projective limits of Banach spaces is much simpler and more "classical" than that for inductive limits. For completeness and easier citation we state the following standard results on projective limits of Banach spaces, the proof of which is straightforward and can be found in the standard text books on functional analysis and generalized functions, e.g., [G-S], [Sch] or [Wil].

Let there be given a sequence of Banach spaces $\{(F_n , \|\cdot\|_n) \mid n \in \mathbb{N}_0\}$ such that $F_n \supset F_{n+1}$ for all $n \in \mathbb{N}_0$ and

$$\| u \|_n \leq M_n \| u \|_{n+1} \quad \forall u \in F_{n+1} , \forall n \in \mathbb{N}_0. \tag{4}$$

If we set

$$\begin{cases} |\cdot|_0 = \|\cdot\|_0 \\ |\cdot|_{n+1} = M_0 M_1 \cdots M_n \|\cdot\|_{n+1} \quad n \in \mathbb{N}_0 \end{cases} \tag{5}$$

then each of the new norms $|\cdot|_n$ on F_n is equivalent to the original $\|\cdot\|_n$ and they are monotone increasing:

$$|\cdot|_n \leq |\cdot|_{n+1} \quad n \in \mathbb{N}_0. \tag{6}$$

Let us assume that $F^- = \bigcap_{n \in \mathbb{N}_0} F_n$ be not empty and equip F^- with the locally convex topology τ_{proj} generated by the sequence of norms $\{\|\cdot\|_n \mid n \in \mathbb{N}_0\}$. Then we have

Theorem I.1.5.

(i) F^- is a Frechét space, i.e., a complete metrizable locally convex space.

(ii) A sequence $\{u_n \mid n \in \mathbb{N}_0\}$ in F^- converges (to zero) in $(F^-, \tau_{\text{proj}})$ iff it converges (to zero) in all the spaces F_n.

(iii) A sequence $\{\| u_n \mid n \in \mathbb{N}_0\}$ in F^- is a Cauchy sequence in $(F^-, \tau_{\text{proj}})$ iff it is a Cauchy sequence in each of the spaces F_n.

(iv) A set G in F^- is compact in $(F^-, \tau_{\text{proj}})$ iff it is compact in each of the spaces F_n. The same applies to relative compactness. □

Now we proceed to characterize continuous mappings between spaces which are inductive limits or projective limits of sequences of Banach spaces.

Theorem I.1.6. Let $\{(E_n, \|\cdot\|_{E_n}) \mid n \in \mathbb{N}_0\}$ and $\{(\tilde{E}_n, \|\cdot\|_{\tilde{E}_n}) \mid n \in \mathbb{N}_0\}$ be two sequences of Banach spaces with inductive limits E^+ and $\tilde{E}^+$ respectively, and let $\tilde{E}^+$ satisfy the conditions in Theorem I.1.4. Let $\{(F_n, \|\cdot\|_{F_n}) \mid n \in \mathbb{N}_0\}$ and $\{(\tilde{F}_n, \|\cdot\|_{\tilde{F}_n}) \mid n \in \mathbb{N}_0\}$ be two sequences of Banach spaces with nonempty projective limits F^- and $\tilde{F}^-$ respectively.

(i) For a linear mapping $T : E^+ \to \tilde{E}^+$ the following are mutually equivalent:

1) T is continuous.

2) If a sequence $\{u_m \mid m \in \mathbb{N}_0\}$ is contained and converges (to zero) in E_n for some $n \in \mathbb{N}_0$, then $\{T u_m \mid m \in \mathbb{N}_0\}$ is contained and converges (to zero) in some $\tilde{E}_{\tilde{n}}$.

3) For any $n \in \mathbb{N}_0$ there exists an $\tilde{n} \in \mathbb{N}_0$ such that

$$\|T u\|_{\tilde{E}_{\tilde{n}}} \le M_{n,\tilde{n}} \|u\|_{E_n}, \quad \forall u \in E_n \tag{7}$$

where (and below) $M_{n,\tilde{n}}$ is a positive constant.

(ii) For a linear mapping $T : F^- \to \tilde{F}^-$ the following are mutually equivalent:

1) T is continuous.

2) If a sequence $\{u_m \mid m \in \mathbb{N}_0\}$ converges (to zero) in each F_n, then $\{T u_m \mid m \in \mathbb{N}_0\}$ converges (to zero) in each $\tilde{F}_{\tilde{n}}$.

3) For any $\tilde{n} \in \mathbb{N}_0$ there exists an $n \in \mathbb{N}_0$ such that

$$\|T u\|_{\tilde{F}_{\tilde{n}}} \le M_{n,\tilde{n}} \|u\|_{F_n}, \quad \forall u \in F^-. \tag{8}$$

(iii) For a linear mapping $T : E^+ \to F^-$ the following are mutually equivalent:

(1) T is continuous.

(2) If a sequence $\{u_m \mid m \in \mathbb{N}_0\}$ is contained and converges (to zero) in some E_n, then $\{T u_m \mid m \in \mathbb{N}_0\}$ converges (to zero) in all the spaces $F_k (k \in \mathbb{N}_0)$.

(3) For each n and k in $\mathbb{N}_0$ holds that

$$\|T u\|_{F_k} \le M_{n,k} \|u\|_{E_n}, \quad \forall u \in E_n. \tag{9}$$

(iv) For a linear mapping: $T : F^- \to E^+$ the following are mutually equivalent:

(1) T is continuous.

(2) If a sequence $\{u_m \mid m \in \mathbb{N}_0\}$ converges (to zero) in all the spaces $F_n (n \in \mathbb{N}_0)$, then $\{T u_m \mid m \in \mathbb{N}_0\}$ is contained and converges (to zero) in some E_k.

(3) There exist n and k in $\mathbb{N}_0$ such that

$$\|T u\|_{E_k} \le M_{n,k} \|u\|_{F_n} \quad \forall u \in F^-. \tag{10}$$

Proof. Since the arguments in the proofs to the four cases are quite similar, here we only include the proofs to cases (i) and (iv).

(i) 3) $\Rightarrow$ 2) is trivial. 2) $\Rightarrow$ 3): Suppose that there exists an $n \in \mathbb{N}_0$ such that

$$\sup_{\|u\|_{E_n}=1} \|T u\|_{\tilde{E}_i} = \infty \quad \forall i \in \mathbb{N}_0. \tag{10}$$

Then, for each $i \in \mathbb{N}_0$ we can find a sequence $\{u_{ij} \mid j \in \mathbb{N}_0\}$ such that

$$\|u_{ij}\|_{E_n} = 1 \text{ and } \|T u_{ij}\|_{\tilde{E}_i} \geq j^2, \; j \in \mathbb{N}_0.$$

Form the diagonal sequences $\{T u_{ii} \mid i \in \mathbb{N}_0\}$ and $\{u_{ii} \mid i \in \mathbb{N}_0\}$. We have for each $j \in \mathbb{N}_0$ and $i > j$

$$\|T u_{ii}\|_{\tilde{E}_j} \geq \|T u_{ii}\|_{\tilde{E}_i} \geq i^2$$

and consequently

$$\|T(u_{ii}/i)\|_{\tilde{E}_j} \to \infty \text{ as } i \to \infty \text{ for each } j \in \mathbb{N}_0.$$

However

$$\|u_{ii}/i\|_{E_n} = \frac{1}{i} \to 0 \text{ as } i \to \infty.$$

The above two relations contradict with the statement in 2). Thus we have proved the equivalence of 2) to 3).

1) $\Rightarrow$ 2): If a sequence $\{u_m \mid m \in \mathbb{N}_0\}$ is contained and converges to zero in some E_n, then it converges to zero in E^+ and from the continuity of T follows that $\{T u_m \mid m \in \mathbb{N}_0\}$ converges to zero in $\tilde{E}^+$. Theorem I.1.4 (i) ensures the existence of some $\tilde{n} \in \mathbb{N}_0$ such that $\{T u_m \mid m \in \mathbb{N}_0\}$ is contained and converges to zero in $\tilde{E}_{\tilde{n}}$.

3) $\Rightarrow$ 1): Consider the restrictions of T to the subspace $E_n (n \in \mathbb{N}_0)$. For a sequence $\{u_m \mid m \in \mathbb{N}_0\}$ lying and converging to zero in E_n, since there exists an $\tilde{n} \in \mathbb{N}_0$ such that (7) holds, $\{T u_m \mid m \in \mathbb{N}_0\}$ converges to zero in $\tilde{E}_{\tilde{n}}$ and therefore in $\tilde{E}^+$. Thus all the restrictions are continuous. Now let U be a convex neighbourhood of zero in $\tilde{E}^+$. Obviously $T^{-1}(U)$ is convex in E^+ and $E_n \cap T^{-1}(U) = (T\restriction_{E_n})^{-1}(U)$ is a neighbourhood of zero in E_n. So $T^{-1}(U)$ is a neighbourhood of zero in E^+ and T is continuous.

Thus we have completed the proof for (i).

(iv) (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Suppose that there be no pair of n and k such that (10) is valid. Then for any $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ we can find a sequence $\{u_{n,k,j} \mid j \in \mathbb{N}_0\}$ in $Fsup-$ such that

$$\|u_{n,k,j}\|_{F_n} = 1, \; \|T u_{n,k,j}\|_{E_k} \geq j^2, \; \forall j \in \mathbb{N}_0. \tag{11}$$

Now we form the sequence $\{v_j \mid j \in \mathbb{N}_0\}$ with $v_j = u_{j,j,j}$, which is the diagonal of the "cube" of elements $\{u_{n,k,j} \mid n,k,j \in \mathbb{N}_0\}$. For each $n \in \mathbb{N}_0$ we have, in view of the first relation in (11)

$$\| v_j / j \|_{F_n} \leq \| v_j / j \|_{F_j} = \frac{1}{j}, \quad \forall j \geq n. \tag{12}$$

For each $k \in \mathbb{N}_0$ we have, in view of the second relation in (11)

$$\| T(v_j / j) \|_{E_k} \geq \| T(v_j / j) \|_{E_j} \geq j, \quad \forall j \geq k. \tag{13}$$

(12) and (13) readily lead to the conclusion that

$$\left.\begin{array}{l} \| v_j / j \|_{F_n} \to 0 \; (j \to \infty), \quad \forall n \in \mathbb{N}_0 \\ \| T(v_j / j) \|_{E_k} \to \infty \; (j \to \infty), \quad \forall k \in \mathbb{N}_0 \end{array}\right\} \tag{14}$$

which contradicts the statement in (2).

(3) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2) follows from Theorem I.1.4 (i). □

Note that the regularity of $\tilde{E}^+$ and its satisfying an interpolation type inequality are used only in the arguments of (1) $\Rightarrow$ (2) above, not in the other implications. By the way we also remark that for a specific case of (iv) above both [Gr3] and [E-G1] gave an incorrect proof and the proof given in [E-G2] is quite lenghy.

Corollary I.1.7. In each of the four cases in the above theorem, a linear operator T is continuous iff it is bounded, i.e. it maps bounded subsets into bounded subsets. □

Corollary I.1.8. $(E^+)^* = \bigcap_{n \in \mathbb{N}_0} E_n^*$

$$(F^-)^* = \bigcup_{n \in \mathbb{N}_0} F_n^*.$$
□

Corollary I.1.9.

(i) If $E^+ = \tilde{E}^+$ are vector spaces, then the following are mutual equivalent:

1) $E^+ = \tilde{E}^+$ as topological vector spaces, and both $\tilde{E}^+$ and E^+ satisfy the mentioned conditions in Theorem I.1.6.

2) If a sequence $\{u_m \mid m \in \mathbb{N}_0\}$ is contained and converges (to zero) in some E_n, so does it in some $\tilde{E}_{\tilde{n}}$, and vice versa.

3) For any $n \in \mathbb{N}_0$ there exists some $\tilde{n} \in \mathbb{N}_0$ and a positive constant $M_{n,\tilde{n}}$ such that T maps E_n into $\tilde{E}_{\tilde{n}}$ and

$$\| u \|_{\tilde{E}_{\tilde{n}}} \leq M_{n,\tilde{n}} \| u \|_{E_n}, \quad \forall u \in E_n,$$

and vice versa. Note that in this case the regularity (and an interpolation type inequality) of $\tilde{E}^+$ implies that of E^+.

(ii) If $F^- = \tilde{F}^-$ as vector spaces, then the following are mutually equivalent:

1) $F^- = \tilde{F}^-$ as topological vector spaces.

2) If a sequence $\{u_m \mid m \in \mathbb{N}_0\}$ converges (to zero) in each of the spaces F_n $(n \in \mathbb{N}_0)$, then it is convergent in each of the spaces $\tilde{F}_{\tilde{n}}$ $(\tilde{n} \in \mathbb{N}_0)$.

3) For any $\tilde{n} \in \mathbb{N}_0$ there exists some $n \in \mathbb{N}_0$ and a constant $M_{n,\tilde{n}} > 0$ such that

$$\| u \|_{\tilde{F}_{\tilde{n}}} \le M_{n,\tilde{n}} \| u \|_{F_n} \quad \forall u \in F^+ ,$$

and vice versa.

□

We point out that Theorem I.1.6 (ii) and its counterparts in the above corollaries (the case of projective limits) are somewhat standard (see, e.g., [G-S], [Sch] and [Wil]). They are included here mainly for completeness. It seems that Theorem I.1.6 (i) and its corollaries are not current in this general setting. Gelfand and Shilov made the statement, in [G.S.], vol. II, Chapter 1 Section 8, that sequential continuity and sequential boundedness for a linear mapping between two spaces of inductive limit are equivalent to each other; however their proof is not correct. The advantage of the above theorem and its corollaries consists in characterizing the continuity of linear mappings in terms of inequality estimates rather than in topologies, which are the usual tools of analysis.

At the conclusion of this section we remark that in many instances of inductive limits interpolation type inequalities as (4) indeed hold. It will be the case in the definition and discussion of regular spaces and hyper-spaces involving fractional powers of operators; see the next section. This applies also to the spaces defined by ter Elst as intersections of Grevery spaces ([tE]). Here we give another example.

Example I.1.10. Let $q \ge 2$ be a natural number and $a,b > 0$. The space $HA_e^q(a,b)$ consists of all harmonic functions $u(x)$ on $\mathbb{R}^q$ such that

$$\| u \|_a = \Big(\int_{\mathbb{R}^q} | u(x) |^2 \, e^{-a|x|^b} dx \Big)^{1/2} < \infty. \tag{15}$$

It is a Hilbert space with the given norm and its corresponding inner product. In the following q and b are fixed. It is easy to see that if $a_1 > a_2 > 0$ then

$$HA_e^q(a_1,b) \supset HA_e^q(a_2,b) \quad \text{and} \quad \| u \|_{a_1} \le \| u \|_{a_2}$$

for $u \in HA_e^q(a_2,b)$. We have an interpolation type inequality for this scale of spaces.

Lemma I.1.11. For $a_1 > a_2 > a_3 > 0$ we have

$$\| u \|_{a_2} \le \| u \|_{a_1}^{\frac{a_2-a_3}{a_1-a_3}} \cdot \| u \|_{a_3}^{\frac{a_1-a_2}{a_1-a_3}} , \quad u \in HA_e^q(a_3,b) \tag{16}$$

Proof. It follows from Hölder's inequality that

$$\| u \|_{a_2}^2 = \int_{\mathbb{R}^q} | u(x) |^2 e^{-a_2|x|^b} dx$$

$$= \int_{\mathbb{R}^q} | u(x) |^{2\frac{a_2-a_3}{a_1-a_3}} e^{-a_1\frac{a_2-a_3}{a_1-a_3}|x|^b} \cdot | u(x) |^{2\frac{a_1-a_2}{a_1-a_3}} e^{-a_3\frac{a_1-a_2}{a_1-a_3}|x|^b} dx$$

$$\le \left(\int_{\mathbb{R}^q} | u(x) |^2 e^{-a_1|x|^b} dx \right)^{\frac{a_2-a_3}{a_1-a_3}} \left(\int_{\mathbb{R}^q} | u(x) |^2 e^{-a_3|x|^b} dx \right)^{\frac{a_1-a_2}{a_1-a_3}} .$$

□

Given $a \in (0,\infty]$. Take a sequence of increasing positive numbers $\{a_m \mid m \in \mathbb{N}_0\}$ such that $a_m < a$ for all $m \in \mathbb{N}_0$ and $a_m \to a$ as $m \to \infty$. Corollary I.1.2 implies that the inductive sequence $\{HA_e^q(a_m,b) \mid m \in \mathbb{N}_0\}$ is regular; the corresponding inductive limit does not depend on $\{a_m\}$ and is denoted by $HA_e^q(a+,b)$. For $a < \infty$ the above Lemma I.1.11 ensured that Theorem I.1.4 applies here. In particular, a sequence $\{u_m\} \subset HA_e^q(a+,b)$ converges to zero iff it converges to zero in some $HA_e^q(a',b)$ with $a' < a$. It is not clear how we could directly apply Theorem I.1.4 to the inductive limit space $HA_e^q(\infty,b)$. It turns out that all the conclusions in Theorem I.1.4 remain valid for the space $HA_e^q(\infty,b)$. Indeed, in stead of a Banach space we can use the Frechét space $HA(\mathbb{R}^q)$, the topology of which means uniform convergence on each compact set of $\mathbb{R}^q$. It is then obvious that each space $HA_e^q(a,b)$ is continuously embedded in $HA(\mathbb{R}^q)$, so is the inductive limit $HA_e^q(\infty,b)$. Let $\{u_m \mid m \in \mathbb{N}_0\}$ be a sequence converging to zero in $HA_e(\infty,b)$. So it is bounded in $HA_e(\infty,b)$ and hence in some $HA_e(a,b)$, for $HA_e(\infty,b)$ is regular. It is also a zero-sequence in $HA(\mathbb{R}^q)$. Therefore we conclude that $\{u_m\}$ is a zero sequence in $HA_e(a',b)$ for $a' > a$. Indeed

$$\| u \|_{a'}^2 = \int_{\mathbb{R}^q} | u(x) |^2 e^{-a'|x|^b} dx$$

$$\le e^{-a'R^b} m(B^q(R)) \sup_{|x| \le R} | u(x) |^2 + e^{-(a'-a)R^b} \| u \|_a^2$$

where $m(B^q(R))$ denotes the Lebesque measure of the ball $\{x \in \mathbb{R}^q \mid |x| \le R\}$.

In the above we have considered spaces of harmonic functions. Actually the harmonicity of functions in the spaces play no role here; it is assumed here only for easier citation in Chapter IV.

I.2. Regular Spaces and Hyper-spaces: General Theory

First we give a brief survey on the definition and basic properties of fractional powers of linear operators. For the details the reader is referred to [Ka3], [Kr], [Ko2], [Ta], [Fr] and [Pa].

Let X be a Banach space with norm $\|\cdot\|$. For given constants $\omega \in [0,\pi)$ and $M \geq 1$, a densely defined closed linear operator $B : D(B) \subset X \to X$ is said to be of type $P(\omega,M)$ if $\Sigma_\omega \subset \rho(B)$, $\|(\lambda I - B)^{-1}\| \leq M\,|\lambda|^{-1}$ for all $\lambda < 0$, and for each $\varepsilon \in (0,\pi-\omega)$ there exists an $M_\varepsilon \geq 1$ such that $\|(\lambda I - B)^{-1}\| \leq M_\varepsilon\,|\lambda|^{-1}$ for all $\lambda \in \Sigma_{\omega+\varepsilon}$. Here for an arbitrary $\alpha \in [0,\pi)$, Σ_α stands for the cone in the complex plane $\{\lambda \in \mathbb{C} \mid |\arg\lambda| > \alpha\}$. We say that the operator B is of type $P(\omega)$ if it is of type (ω,M) for some $M \geq 1$. A densely defined closed operator B in X is of type $P(\pi/2,1)$ iff both B itself and its dual operator B^* are accretive, or equivalently, iff B is m-accretive, i.e., accretive and $\lambda \in \rho(B)$ for some (hence all) $\lambda < 0$. It is well-known that for $\omega \in [0,\pi/2)$ the operator B is of type $P(\omega)$ iff it generates an analytic semigroup in the cone $\Sigma'_{\pi/2-\omega} = \mathbb{C} - \overline{\Sigma}_{\pi/2-\omega}$. Here by an analytic semigroup we mean a family of operators $\{T(t) \mid t \in \Sigma'_{\pi/2-\omega}\} \subset L(X)$ such that $T(t)\,T(s) = T(s+t)$ for all $t,s \in \Sigma'_{\pi/2-\omega}$, for each $u \in X$ and each $\varepsilon > 0$ $\lim\limits_{t\to 0} T(t)u = u$, and the mapping $T : \Sigma'_{\pi/2-\omega} \to L(X)$ is analytic. (The analycity is equivalent to $T(\cdot)u : \Sigma'_{\lambda/2-\omega} \to X$ being analytic for all $u \in X$, or still, equivalent to $(T(\cdot)u, v^*) : \Sigma'_{\pi/2-\omega} \to \mathbb{C}$ being analytic for all $u \in X$ and $v^* \in X^*$.)

Assume that $B : D(B) \subset X \to X$ is an operator of type $P(\omega,M)$ and $0 \in \rho(B)$. We can define its fractional powers $B^\sigma (\sigma \in \mathbb{R})$ as follows. Take a neighbourhood Ω of the origin in $\mathbb{C}$ such that $\Omega \cup \Sigma_\omega \subset \rho(B)$. Choose $a > 0$ and $\phi \in (\omega,\pi)$ such that

$$\Gamma = \{\lambda \in \mathbb{C} \mid \arg(\lambda - a) = \pm\phi\} \subset \Omega \cup \Sigma_\omega.$$

For $\sigma > 0$ define

$$B^{-\sigma} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\sigma} (\lambda I - B)^{-1}\, d\lambda \tag{17}$$

where $\lambda^{-\sigma} = |\lambda|^{-\sigma} e^{-\sigma\arg\lambda \cdot i}$ is analytic in $\mathbb{C} - \{\lambda \mid \lambda \leq 0\}$ and the path of integration is orientated so that $\arg\lambda$ is decreasing along Γ. $B^{-\sigma}$ thus defined does not depend on the admissible pathes Γ. In particular, for $\sigma \in (0,1)$ we can transfer the path Γ to the upper and lower left-half real axis and obtain

$$B^{-\sigma} = \frac{\sin\pi\sigma}{\sigma} \int_0^\infty \lambda^{-\sigma} (\lambda I + B)^{-1}\, d\lambda. \tag{18}$$

Similarly it is easy to see that $B^{-n} = (B^{-1})^n$. Moreover, if B is a positive self-adjoint operator in a Hilbert space X, then

$$B^{-\sigma} u = \int_0^\infty \lambda^{-\sigma} \, dE(\lambda) u , \quad u \in X \tag{19}$$

where $\{E(\lambda)\}$ is the spectral resolution of B. Still other formulations of the fractional powers exist, especially when B is a generator of a C_0 semigroup (cf. [Yo]).

Fractional powers of operators enjoy very nice properties. Some of these properties are listed below:

(i) $B^{-\sigma}$ is injective for each $\sigma > 0$; we define $B^{\sigma} = (B^{-\sigma})^{-1}$ for $\sigma > 0$ and $B^0 = I$.

(ii) For all $\sigma > 0$ the operator B^{σ} is densely defined and closed.

(iii) If $0 \le \tau \le \sigma$ then $D(B^{\sigma}) \subset D(B^{\tau})$ and $D(B^{\sigma})$ is a core for B^{τ}.

(iv) $\{B^{-\sigma} \mid \sigma \ge 0\}$ is a C_0 semigroup of operators on X which can be analytically extended to the whole right-half plane. In particular, for all $\sigma, \tau \in \mathbb{R}$

$$B^{\sigma+\tau} u = B^{\sigma} B^{\tau} u , \quad u \in D(B^{\theta})$$

where $\theta = \max\{\sigma, \tau, \sigma + \tau\}$.

(v) (Interpolation inequality) For any $\sigma < \tau < \theta$ there exists a constant $C(\sigma,\tau,\theta)$ such that

$$\| B^{\tau} u \| \le C(\sigma,\tau,\theta) \, \| B^{\sigma} u \|^{\frac{\theta-\tau}{\theta-\sigma}} \, \| B^{\theta} u \|^{\frac{\tau-\sigma}{\theta-\sigma}} , \quad u \in D(B^{\theta}).$$

(vi) If B is a generator of a C_0 semigroup, then $D(B^{\infty}) = \bigcap_{\sigma>0} D(B^{\sigma})$ is dense in X.

We are now in a position to define a scale of graded Banach spaces. Given a linear operator B of type (ω, M) in a Banach space $(X, \|\cdot\|)$ whose resolvent set contains the origin 0, then the same properties are satisfied for its dual B^* in $(X^*, \|\cdot\|_*)$.

Definition I.2.1. Let $\sigma \in (0,\infty)$.

(i) $X_B^{\sigma} = (D(B^{\sigma}), \|\cdot\|_{B,\sigma})$ where $\| u \|_{B,\sigma} = \| B^{\sigma} u \|$ for $u \in D(B^{\sigma})$. $X_B^0 = (X, \|\cdot\|)$. $\|\cdot\|_{B,\sigma}$ is often abbreviated to $\|\cdot\|_{\sigma}$; $\|\cdot\|_{B,0} = \|\cdot\|_0 = \|\cdot\|$.

(ii) $X_B^{-\sigma}$ is the completion of $(X, \|\cdot\|_{B,-\sigma})$ where

$$\| u \|_{B,-\sigma} = \| u \|_{-\sigma} = \| B^{-\sigma} u \| \quad \text{for } u \in X.$$

$\square$

From this definition and the properties of fractional powers immediately follows

Proposition I.2.2. The scale of spaces $\{X_B^{\sigma} \mid \sigma \in \mathbb{R}\}$ is a scale of graded Banach spaces. For any $\tau > \sigma > 0$ we have the relation

$$X_B^{-\tau} \supset X_B^{-\sigma} \supset X \supset X_B^{\sigma} \supset X_B^{\tau} \tag{20}$$

where each smaller space is densely and continuously embedded into another bigger one.

Proof. We omit the details of the somewhat routine proof of this proposition. Nevertheless we emphasize the important role played by the fact that for $0 < \sigma < \tau$, $D(B^{\tau})$ is a core for the operator B^{σ} in X (Property (iii) of fractional powers listed above). Since no literature, which is available to us, states this explicitly, we give a brief proof here. Without loss of generality we can assume that τ is an integer. Let $u \in D(B^{\sigma})$. Then, for $\lambda > 0$, $u_{\lambda} = \lambda^{\tau}(\lambda I + B)^{-\tau} u \in D(B^{\tau})$. It is easy to see that $(\lambda I + B)^{-\tau} B^{-\sigma} = B^{-\sigma}(\lambda I + B)^{-\tau}$. From this follows it that $B^{\sigma} u_{\lambda} = \lambda^{\tau}(\lambda I + B)^{-\tau} B^{\sigma} u$. In view of the standard fact $(I + \lambda^{-1} B)^{-1} u \to u$ as $\lambda - \infty$ for all $u \in X$ and the continuity of the operator $(I + \lambda^{-1} B)^{-\tau}$ we have $u_{\lambda} \to u$ and $B^{\sigma} u_{\lambda} \to B^{\sigma} u$. This completes our proof. □

From the scale of Banach spaces $\{X_B^{\sigma} \mid \sigma \in (-\infty, +\infty)\}$ we can construct their inductive limits and projective limits.

Definition I.2.3.

(i) For $\sigma \in [-\infty, +\infty)$, $X_B^{\sigma+} = \bigcup_{\tau > \sigma} X_B^{\tau}$ with inductive limit topology; $X_B^{(-\infty)+} \equiv X_B^{-\infty}$.

(ii) For $\sigma \in (-\infty, +\infty]$, $X_B^{\sigma-} = \bigcap_{\tau < \sigma} X_B^{\tau}$ with projective limit topology; $X_B^{(+\infty)-} \equiv X_B^{\infty}$. □

We remark that because of Proposition I.2.2 above the inductive limits and projective limits are well defined by any sequence of Banach spaces with monotone indices converging to the right limits.

Proposition I.2.4. Among the Banach spaces $\{X_B^{\sigma} \mid \sigma \in \mathbb{R}\}$, the inductive limits $\{X_B^{\sigma+} \mid \sigma [-\infty, \infty)\}$ and the projective limits $\{X_B^{\sigma-} \mid \sigma \in (-\infty, \infty]\}$ there holds the following relation $(0 < \sigma < \tau < \infty)$:

$$X_B^{-\infty} \supset X_B^{-\tau} \supset X_B^{-\tau} \supset X_B^{-\tau+} \supset X_B^{-\sigma-} \supset X_B^{-\sigma} \supset X_B^{-\sigma+} \supset X_B^{0-} \supset X$$

$$X \supset X_B^{0+} \supset X_B^{\sigma-} \supset X_B^{\sigma} \supset X_B^{\sigma+} \supset X_B^{\tau-} \supset X_B^{\tau} \supset X_B^{\tau+} \supset X_B^{\infty}. \tag{21}$$

Here each smaller space is densely and continuously embedded in another bigger one except for X_B^{∞} which is only known to be dense in another bigger space when B is of type $(\lambda/2, 1)$. □

We omit the proof of this proposition again only mentioning that $X_B^{\tau}(\tau > 0)$ is shown to be dense in $X_B^{\tau-}$ via the same procedure as we used to prove that $D(B^{\tau})$ is a core for B^{σ} $(0 < \sigma < \tau)$.
The next two theorems clarify the topological properties of these spaces of inductive limits and projective limits.

Theorem I.2.5. Assume that either the Banach space X is reflexive or all the operators $B^{-\sigma}$ $(\sigma > 0)$ are compact in X. Then we have

1) All the inductive limit spaces $X_B^{\sigma+}$ $(\sigma \in [-\infty, +\infty))$ are regular.

2) For any $\sigma \in \mathbb{R}$ a bounded net $\{u_\alpha \mid \alpha \in I\}$ in $X_B^{\sigma+}$ converges (to zero) iff it converges (to zero) in some X_B^τ $(\tau > \sigma)$.

3) For any $\sigma \in \mathbb{R}$ a bounded net $\{u_\alpha \mid \alpha \in I\}$ in $X_B^{\sigma+}$ is a Cauchy net in $X_B^{\sigma+}$ iff it is a Cauchy net in some X_B^τ $(\tau > \sigma)$. All the spaces $X_B^{\sigma+}$ $(\sigma \in \mathbb{R})$ are complete.

4) For any $\sigma \in \mathbb{R}$ a set G in $X_B^{\sigma+}$ is compact iff it is a compact set in some X_B^τ $(\tau > \sigma)$. The same is true for relative compactness.

5) Each of the spaces $X_B^{\sigma+}$ $(\sigma \in \mathbb{R})$ is barreled and bornological; it is Montel iff all the mappings $B^{-\tau} : X \to X (\tau > 0)$ are compact.

Proof. It is readily seen that for each $\sigma > 0$ the isometry operator $B^{-\sigma} : X \to X_B^\sigma$ extends uniquely to an isometry operator from $X_B^{-\sigma}$ onto X, still denoted $B^{-\sigma}$; its inverse extends $B^\sigma : X_B^\sigma \to X$ and is denoted by B^σ again. In this way, via X, we have an isometric operator from X_B^σ onto X_B^τ, denoted by $B^{\sigma-\tau}$, for each $\sigma, \tau \in \mathbb{R}$.

1) If the Banach space X is reflexive, so is each X_B^σ $(\sigma \in \mathbb{R})$. It is also evident that $B^{-\tau} : X \to X$ $(\tau > 0)$ is compact iff the inclusion mapping from X_B^σ into $X_B^{\sigma-\tau}$ is compact. Then the regularity for each of the inductive limit spaces $X_B^{\sigma+}$ $(\sigma \in [-\infty, \infty))$ follows from Corollary I.1.2 and the remarks following it.

2), 3) and 4) are consequences of 1) and Theorem I.1.4 since now we have the interpolation inequality

$$\| u \|_\tau \le C(\sigma,\tau,\theta) \| u \|_\sigma^{\frac{\theta-\tau}{\theta-\sigma}} \| u \|_\theta^{\frac{\tau-\sigma}{\theta-\sigma}} \quad (\sigma < \tau < \theta) \tag{22}$$

which is just the corresponding property for fractional powers. Here for fixed σ we take $E = X_B^\sigma$ in Theorem I.1.4.

$X_B^{\sigma+}$ is barreled and bornological since it is regular. If all the mappings $B^{-\tau} : X \to X$ $(\tau > 0)$ are compact, then, equivalently, all the inclusion mappings $i : X_B^\theta \to X_B^{\theta-\tau}$ $(\theta \in \mathbb{R})$ are compact. Thus, if θ is a closed and bounded set in $X_B^{\sigma+}$, then by 1) there exists $\theta > \sigma$ such that G is bounded in X_B^θ. Thence it is compact in $X_B^{(\theta+\sigma)/2}$, so is it in $X_B^{\sigma+}$. This shows that $X_B^{\sigma+}$ is Montel. Conversely, assume that $X_B^{\sigma+}$ is Montel for some $\sigma \in \mathbb{R}$. For $\tau > 0$ and a bounded set G in X, $B^{-(\sigma+\tau)} G$ is bounded in $X_B^{\sigma+\tau}$, so is it in $X_B^{\sigma+}$. Since $X_B^{\sigma+}$ is Montel, $B^{-(\sigma+\tau)} G$ is relatively compact in $X_B^{\sigma+}$. By 4) above there exists some $\delta \in (0,1)$ such that $B^{-(\sigma+\tau)} G$ is relatively compact in $X_B^{(\sigma+\delta\tau)}$. This in turn is equivalent to $B^{(\sigma+\delta\tau)} B^{-(\sigma+\tau)} G = B^{-(1-\delta)\tau} G$ being relatively compact in X, so is $B^{-\tau} G = B^{-\delta} B^{-(1-\delta)\tau} G$. This proves the compactness of $B^{-\tau}$ for each $\tau > 0$. □

Open problem I.2.6. For a sequence $\{u_n \mid n \in \mathbb{N}_0\}$ converging to zero in $X_B^{-\infty}$, is there some σ such that it converges to zero in the space X_B^σ? □

Theorem I.2.7. Let $\sigma \in (-\infty,+\infty]$. Then we have

1) $X_B^{\sigma-}$ is a Frechet space.
2) A sequence in $X_B^{\sigma-}$ converges (to zero) iff it converges (to zero) in each of the spaces X_B^{τ} $(\tau<\sigma)$.
3) A sequence in $X_B^{\sigma-}$ is a Cauchy sequence iff it is a Cauchy sequence in each of the spaces X_B^{τ} $(\tau<\sigma)$.
4) A set G in $X_B^{\sigma-}$ is bounded (compact) iff it is bounded (compact) in each of the spaces X_B^{τ} $(\tau<\sigma)$. The same applies to relative compactness.
5) $X_B^{\sigma-}$ is Montel iff all the mappings $B^{-\tau}: X \to X$ $(\tau>0)$ are compact.

Proof. Assertions 1) to 4) follow directly from Theorem I.1.5. The proof for 5) is similar to that for Theorem I.2.5 5) above and is omitted. □

If the Banach space $(X, \|\cdot\|)$ and the operator B are replaced by the dual space $(X^*, \|\cdot\|_*)$ and the dual operator B^*, then we obtain another scale of Banach spaces $(X^*)_{B^*}^{\sigma}$ $(\sigma\in \mathbb{R})$, and their inductive limits $(X^*)_{B^*}^{\sigma+}$ $(\sigma\in[-\infty,\infty))$ and projective limits $(X^*)_{B^*}^{\sigma-}$ $(\sigma\in(-\infty,\infty])$. The norm of $(X^*)_B^{\sigma}$ is denoted $\|\cdot\|_{*,B,\sigma}$, sometimes abbreviated to $\|\cdot\|_{*,\sigma}$. There is a natural duality relation between the two scales of spaces.

Theorem I.2.8. $(X_B^{\sigma})^* = (X^*)_{B^*}^{-\sigma}$ and $X_B^{\sigma} \hookrightarrow [(X^*)_{B^*}^{-\sigma}]^*$ $(\sigma>0)$ isometrically via the duality pairing $<\cdot,\cdot>_\sigma : X_B^{\sigma}\times (X^*)_{B^*}^{-\sigma}$

$$<u,f>_\sigma = (B^\sigma u, (B^*)^{-\sigma} f),\ u\in X_B^{\sigma},\ f\in (X^*)_{B^*}^{-\sigma} \tag{23}$$

where $(\cdot,\cdot)$ is the duality pairing between X and X^*. If, furthermore, the space X is reflexive, then $((X^*)_{B^*}^{-\sigma})^* = X_B^{\sigma}$ isometrically via the same duality pairing (23).

Proof. For $u\in X_B^{\sigma}$ and $f\in (X^*)_{B^*}^{-\sigma}$ we have

$$|<u,f>_\sigma| = |(B^\sigma u, (B^*)^{-\sigma} f)| \le \|u\|_\sigma \|f\|_{*,-\sigma}. \tag{24}$$

Now let $f\in (X^*)_{B^*}^{-\sigma}$ be given and set $g=(B^*)^{-\sigma} f\in X^*$. There exists a sequence $\{v_n\}\subset X$ such that $\|v_n\|=1$ and $|(v_n,g)|\to \|g\|_*$. Putting $u_n = B^{-\sigma} v_n$, then $u_n\in X_B^{\sigma}$, $\|u_n\|_\sigma=1$ and

$$|<u_n,f>_\sigma| = |(v_n,g)| \to \|g\|_* = \|f\|_{*,-\sigma}. \tag{25}$$

(24) and (25) together then implies that $F=<\cdot,f>_\sigma$: $X_B^{\sigma}\to \mathbb{C}$ belongs to $(X_B^{\sigma})^*$ and $\|F\|=\|f\|_{*,-\sigma}$.

Conversely, if $F: X_B^{\sigma}\to\mathbb{C}$ is in $(X_B^{\sigma})^*$, then, since X_B^{σ} and X are isometric to each other under the mapping B^σ, there exists a unique $g\in X^*$ such that $F(u)=(B^\sigma u,g)$ for $u\in X_B^{\sigma}$. Putting $f=(B^*)^\sigma g$ we have $f\in (X^*)_{B^*}^{-\sigma}$ and $F(u) = <u,f>_\sigma$. Thus we have shown that $(X_B^{\sigma})^* = (X^*)_{B^*}^{-\sigma}$ isometrically via the duality pairing (23).

Now let $u \in X_B^\sigma$ be given and put $v = B^\sigma u \in X$. Then, the Hahn-Banach theorem ensures the existence of some $g \in X^*$ such that $(v,g) = \| v \| \| g \|_*$. Putting $f = (B^*)^{-\sigma} g$ we have $f \in (X^*)_{B^*}^{-\sigma}$ and

$$| < u , f >_\sigma | = | (v,g) | = \| u \|_\sigma \| f \|_{*,-\sigma}. \tag{26}$$

(26) and (24) together imply that $U = < u , \cdot >_\sigma : (X^*)_{B^*}^{-\sigma} \to \mathbb{C}$ belongs to $[(X^*)_{B^*}^{-\sigma}]^*$ and $\| U \| = \| u \|_\sigma$. This shows that $X_B^\sigma \hookrightarrow [(X^*)_{B^*}^{-\sigma}]^*$ isometrically.

Let now X be reflexive. For given $U \in [(X_{B^*}^*)^{-\sigma}]^*$, since $(X_{B^*}^*)^{-\sigma}$ and X^* are isometric to each other, there exists a unique $v \in X^{**} = X$ such that

$$U(f) = (v, (B^*)^{-\sigma} f) = (B^\sigma u , (B^*)^{-\sigma} f) = < u , f >_\sigma .$$

where $u = B^{-\sigma} v \in X_B^\sigma$. Thus, if X is reflexive then $X_B^\sigma = [(X^*)_{B^*}^{-\sigma}]^*$ isometrically via the duality pairing (23). □

Lemma I.2.9.

(i) Given $\sigma \geq 0$. Then

$$< u , f >_\sigma = (u,f) \quad \text{for} \quad u \in X_B^\sigma \text{ and } f \in X^*. \tag{27}$$

(ii) Given $0 < \sigma < \tau < \infty$. Then

$$< u , f >_\sigma = < u , f >_\tau \quad \text{for} \quad u \in X_B^\tau \text{ and } f \in (X^*)_{B^*}^{-\sigma}. \tag{28}$$

Thus the mapping $< \cdot , \cdot > : \bigcup\limits_{\sigma > 0} X_B^\sigma \times (X^*)_{B^*}^{-\sigma} \to \mathbb{C}$ is well defined in the natural way.

Proof.

(i) $$\begin{aligned} < f , g >_\sigma &= (B^\sigma u , (B^*)^{-\sigma} f) \\ &= (B^\sigma u , (B^{-\sigma})^* f) \\ &= (B^{-\sigma} B^\sigma u , f) \\ &= (u , f). \end{aligned}$$

(ii) $$\begin{aligned} (u , f)_\tau &= (B^\tau u , (B^*)^{-\tau} f) \\ &= (B^{\tau-\sigma} B^\sigma u , (B^*)^{-(\tau-\sigma)} (B^*)^{-\sigma} f) \\ &= (B^{\tau-\sigma} B^\sigma u , (B^{-(\tau-\sigma)})^* (B^*)^{-\sigma} f) \\ &= (B^\sigma u , (B^*)^{-\sigma} f) \\ &= (u , f)_\sigma . \end{aligned}$$

□

Theorem I.2.10.

(i) Let $\sigma \in [0,\infty)$. Then

$$(X_B^{\sigma+})^* = (X^*)_{B^*}^{(-\sigma)-}, \; [(X^*)_{B^*}^{(-\sigma)-}]^* \hookrightarrow X_B^{\sigma+} \tag{29}$$

and if furthermore X is reflexive, equality "=" holds instead of "$\hookrightarrow$" in (29).

(ii) Let $\sigma \in (0,\infty]$. Then

$$(X_B^{\sigma-})^* = (X^*)_{B^*}^{(-\sigma)+}, \; [(X^*)_{B^*}^{(-\sigma)+}]^* \hookrightarrow X_B^{\sigma-} \tag{30}$$

and if furthermore X is reflexive, equality "=" holds instead of "$\hookrightarrow$" in (30).

All the equality relations above hold via the duality pairing $<\cdot,\cdot>$.

Proof. The conclusions here directly follow from Theorem I.2.8, Lemma I.2.9 and Corollary I.1.8. □

We now turn to the study of extendibility of an operator A in X to spaces $X_B^{-\sigma} (\sigma > 0)$ or their inductive limits or projective limits.

Theorem I.2.11. Suppose that the space X is reflexive. Let $A : D(A) \subset X \to X$ be a densely defined operator and $A^* : D(A^*) \subset X^* \to X^*$ its dual operator. Then

(i) For given $\sigma, \tau > 0$ the operator A extends uniquely to a continuous operator from $X_B^{-\sigma}$ to $X_B^{-\tau}$ iff $(X^*)_{B^*}^{\tau} \subset D(A^*)$, $A^*(X^*)_{B^*}^{\tau} \subset (X^*)_{B^*}^{\sigma}$ and $A^* \restriction_{(X^*)_{B^*}^{\tau}}$ is continuous from $(X^*)_{B^*}^{\tau}$ to $(X^*)_{B^*}^{\sigma}$.

(ii) For given $\sigma, \tau \in (0,\infty)$ the operator A extends uniquely to a continuous operator from $X_B^{-\sigma+}$ to $X_B^{-\tau+}$ iff $(X^*)_{B^*}^{\tau-} \subset D(A^*)$, $A^*(X^*)_{B^*}^{\tau-} \subset (X^*)_{B^*}^{\sigma-}$ and $A^* \restriction_{(X^*)_{B^*}^{\tau-}}$ is continuous from $(X^*)_{B^*}^{\tau-}$ to $(X^*)_{B^*}^{\sigma-}$.

(iii) For given $\sigma, \tau \in [0,\infty)$ the operator A extends uniquely to a continuous operator from $X_B^{-\sigma-}$ to $X_B^{-\tau-}$ iff $(X^*)_{B^*}^{\tau+} \subset D(A^*)$, $A^*(X^*)_{B^*}^{\tau+} \subset (X^*)_{B^*}^{\sigma+}$ and $A^* \restriction_{(X^*)_{B^*}^{\tau+}}$ is continuous from $(X^*)_{B^*}^{\tau+}$ to $(X^*)_{B^*}^{\sigma+}$.

Proof.

(i) " $\Leftarrow$ ". Set $\bar{A}_{\sigma,\tau} = (A^* \restriction_{(X^*)_{B^*}^{\tau}})^*$. Then, since $[(X^*)_{B^*}^{\tau}]^* = X_B^{-\tau}$ and $[(X^*)_{B^*}^{\sigma}]^* = X_B^{-\sigma}$ by Theorem I.2.8 applied to X^* and B^* and by the reflexitivity to X, the standard theorem on the dual of a continuous operator from a Banach space to another (cf. Theorem 0.8) implies that $\bar{A}_{\sigma,\tau}$ is a continuous operator from $X_B^{-\sigma}$ to $X_B^{-\tau}$ and $\| \bar{A}_{\sigma,\tau} \| = \| A^* \restriction_{(X^*)_{B^*}^{\tau}} \|$. Let us show that $\bar{A}_{\sigma,\tau}$ is indeed an extension of A. In the following $<\cdot,\cdot>_{\sigma,*} : X_B^{-\sigma} \times (X^*)_{B^*}^{\sigma}$ $(\sigma > 0)$ stands for the duality pairing between $(X^*)_{B^*}^{\sigma}$ and $X_B^{-\sigma}$, and, of course, it has similar properties of $<\cdot,\cdot>_{\sigma}$ as are stated in Lemma I.2.9 above; $<\cdot,\cdot>_*$ is understood similarly to $<\cdot,\cdot>$. By definition we have

$$<u, A^* f>_{\sigma,*} = <\bar{A}_{\sigma,\tau} u, f>_{\tau,*}, \; u \in X_B^{-\sigma}, \; f \in (X^*)_{B^*}^{\tau}. \tag{31}$$

If $u \in D(A) \subset X \subset X_B^{-\sigma}$ and $f \in (X^*)_{B^*}^{\tau}$ then

$$<u\,, A^* f>_{\sigma,*} = (u\,, A^* f) = (A\,u\,, f) = <A\,u\,, f>_{\tau,*}. \tag{32}$$

Thus

$$<A\,u\,, f>_{\tau,*} = <\bar{A}_{\sigma,\tau}\,u\,, f>_{\tau,\sigma}\,,\ f \in (X^*)^{\tau}_{B^*}$$

which implies that $\bar{A}_{\sigma,\tau}\,u = A\,u$. The uniqueness follows from the denseness of $D(A)$ in X and X in $X_B^{-\sigma}$.

" $\Rightarrow$ ". Assume that A extends to a continuous operator from $X_B^{-\sigma}$ to $X_B^{-\tau}$, denoted $\bar{A}_{\sigma,\tau}$. Set $A^*\restriction = (\bar{A}_{\sigma,\tau})^*$. Then, since $(X_B^{-\sigma})^* = (X^*)^{\sigma}_{B^*}$ and $(X_B^{-\tau})^* = (X^*)^{\tau}_{B^*}$ by Theorem I.2.8 applied to X^* and B^* and by the reflexitivity of X, $A^*\restriction$ is a continuous operator from $(X^*)^{\tau}_{B^*}$ to $(X^*)^{\tau}_{B^*} \subset D(A^*)$ and $\| A^*\restriction \| = \| \bar{A}_{\sigma,\tau} \|$. If we can show that $(X^*)^{\tau}_{B^*} \subset D(A^*)$ and $A^*\restriction (X^*)^{\tau}_{B^*} = A^*\restriction$, we have completed the proof. By definition

$$<\bar{A}_{\sigma,\tau}\,u\,, f>_{\tau,*} = <u\,, A^*\restriction f>_{\sigma,*}\,,\ u \in X_B^{-\sigma}\,,\ f \in (X^*)^{\tau}_{B^*}. \tag{33}$$

If $u \in D(A)$, then

$$(A\,u\,, f) = <\bar{A}_{\sigma,\tau}\,u\,, f>_{\tau,*} = <u\,, A^*\restriction f>_{\sigma,*}\,,\ f \in (X^*)^{\tau}_{B^*}$$

which implies that $(A\,\cdot\,, f) : D(A) \subset X \to \mathbb{C}$ is continuous and therefore $f \in D(A^*)$; $(X^*)^{\tau}_{B^*} \subset D(A^*)$. Furthermore, for $f \in (X^*)^{\tau}_{B^*}$ the above equation can be rewritten as

$$<u\,, A^* f>_{\sigma,*} = <u\,, A^*\restriction f>_{\sigma,*}\,,\ u \in D(A).$$

This together with the denseness of $D(A)$ in X and $X_B^{-\sigma}$ implies that $A^* f = A^*\restriction f$. Thus $A^*\restriction_{(X^*)^{\tau}_{B^*}} = A^*\restriction$.

(ii) " $\Leftarrow$ ". Set $A^*\restriction_{(X^*)^{\tau-}_{B^*}} = S$. Then, as is assumed, S is a continuous operator from $(X^*)^{\tau-}_{B^*}$ to $(X^*)^{\sigma-}_{B^*}$. By Theorem I.2.10 we have $[(X^*)^{\tau-}_{B^*}]^* = X_B^{-\tau+}$ and $[(X^*)^{\sigma-}_{B^*} = X_B^{-\sigma+}$. The dual operator $S^* : X_B^{-\sigma+} \to X_B^{-\tau+}$ is well defined via the duality pairing

$$<u\,, S f> = <S^*\,u\,, f>\,,\ u \in X_B^{-\sigma+}\,,\ f \in (X^*)^{\tau-}_{B^*}. \tag{34}$$

A proof similar to the corresponding part of (i) above shows that $S^*\restriction_{D(A)} = A$. Let us prove the continuity of S^*. Given $\sigma' \in (0,\sigma)$. By the continuity of S, Theorem I.1.6 (ii) implies the existence of some $\tau' \in (0,\tau)$ such that

$$\| S f \|_{\sigma',*} \le C_{\sigma',\tau'} \| f \|_{\tau',*}\,,\ f \in (X^*)^{\tau-}_{B^*}. \tag{35}$$

Let $u \in X_B^{-\sigma'} \subset X_B^{-\sigma+}$. Then, (34) and (35) implies $S^*\,u \in X_B^{-\tau'}$ and

$$\| S^*\,u \|_{-\tau'} = \sup_{\substack{f \in (X^*)^{\tau-}_{B^*} \\ \| f \|_{\tau',*}}} | <S^*\,u\,, f>_{\tau',*} | \quad (\text{since } (X^*)^{\tau-}_{B^*} \text{ is dense in } (X^*)^{\tau'}_{B^*})$$

$$= \sup_{\substack{f \in (X^*)_{B^*}^{\tau-} \\ \|f\|_{\tau',*}=1}} |<S^* u, f>|$$

$$= \sup_{\substack{f \in (X^*)_{B^*}^{\tau-} \\ \|f\|_{\tau',*}=1}} |<u, Sf>| \quad \text{(by (34))}$$

$$\le C_{\sigma',\tau'} \|u\|_{-\sigma'}. \quad \text{(by (35))}$$

Thus Theorem I.1.6 (i) is invoked to ensure the continuity of S^*. The uniqueness of the extension follows from the denseness of $D(A)$ in X and X in $X_B^{-\sigma+}$.

" $\Rightarrow$ ". Assume, conversely, that A extends to a continuous operator from $X_B^{-\sigma+}$ to $X_B^{-\tau+}$, denoted T. Define its dual T^* via the duality pairing

$$<Tu, f>_* = <u, T^* f>_*, \; u \in X_B^{-\sigma+}, f \in (X^*)_{B^*}^{\tau-}. \tag{36}$$

It is a well defined operator from $(X^*)_{B^*}^{\tau-}$ to $(X^*)_{B^*}^{\sigma-}$. If $u \in D(A)$ and $f \in (X^*)_{B^*}^{\tau-}$ then

$$(Au, f) = <Au, f>_* = <Tu, f>_* = <u, T^* f>_* = (u, T^* f)$$

which implies that $f \in D(A^*)$ and $A^* f = T^* f$. This proves that $(X^*)_{B^*}^{\tau-} \subset D(A^*)$ and $A^* \restriction_{(X^*)_{B^*}^{\tau-}} = T^*$.

Let us show that T^* is continuous. Given $\sigma' \in (0,\sigma)$. By virtue of the continuity of $T : X_B^{-\sigma+} \to X_B^{-\tau+}$, Theorem I.1.6 (i) guarantees the existence of some $\tau' \in (0,\tau)$ such that

$$\|Tu\|_{-\tau'} \le C_{\sigma',\tau'} \|u\|_{-\sigma'}, \; u \in X_B^{-\sigma'}. \tag{37}$$

For $f \in (X^*)_{B^*}^{\tau'}$, (36) and (37) imply that $T^* f \in (X^*)_{B^*}^{\sigma'}$ and

$$\|T^* f\|_{\sigma',*}.$$

$$= \sup_{\substack{u \in X_B^{-\sigma'} \\ \|u\|_{-\sigma'}=1}} |<u, T^* f>_{\sigma',*}|$$

$$= \sup_{\substack{u \in X_B^{-\sigma'} \\ \|u\|_{-\sigma'}=1}} |<Tu, f>_{\tau',*}| \quad \text{(by (36))}$$

$$\le C_{\sigma',\tau'} \|f\|_{\tau',*} \quad \text{(by (37))}.$$

By Theorem I.1.6 (ii) we have the continuity of T^*.

(iii) The proof is completely parallel on that for (ii) above and is omitted. □

We have a few remarks to the above theorem:

a) The conclusion in ii) above is still true for $\sigma=\infty$ or $\tau=\infty$ if we assume that $(X^*)_{B^*}^{\tau}$ is dense in X^* in case $\tau=\infty$ and if we replace the concept of continuity of operators by a formally stronger one, as is described in Theorem I.1.6 (i) (2) (or equivalently (3)).

b) Similarly to ii) and iii) above we have also characterizations of operators A in X which are extendible continuously to one from $X_B^{-\sigma+}$ to $X_B^{-\tau-}$ or from $X_B^{-\sigma-}$ to $X_B^{-\tau+}$ in terms of its dual operator A^*.

c) Instead of one space X and one operator B we have entirely similar results on the continuous extendibility of an operator $A : D(A) \subset X \to Y$ to one from $X_B^{-\sigma}$ to $Y_C^{-\tau}$, et al, for two spaces X and Y and operators B and C.

To conclude the present section we put the results above in perspective. Given a reflexive Banach space X and an operator B of type $P(\omega,M)$ therein such that $0 \in \rho(B)$. Along with the space X and the operator B we have the dual space X^* and the dual operator B^* having similar properties. Using the domains of the fractional powers B^σ and $(B^*)^\sigma$ we construct the scales of Banach spaces X_B^σ $(\sigma \in \mathbb{R})$ and $(X^*)_{B^*}^\sigma$ and we form the scales of spaces of their inductive limits and projective limits, namely, $X_B^{\sigma+}$ and $(X^*)_{B^*}^{\sigma+}$ $(\sigma \in [-\infty,\infty))$, $X_B^{\sigma-}$ and $(X^*)_{B^*}^{\sigma-}$ $(\sigma \in (-\infty,\infty])$.
Thus we have the following diagram $(\sigma>0)$:

$$X \xrightarrow{\;B^\sigma\;} X_B^{-\sigma}$$

$$X_B^{-\infty} \supset X_B^{-\sigma+} \supset X_B^{-\sigma} \supset X_B^{-\sigma+} \supset X_B^{0-} \supset X$$

$$(X^*)_{B^*}^{-\infty} \supset (X^*)_{B^*}^{-\sigma+} \supset (X^*)_{B^*}^{-\sigma} \supset (X^*)_{B^*}^{-\sigma+} \supset (X^*)_{B^*}^{0-} \supset X^*$$

$$X^* \xrightarrow{\;(B^*)^\sigma\;} (X^*)_{B^*}^{-\sigma}$$

$$X_B^{\sigma} \xrightarrow{\;B^{-\sigma}\;} X$$

$$X \supset X_B^{0+} \supset X_B^{\sigma-} \supset X_B^{\sigma} \supset X_B^{\sigma+} \supset X_B^{\infty}$$

$$X^* \supset (X^*)_{B^*}^{0+} \supset (X^*)_{B^*}^{\sigma-} \supset (X^*)_{B^*}^{\sigma} \supset (X^*)_{B^*}^{\sigma+} \supset (X^*)_{B^*}^{\infty}$$

$$(X^*)_{B^*}^{\sigma} \xrightarrow{\;(B^*)^{-\sigma}\;} X^*$$

If X and X^* are suitable spaces of functions and the operators B and B^* are appropriately taken (usually differential operators), then various classical function spaces appear as spaces X_B^σ or $(X^*)_{B^*}^\sigma$ $(\sigma>0)$, and different test function spaces and their corresponding generalized function spaces emerge as the spaces of inductive limits or projective limit with nonnegative indices and nonpositive indices respectively. Thus, we call the spaces to the right of X and X^* in the diagram regular spaces, and those to the left hyper-spaces. Theorems I.2.5 and I.2.7 then clarify the topological structures of all the spaces of inductive limit and projective limit. Theorem I.1.6 can be directly invoked to give characterizations of continuous operators between those spaces. Theorems I.2.8 and I.2.10 establish the duality between the two scales of spaces in the above diagram (i.e. between spaces of smooth functions and generalized functions). And Theorem

I.2.11 gives criteria which ensures that an operator initially acting on smooth functions could be extended to spaces of generalized functions. In short our frame is a kind of Gelfand-Shilov triple in a Banach space setting. We notice that in general the spaces to the right of X, in the above diagram, cannot be embedded into their dual spaces, i.e., the ones to the left of X^*. If, however, X is a Hilbert space, this can always be done as long as we identify the dual of the Hilbert space with itself. Also, in some instances, spaces "enough" right to X can be embedded to spaces "enough" left to X^*. Anyway, the spaces right to the spaces X and X^* together are included in the total of the spaces left to X and X^* together.

I.3. Regular Spaces and Hyper-spaces: Examples

In this section we present some concrete regular spaces and hyper-spaces, i.e., classical function spaces, test function spaces and generalized function spaces, which can be identified with the spaces X_B^σ or $X_B^{\sigma\pm}$ with an appropriate Banach space X and a suitable operator B.

Example I. Let X be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and $B : D(B) \subset X \to X$ a nonnegative self-adjoint operator. Then both the operators B and e^B are of type $P(0,1)$, and e^B is invertible. Let $\{E(t)\}_{t\geq 0}$ be the spectral resolution of B. Then, by definition

$$\begin{aligned}
(e^B)^{-\sigma} u &= \frac{1}{2\pi i} \int_\Gamma \lambda^{-\sigma} (\lambda I - e^B)^{-1} u \, d\lambda \\
&= \frac{1}{2\pi i} \int_\Gamma \lambda^{-\sigma} \Big(\int_0^\infty (\lambda - e^t)^{-1} \, dE(t) u \Big) d\lambda \\
&= \int_0^\infty \Big(\frac{1}{2\pi i} \int_\Gamma \lambda^{-\sigma} (\lambda - e^t)^{-1} \, d\lambda \Big) dE(t) u \\
&= \int_0^\infty (e^t)^{-\sigma} \, dE(t) u \\
&= e^{-\sigma B} u \quad (u \in X,\ \sigma > 0). \qquad (38)
\end{aligned}$$

Thus $(e^B)^\sigma = e^{\sigma B}$ for all $\sigma \in I\!R$. In particular we have $X_{e^B}^{0+} = S_{X,B}$, $X_{e^B}^{0-} = T_{X,B}$, $X_{e^B}^{\infty} = \tau_{X,B}$ and $X_{e^B}^{-\infty} = \sigma_{X,B}$, the spaces of De Graaf and Van Eijndhoven. See [Gr 2], [Gr 3], [Ei] and [E-G] where these spaces are defined and studied and a number of classical test function spaces and generalized function spaces have been realized as spaces of these types for suitable Hilbert spaces and operators. In fact, the present work is very much inspired by theirs.

Example II. Let $X = l^p (1 \leq p \leq \infty)$ with norm $\| u \|_p = (\sum_{k=0}^{\infty} | u_k |^p)^{1/p}$ for $u = (u_k) \in l^p$. Given a sequence of complex numbers $\{\lambda_k \mid k \in \mathbb{N}_0\}$ such that

$$|\arg \lambda_k| \le \omega \quad \text{for some } \omega \in [0,\pi) \tag{39}$$

and

$$|\lambda_k| \ge a \quad \text{for some constant } a > 0 \tag{40}$$

for all $k \in \mathbb{N}_0$. With this sequence of numbers we define an operator B as follows: $u = (u_k) \in D(B)$ iff $(\lambda_k u_k) \in l^p$ and $B\,u = (\lambda_k u_k)$.

It is easy to see that for all $k \in \mathbb{N}_0$.

$$|\lambda - \lambda_k| \ge \begin{cases} |\lambda| \ \text{ if } 0 \le \omega \le \pi/2 \\ & (\lambda < 0) \\ |\lambda| \sin\omega \ \text{ if } \pi/2 < \omega < \pi \end{cases} \tag{41}$$

and

$$|\lambda - \lambda_k| \ge |\lambda| \sin\omega \quad \text{for } \lambda \in \Sigma_{\omega+\varepsilon}. \tag{42}$$

Therefore the operator B is of type $(\omega, 1)$ if $0 \le \omega \le \pi/2$ and type $(\omega, 1/\sin\omega)$ if $\pi/2 < \omega < \pi$. In particular it is m-accretive if $\omega = \pi/2$. Moreover

$$(\lambda I - B)^{-1} u = ((\lambda - \lambda_k)^{-1} u_k) \quad \text{for } \lambda \in \rho(B).$$

Condition (40) implies that $\{\lambda \in \mathbb{C} \mid |\lambda| < a\} \subset \rho(B)$. Thus, for $\sigma > 0$ by definition

$$B^{-\sigma} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\sigma} (\lambda I - B)^{-1} u \, d\lambda = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\sigma}((\lambda - \lambda_k)^{-1} u_k) \, d\lambda$$

$$= (\frac{1}{2\pi i} \int_\Gamma \lambda^{-\sigma} (\lambda - \lambda_k)^{-1} \, d\lambda \cdot u_k) = (\lambda_k^{-\sigma} u_k)$$

where the integration path Γ can be taken $\{\lambda \in \mathbb{C} \mid \arg(\lambda - a/2) = \phi\} \subset \rho(B)$ for appropriate $\phi \in (\omega, \pi)$. Therefore, by definition, for $\sigma > 0$

$$X_B^\sigma = l^{p,\sigma}\{\lambda_k\} \equiv \{u = (u_k) \in l^p \mid \|u\|_{p,\sigma} = (\sum_{k=0}^\infty (|\lambda_k|^\sigma |u_k|)^p)^{1/p} < \infty\}. \tag{43}$$

For $-\sigma < 0$ let

$$l^{p,-\sigma}\{\lambda_k\} = \{u = (u_k) \mid \|u\|_{p,-\sigma} = (\sum_{k=0}^\infty (|\lambda_k|^{-\sigma} |u_k|)^p)^{1/p} < \infty\}. \tag{44}$$

Then it is readily seen that $l^{p,-\sigma}\{\lambda_k\}$ is a normed space isometric to l^p under the mapping $l^p \ni (u_k) \mapsto (\lambda_k^\sigma u_k) \in l^{p,-\sigma}\{\lambda_k\}$. So $l^{p,-\sigma}\{\lambda_k\}$ is a Banach space. Moreover, since obviously l^p is dense in $l^{p,-\sigma}$, we have $\overline{l^p} = l^{p,-\sigma}$. Namely $X_B^{-\sigma} = l^{p,-\sigma}\{\lambda_k\}$. We set $X_B^0 = l^p = l^{p,0}\{\lambda_k\}$. If no confusion is incurred, $l^{p,\sigma}\{\lambda_k\}$ $(\sigma \in \mathbb{R})$ is abbreviated to $l^{p,\sigma}$. This applies, of course, also to the inductive limits $X_B^{\sigma+} = l^{p,\sigma+}\{\lambda_k\} = \bigcup_{\tau>\sigma} l^{p,\tau}\{\lambda_k\}$ $(\sigma \in [-\infty,\infty))$ and the projective limits $X_B^{\sigma-} = l^{p,\sigma-}\{\lambda_k\} = \bigcup_{\tau<\sigma} l^{p,\tau}\{\lambda_k\}$ $(\sigma \in (-\infty,\infty])$.

For $1 \le p < \infty$, $X^* = (l^p)^* = l^q$ where $q = p^* = \frac{p}{p-1}$, and $B^* : D(B^*) \subset l^q \to l^q$ is defined by

$$D(B^*) = \{u = (u_k) \in l^q \mid (\bar{\lambda}_k u_k) \in l^q\}$$

$$B^* u = (\bar{\lambda}_k u_k).$$

Thus, in accordance with the above notations, $(X^*)_{B^*}^{\sigma} = l^{q,\sigma}\{\bar{\lambda}_k\}$ $(\sigma \in \mathbb{R})$. Theorems I.2.8 and I.2.10 then imply the following: $(l^{p,\sigma})^* = l^{q,-\sigma}$ for all $1 \le p < \infty$ and $\sigma > 0$; $(l^{q,-\sigma})^* = l^{p,\sigma}$ for all $1 < p < \infty$; $l^{1,\sigma} \hookrightarrow (l^{\infty,-\sigma})^*$; and similarly for the inductive and projective limits.

We notice that for $p > q = p^*$, if $\mu = (\lambda_k^{(\tau-\sigma)}) \in l^{\frac{pq}{p-q}}$ $(\tau,\sigma \in \mathbb{R})$ then $l^{p,\sigma} \hookrightarrow l^{q,\tau}$ and $\| u \|_{q,\tau} \le \| \mu \|_{\frac{pq}{p-q}} \| u \|_{p,\sigma}$. Thus, if, for instance, $\lambda_k = e^{\alpha k^{\nu}}$ $(\alpha > 0, \nu > 0)$, then $l^{p,0+} \hookrightarrow l^{q,0-}$ (conf. Theorem I.1.6 (iii)). Indeed, by Hölder' inequality we have

$$\| u \|_{q,\tau} = (\Sigma (|\lambda_k|^{\tau} |u_k|)^q)^{1/q} = (\Sigma |\lambda_k|^{(\tau-\sigma)q} (|\lambda_k|^{\sigma} |u_k|)^q)^{1/q}$$

$$\le (\Sigma |\lambda_k|^{(\tau-\sigma)q \cdot \frac{p}{p-q}})^{\frac{p-q}{pq}} (\Sigma (|\lambda_k|^{\sigma} u_k)^p)^{1/p} = \| \mu \|_{\frac{pq}{p-q}} \| u \|_{p,\sigma}. \tag{45}$$

Let us now consider linear operators. Given an infinite matrix (a_{kj}), $p \in [1,\infty]$, $q = p^*$ and $\sigma,\tau \in \mathbb{R}$. Formally we have, for $u = (u_k)$ and $v = (v_k)$ with $v_k = \sum_j a_{kj} u_j$, the following estimates.

$$\| v \|_{p,\sigma} = (\sum_k |\lambda_k|^{\sigma p} |\sum_j a_{kj} u_j|^p)^{1/p}$$

$$\le [\sum_k |\lambda_k|^{\sigma p} \cdot (\| a_{k\cdot} \|_{q,-\tau} \cdot \| u \|_{p,\tau})^p]^{1/p}$$

$$= \| \| a_{k\cdot} \|_{q,-\tau} \|_{p,\sigma} \| u \|_{p,\tau}. \tag{46}$$

If $a_{kj} = b_{kj} c_{kj}$ then

$$\| v \|_{p,\sigma} = [\sum_k |\lambda_k|^{\sigma p} |\sum_j b_{kj} c_{kj} u_j|^p]^{1/p}$$

$$\le [\sum_k |\lambda_k|^{\sigma p} \| b_{k\cdot} \|_{q,-\tau}^p \sum_j (|\lambda_k|^{\tau} |c_{kj} u_j|)^p]^{1/p}$$

$$\leq \sup_k \| b_{k\cdot} \|_{q,-\tau} \, [\sum_j \sum_k (| \lambda_j |^\tau | u_j | \, | \lambda_k |^\sigma | c_{kj} |)^p]^{1/p}$$

$$\leq \sup_k \| b_{k\cdot} \|_{q,-\tau} \cdot \sup_j \| c_{\cdot j} \|_{p,\sigma} \| u \|_{p,\tau}. \tag{47}$$

With the estimates (46) and (47) we have

Proposition I.3.1. Given an infinite matrix (a_{kj}), $p \in [1,\infty]$, $q = p^*$ and $\sigma,\tau \in \mathbb{R}$. Then the operator $A : A\, u = v$ with $u = (u_k)$ and $v_k = \sum_j a_{kj} u_j$ is well defined and continuous from $l^{p,\tau}$ to $l^{p,\sigma}$ if one of the following conditions are satisfied:

1) $\| \| a_{k\cdot} \|_{q,-\tau} \|_{p,\sigma} < \infty.$ (48)

2) $a_{kj} = b_{kj} c_{kj}$ and

$$\sup_{k \in \mathbb{N}_0} \| b_{k\cdot} \|_{q,-\tau} < \infty, \ \sup_{j \in \mathbb{N}_0} \| c_{\cdot j} \|_{p,\sigma} < \infty. \tag{49}$$

Moreover, the norm of the operator A has the following estimates respectively

$$\| A \|_{p,\tau;p,\sigma} \leq \| \| a_{k\cdot} \|_{q,-\tau} \|_{p,\sigma}$$

$$\| A \|_{p,\tau;p,\sigma} \leq \sup_{k \in \mathbb{N}_0} \| b_{k\cdot} \|_{q,-\tau} \cdot \sup_{j \in \mathbb{N}_0} \| c_{\cdot j} \|_{p,\sigma}.$$

□

Corollary I.3.2. The operator A given above is well defined and continuous from $l^{p,\tau}$ to $l^{p,\sigma}$ if

$$\sup_{k \in \mathbb{N}_0} \| a_{k\cdot} \|_{1,-\tau q} < \infty, \ \sup_{j \in \mathbb{N}_0} \| a_{\cdot j} \|_{1,\sigma p} < \infty, \tag{50}$$

and with norm

$$\| A \|_{p,\tau;p,\sigma} \leq (\sup_{k \in \mathbb{N}_0} \| a_{k\cdot} \|_{1,-\tau q})^{1/q} (\sup_{j \in \mathbb{N}_0} \| a_{\cdot j} \|_{1,\sigma p})^{1/p}.$$

Proof. Take $b_{kj} = | a_{kj} |^{1/q}$ and $c_{kj} = | a_{kj} |^{1/p} e^{i \arg(a_{kj})}$ in case 1) above. □

The above proposition and its corollary are generalizations of the well known criteria of the boundedness of operators from l^2 to l^2. See e.g., Chapter 6 Section 3 of the book [We].

From the above Proposition I.3.3 and Theorem I.1.6 immediately follows the following criterion for a linear operator to be continuous in the inductive/projective spaces $l^{p,\sigma\pm}$.

Proposition I.3.3. Given (a_{kj}), $p \in [1,\infty)$ and $q = p^*$.

(i) For $\tau_0, \sigma_0 \in [-\infty,\infty)$, the operator A given above is well defined and continuous from l^{p,τ_0+} to l^{p,σ_0+} if for any $\tau > \tau_0$ there exists a $\sigma > \sigma_0$ such that one of the conditions 1) and 2) in Proposition I.3.1 is satisfied.

(ii) For $\tau_0, \sigma_0 (-\infty,\infty]$, the operator A is well defined and continuous from l^{p,τ_0-} to l^{p,σ_0-} if for any $\sigma < \sigma_0$ there exists a $\tau < \tau_0$ such that one of the conditions 1) and 2) in Proposition I.3.1 is satisfied. □

We have omitted other two cases: operators from l^{p,τ_0+} to l^{p,σ_0-} and from l^{p,τ_0-} to l^{p,σ_0+}.

We continue by discussing the problem of extendibility of linear operators in l^p to spaces of types $l^{p,-\sigma}$ and $l^{p,(-\sigma)\pm}$. According to Theorem I.2.11 we need to study the behaviour of the dual operators in l^q.

With a given infinite matrix (a_{kj}) we associate an operator A in l^p $(1 \le p < \infty)$ as follows: $u = (u_k) \in D(A)$ iff $v_k = \sum_{j \in \mathbb{N}_0} a_{kj} u_j$ converges for each $k \in \mathbb{N}_0$ and $v = (v_k) \in l^p$; in this case $A u = v$. The following proposition is about the denseness of the domains, the closedness and the duals of such operators.

Proposition I.3.4.

(i) If $(a_{.j}) \in l^p$ for each $j \in \mathbb{N}_0$ then A is densely defined and $A^* \subset A^+$, which is an operator in l^q defined as the operator A with the conjugate transpose matrix $(a^+_{kj}) = (\bar{a}_{jk})$.

(ii) If both $(a_{.j}) \in l^p$ for each $j \in \mathbb{N}_0$ and $(a_{k.}) \in l^p$ for each $k \in \mathbb{N}_0$, then A is closed. □

We omit the somewhat standard proof of this proposition. Conf. [We] Chapter 6 Section 3, where the l^2 case corresponding to (i) and (ii) above is studied. Applying the above Propositions I.3.1 and I.3.4 and Theorem I.2.11 we can easily write down some conditions which ensure the extendibility of the operator A in l^p associated with a matrix (a_{kj}) to spaces $l^{p,-\sigma}$ or $l^{p,(-\sigma)\pm}$.

For any two sequences $u = (u_k)$ and $v = (v_k)$ we define their convolution product to be the sequence $w = (w_k)$ where

$$w_k = \sum_{j=0}^{k} u_j \, v_{k-j} \quad , k \in \mathbb{N}_0 .$$

We use the notation $w = u * v = v * u$.

Proposition I.3.5. Let $p \in [1,\infty)$ and $\sigma, \tau, \theta \in \mathbb{R}$.
Define a sequence $\mu = \mu(\sigma,\tau) = (\mu_k(\sigma,\tau))$

$$\mu_k(\sigma,\tau) = \max\{\,|\lambda_j|^{-\sigma}\,|\lambda_{k-j}|^{-\tau}\mid 0\le j\le k\}. \tag{50}$$

Assume that $\|\mu\|_{1,\theta} = \sum_{k\in \mathbb{N}_0} |\lambda_k|^\theta \mu_k < \infty$. Then for $u\in l^{p,\sigma}$ and $v\in l^{q,\tau}$ we have $w = u * v \in l^{1,\theta}$ and

$$\|w\|_{1,\theta} \le \|\mu\|_{1,\theta}\,\|u\|_{p,\sigma}\,\|v\|_{q,\tau}. \tag{51}$$

Thus, the convolution product is a continuous mapping from $l^{p,\sigma}\times l^{q,\tau}$ to $l^{1,\theta}$.

Proof. We have, by Hölder's inequality,

$$|w_k| \le \mu_k(\sigma,\tau)\sum_{j=0}^{k} |\lambda_j|^\sigma\,|u_j|\cdot|\lambda_{k-j}|^\tau\,|v_{k-j}|$$

$$\le \mu_k(\sigma,\tau)\,\|u\|_{p,\sigma}\,\|v\|_{q,\tau}$$

from which readily follows (51). □

Corollary I.3.6. If for any $\sigma,\tau>0$ there exists a $\theta>0$ such that $\|\mu\|_{1,\theta}<\infty$, then $l^{2,0+}$ is a topological algebra under the ordinary linear operations and the convolution product.

Proof. Note that $\|w\|_{r,\theta}\le\|w\|_{1,\theta}$ for all $r\ge 1$ and $w\in l^{r,\theta}$. □

To conclude the present example we give simple illustrations of some of the above results. Take $\lambda=(\lambda_k)$ with $\lambda_k = e^{k^\nu}$ $(\nu>0)$ and

$$(a_{kj}) = \begin{bmatrix} 0 & \sqrt{1} & & & \\ & 0 & \sqrt{2} & & 0 \\ 0 & & 0 & \sqrt{3} & \\ & & & \ddots & \ddots \end{bmatrix}$$

which is the matrix representation (with the Hermite functions as a basis) of the anihlation operator in quantum mechanics. Obviously

$$\|\,\|a_k\|_{q,-\tau}\|^p_{p,\sigma} = \sum_{k\in\mathbb{N}_0} \sqrt{k+1}\; e^{p[-\tau(k+1)^\nu+\sigma k^\nu]}\quad(\tau,\sigma\in\mathbb{R},\,p\in[1,\infty]).$$

Therefore from Propositions I.3.1 and I.3.3 it readily follows that: If $\nu>1$ then the anihlation operator A is continuous on $l^{p,\sigma}$ $(\sigma>0)$; if $\nu>0$ it is continuous on $l^{p,\sigma+}$ $(\sigma\in[-\infty,\infty)$ and on $l^{p,\sigma-}$ $(\sigma\in(-\infty,\infty])$.

From the inequality

$$\sigma j^{\nu} + \tau(k-j)^{\nu} \geq 2(\sigma\tau)^{1/2} [j(k-j)]^{\nu/2} \geq 2(\sigma\tau)^{1/2} (k/2)^{\nu} \ (\sigma,\tau > 0)$$

follows that $\mu_k(\sigma,\tau) \leq e^{-2(\sigma\tau)^{1/2}(k/2)^{\nu}}$. Then Proposition I.3.5 and Corollary I.3.6 imply that the harmonic product is continuous from $l^{p,\sigma} \times l^{q,\tau}$ to $l^{1,\theta}$ for $\theta < 2^{1-\nu}(\sigma\tau)^{1/2}$. Thence $l^{2,0+}$ and $l^{2,\infty}$ are topological algebras.

Example III. Let $X = L^p(I\!R^n)$ $(1 \leq p \leq \infty,\ q = p^*)$ with the usual norm $\| u \|_p = (\int_{I\!R^n} | u |^p dx)^{1/p}$. Let $\Lambda(x)$ denote a complex-valued measurable function on $I\!R^n$ satisfying the conditions

$$\Lambda(x) \in L^p_{loc}(I\!R^n) \tag{52}$$

$$| \Lambda(x) | \geq a > 0 \ \text{ for a.e. } x \in I\!R^n \tag{53}$$

and

$$| \arg \Lambda(x) | \leq \omega \text{ for a.e. } x \in I\!R^n \text{ with } \omega \in [0,\pi). \tag{54}$$

With the given function $\Lambda(x)$ we define an operator B as follows:

$$D(B) = \{u \in L^p(I\!R^n) \mid \Lambda(x)\, u(x) \in L^p(I\!R^n)\}$$

$$(Bu)\,(x) = \Lambda(x)\, u(x).$$

Since $\Lambda(x) \in L^p_{loc}$, $C_0^{\infty}(I\!R^n) \subset D(B)$ and B is densely defined. It is easily seen that

$$| \lambda - \Lambda(x) | \geq \begin{cases} | \lambda | & \text{if } 0 \leq \omega \leq \pi/2 \\ | \lambda | \sin\omega & \text{if } \pi/2 < \omega < \pi \end{cases} \qquad (\lambda < 0)$$

and

$$| \lambda - \Lambda(x) | \geq | \lambda | \sin\varepsilon \quad (\lambda \in \Sigma_{\omega+\varepsilon}.$$

Consequently B is an operator of type $P(\omega,1)$ if $0 \leq \omega \leq \pi/2$ and of type $P(\omega, 1/\sin\omega)$ if $\pi/2 < \omega < \pi$. Condition (53) implies that $\{\lambda \in C\!\!\!\!I \mid | \lambda | < a\} \subset \rho(B)$. Moreover

$$[(\lambda I - B)^{-1} u]\,(x) = (\lambda - \Lambda(x))^{-1}\, u(x) \text{ for } \lambda \in \rho(B).$$

Thus, for $\sigma > 0$, by definition

$$(B^{-\sigma} u)\,(x) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-\sigma} (\lambda - \Lambda(x))^{-1}\, u(x)\, d\lambda$$

$$= [\Lambda(x)]^{-\sigma}\, u(x)$$

where the path can be $\{\lambda \in C\!\!\!\!I \mid \arg(\lambda - a/2) = \phi\} \subset \rho(B)$ for suitable $\phi \in (\omega,\pi)$. Hence, by definition

$$X_B^\sigma = L^{p,\sigma}\{\Lambda(x)\} \equiv \{u \in L^p \mid \|u\|_{p,\sigma} = \int (|\Lambda(x)|^\sigma\ |u(x)|)^p\, dx)^{1/p} < \infty\}.$$

For $-\sigma < 0$ set

$$L^{p,-\sigma}\{\Lambda(x)\} = \{u \text{ measurable on } \mathbb{R}^n \mid \|u\|_{p,-\sigma} = (\int(|\Lambda(x)|^{-\sigma}\ |u(x)|)^p\, dx)^{1/p} < \infty\}.$$

$L^{p,-\sigma}\{\Lambda(x)\}$ thus defined is a normed space which is isometric to the Banach space L^p under the mapping $L^p \ni u(x) \mapsto [\Lambda(x)]^{-\sigma} u(x) \in L^{p,-\sigma}\{\Lambda(x)\}$ and hence is a Banach space itself. Moreover it is easily seen that L^p is dense in $L^{p,-\sigma}\{\Lambda(x)\}$. Thus $X_B^{-\sigma} = L^{p,-\sigma}\{\Lambda(x)\}$. We set $X_B^0 = L^p = L^{p,0}\{\Lambda(x)\}$. Furthermore,

$$X_B^{\sigma+} = L^{p,\sigma+}\{\Lambda(x)\} = \bigcup_{\tau>\sigma} L^{p,\tau}\{\Lambda(x)\} \quad (\sigma \in [-\infty,\infty))$$

$$X_B^{\sigma-} = L^{p,\sigma-}\{\Lambda(x)\} = \bigcap_{\tau<\sigma} L^{p,\tau}\{\Lambda(x)\} \quad (\sigma \in (-\infty,\infty])$$

are the corresponding inductive limits and projective limits respectively.

Obviously we have

$$(B^* u)(x) = \overline{\Lambda(x)}\, u(x)$$

$$D(B^*) = \{u \in L^q \mid \overline{\Lambda}(x)\, u(x) \in L_q\}.$$

Therefore, $(X^*)_{B^*}^\sigma = L^{q,\sigma}\{\overline{\Lambda}(x)\}$, $(X^*)_{B^*}^{\sigma\pm} = L^{q,\sigma\pm}\{\overline{\Lambda}(x)\}$. Theorems I.2.8 and I.2.10 then imply the following: $(L^{p,\sigma})^* = L^{q,-\sigma}$ for all $1 \le p < \infty$ and $\sigma \ge 0$; $(L^{q,-\sigma})^* = L^{p,\sigma}$ for all $1 < p < \infty$; $L^{1,\sigma} \hookrightarrow (L^{\infty,-\sigma})^*$; similar assertions hold for the inductive and projective limits.

Most of the conclusions for l^p in the above Example II remain valid for the L^p case treated here. We omit the details.

Example IV. Let $X = L^p(\mathbb{R}^n)$ $(1 \le p \le \infty\ q = p^* = \dfrac{p}{p-1})$ with the usual norm $\|\cdot\|_p$. The operator B in L^p is given by

$$D(B) = \{u \in L^p \mid \Delta u \in L^p\}$$

$$B\,u = (I - \Delta)\,u$$

where I is the identity operator on L^p and Δ is the ordinary Laplace operator. We note that the operator B is well defined on S and S', the Schwartz test function space and tempered distribution space. Let F and F^{-1} be the Fourier transform and its inverse, which act on S or S' continuously. We have

$$F(\lambda I - B)\,u = (\lambda - 1 - x^2)F\,u\,,\ u \in S'. \tag{55}$$

Therefore, if $\lambda \,\overline{\in}\, [1,\infty)$, then $(\lambda I - B) : S' \to S'$ is invertible and

$$(\lambda I - B)^{-1} u = F^{-1}[(\lambda - 1 - x^2)^{-1} F u]$$

$$= \frac{1}{(2\pi)^{n/2}} F^{-1}[(\lambda - 1 - x^2)^{-1}] * u. \tag{57}$$

We note that (by Fubini's Theorem)

$$(-\lambda + 1 + x^2)^{-1} = \int_0^\infty e^{-(-\lambda+1+x^2)\delta}\, d\delta$$

in the sense of tempered distributions. Then

$$K_\Lambda(x) \equiv \frac{1}{(2\pi)^{n/2}} F^{-1}(\lambda - 1 - x^2)^{-1}$$

$$= \frac{1}{(2\pi)^{n/2}} \int_0^\infty F^{-1} e^{-(-\lambda+1+x^2)\delta}\, d\delta$$

$$= \frac{1}{(2\pi)^{½}} \int_0^\infty e^{(\lambda-1)\delta - \frac{1}{4\delta}x^2} (2\delta)^{-n/2}\, d\delta.$$

And

$$\int_{I\!R^n} | K_\Lambda(x) |\, dx$$

$$\leq \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{(\mathrm{Re}\lambda - 1)\delta} (2\delta)^{-n/2}\, d\delta \int_{I\!R^n} e^{\frac{1}{4\delta}x^2}\, dx$$

$$= (-\mathrm{Re}\,\lambda + 1)^{-1}.$$

Thus, according to Young's inequality, we have

$$\| (\lambda I - B)^{-1} u \|_p \leq (-\mathrm{Re}\,\lambda + 1)^{-1} \| u \|_p,\ u \in L^p.$$

From this inequality it is readily seen that the operator B is of type $P(\pi/2, 1)$ and $0 \in \rho(B)$. [More detailed analysis actually shows that B is of type $P(0,1)$.]

For $\sigma > 0$ and $u \in L^p$, by definition

$$(B^{-\sigma} u)(x) = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\sigma}[(\lambda I - B)^{-1} u](x)\, d\lambda$$

$$= \frac{1}{2\pi i} \int_\Gamma \lambda^{-\sigma} F^{-1}[(\lambda - 1 - x^2)^{-1} F u]\, d\lambda$$

$$= F^{-1}[(\frac{1}{2\pi i}\int_\Gamma \lambda^{-\sigma}(\lambda-1-x^2)^{-1}\,d\lambda)\,F\,u]$$

$$= F^{-1}[(1+x^2)^{-\sigma}F\,u]. \tag{58}$$

Therefore

$$X_B^{\sigma/2} = W^{p,\sigma} \equiv \{u \in L^p \mid \|u\|_{p,\sigma} = \|F^{-1}(1+x^2)^{\sigma/2}F\,u\|_p < \infty\}. \tag{59}$$

The spaces $W^{p,\sigma}$ ($\sigma > 0$) are exactly the ones proposed and studied by Aronsjain and Smith [A-S] and Calderón [Ca]. See also [St].

For $-\sigma < 0$ set

$$W^{p,-\sigma} = \{u \mid u \in S',\, F^{-1}(1+x^2)^{-\sigma/2}F\,u \in L^p\}$$

with norm

$$\|u\|_{p,-\sigma} = \|F^{-1}(1+x^2)^{-\sigma/2}F\,u\|_p.$$

Since each $W^{p,-\sigma}$ is isometric to L^p under the mapping $F^{-1}(1+x^2)^{-\sigma/2}F : W^{p,-\sigma} \to L^p$, it is a Banach space itself. And it is not difficult to show the denseness of L^p in $W^{p,-\sigma}$. Therefore $X_B^{-\sigma/2} = W^{p,-\sigma}$.

For $\sigma \in [-\infty,\infty)$ we set $W^{p,\sigma+} = \bigcup_{t>\sigma} W^{p,\tau}$ with inductive limit topology. For $\sigma \in (-\infty,\infty]$ we set $W^{p,\sigma-} = \bigcap_{\tau<\sigma} W^{p,\tau}$ with projective limit topology. Then, of course, $X_B^{\sigma/2+} = W^{p,\sigma+}$ and $X_B^{\sigma/2-} = W^{p,\sigma-}$.

We can easily see that $B^* = I - \Delta$ in L^q with domain $D(B^*) = \{u \in L^q \mid \Delta u \in L^q\}$. Thus, we have $(X^*)_{B^*}^{\sigma/2} = W^{q,\sigma}$, $(X^*)_{B^*}^{(\sigma/2)+} = W^{q,\sigma+}$ and $(X^*)_{B^*}^{(\sigma/2)-} = W^{q,\sigma-}$. Theorems I.2.8 and I.2.10 then imply that $(W^{p,\sigma})^* = W^{q,-\sigma}$ ($\sigma \geq 0$ and $1 \leq p < \infty$), $(W^{q,-\sigma})^* = W^{p,\sigma}$ ($\sigma > 0$, $1 < p < \infty$), $W^{1,\sigma} \hookrightarrow (W^{\infty,-\sigma})^*$ ($\sigma > 0$), and similar relations for the inductive and projective limits.

Proposition I.3.7. If $F^{-1}(1+x^2)^{(\tau-\sigma)/2} \in L^{q/2}$, then $W^{p,\sigma} \hookrightarrow W^{q,\tau}$.

Proof. We have the identity

$$F^{-1}(1+x^2)^{\tau/2}F\,u = F^{-1}(1+x^2)^{(\tau-\sigma)/2}F\,F^{-1}(1+x^2)^{\sigma/2}F\,u$$

$$= [F^{-1}(1+x^2)^{(\tau-\sigma)/2}] * [F^{-1}(1+x^2)^{\sigma/2}F\,u]\ (u \in S').$$

So our assumption and Young's inequality give rise to the conclusion. □

Corollary I.3.8. If $p \leq 4/3$ ($q \geq 4$) and $\dfrac{(\sigma-\tau)p}{n(2-p)} > 1$, then $W^{p,\sigma} \hookrightarrow W^{q,\tau}$.

Proof. If $\frac{(\sigma-\tau)p}{n(2-p)} > 1$, then $(1+x^2)^{(\tau-\sigma)/2} \in L^{\frac{q}{q-2}}$. Thus, Hausdoff-Young inequality implies that $F^{-1}(1+x^2)^{(\tau-\sigma)/2} \in L^{q/2}$ and the desired conclusion follows from that of the last proposition.

□

We now discuss operators in those spaces. Recall that a bounded measurable function $m(x)$ on $\mathbb{R}^n$ is said to be an L^p-multiplier if $T_m u = F^{-1} m(x) F u \in L^p$ for all $u \in L^2 \cap L^p$ and $\| T_m u \|_p \leq M_p \| u \|_p$ for all $u \in L^2 \cap L^p$ with a positive constant M_p; the greatest lower bound of such M_p's is called the norm of the multiplier. Each bounded measurable function $m(x)$ is an L^2-multiplier with norm identical to the L^∞-norm of $m(x)$. The Fourier transforms of all the finite Borel measures on $\mathbb{R}^n$ constitute the L_1-multipliers with norm identical to those of the corresponding Borel measures. The bounded measurable function $m(x)$ is an L^p-multiplier iff it is an L^q-multiplier. We cite the following condition for a function $m(x)$ to be a L^p-multiplier; for its proof and the above assertions we refer to [St] Chapter IV Section 3.

Theorem I.3.9. Let $m(x) \in C^k(\mathbb{R}^n - \{0\})$, k an integer $> n/2$. If

$$\left| \left[\frac{\partial}{\partial x} \right]^\alpha m(x) \right| \leq C \, | x |^{-|\alpha|}, \quad | \alpha | \leq k, \; x \neq 0$$

for a constant C. Then $m(x)$ is an L^p-multiplier for all $p \in (1,\infty)$. □

From the above theorem immediately follows

Theorem I.3.10. For $p \in (1,\infty)$ and r a positive integer we have

$$W^{p,r} = \{ u \in L^p \mid \partial^\alpha u \in L^p, \; | \alpha | \leq r \}$$

with equivalent norm

$$\sum_{|\alpha| \leq r} \| \partial^\alpha u \|_p .$$

Proof. Let $u \in W^{p,r}$, then $F^{-1}(1+x^2)^{r/2} F u \in L^p$. For any α with $| \alpha | \leq r$, $m(x) = \frac{x^\alpha}{(1+x^2)^{r/2}}$ satisfies the conditions in the above theorem and therefore is an L^p-multiplier. So the identity

$$F^{-1} x^\alpha F u = F^{-1} m(x) F \, F^{-1}(1+x^2)^{r/2} F u$$

implies that $F^{-1} x^\alpha F u \in L^p$, i.e., $\partial^\alpha u \in L^p$.

Conversely, assume that $\partial^\alpha u \in L^p$ for all α such that $| \alpha | \leq r$. Obviously this is equivalent with the fact that $F^{-1} x^\alpha f u \in L^p$ for all α such that $| \alpha | \leq r$. If $r = 2s$ is even, then $(1+x^2)^{r/2} = (1+x^2)^s$. It is then readily follows that $F^{-1}(1+x^2)^{r/2} F u \in L^p$. If $v = 2s+1$ is odd, then $(1+x^2)^{r/2} = (1+x^2)^s \, (1+x^2)^{1/2}$. For any $1 \leq k \leq n$ $F^{-1} x^\alpha F v_k \in L^p$ for all α such that $| \alpha | \leq r-1 = 2s$, where $v_k = F^{-1} x_k F u \in W^{p,2s}$.

Note that $(1+x^2)^{½} = \frac{1+x^2}{(1+x^2)^{½}} = \frac{1}{(1+x^2)^{½}} + \sum_{k=1}^{n} \frac{x_k}{(1+x^2)^{½}} x_k$. And each of the functions $m_0(x) = \frac{1}{(1+x^2)^{½}}$ and $m_k(x) = \frac{x_k}{(1+x^2)^{½}}$ is an L^p-multiplier by Theorem I.3.9. Thus the identity

$$F^{-1}(1+x^2)^{r/2} F u = \sum_{k=0}^{n} F^{-1} m_k F\, F^{-1}(1+x^2)^s F v_k$$

implies that $F^{-1}(1+x^2)^{r/2} F u \in L^p$.

The equivalence of the norms is seen readily in the above. □

Theorem I.3.10 is due to Calderon; see [St] Chapter V Section 3.3. Here we have given a shorter proof using directly Theorem I.3.9. The spaces $W^{p,\sigma}$ are hence called fractional Sobolev spaces.

For $a \in \mathbb{R}^n$ and A an invertible $n \times n$ real matrix we define the operators τ_a, θ_a and L_A as follows:

$$(\tau_a u)(x) = u(x-a)$$

$$(\theta_a u)(x) = e^{iax} u(x) \quad u \in L^p$$

$$(L_A u)(x) = u(A^{-T} x).$$

It is easy to see that they are all bounded operators on L^p. Moreover $\tau_a^* = \tau_{-a}$, $\theta_a^* = \theta_{-a}$, $L_A^* = |A|\, L_{A^{-1}}$, all as operators in $L^q (1 < p < \infty)$. They are related via the Fourier transform in the following way:

$$F \tau_a = \theta_{-a} F$$

$$F \theta_a = \tau_a F \qquad (\text{in } S)$$

$$F L_A = |A|\, L_{A^{-T}} F.$$

Note that since all the operators τ_{-a}, θ_{-a} and $|A|\, L_{A^{-1}}$ are continuous on S, Theorem I.2.11 ensures that the operators τ_a, θ_a, L_A extend continuously to S' and the above relations are valid on S'.

Proposition I.3.11. Let $1 \le p < \infty$ and $a \in \mathbb{R}^n$, A an invertible $n \times n$ matrix. Then all the operators τ_a, θ_a and L_A are continuous on each of the spaces $W^{p,\sigma}$ ($\sigma \in \mathbb{R}$).

Proof. We have the following identities:

$$F^{-1}[(1+x^2)^{\sigma/2} F(\tau_a u)]$$

$$= F^{-1}[(1+x^2)^{\sigma/2}\, \theta_{-a} F u]$$

$$= F^{-1}[\theta_{-a}\, (1+x^2)^{\sigma/2} F u]$$

$$= \tau_a F^{-1}[(1+x^2)^{\sigma/2} F u]\,;$$

$$F^{-1}[(1+x^2)^{\sigma/2} F(\theta_a u)]$$

$$= F^{-1}[(1+x^2)^{\sigma/2}\, \tau_a F u]$$

$$= F^{-1}\, \tau_a[\tau_{-a}(1+x^2)^{\sigma/2} \cdot F u]$$

$$= \theta_a F^{-1}[\tau_{-a}(1+x^2)^{\sigma/2} \cdot F u]$$

$$= \theta_a F^{-1} \frac{\tau_{-a}(1+x^2)^{\sigma/2}}{(1+x^2)^{\sigma/2}} F \cdot F^{-1}(1+x^2)^{\sigma/2} F u\,;$$

$$F^{-1}[(1+x^2)^{\sigma/2} F L_A u]$$

$$= F^{-1}[(1+x^2)^{\sigma/2}\, |A|\, L_{A^{-T}} F u]$$

$$= F^{-1} L_{A^{-T}}\, [\,|A|\, (L_{A^T}(1+x^2)^{\sigma/2}) \cdot F u]$$

$$= \frac{1}{|A|} L_A F^{-1}\, [\,|A|\, (L_{A^T}(1+x^2)^{\sigma/2}) \cdot F u]$$

$$= L_A F^{-1}\, \{[L_{A^T}(1+x^2)^{\sigma/2}]\, (1+x^2)^{-\sigma/2} F F^{-1}(1+x^2)^{\sigma/2} F u\}.$$

Since the functions $[\tau_{-a}(1+x^2)^{\sigma/2}]\,(1+x^2)^{-\sigma/2}$ and $[L_{A^T}(1+x^2)^{\sigma/2}]\,(1+x^2)^{-\sigma/2}$ satisfy the conditions in the above Theorem I.3.9, they are L_p-multipliers. Then the desired conclusions follow readily. □

Proposition I.3.12. Let $m(x)$ be a function on $\mathbb{R}^n$ with polynomial groth, i.e., it is infinitely differentiable, and for any multi-index α there exist $M_\alpha > 0$ and $r_\alpha > 0$ such that $|\partial^\alpha m(x)| \le M_r(1+x^2)^{r_\alpha}$ for all $x \in \mathbb{R}^n$. Then for $\sigma,\tau \in \mathbb{R}$, the convolution operator $T_m u = F^{-1} m(x) F u$ is continuous from $W^{p,\tau}$ to $W^{p,\sigma}$ if $(1+x^2)^{(\sigma-\tau)/2}\, m(x)$ is an L_p multiplier.

Proof. The desired result follows directly from the assumptions and the identity

$$F^{-1}(1+x^2)^{\sigma/2} F \cdot F^{-1} m(x) F u$$

$$= F^{-1}(1+x^2)^{(\sigma-\tau)/2} m(x)\, F F^{-1}(1+x^2)^{\tau/2} F u.$$

□

Corollary I.3.13.

(i) A differential operator with constant coefficients of order r is continuous from $W^{p,\tau}$ to $W^{p,\tau-r}$ for each $\tau \in \mathbb{R}$.

(ii) The convolution operator $T_{t,\nu} = F^{-1}\, e^{-t(x^2)^\nu}\, F\, u$ $(t, \nu > 0)$ is continuous from $W^{p,-\infty}$ to $W^{p,\infty}$.

Proof. In each of the two cases $(1+x^2)^{(\sigma-\tau)/2}\, m(x)$ satisfies the conditions in Theorem I.3.9 (in case (i) $\sigma = \tau - r$, and in case (ii) for any $\tau, \sigma \in \mathbb{R}$). □

Corollary I.3.13. (ii), in particular, demonstrates the smoothing effect of the solution operator of the heat-diffusion equation $\dfrac{\partial u}{\partial t} = -(-\Delta)^\nu\, u$.

Example V. Let $X = L^2((0,2\pi)^n)$, the space of L^2-functions $u(x)$ on the cube $[0,2\pi]^n$, with the usual norm $\|u\| = (\int_{[0,2\pi]^n} |u(x)|^2\, dx)^{1/2}$. As is well known the set of functions $\{e_l \mid e_l = (2\pi)^{-n/2}\, e^{il\cdot x},\ l = (l_1,\dots,l_n) \in \mathbb{N}_0^n\}$ is an orthonormal basis in $L^2((0,2\pi)^n)$. Thus, the space L^2 is identified with the space l^2 isometrically via the mapping: $L^2 \ni u \mapsto ((u,e_l))_l \in l^2$. Note that $\Delta(e_l) = -|l|^2\, e_l$ for $l \in \mathbb{N}_0^n$, where Δ is the Laplacian. The operator $B_0 = I - \Delta$ defined on the span of $\{e_l \mid_l \in \mathbb{N}_0^n\}$ is closable in L^2 and its closure B is the operator:

$$B\,u = \sum_{l \in \mathbb{N}_0^n} (1+|l|^2)\, c_l\, e_l, \quad u = \sum_{l \in \mathbb{N}_0^n} c_l\, e_l$$

$$D(B) = \{u \in L^2 \mid \sum_{l \in \mathbb{N}_0^n} (1+|l|^2)^2\, |c_l|^2 < \infty\}.$$

The operator B is a positive self-adjoint operator, in particular, of type $P(0,1)$ and $0 \in \rho(B)$. So, as has been derived in Example II above, $B^{-\sigma} u = \sum_{l \in \mathbb{N}_0^n} (1+|l|^2)^{-\sigma}\, (u,e_l)\, e_l$ for $\sigma > 0$ and $u \in L^2$, and

$$X_B^{\sigma/2} = W_{\text{per}}^{\sigma} = \{u \mid \sum_{l \in \mathbb{N}_0^n} (1+|l|^2)^{\sigma}\, |(u,e_l)|^2 \equiv \|u\|_\sigma < \infty\}.$$

For $-\sigma < 0$ the spaces can be identified with spaces of formed Fourier series

$$X_B^{-\sigma/2} = W_{\text{per}}^{-\sigma} = \{u = \sum_{l \in \mathbb{N}_0^n} c_l\, e_l \mid \sum_{l \in \mathbb{N}_0^n} (1+|l|^2)^{-\sigma}\, |c_l|^2 \equiv \|u\|_{-\sigma} < \infty\}.$$

The isometric mapping $B^{-\sigma/2} : W_{\text{per}}^{-\sigma} \to L^2$ is given by $B^{-\sigma/2} u = \sum_{l \in \mathbb{N}_0^n} (1+|l|^2)^{-\sigma/2}\, c_l\, e_l$. For $\sigma \in [-\infty,\infty)$ we form the inductive limit $X_B^{\sigma/2+} = W_{\text{per}}^{\sigma+} = \bigcup_{\tau > \sigma} W_{\text{per}}^{\tau}$, and for $\sigma \in (-\infty,\infty]$ the projective limit $X_B^{\sigma/2-} = W_{\text{per}}^{\sigma-} = \bigcap_{\tau < \sigma} W_{\text{per}}^{\tau}$. Theorems I.2.8 and I.2.10 then imply that $(W_{\text{per}}^{\sigma})^* = W_{\text{per}}^{-\sigma}$,

$(W_{per}^{-\sigma})^* = W_{per}^{\sigma}$ ($\sigma \geq 0$) and similarly for the inductive and projective limits, under the duality pairing $<u, v> = (B^{\sigma/2} u, B^{-\sigma/2} v) = \sum_{l \in \mathbb{N}_0^n} c_l \bar{d}_l$ for $u = \Sigma c_l e_l$ and $v = \Sigma d_l e_l$. It is easily seen that each of the operators $B^{-\sigma} : L^2 \to L^2$ ($\sigma > 0$) is compact. Hence W_{per}^{σ} is embedded in W_{per}^{τ} compactly if $\sigma > \tau$ and all the inductive and projective limit spaces are Montel.

For $k \in \mathbb{N}_0$ let $c_{per}^k \equiv c_{per}^k((0,2\pi)^n)$ be the Banach space of 2π-periodic c^k-functions on $\mathbb{R}^n$ with norm

$$| u |_k = \max_{|l| \leq k} \max_{x \in [0,2\pi]^n} | D^l u |.$$

We have

Proposition I.3.14. (Sobolev) $W_{per}^{\sigma} \hookrightarrow C_{per}^k$ if $\sigma > n/2 + k$.

Proof. Let $u = \sum_{l \in \mathbb{N}_0^n} c_l e_l \in W_{per}^{\sigma}$ and set $u_j = \sum_{\substack{l \in \mathbb{N}_0^n \\ |l| \leq j}} c_l e_l$ ($j \in \mathbb{N}_0$). For any $\alpha \in \mathbb{N}_0^n$ we have

$$D^{\alpha} u_j = \sum_{\substack{l \in \mathbb{N}_0^n \\ |l| \leq j}} l^{\alpha} c_l e_l. \tag{60}$$

From the inequality

$$\Big(\sum_{l \in \mathbb{N}_0^n} l^{\alpha} | c_l |\Big)^2 \leq \sum_{l \in \mathbb{N}_0^n} (1 + |l|^2)^{\sigma} | c_l |^2 \cdot \sum_{l \in \mathbb{N}_0^n} [l^{\alpha}(1 + |l|^2)^{-\sigma/2}]^2$$

and the assumption that $\sigma > n/2 + k$ it follows that for each α with $| \alpha | \leq k$ the series in (60) converges uniformly on the cube $[0, 2\pi]^n$. Hence $D^{\alpha} u \in C_{per}^k$ and

$$| u |_k \leq \Big(\sum_{|l| \in \mathbb{N}_0^n} (1 + |l|^2)^{(k-\sigma)/2}\Big)^{1/2} \| u \|_{\sigma}.$$

□

Proposition I.3.15. Let $C_{per}^{\infty} = \bigcap_{k \in \mathbb{N}_0} C_{per}^k$ be the projective limit of the sequence of Banach spaces $(C_{per}^k)_{k \in \mathbb{N}_0}$. Then $W_{per}^{\infty} = C_{per}^{\infty}$ topologically.

Proof. The above proposition implies that $W_{per}^{\sigma} \hookrightarrow C_{per}^k$ if $\sigma > n/2 + k$. Let us show the converse. Given $\sigma \in \mathbb{N}$. Let $u \in C_{per}^{\infty}$. Then, as is well known in the theory of Fourier series

$$u = \sum_{l \in \mathbb{N}_0^n} c_l e_l$$

$$D^\alpha u = \sum_{l \in \mathbb{N}_0^n} l^\alpha c_l e_l, \quad |\alpha| \leq \sigma$$

where the convergence of all the summations is uniform with respect to the cube $[0,2\pi]^n$, and hence in the norm of L^2. Thus

$$\left(\sum_{l \in \mathbb{N}_0^n} l^{2\alpha} |c_l|^2\right)^{1/2} = \|D^\alpha u\| \leq M \, |u|_\sigma, \quad |\alpha| \leq \sigma$$

from which readily follows that $u \in W^\sigma_{\text{per}}$ and

$$\|u\|_\sigma \leq M' \, |u|_\sigma \tag{61}$$

where M and M' are suitable absolute constant. Since C^∞_{per} is dense in C^σ_{per}, the above holds for all u in C^σ_{per}. So $C^\sigma_{\text{per}} \hookrightarrow W^\sigma_{\text{per}}$. Corollary I.1.9 (ii) then implies that $C^\infty_{\text{per}} = W^\infty_{\text{per}}$ topologically. □

Corollary I.3.16. $W^{-\infty}_{\text{per}}$ is the space of periodic distributions, i.e., $W^{-\infty}_{\text{per}} = (C^\infty_{\text{per}})^*$.

Proof. $(C^\infty_{\text{per}})^* = (W^\infty_{\text{per}})^* = W^{-\infty}_{\text{per}}$. □

Proposition I.3.17.

(i) For each $1 \leq k \leq n$, the differential operator $D_k : C^\infty_{\text{per}} \to C^\infty_{\text{per}}$ extends uniquely to a continuous operator from W^σ_{per} to $W^{\sigma-1}_{\text{per}}$ $(\sigma \in \mathbb{R})$.

(ii) Let $\phi(x) \in C^\infty_{\text{per}}$. Then the multiplication operator $M_\phi : C^\infty_{\text{per}} \in u(x) \mapsto \phi(x)\, u(x) \in C^\infty_{\text{per}}$ extends uniquely to a continuous operator on W^σ_{per} $(\sigma \in \mathbb{R})$.

Proof. (i) If $u = \sum_{l \in \mathbb{N}_0^n} c_l e_l \in C^\infty_{\text{per}}$ then $D_k u = \sum_{l \in \mathbb{N}_0^n} c_l l_k e_l \in C^\infty_{\text{per}}$ and for $\sigma \in \mathbb{R}$

$$\|D_k u\|_{\sigma-1} = \left(\sum_{l \in \mathbb{N}_0^n} (1+|l|^2)^{(\sigma-1)} l_k^2 \, |c_l|^2\right)^{1/2}$$

$$\leq \left(\sum_{l \in \mathbb{N}_0^n} (1+|l|^2)^\sigma \, |c_l|^2\right)^{1/2} = \|u\|_\sigma.$$

Since C^∞_{per} is dense in W^σ_{per}, D_k extends uniquely to a continuous operator from W^σ_{per} to $W^{\sigma-1}_{\text{per}}$.

(ii) First we assume that $\sigma \geq 0$. Then Leibnitz's rule and the inequalities

$$\|D^\alpha u\|_0 \leq \|u\|_{|\alpha|}$$

and

$$\| u \|_\sigma \le M_\sigma \sum_{|\alpha| \le \sigma} \| D^\alpha u \|$$

imply that

$$\| M_\phi u \|_\sigma \le \tilde{M}_0 \| u \|_\sigma \quad (u \in C^\infty_{\text{per}})$$

where M and $\tilde{M}_\sigma$ are constants. Therefore M_ϕ extends uniquely to a continuous operator on W^σ_{per}.

Since obviously $(M_\phi)^* = M_{\bar{\phi}}$, Theorem I.2.11 (i) leads to the wanted result for $\sigma < 0$. □

Most of the results above are quite standard (see e.g., [BJS], Chapter 3). In the present treatment however the various results follow as special cases.

Example VI. The Range of the propagation operator of the Ordinary Heat-Diffusion Equation. Let $X = L^p(\mathbb{R})$ $(1 \le p < \infty)$ with the usual norm $\| u \|_p = (\int_{\mathbb{R}} | u(x) |^p dx)^{1/p}$ for $u \in L^p(\mathbb{R})$. Consider the heat-diffusion equation in the space of tempered distributions S'

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{62}$$

Here the differentiation with respect to x is in the sense of tempered distributions, while that with respect to t is of the topology of S'. Since the Fourier transform F and its inverse are continuous on S', the above equation (62) is equivalent to

$$\frac{\partial (F u)}{\partial t} = -x^2 F u. \tag{63}$$

From this easily follows the solution of the initial value problem of (62):

$$e^{t \frac{\partial^2}{\partial x^2}} u = F^{-1} e^{-tx^2} F u = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} * u. \tag{64}$$

If $u \in L^p$, then $e^{t \frac{\partial^2}{\partial x^2}} u = \mathrm{v}(x)$ extends to an entire function $\mathrm{v}(\zeta)$ $(\zeta = x + iy)$ which is given by

$$\mathrm{v}(\zeta) = \mathrm{v}(x+iy) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x+iy)^2}{4t}} *_{(x)} u(x)$$

$$= \frac{1}{2\sqrt{\pi t}} e^{\frac{y^2}{4t}} \int_{\mathbb{R}} e^{-\frac{(x-\zeta)^2 + 2i(x-\zeta)y}{4t}} u(\zeta) \, d\zeta.$$

Furthermore, Young's inequality leads to that

$$\| \mathrm{v}(x+iy) \|_{p,x} \le e^{y^2/4t} \| u \|_p. \tag{65}$$

Indeed

$$\frac{1}{2\sqrt{\pi t}}\int_{I\!R} |e^{-\frac{(x-\zeta)^2+2i(x-\zeta)y}{4t}}|\,d\zeta=\frac{1}{2\sqrt{\pi t}}\int_{I\!R} e^{-\zeta^2/4t}\,d\zeta=1.$$

Put $B=(e^{\frac{\partial^2}{\partial x^2}})^{-1}$ with $D(B)=R(e^{\frac{\partial^2}{\partial x^2}}\restriction_{L^p})$. It is not difficult to show that the operator B is of type $P(0,M)$ for some $M\geq 1$, and for $t>0$, $B^t=(e^{t\frac{\partial^2}{\partial x^2}}\restriction_{L^p})^{-1}$. Therefore, by definition, $X_B^t=R(e^{t\frac{\partial^2}{\partial x^2}}\restriction_{L^p})$ for $t>0$. Thus, we need to characterize $R(e^{t\frac{\partial^2}{\partial x^2}}\restriction_{L^p})$ explicitly.

In view of the above consideration let us assume that $v(\zeta)=v(x+iy)$ be an entire function such that

$$\| v(x+iy)\|_{p,x}\leq M\,e^{sy^2},\quad x,y\in I\!R \tag{66}$$

where M and s are nonnegative constants. We intend to find some $t>0$ and $u\in L^p$ such that $v(x)=e^{t\frac{\partial^2}{\partial x^2}}u(x)$. A heuristic consideration suggests the following candidate for u:

$$u(x)=\frac{1}{2\sqrt{\pi t}}\int_{I\!R} e^{-\frac{1}{4t}(\eta+ix)^2}\,v(i\eta)\,d\eta$$

$$=\frac{1}{2\sqrt{\pi t}\,i}\int_{-\infty i}^{+\infty i} e^{\frac{1}{4t}(x-\zeta)^2}\,v(\zeta)\,d\zeta. \tag{67}$$

Of course, we must actually prove the convergence of the above integrals and $u\in L^p$ and $e^{t\frac{\partial^2}{\partial x^2}}u(x)=v(x)$.

Lemma I.3.18. If $v(\zeta)$ is an entire function satisfying the condition (66) above, then, for any $s'>s$ there exists α depending only on p,s and s' such that

$$\sup_{x+iy\in\mathbb{C}} |v(x+iy)|\leq \alpha M\,e^{s'y^2}. \tag{68}$$

Proof. By the mean value theorem we have, $(R>0)$,

$$v(x+iy)=\frac{1}{\pi R^2}\int_{|\zeta+i\eta|<R} v[(x+\zeta)+i(y+\eta)]\,d\zeta\,d\eta.$$

An application of Hölder's inequality leads to the estimate

$$|v(x+iy)|\leq\frac{1}{\pi R^2}(\pi R^2)^{1-\frac{1}{p}}\,(\int_{|\zeta+i\eta|<R} |v[(x+\zeta)+i(y+\eta)]|^p\,d\zeta\,d\eta)^{1/p}$$

$$\leq \frac{1}{\pi R^2} (\pi R^2)^{\frac{1-1}{p}} \Big(\int_{|\zeta+i\eta|<R} |\, v[(x+\zeta)+i(y+\eta)]\,|^p \, d\zeta d\eta \Big)^{1/p}$$

$$\leq (\pi R^2)^{-\frac{1}{p}} \Big(2R \sup_{|\zeta-y|<R} \int_{-\infty}^{+\infty} |\, v(x+i\eta)\,|^p \, dx\Big)^{1/p}$$

$$\leq \left[\frac{2}{\pi R}\right]^{1/p} M\, e^{s(|y|+R)^2}.$$

From this estimate follows the wanted assertion. □

According to the above lemma we are now certain that for an entire function v satisfying (66) for some $s > 0$ the integrals in (67) converge as long as $0 < t < \frac{1}{4s}$. Furthermore, the Cauchy integral theorem enables us to transfer the path of integration so that ($c \in \mathbb{R}$)

$$u(x) = \frac{1}{2\sqrt{\pi t}\, i} \int_{c-\infty i}^{c+\infty i} e^{\frac{1}{4t}(x-\zeta)^2}\, v(\zeta)\, d\zeta. \tag{69}$$

In particular, for $c = x$ we have

$$u(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{4t}\eta^2}\, v(x+i\eta)\, d\eta. \tag{70}$$

Lemma I.3.19. As in the above lemma let $v(\zeta)$ be an entire function satisfying the condition (66). Then for $t < \frac{1}{4s}$ the function $u(x)$ in (67) is well defined and is equivalently given by equations (69) or (70). Moreover, $u(x) \in L^p$, and for any $a \in (s, \frac{1}{4t})$ holds that

$$\| u \|_p \leq \frac{M}{2\sqrt{\pi t}} \| e^{-(\frac{1}{4t}-a)\eta^2} \|_q \, \| e^{-(a-s)\eta^2} \|_p. \tag{71}$$

Proof. As already observed above the function $u(x)$ is well defined and is equivalently given by (67), (69) and (70). Fix $a \in (s, \frac{1}{4t})$. By Hölder's inequality and Fubini's theorem we have

$$(2\sqrt{\pi t})^p \int_{-\infty}^{+\infty} |\, u(x)\,|^p \, dx$$

$$= \int_{-\infty}^{+\infty} \Big|\int_{-\infty}^{+\infty} e^{-\frac{1}{4t}\eta^2}\, v(x+i\eta)\, d\eta\, \Big|^p \, dx$$

$$\leq \int_{-\infty}^{+\infty} dx \Big(\int_{-\infty}^{+\infty} e^{-q(\frac{1}{4t}-a)\eta^2}\, d\eta\Big)^{p/q} \Big(\int_{-\infty}^{+\infty} e^{-pa\eta^2}\, |\, v(x+i\eta)\,|^p \, d\eta\Big)$$

$$= (\| e^{-(\frac{1}{4t}-a)\eta^2} \|_q)^p \int_{-\infty}^{+\infty} e^{-pa\eta^2} d\eta \int_{-\infty}^{+\infty} | v(x+i\eta) |^p dx$$

$$\leq (\| e^{-(\frac{1}{4t}-a)\eta^2} \|_q)^p M^p \int_{-\infty}^{+\infty} e^{-p(a-s)\eta^2} d\eta$$

$$= (\| e^{-(\frac{1}{4t}-a)\eta^2} \|_q)^p (\| e^{-(a-s)\eta^2} \|_p)^p M^p$$

from which readily follows (71). □

Lemma I.3.20. Assume that all the conditions in the above Lemma I.3.19 are satisfied. Then $e^{t\frac{\partial^2}{\partial x^2}} u(x) = v(x)$.

Proof. Put $w(x) = e^{t\frac{\partial^2}{\partial x^2}} u(x)$. Then, by (64) and (76)

$$w(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} * u(x)$$

$$= \frac{1}{4\pi t} \iint_{\mathbb{R}^2} e^{-\frac{(x-\zeta)^2}{4t} - \frac{1}{4t}\eta^2} v(\zeta + i\eta) \, d\zeta \, d\eta$$

$$= \frac{1}{4\pi t} \int_0^{\infty} e^{-\frac{r^2}{4t}} r dr \int_0^{2\pi} v(x + re^{i\theta}) \, d\theta$$

$$= \frac{1}{4\pi t} \int_0^{\infty} 2\pi v(x) r e^{-\frac{r^2}{4t}} dr$$

$$= v(x) \quad (x \in \mathbb{R}).$$

□

Definition I.3.21. Given $p \geq 1$ and $s > 0$. Let $A^{p,s}$ denote the normed space of entire function $v(\zeta)$ such that

$$| v |_{p,s} = \sup_{y \in \mathbb{R}} e^{-sy^2} \left(\int_{\mathbb{R}} | v(x+iy) |^p dx \right)^{1/p} < \infty. \tag{72}$$

□

Proposition I.3.22. For each $p \geq 2$ and $s > 0$, the space $A^{p,s}$ is a Banach space.

Proof. Let (v_n) be a Cauchy sequence in $A^{p,s}$. Then for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$| \mathrm{v}_n - \mathrm{v}_m |_{p,s} = \sup_{y \in \mathbb{R}} e^{-sy^2} \Big(\int_{\mathbb{R}} | \mathrm{v}_n(x+iy) - \mathrm{v}_m(x+iy) |^p \, dx \Big)^{1/p} \le \varepsilon \text{ for } n,m \ge N. \quad (73)$$

Then for any given $s' < s$ Lemma I.3.18 implies that

$$\sup_{x+iy \in \mathbb{C}} | \mathrm{v}_n(x+iy) - \mathrm{v}_m(x+iy) | \le \alpha \, \varepsilon \, e^{s'y^2}. \quad (74)$$

This estimate shows that the sequence of functions $\{\mathrm{v}_m(x+iy)\}$ converges to an entire function $\bar{\mathrm{v}}(x+iy)$ uniformly on each strip $\{x+iy \mid |y| \le b\}$ $(b>0)$. On fixing n and letting $m \to \infty$ in (73), in view of Lebesque's dominance convergence theorem we conclude that $\bar{\mathrm{v}} \in A^{p,s}$ and $\mathrm{v}_n \to \bar{\mathrm{v}}$ in $A^{p,s}$. Thus the space $A^{p,s}$ is complete. □

Definition I.3.23. For $s \in (0,\infty]$ let $A^{p,s+} = \bigcup_{\sigma<s} A^{p,\sigma}$ be the inductive limit of the family of Banach spaces $\{A^{p,\sigma} \mid \sigma<s\}$. For $s \in [0,\infty)$, let $A^{p,s} = \bigcap_{\sigma>s} A^{p,\sigma}$ be the projective limit of the family of Banach spaces $\{A^{p,\sigma} \mid \sigma > s\}$. □

In summary of the above discussion we obtain

Theorem I.3.24. For $t \in [0,\infty)$

$$(L^p)_B^{t+} = A^{p,(\frac{1}{4t})+} \quad \text{topologically.}$$

For $t \in (0,\infty]$

$$(L^p)_B^{t-} = A^{p,(\frac{1}{4t})-} \quad \text{topologically.}$$

□

In the case $p=2$ we can characterize each of the Hilbert spaces $(L^2)_B^t$ exactly. In fact, for $u \in L^2$ and $\mathrm{v} = e^{t\frac{\partial^2}{\partial x^2}} u$, using Plancherel's theorem we have

$$\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} | \mathrm{v}(x+iy) |^2 \, e^{-\frac{1}{2t}y^2} \, dx \, dy$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}y^2} dy \int_{\mathbb{R}} dx \, \Big| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixk} \, e^{-ky-tk^2} \, (F u) \, dk \Big|^2$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{I\!R} e^{-2tk^2} \mid Fu \mid^2 dk \int_{I\!R} e^{-\frac{1}{2t}y^2 - 2ky} dy$$

$$= \int_{I\!R} \mid Fu \mid^2 dk$$

$$= \int_{I\!R} \mid u \mid^2 dx. \tag{75}$$

The above identity and a theorem of Zalik and Saad [Z-S] yield

Theorem I.3.26. $(L^2)_B^t$ is isometrically equivalent to the Hilbert space of entire functions v such that

$$\| v \|^2 = \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} \mid v(x+iy) \mid^2 e^{-\frac{y^2}{2t}} dx\, dy < \infty.$$

□

Example VII. If we take $X = L^2(I\!R^n)$ and $B = x^2 - \Delta$ then $X_B^{\sigma/2} = (LW)^{2,\sigma}$. These spaces can be characterized by the asymptotic behaviour of the expansion coefficients of a function u in L^2 in terms of the basis consisting of Hermite functions. Moreover, for σ a nonnegative integer

$$(LW)^{2,\sigma} = \{ u \in L^2(I\!R^n) \mid x^\alpha \partial^\beta u \in L^2(I\!R^n),\ \forall \mid\alpha\mid \le \sigma,\ \mid\beta\mid \le \sigma \}.$$

Of course we have the inductive limits $(LW)^{2,\sigma+}$ and projective limits $(LW)^{2,\sigma-}$. And $(LW)^{2,\infty} = S$, the Schwartz test function space, and $(LW)^{2,-\infty} = S'$, the space of tempered distributions. For details we refer to [R-S], Vol. 1, p. 141.

If we take $X = L^2(I\!R)$ and $B = e^{x^2 - \frac{d^2}{dx^2}}$, then $X_B^{\sigma+} = S_{1/2}^{1/2}$, one of the test spaces introduced by Gelfand and Shilov, cf. [Zh]. Moreover $X_B^\sigma = \{ u \mid u$ extends to an entire analytic function $u(x+iy)$ such that $\| u \|_\sigma^2 = \iint_{I\!R^2} \mid u(x+iy) \mid^2 \exp[\tanh(\frac{\sigma}{2}) x^2 - \frac{1}{\tanh(\frac{\sigma}{2})} y^2]\, dx\, dy < \infty \}$; cf. [E-M].

II. Regularity and Extendibility of Solutions to Autonomous Evolution Equations

Recall that in Chapter I we constructed a scale of Banach spaces X_B^ν ($\nu \in \mathbb{R}$) for a fixed positive operator B in a Banach space $(X, \|\cdot\|)$. In the first section of this chapter we assume that an operator $A : D(A) \subset X$ generates a c_0 semigroup e^{-tA} on X and we discuss its regularity and extendibility with respect to the spaces X_B^ν ($\nu \in \mathbb{R}$). More precisely, both necessary-sufficient conditions and sufficient conditions involving the operators A and B^ν or A^* and $(B^*)^\nu$ are given such that e^{-tA} restricts to a c_0 semigroup on X_B^ν ($\nu > 0$) or e^{-tA} extends to a c_0 semigroup on $X_B^{-\nu}$ ($-\nu < 0$).

In the first part of the second section two criteria are given for an infinite complex matrix (a_{jk}) to generate a c_0 semigroup on l^2. One of these criteria deals with diagonal-dominant matrices in some sense using the well known perturbation theorem of Rellich-Kato-Gustafson-Chernoff. The other aims at skew-symmetric matrices; here essential is the technique of the auxiliary operator of De Graaf. In the final part of the section the results in the previous section are applied so that these c_0 semigroups on l^2 are regular or extendible with respect to the scale of weighted l^2 spaces $l^{2,\nu}$ ($\nu \in \mathbb{R}$). Several concrete examples are given to illustrate the general results.

In the last section the theory presented in the first section is applied to the second order partial differential operator

$$Au = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

in $L^2(\mathbb{R}^n)$ and to the operator $B = I - \Delta$ or $x^2 - \Delta$. So regularity and extendibility of the solutions of the corresponding evolution equation $\frac{\partial u}{\partial t} = -Au$ with respect to the Sobolev spaces $H^\nu(\mathbb{R}^n)$ and modified Sobolev spaces $(LW)^{2,\nu}(\mathbb{R}^n)$ are obtained under certain growth conditions of the coefficients. Some estimates involving commutants of two differential operators play an important role. In consistence with this the technique of pseudo-differential operators is also applied.

II.1. General Theory

Theorem II.1.1. Let e^{-tA} be a c_0 semigroup of operators generated by the operator $-A$ in X. Then, the following two properties are equivalent ($\nu > 0$):

(i) $e^{-tA} X_B^\nu \subset X_B^\nu$ for all $t \geq 0$ and $e^{-tA} \restriction_{X_B^\nu}$ is a c_0 semigroup on X_B^ν.

(ii) The operator $B^\nu A B^{-\nu} \equiv A_\nu$ is a generator of a c_0 semigroup e^{-tA_ν} in X.

Proof. (ii) $\Rightarrow$ (i). Let $u \in X_B^\nu$ and set $v = B^\nu u \in X$. Since A_ν generates a c_0 semigroup in X, there must exist a sequence of points $v_n \in D(A_\nu)$ such that $\| v_n - v \| \to 0$ and

$$\frac{d(e^{-tA_\nu}\,\mathrm{v}_\nu)}{dt} = A_\nu(e^{-tA_\nu}\,\mathrm{v}_n),\ \ t \geq 0. \tag{2}$$

An application of $B^{-\nu}$ on both sides yields that

$$\frac{d}{dt}\,[B^{-\nu}(e^{-tA_\nu}\,\mathrm{v}_n)] = A[B^{-\nu}(e^{-tA_\nu}\,\mathrm{v}_n)]. \tag{3}$$

Obviously $B^{-\nu}(e^{-tA_\nu}\,\mathrm{v}_n)\,|_{t=0} = B^{-\nu}\,\mathrm{v}_n \equiv u_n$. This, together with (3) implies that $B^{-\nu}[e^{-tA_\nu}\,\mathrm{v}_n] = e^{-tA}\,u_n$. Here we have used the uniqueness of the Cauchy problem for the equation (1). Further from the inequalities

$$\begin{aligned} &\| e^{-tA}\,u - B^{-\nu}\,e^{-tA_\nu}\,\mathrm{v} \| \\ &\leq \| e^{-tA} \|\ \| u - u_n \| + \| e^{-tA}\,u_n - B^{-\nu}\,e^{-tA_\nu}\,\mathrm{v}_n \| + \| B^{-\nu}\,e^{-tA_\nu}\,(\mathrm{v}_n - \mathrm{v}) \| \\ &\leq \| e^{-tA} \|\ \| u - u_n \| + \| B^{-\nu} \|\ \| e^{-tA_\nu} \|\ \| \mathrm{v}_n - \mathrm{v} \| \to 0 \ \text{ as } \ n \to \infty \end{aligned}$$

it follows that

$$e^{-tA}\,u = B^{-\nu}\,e^{-tA_\nu}\,B^{\nu}\,u \tag{4}$$

which in turn implies (i).

(i) $\Rightarrow$ (ii) If property (i) holds, then the family $S(t) = B^{\nu}\,e^{-tA}\,B^{-\nu}$ is a c_0 semigroup on X. It is easy to see that the generator of $S(t)$ is precisely $B^{\nu}\,A\,B^{-\nu}$. □

Thus we find essentially the same result as was obtained quite a time ago in [Ka1]. See also [Pa] and [Ta]. A comparison of the two proofs shows however that they are very different from each other.

The above theorem deals with the regularity of the semigroup e^{-tA} with respect to the space X_B^{ν}. The next one is about the extendibility of the semigroup e^{-tA} to the spaces $X_{\bar{B}}^{-\nu}$ ($\nu > 0$).

Theorem II.1.2. Assume that e^{-tA} is a c_0 semigroup of linear operators with generator $-A$ in a reflexive Banach space $(X, \|\cdot\|)$. Let $\nu > 0$. Then the following two statements are mutually equivalent:

(i) e^{-tA} extends (uniquely) to a c_0 semigroup of linear operators on $X_{\bar{B}}^{-\nu}$.

(ii) The operator $(B^*)^{\nu}\,A^*(B^*)^{-\nu}$ generates a c_0 semigroup of linear operators on X^*.

Proof. (i) $\Rightarrow$ (ii). Put $e^{-tA} = T(t)$ and let the extension of $T(t)$ onto the space $X_{\bar{B}}^{-\nu}$ be $\bar{T}_\nu(t)$ with generator $-\bar{A}_\nu$. According to the standard result on dual semigroups (conf. e.g. Theorem 0.8 or [Ta], Theorem 3.1.6), both the operators $-A^*$ and $-(\bar{A}_\nu)^*$ generate semigroups of operators on the spaces X^* and $(X^*)_{B^*}^{\nu}$, respectively. In fact, $e^{-tA^*} = (e^{-tA})^*$ and $e^{-t(\bar{A}_\nu)^*} = (e^{-\bar{A}_\nu})^*$. We want to show that $e^{-tA^*}\restriction_{(X^*)_{B^*}^{\nu}} = e^{-t(\bar{A}_\nu)^*}$. Let $<\cdot,\cdot>_{*,\nu}$ be the duality pairing between $(X^*)_{B^*}^{\nu}$ and

$X_B^{-\nu}$, i.e. $<u,f>_{*,\nu} = (B^{-\nu} u, (B^*)^\nu f)$ for $u \in X_B^{-\nu}$ and $f \in (X^*)_{B^*}^\nu$, where $(\cdot\,,\cdot)$ is the duality pairing between X and X^*. Assume that $u \in D(A)$ and $f \in D((\bar{A}_\nu)^*)$. Then we have

$$\frac{d}{dt}\,[e^{-t(\bar{A}_\nu)^*} f] = -(\bar{A}_\nu)^*\,[e^{-t(\bar{A}_\nu)^*} f]. \tag{5}$$

Thus

$$<u,\ \frac{d}{dt}\,[e^{-t(\bar{A}_\nu)^*} f]>_{*,\nu} = <u,\ (\bar{A}_\nu)^*\,[e^{-t(\bar{A}_\nu)^*} f]>_{*,\nu} \tag{6}$$

or

$$(u,\ \frac{d}{dt}\,[e^{-t(\bar{A}_\nu)^*} f]) = <-\bar{A}_\nu u\,,\ e^{-t(\bar{A}_\nu)^*} f>_{*,\nu} \tag{7}$$

or

$$(u,\ \frac{d}{dt}\,[e^{-t(\bar{A}_\nu)^*} f]) = (-A\,u\,,\ e^{-t(\bar{A}_\nu)^*} f)_{*,\nu} \tag{8}$$

from which it follows that $e^{-t(\bar{A}_\nu)^*} f \in D(A^*)$ and that

$$(u,\ \frac{d}{dt}\,[e^{-t(\bar{A}_\nu)^*} f]) = (u,\ -A^*\,e^{-t(\bar{A}_\nu)^*} f). \tag{9}$$

Since $D(A)$ is dense we have

$$\frac{d}{dt}\,[e^{-t(\bar{A}_\nu)^*} f] = -A^*\,e^{-t(\bar{A}_\nu)^*} f. \tag{10}$$

The differentiation with respect to t in the above equation is originally in the norm of the space $(X^*)_{B^*}^\nu$, and hence also in the weaker norm of X^*. By the uniqueness of solutions we have finally arrived at $e^{-t(\bar{A}_\nu)^*} f = e^{-tA^*} f$ for all $f \in D((\bar{A}_\nu)^*)$. Since $D((\bar{A}_\nu)^*)$ is dense in $(X^*)_{B^*}^\nu$ it immediately follows that the above relation holds for all $f \in (X^*)_{B^*}^\nu$; in other words, we have finally proved

$$e^{-tA^*} \restriction_{(X^*)_{B^*}^\nu} = e^{-t(\bar{A}_\nu)^*}. \tag{11}$$

Then, from Theorem II.1.1 follows readily assertion (ii).

(ii) $\Rightarrow$ (i). Conversely, if the operator $(B^*)^\nu A^* (B^*)^{-\nu}$ generates a c_0 semigroup on X^*, then, according to Theorem II.1.1, $e^{-tA^*} \restriction_{(X^*)_{B^*}^\nu}$ is a c_0 semigroup on $(X^*)_{B^*}^\nu$ with generator $-\tilde{A}^*$ as the part of $-A^*$ in $(X^*)_{B^*}^\nu$. Of course, $-(\tilde{A}^*)^*$ generates a c_0 semigroup $e^{-t(\tilde{A}^*)^*}$ on $[(X^*)_{B^*}^n]^* = X_B^{-\nu}$, i.e. $(e^{-t\tilde{A}^*})usp*$. It remains to prove that

$$e^{-t(\tilde{A}^*)^*} \restriction_X = e^{-tA}. \tag{12}$$

Let $u \in D(A)$ and $f \in D(\tilde{A}^*)$. Then

$$\frac{d}{dt}(e^{-tA}u) = -A\, e^{-tA}\, u. \tag{13}$$

Further

$$(\frac{d}{dt}(e^{-tA}u), f) = (-A\, e^{-tA}\, u, f) \tag{14}$$

or

$$(\frac{d}{dt}(e^{-tA}u), f) = (e^{-tA}\, u, -\tilde{A}^* f) \tag{15}$$

or

$$< \frac{d}{dt}(e^{-tA}u), f >_{*,\nu} = < e^{-tA}\, u, -\tilde{A}^* f >_{*,\nu}. \tag{15}$$

The last equality shows that $e^{-tA}\, u \in D((\tilde{A}^*)^*)$ and

$$< \frac{d}{dt}(e^{-tA}u), f >_{*,\nu} = < -(\tilde{A}^*)^*\, e^{-tA}\, u, f >_{*,\nu}. \tag{18}$$

Since $D(\tilde{A}^*)$ is dense in $(X^*)^{\nu}_{B^*}$ it follows that

$$\frac{d}{dt}(e^{-tA}u) = -(\tilde{A}^*)^*\,(e^{-tA}\, u). \tag{19}$$

Differentiation to t in the last equation is originally in terms of the norm in X, and hence also in the weaker norm of $X_B^{-\nu}$. Therefore, from the uniqueness of the solutions follows that $e^{-tA}\, u = e^{-t(\tilde{A}^*)^*}\, u$. Since this is true for all $u \in D(A)$ which is dense in X, we finally have proved the equation (12). □

The above Theorems II.1.1 and II.1.2 give necessary and sufficient conditions for a c_0 semigroup e^{-tA} to be regular with respect to the space X_B^{ν} or to be extendible to the space $X_B^{-\nu}$, respectively. In the following we present sufficient conditions which are easier to check in applications.

Theorem II.1.3. Let $\nu > 0$ be fixed and a generator Q in X given. If there exists a core $\Delta \subset D(Q)$ of the generator Q such that

$$B^{-\nu}\Delta \subset D(A) \tag{20}$$

$$A\, B^{-\nu}\Delta \subset D(B^{\nu}) \tag{21}$$

$$(B^{\nu} A\, B^{-\nu} - Q)\restriction_{\Delta} \text{ quasi-accretive} \tag{21'}$$

and

$$\| (B^{\nu} A\, B^{-\nu} - Q)\, u \| \le M_{\nu} \| u \| + \| Q\, u \|, \quad u \in \Delta \tag{22}$$

where M_{ν} is a constant. Then, the above conditions (20), (21), (21') and (22) are actually valid for $D(Q)$ instead of Δ, and $B^{\nu} A\, B^{-\nu}$ generates a c_0 semigroup of operators on X.

Proof. Let $u \in D(Q)$. Then, there exists a sequence $\{u_k\} \subset \Delta$ such that $u_k \to u$ and $Q\,u_k \to Q\,u$. From the inequality (22) follows that

$$\| (B^{\nu} A B^{-\nu} - Q)(u_k - u_m) \| \le M_{\nu} \| u_k - u_m \| + \| Q\,u_k - Q\,u_m \| \tag{23}$$

which shows that $\{(B^{\nu} A B^{-\nu} - Q)\,u_k\}$ is a Cauchy sequence and hence convergent. Now we have

$$D(A) \ni B^{-\nu} u_k \to B^{-\nu} u \tag{24}$$

$$D(B^{\nu}) \ni A B^{-\nu} u_k = B^{-\nu} Q\,u_k + B^{-\nu}(B^{\nu} A B^{-\nu} - Q)\,u_k \text{ converges} \tag{25}$$

$$B^{\nu} A B^{-\nu} u_k = Q\,u_k + (B^{\nu} A B^{-\nu} - Q)\,u_k \text{ converges.} \tag{26}$$

Therefore, the closedness of A and B^{ν} implies that

$$B^{-\nu} u \in D(A) \tag{27}$$

$$A B^{-\nu} u_k \to A B^{-\nu} u \in D(B^{\nu}) \tag{28}$$

$$B^{\nu} A B^{-\nu} u_k \to B^{\nu} A B^{-\nu} u. \tag{29}$$

Further from (29) and

$$\| (B^{\nu} A B^{-\nu} - Q)\,u_k \| \le M_{\nu} \| u_k \| + \| Q\,u_k \|$$

it follows that

$$\| (B^{\nu} A B^{-\nu} - Q)\,u \| \le M_{\nu} \| u \| + \| Q\,u \|. \tag{30}$$

So conditions (20) - (22) are actually valid for $D(Q)$ instead of Δ.

Then, according to the standard result on the perturbation of infinitesimal generators (see, e.g., Theorem 0.9 or [Pa], Chapter 3, Sect. 3), the closure of the operator $B^{\nu} A B^{-\nu} = Q + (B^{\nu} A B^{-\nu} - Q)$ as defined on $D(Q)$ generates a c_0 semigroup. Our proof is complete if we can show the closedness of the operator $B^{\nu} A B^{-\nu}$. In fact, for a sequence $u_k \in D(B^{\nu} A B^{-\nu})$ i.e. $B^{-\nu} u_k \in D(A)$ and $A B^{-\nu} u_k \in D(B^{\nu})$ such that $u_k \to u$ and $B^{\nu} A B^{-\nu} u_k \to \mathrm{v}$, since $B^{-\nu}$ is continuous, we have $B^{-\nu} u_k \to B^{-\nu} u$ and $A B^{-\nu} u_k \to B^{-\nu} \mathrm{v}$. Therefore from the closedness of A follows that $B^{-\nu} u \in D(A)$ and $A B^{-\nu} u = B^{-\nu} \mathrm{v}$, i.e., $A B^{-\nu} u \in D(B^{\nu})$ and $B^{\nu} A B^{-\nu} u = \mathrm{v}$. □

Of course, if conditions (20) - (22) are satisfied for A^* instead of A and B^* instead of B, then $(B^*)^{\nu} A^* (B^*)^{-\nu}$ generates a c_0 semigroup on X^*. We emphasize that only having to check conditions (20) - (22) for a core instead of the whole domain $D(Q)$ greatly facilitates the applications to concrete problems. Recalling the structures of the spaces $X_B^{\nu\pm}$ the above Theorem II.1.1 - II.1.3 immediately give rise to the following two results on the regularity of a semigroup with respect to the regular spaces of inductive or projective type and the extendibility to hyper-spaces of inductive or projective type.

Theorem II.1.4.

(i) For some $\nu \geq 0$ suppose there exists a sequence $\{\nu_i\}$ such that $\nu_i > \nu$ and $\nu_i \to \nu$ and each $B^{\nu_i} A B^{-\nu_i}$ generates a c_0 semigroup on X (in particular if there exists a core Δ of some generator Q such that conditions (20) - (22) are valid for all the $\nu = \nu_i$), then $e^{-tA} X_B^{\nu+} \subset X_B^{\nu+}$ and $e^{-tA} \restriction X_B^{\nu+}$ is a continuous semigroup of operators on $X_B^{\nu+}$.

(ii) For ν fixed, $0 < \nu \leq \infty$ suppose there exists a sequence $\{\nu_i\}$ such that $\nu_i < \nu$ and $\nu_i \to \nu$ and each $B^{\nu_i} A B^{-\nu_i}$ generates a c_0 semigroup on X (in particular if there exists a core Δ such that conditions (20) - (22) are valid for all $\nu = \nu_i$). Then, $e^{-tA} X_B^{\nu-} \subset X_B^{\nu-}$ and $e^{-tA} \restriction X_B^{\nu-}$ is a continuous semigroup of operators on $X_B^{\nu-}$. □

Theorem II.1.5.

(i) For some ν, $0 \leq \nu < \infty$, if all the conditions in the above Theorem II.1.4 (i) are satisfied for X^*, B^* and A^* instead of X,B and A, then the semigroup e^{-tA} extends (uniquely) to a continuous semigroup on the space $X_B^{(-\nu)-}$.

(ii) For some ν, $0 < \nu \leq \infty$ if all the conditions in the above Theorem II.1.4 (ii) are satisfied for X^*, B^* and A^* instead of X,B and A, then the semigroup e^{-tA} extends uniquely to a continuous semigroup on the space $X_B^{(-\nu)+}$. □

We note that in general the restricted semigroups $e^{-tA} \restriction X_B^{\nu\pm}$ or the extended semigroups $e^{-tA} \restriction X_B^{(-\nu)\pm}$ are not necessarily equicontinuous on the respective spaces (even if after multiplication by a factor $e^{-\beta t}$) under the assumptions of the above Theorems. However, if conditions as (20) - (22) are met without the term $\| A\, u \|$ in (22) and with a bounded sequence $\{M_{\nu_i}\}$, then they are indeed equicontinuous after multiplication by a suitable factor $e^{-\beta t}$.

II.2. c_0 Semigroups in l^2 and Their Regularity and Extendibility

Let $(a_{jk})_{j,k \in \mathbb{N}_0}$ be an infinite matrix of complex numbers with j,k the row index and column index respectively. We suppose that

$$\{a_{\cdot k}\} \in l^2, \ \forall k \in \mathbb{N}_0 \, ; \{a_{j\cdot}\} \in l^2, \ \forall j \in \mathbb{N}_0 .$$

Corresponding to the matrix (a_{jk}) we can define an operator $A_{\max} \equiv \mathrm{Op}(a_{jk})$ in l^2 as follows:

$$A_{\max}\, u = \{ \sum_{k=0}^{\infty} a_{jk}\, u_k \}_{j \in \mathbb{N}_0}$$

with

$$D(A_{\max}) = \{ u = (u_k) \in l^2 \mid \sum_{k=0}^{\infty} a_{jk}\, u_k \text{ converges for all}$$

$j \in \mathbb{N}_0$ and $\{\sum_{k=0}^{\infty} a_{jk} u_k\}_{j \in \mathbb{N}_0} \in l^2\}$.

If instead of the matrix (a_{jk}) we use its complex conjugate $(\overline{a}_{kj})$, then we obtain another operator on l^2, denoted by $A^+_{\max}$. Under the above conditions it is easy to see that $l^2_c \subset D(A_{\max})$ and $l^2_c \subset D(A^+_{\max})$, where $l^2_c = \{u = (u_k) \mid u_k = 0$ if $k \geq K$ depending on $u\}$, the subspace of finite sequences. We fix the notations $A_{\max} \restriction l^2_c = A_{\min}$ and $A^+_{\max} \restriction l^2_c = A^+_{\min}$. Thus both the operators $A_{\max}$ and $A^+_{\max}$ are densely defined. Actually they are also closed and obey the following relations:

$$(A_{\max})^* \subset (A_{\min})^* = A^+_{\max} \; ; \quad (A^+_{\max})^* \subset (A^+_{\min})^* = A_{\max}.$$

The above facts are simple to prove and can be found in standard text books, e.g., [We]. In this section we first give two criteria under which a matrix (a_{jk}) (actually a appropriate operator corresponding to it) generates a c_0 semigroup in l^2, and finally examine its regularity and extendibility with respect to the weighted l^2 spaces $l^{2,\sigma}\{\lambda_k\} = l^{2,\sigma}$ constructed in I.3.

For the sake of simplicity in formulation from now on we always assume that the matrices discussed are tridiagonal. Extensions to cases involving more general matrices are immediate.

Theorem II.2.1. Suppose that (a_{jk}) is a tridiagonal matrix with the diagonal elements $a_{kk} \to +\infty$ $(k \to \infty)$ and $(A_{\max} - Q) \restriction l^2_c$ quasi-accretive. Here $Q = \mathrm{Op}[\mathrm{diag}(a_{kk})]$. Assume that there exist constants $K \in \mathbb{N}_0$ and $c_l \geq 0$ and $c_u \geq 0$ with $c_l + c_u \leq 1$ such that

$$| a_{k+1,k} | \, a_{kk}^{-1} \leq c_l \, , \quad | a_{k-1,k} | \, a_{kk}^{-1} \leq c_u \, , \quad \forall k \geq K. \tag{31}$$

Then the closure of $A_{\max} \restriction l^2_c$ generates a c_0 semigroup of operators on l^2.

Proof. From the condition $a_{kk} \to +\infty$ $(k \to \infty)$ it follows easily that the operator Q generates a c_0 semigroup on l^2 with l^2_c as a core of Q.

For $u \in D(Q)$ we have the estimate

$$\Big(\sum_{k=0}^{\infty} | a_{k,k-1} u_{k-1} + a_{k,k+1} u_{k+1} |^2\Big)^{1/2}$$

$$\leq \Big(\sum_{k=0}^{\infty} | a_{k,k-1} u_{k-1} |^2\Big)^{1/2} + \Big(\sum_{k=0}^{\infty} | a_{k,k+1} u_{k+1} |^2\Big)^{1/2}$$

$$\leq M_k \| u \| + (c_l + c_u) \, \| Q u \|$$

where

$$M_k = 2 \max_{0 \leq k \leq K} \{ | a_{k,k-1} | , \; | a_{k,k+1} | \}.$$

From this estimate we have $D(Q) \subset D(A_{\max})$ and for $u \in D(Q)$

$$\| (A-Q)u \| \le 2M_K \| u \| + (c_l + c_u) \| Q u \| \tag{32}$$

and

$$\| A u \| \le 2M_K \| u \| + (1 + c_l + c_u) \| Q u \| . \tag{33}$$

By virtue of (32) with $c_l + c_u \le 1$ the well known perturbation theorem (cf. e.g., Theorem 0.9) ensures that the closure of $A_{\max} \restriction D(Q)$ generates a c_0 semigroup on l^2. Moreover (33) implies that l_c^2 is a core for $A_{\max} \restriction D(Q)$, and hence also for its closure. □

Corollary II.2.2. If in the above theorem instead of condition (31) we assume the stronger condition that

$$\limsup | a_{k+1,k} | \, a_{kk}^{-1} + \limsup | a_{k,k-1} | \, a_{kk}^{-1} < 1 \tag{34}$$

then $A_{\max} \restriction D(Q)$ generates a c_0 semigroup in l^2 with l_c^2 as a core for $A_{\max} \restriction D(Q)$. □

We remark that if $(a_{jk}) = \mathrm{diag}(a_{kk}) + (a_{jk}^{(1)}) + (a_{jk}^{(2)})$ with $(a_{jk}^{(1)})$ a bounded matrix on l^2 and $(a_{jk}^{(2)})$ skew symmetric, then $(A_{\max} - Q) \restriction l_c^2$ is quasi-accretive.

Example II.2.3. With (a_{jk}) given by $(\mu, \nu < 1)$

$$\mathrm{diag}(1,2,3,\ldots) + \begin{bmatrix} 0 & -1^\nu & & & \\ 1^\nu & 0 & -2^\nu & & \\ & 2^\nu & 0 & -3^\nu & \\ & & 3^\nu & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

$$+ i \begin{bmatrix} 0 & 1^\mu & & & \\ 1^\mu & 0 & 2^\mu & & \\ & 2^\mu & 0 & 3^\mu & \\ & & 3^\mu & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 1-\frac{1}{1} & & & \\ 2+\frac{1}{1} & 0 & 1-\frac{1}{2} & & \\ 0 & 2+\frac{1}{2} & 0 & 1-\frac{1}{3} & \\ & & 2+\frac{1}{3} & \ddots & \ddots \\ & & & \ddots & \end{bmatrix}$$

the operator $A_{\max} \restriction D(Q)$ generates a c_0 semigroup in l^2 with l_c^2 as a core of it. Here Q is the maximal operator corresponding to $\mathrm{diag}(1,2,3,\ldots)$. □

The matrices corresponding to the generator in the above discussion are diagonal-dominant in a certain sense. The next theorem avoids such a requirement.

Theorem II.2.4. For an infinite matrix (a_{jk}) assume there exists a diagonal matrix diag $(q_0, q_1, \ldots)$ with $0 < q_k \to \infty$ such that its corresponding maximal operator Q and the operators $A_{\max}$ and $A^{+}_{\max}$ satisfy the following conditions:

$$\mathrm{Op}(a_{jk}\, q_k^{-1}) \text{ is a Hilbert–Schmidt operator on } l^2 \tag{35}$$

$$\mathrm{Op}(a_{jk} + \bar{a}_{kj}) \restriction D(Q) \text{ is quasi–accretive} \tag{36}$$

$$\mathrm{Op}(q_j^{1/2}\, a_{jk}\, q_k^{-1/2} + q_j^{-1/2}\, \bar{a}_{kj}\, q_k^{1/2}) \restriction D(Q^{1/2}) \text{ is quasi–accretive.} \tag{37}$$

Then $A_{\max} \restriction D(Q)$ is closable and its closure generates a c_0 semigroup on l^2.

Proof. Condition (35) implies in particular that $D(Q) \subset D(A_{\max})$. Theorem 0.10 will yield the wanted conclusion if we can check the conditions that

$$\mathrm{Re}(u\,,A_{\max}\, u) \ge \beta(u,u)\,,\ \ \forall u \in D(Q) \tag{38}$$

$$\mathrm{Re}(Q\, u\,,A_{\max}\, u) \ge \beta(Q\, u,u)\,,\ \ \forall u \in D(Q) \tag{39}$$

where β is a constant.

Let $u \in D(Q)$. We have

$$(u\,,A_{\max}\, u) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j\, \bar{a}_{jk}\, \bar{u}_k$$

and

$$\begin{aligned}\overline{(u\,,A_{\max}\, u)} &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \bar{u}_j\, a_{jk}\, u_k \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \overline{(q_j\, u_j)}\, (q_j^{-1}\, a_{jk}\, q_k^{-1})\, (q_k\, u_k) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overline{(q_j\, u_j)}\, (q_j^{-1}\, a_{jk}\, q_k^{-1})\, (q_k\, u_k) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \bar{u}_j\, a_{jk}\, u_k \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j\, a_{kj}\, \bar{u}_k .\end{aligned}$$

Note that the double summations above are interchangable because of the fact that $(q_j^{-1}\, a_{jk}\, a_k^{-1})$ is a Hilbert-Schmidt matrix in l^2. Consequently

$$\mathrm{Re}(u\,,A_{\max}\, u) = \frac{1}{2}\, [(u\,,A_{\max}\, u) + \overline{(u\,,A_{\max}\, u)}]$$

$$= \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_j (\overline{a}_{jk} + a_{kj}) \overline{u}_k$$

$$= \frac{1}{2} (u, \mathrm{Op}(a_{jk} + \overline{a}_{kj}) u), \quad u \in D(Q).$$

This together with the condition (36) implies (38).

Similarly, for $v \in D(Q^{1/2})$ we have

$$(Q^{1/2} v, A_{\max} Q^{-1/2} v) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_j^{1/2} v_j \overline{a}_{jk} q_k^{-1/2} \overline{v}_k.$$

And

$$\overline{(Q^{1/2} v, A_{\max} Q^{-1/2} v)} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (q_j^{1/2} \overline{v}_j)(a_{jk} q_k^{-1}) q_k^{1/2} v_k$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (q_j^{1/2} \overline{v}_j)(a_{jk} q_k^{-1}) q_k^{1/2} v_k$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_j^{-1/2} v_j a_{kj} q_k^{1/2} \overline{v}_k.$$

The interchange of summations above is allowable since $\mathrm{Op}(a_{jk} q_k^{-1})$ is Hilbert-Schmidt. Therefore for $u = Q^{-1/2} v \in D(Q)$

$$\mathrm{Re}(Q u, A_{\max} u) = \mathrm{Re}(Q^{1/2} v, A_{\max} Q^{-1/2} v)$$

$$= \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} v_j (q_j^{1/2} \overline{a}_{jk} q_k^{-1/2} + q_k^{-1/2} a_{kj} q_k^{1/2}) \overline{v}_k$$

$$= \frac{1}{2} (v, \mathrm{Op}(q_j^{1/2} a_{jk} q_k^{-1/2} + q_j^{-1/2} \overline{a}_{kj} q_k^{1/2}) v).$$

Condition (39) then follows from (37) and the above relation. []

We have a few remarks on the conditions in the previous theorem. Condition (35) is satisfied if we take

$$q_k = \alpha_k^{-1} \max \{ | a_{k-1,k} |, | a_{kk} |, | a_{k+1,k} | \}$$

with $\{\alpha_k\}$ in l^2. Condition (36) is verified if (a_{jk}) is a sum of a skew-symmetric matrix and a bounded matrix. In case (a_{jk}) s skew-symmetric, then the entry at $(k, k-1)$ of the matrix $(c_{jk}) = (q_j^{1/2} a_{jk} q_k^{-1/2} + q_j^{-1/2} \overline{a}_{kj} q_k^{1/2})$ is

$$a_{k,k-1} \left[\left(\frac{q_k}{q_{k-1}} \right)^{1/2} - \left(\frac{q_k}{q_{k-1}} \right)^{-1/2} \right].$$

Thus if $q_k / q_{k-1} \to 1$ the operator in (37) might be bounded and hence quasi-accretive in spite of

$\{ | a_{k,k-1} | \}$ being unbounded.

Example II.2.5. If

$$(a_{jk}) = \begin{bmatrix} 0 & 1^\nu & & & \\ -1^\nu & 0 & 2^\nu & & \\ & -2^\nu & 0 & 3^\nu & \\ & & -3^\nu & \ddots & \ddots \\ & & & \ddots & \end{bmatrix} \qquad (\nu \le 1)$$

and $Q = \mathrm{Op}[\mathrm{diag}(1^\mu, 2^\mu, 3^\mu, \ldots)]$ $(\mu > \nu + \frac{1}{2})$, then the conditions in the above theorem are satisfied. So $\mathrm{Op}(a_{jk}) \restriction D(Q)$ essentially generates a c_0 semigroup in l^2. □

For a sequence of positive numbers $\{\lambda_k\}$ with $\lambda_k \to \infty$ a corresponding scale of weighted l^2 spaces, $l^{2,\nu}\{\lambda_k\} = l^{2,\nu}$, has been constructed in I.3. In the finally part of the present section we examine the regularity and extendibility with respect to $l^{2,\nu}$ $(\nu \in \mathbb{R})$ of the c_0 semigroups in Theorems II.2.1 and II.2.4. Let $B = \mathrm{Op}[\mathrm{diag}(\lambda_k)]$.

Theorem II.2.6. Suppose that a tridiagonal matrix (a_{jk}) with $a_{kk} \to +\infty$ generates a c_0 semigroup e^{-tA} with l_c^2 as a core. Set $Q = \mathrm{Op}[\mathrm{diag}(a_{kk})]$. For fixed $\nu > 0$ assume further more the following conditions:

$$B^\nu (A_{\max} - Q) B^{-\nu} \restriction l_c^2 \text{ is quasi-accretive} \tag{40}$$

$$\left.\begin{array}{l} | a_{k+1,k} | \, a_{kk}^{-1} (\lambda_{k+1} \lambda_k^{-1})^\nu \le c_{l,\nu} \\ | a_{k-1,k} | \, a_{kk}^{-1} (\lambda_{k-1} \lambda_k^{-1})^\nu \le c_{u,\nu} \end{array}\right. \quad k \ge K \tag{41}$$

with $c_{l,\nu} + c_{u,\nu} \le 1$. Then $e^{-tA}(l^{2,\nu}) \subset l^{2,\nu}$ $(t \ge 0)$ and $e^{-tA} \restriction l^{2,\nu}$ is a c_0 semigroup on $l^{2,\nu}$.

Proof. $\Delta = l_c^2$ is a core for Q. Obviously $B^{-\nu}(l_c^2) \subset l_c^2 \subset D(A)$ and $AB^{-\nu}(l_c^2) \subset A(l_c^2) \subset l_c^2 \subset D(B^\nu)$. Furthermore, entirely similar to the proof of Theorem II.2.1 above, corresponding to (32) we now have

$$\| (B^\nu A B^{-\nu} - Q) u \| \le \tilde{M}_{k,\nu} \| u \| + (c_{l,\nu} + c_{u,\nu}) \| Q u \| , \quad u \in l_c^2$$

where

$$\tilde{M}_{k,\nu} = 2 \max_{0 \le k \le K} \{ | a_{k,k-1} | (\lambda_k \lambda_{k-1}^{-1})^\nu , \ | a_{k,k+1} | (\lambda_k \lambda_{k+1}^{-1})^\nu \}.$$

So, all the conditions in Theorem II.1.3 are satisfied and from this theorem readily follows the wanted conclusion. □

We remark that the condition (41) is satisfied if

$$\limsup |a_{k+1,k}| \, a_{kk}^{-1} (\lambda_{k+1} \lambda_k^{-1})^\nu + \limsup |a_{k-1,k}| \, (\lambda_{k-1} \lambda_k^{-1}) < 1. \tag{42}$$

In case $\lim(\lambda_{k+1} / \lambda_k) = 1$ condition (42) is equivalent to condition (34). Condition (40) holds iff

$$[B^\nu (A_{\max} - Q) B^{-\nu} + B^{-\nu}(A_{\max}^+ - Q) B^\nu] \restriction l_c^2 \text{ quasi–accretive.} \tag{43}$$

In case $(a_{jk}) - \mathrm{diag}(a_{kk})$ is skew-symmetric, the entry at $(k, k-1)$ of the matrix corresponding to the operator in (43) is

$$a_{k,k-1} \, [\left[\frac{\lambda_k}{\lambda_{k-1}} \right]^\nu - \left[\frac{\lambda_k}{\lambda_{k-1}} \right]^{-\nu}]. \tag{44}$$

Corollary II.2.7. Suppose that a tridiagonal matrix (a_{jk}) with $a_{kk} \to +\infty$ generates a c_0 semigroup e^{-tA} on l^2 with l_c^2 as a core. For fixed $\nu > 0$, e^{-tA} extends to a c_0 semigroup on $l^{2,-\nu}$ if the following conditions are verified:

$$(B^\nu A_{\max}^+ B^{-\nu} - Q) \restriction \text{ is quasi–accretive} \tag{45}$$

$$|a_{k,k+1}| \, a_{kk}^{-1} \, (\lambda_{k+1} \lambda_k^{-1})^\nu \leq c_{l,\nu}^+ , \quad k \geq K \tag{46}$$

$$|a_{k,k-1}| \, a_{kk}^{-1} \, (\lambda_{k-1} \lambda_k^{-1})^\nu \leq c_{u,\nu}^+ , \quad k \geq K \tag{47}$$

with $c_{l,\nu}^+ + c_{u,\nu}^+ \leq 1$.

Proof. We have $A^* = (A \restriction l_c^2)^* = A_{\max}^+$. Under the conditions above $B^\nu A_{\max}^+ B^{-\nu}$ generates a c_0 semigroup in l^2, as is shown in the proof of the above theorem. Theorem II.1.2 then leads to the conclusion that e^{-tA} is extendible to $l^{2,-\nu}$. □

Example II.2.8. With $\lambda_k = k + 1$ $(k \in \mathbb{N}_0)$, the semigroup e^{-tA} in Example II.2.3 both restricts to a c_0 semigroup on $l^{2,\nu}$ and extends to a c_0 semigroup on $l^{2,-\nu}$ for any $\nu > 0$. □

From now on we assume that

$$0 < \liminf \lambda_k \lambda_{k-1}^{-1} \leq \limsup \lambda_k \lambda_{k-1}^{-1} < \infty. \tag{48}$$

Theorem II.2.9. Suppose that a tridiagonal matrix (a_{jk}) generates a c_0 semigroup e^{-tA} on l^2 with $D(Q)$ as a core for A, where $Q = \mathrm{Op}[\mathrm{diag}(q_k)]$ with

$$q_k = \max \{ |a_{k-1,k}|, |a_{kk}|, |a_{k+1,k}| \}.$$

If

$$\sup_{j,k \in \mathbb{N}_0} \{ |a_{jk}| \; |(\lambda_j \lambda_k^{-1})^\nu - 1| \} < \infty \tag{49}$$

then e^{-tA} restricts to a c_0 semigroup on $l^{2,\nu}$.

Proof. Because of condition (48) it is easily verified that $B^{-\nu} D(Q) \subset D(A)$ and $AB^{-\nu} D(Q) \subset D(B^{\nu})$. Moreover condition (49) implies that $(B^{\nu} A B^{-\nu} - A)\restriction D(Q)$ is bounded. Theorem II.1.3 then leads us to the wanted conclusion. □

Corollary II.2.10. Suppose that a tridiagonal matrix (a_{jk}) generates a c_0 semigroup e^{-tA} on l^2 with its adjoint semigroup e^{-tA^*} having $D(\hat{Q})$ as a core.
Here $\hat{Q} = \mathrm{Op}[\mathrm{diag}(\hat{q}_k)]$ with

$$\hat{q}_k = \max \{ | a_{k,k-1} | , | a_{kk} | , | a_{k,k+1} | \}.$$

If

$$\sup_{j,k \in \mathbb{N}_0} \{ | a_{kj} | \; | (\lambda_j \lambda_k^{-1})^{\nu} - 1 | \} < \infty \tag{50}$$

then e^{-tA} extends to a c_0 semigroup on $l^{2,-\nu}$.

Proof. A proof similar to the one above shows that $B^{\nu} A^* B^{-\nu}$ generates a c_0 semigroup on l^2. From Theorem II.1.2 it follows readily that e^{-tA} extends to a c_0 semigroup on $l^{2,-\nu}$. □

Example II.2.11. The c_0 semigroup e^{-tA} in Example II.2.5 both restricts to a c_0 semigroup on $l^{2,\nu}$ and extends to a c_0 semigroup on $l^{2,-\nu}$ for any $\nu > 0$ with $\lambda_k = k + 1$ $(k \in \mathbb{N}_0)$.

Example II.2.12. The c_0 semigroup e^{-tA} generated by the matrix

$$(a_{jk}) = \begin{bmatrix} 0 & 1^{1/2} & & & \\ -1^{1/2} & 0 & 2^{1/2} & & \\ & -2^{1/2} & 0 & 3^{1/2} & \\ & & -3^{1/2} & \ddots & \ddots \\ & & & \ddots & \end{bmatrix}$$

as in Example II.2.5 both restricts to a c_0 semigroup on $l^{2,\nu}$ and extends to a c_0 semigroup on $l^{2,-\nu}$ with $\lambda_k = e^{(k+1)^{1/2}}$, for all $\nu > 0$. □

II.3. Application to Second Order Differential Operators

In this section we apply the general results in Section II.1 to the following autonomous evolution equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u \tag{51}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. Here all the a_{ij} are complex constants, and all the $b_i(x)$ and $c(x)$ are given complex functions on $\mathbb{R}^n$ (in the sequel whenever we write $\partial^{\alpha} b_i$ or $\partial^{\alpha} c$ it is implicitly assumed that the respective derivative exists and is continuous). Finally $u = u(x,t)$ is the unknown function. Throughout the whole section we assume that the operator

$$-A\,u = \sum_{i,j=1}^{n} a_{ij}\,\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_{i=1}^{n} b_i(x)\,\frac{\partial u}{\partial x_i} + c(x)\,u$$

generates a c_0 semigroup e^{-tA} in $L^2(\mathbb{R}^n)$ with the Schwartz test function space S as a core for A. (See, e.g., [Ka2] and [Gr1] for such conditions.) Our purpose is to prove the following two theorems.

Theorem II.3.1.

(i) If for an integer N holds that

$$\partial^\alpha b_i = O(1),\ \partial^\alpha c = O(1),\ |\alpha| \le 2N \tag{52}$$

then $e^{-tA}(H^{2N}) \subset H^{2N}$ and $e^{-tA}\restriction_{H^{2N}}$ is a c_0 semigroup of operators on H^{2N}.

(ii) If for an integer N the conditions

$$\partial^\alpha b_i = O(1),\ |\alpha| \le 2N+1;\ \partial^\alpha c = O(1),\ |\alpha| \le 2N, \tag{53}$$

are satisfied, then e^{-tA} extends to a c_0 semigroup of operators on the space H^{-2N}.

(iii) If the conditions

$$\partial^\alpha b_i = O(1),\ \partial^\alpha c = O(1),\ |\alpha| < \infty, \tag{54}$$

are satisfied, then e^{-tA} both restricts to a c_0 semigroup on the space H^∞ and extends to a c_0 semigroup on the space $H^{-\infty}$.

Theorem II.3.2.

(i) If for some integer N hold the conditions

$$b_i = O(1+|x|),\ c = O((1+|x|)^2) \tag{55}$$

and

$$\partial^\alpha b_i = O((1+|x|)^{|\alpha|-1}),\ \partial^\alpha c = O((1+|x|^{|\alpha|}),\ 0 < |\alpha| \le 2N, \tag{56}$$

then $e^{-tA}(LW)_2^{2N} \subset (LW)_2^{2N}$ and $e^{-tA}\restriction_{(LW)_2^{2N}}$ is a c_0 semigroup of operators on $(LW)_2^{2N}$.

(ii) If for an integer N hold the conditions (55) and (56) and moreover

$$D^\alpha b_i = O((1+|x|)^{2N}),\ |\alpha| = 2N+1 \tag{57}$$

then e^{-tA} extends to a c_0 semigroup on the space $(LW)_2^{-2N}$.

(iii) If the conditions (55) and (56) are satisfied for all multi-indices α, then e^{-tA} both restricts to a c_0 semigroup on the Schwartz test function space S and extends to a c_0 semigroup on the Schwartz tempered generalized function spaces S'.

For the proof of these theorems we need the following two simple lemmas, both of which are consequences of Leibnitz's rule. Their proofs are omitted.

Lemma II.3.3. For $c(x) \in C^k(\mathbb{R}^n)$ and $u(x) \in C^k(\mathbb{R}^n)$ holds the following identity

$$\Delta^k(cu) = c\,\Delta^k u + \sum_{\substack{|\alpha+\beta|=2k \\ |\beta|\le 2k-1}} \rho_{\alpha\beta}^{(k)}\,(\partial^\alpha c)\,(\partial^\beta u) \tag{58}$$

where the $\rho_{\alpha\beta}^{(k)}$ are suitable absolute constants. □

Lemma II.3.4. For N a natural number the following identity is valid:

$$(x^2-\Delta)^N = \sum_{k=0}^{N} (-1)^{N-k} \begin{bmatrix} N \\ k \end{bmatrix} x^{2k}\,\Delta^{N-k} + P_N \tag{59}$$

where

$$P_N = \sum_{|\alpha+\beta|\le 2N-2} \delta_{\alpha,\beta}^{(N)}\, x^\alpha\, \partial^\beta \tag{60}$$

while the $\delta_{\alpha,\beta}^{(N)}$ are absolute constants. □

Now we can give the proofs of Theorems II.3.1 and II.3.2.

Proof of Theorem II.3.1. (i) For $B = B_1 = I - \Delta$ we have $X_B^N = H^{2N}$; see Example 4 in Section I.3. By Theorems II.1.1 and II.11.3 we only need to verify the conditions (20) - (22) in the presence of our assumption (52). From the effects of the Fourier transformation on $x^\alpha \partial^\beta u$ and on B_1 it follows immediately that $B_1^{-N}(S) \subset S \subset D(A)$. Also, if the condition (52) is met, then $A(S) \subset H^{2N} = D(B_1^N)$. Thus, the conditions (20) and (21) are verified. Further, since all the a_{ij} are assumed to be constants, Lemma II.3.3 implies that for $u \in S$

$$\begin{aligned} B_1^N A\,u &= \sum_{k=0}^{N} (-1)^k \begin{bmatrix} N \\ k \end{bmatrix} \Delta^k A\,u \\ &= \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i\,\partial x_j} (I-\Delta)^N u + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} (I-\Delta)^N u + c(I-\Delta)^N u \\ &+ \sum_{i=1}^{n} \sum_{\substack{|\alpha+\beta|\le 2N \\ |\beta|\le 2N-1}} \rho_{i,\alpha,\beta}\,(\partial^\alpha b_i)\,(\partial^\beta \frac{\partial u}{\partial x_i}) + \sum_{\substack{|\alpha+\beta|\le 2N \\ |\beta|\le 2N-1}} \rho_{0,\alpha,\beta}\,(\partial^\alpha c)\,(\partial^\beta u) \end{aligned} \tag{61}$$

where all the $\rho_{i,\alpha,\beta}$ $(i = 0, 1, \ldots, n)$ are constants. Therefore from the condition (52) follows

$$\| [A, B_1^N]\,u \| \le M_n\, \| B_1^N u \|, \quad u \in S \tag{62}$$

which is (22) with the term $\| A\,u \|$ dropped.

(ii) Obviously

$$A^* u = \sum_{i,j=1}^{n} \bar{a}_{ji} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} (-\bar{b}_i) \frac{\partial u}{\partial x_i} + (\sum_{i=1}^{n} - \frac{\partial \bar{b}_i}{\partial x_i} + c) u, \quad u \in S.$$

Thus, from the proof of (i) above, we see that under the condition (53), $A^* S \subset D(B_1^{2N})$ and

$$\| [A^*, B_1^N] u \| \leq \bar{M}_n \| B_1^N u \|, \quad u \in S. \tag{63}$$

Then, according to Theorems II.1.3 and II.1.2, e^{-tA} extends to a c_0 semigroup on the space $X_{B_1}^{-N} = H^{-2N}$.

The assertion in (iii) follows from (i) and (ii) directly. □

Proof of Theorem II.3.2. (i) For $B = B_2 = x^2 - \Delta$ we have $X_B^N = (LW)_2^{2N}$; see Example 7 in I.3. Obviously $B_2^{-N} S \subset S$ and in the presence of conditions (55) and (56) we have $A S \subset (LW)_2^{2N} = D(B_2^N)$. For $u \in B_2^{-N} S \subset S$, applying Lemmas II.3.4 and II.3.3 successively we have

$$\begin{aligned} B_2^N A u &= \sum_{k=0}^{N} (-1)^{N-k} \begin{bmatrix} N \\ k \end{bmatrix} x^{2k} \Delta^{N-k} (Au) + P_N A u \\ &= (\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} b_i \frac{\partial}{\partial x_i} + c) \sum_{k=0}^{n} (-1)^{N-k} \begin{bmatrix} N \\ k \end{bmatrix} x^{2k} \Delta^{N-k} u \\ &+ \sum_{k=0}^{N} (-1)^{N-k} \begin{bmatrix} N \\ k \end{bmatrix} [\sum_{i,j=1}^{n} a_{ij} A_{ijk} + \sum_{i=1}^{n} b_i B_{ik}] \\ &+ \sum_{k=0}^{N} (-1)^{N-k} \begin{bmatrix} N \\ k \end{bmatrix} x^{2k} \sum_{i=1}^{n} \sum_{\substack{|\alpha+\beta| \leq 2(N-k) \\ |\beta| \leq 2(N-k)-1}} \rho_{\alpha,\beta}^{(i,k)} (\partial^\alpha b_i) (\partial^\beta \frac{\partial u}{\partial x_i}) \\ &+ \sum_{k=0}^{N} (-1)^{N-k} \begin{bmatrix} N \\ k \end{bmatrix} x^{2k} \sum_{\substack{|\alpha+\beta| \leq 2(N-k) \\ |\beta| \leq 2(N-k)-1}} \rho_{\alpha,\beta}^{(0,k)} (\partial^\alpha c) (\partial^\beta u) \\ &+ P_N A u \end{aligned} \tag{64}$$

where

$$A_{ijk} u = - \frac{\partial(x^{2k})}{\partial x_i \partial x_j} \cdot \Delta^{N-k} u - \frac{\partial(x^{2k})}{\partial x_i} \cdot \frac{\partial(\Delta^{N-k} u)}{\partial x_j} - \frac{\partial(x^{2k})}{\partial x_j} \frac{\partial(\Delta^{N-k} u)}{\partial x_i} \tag{65}$$

and

$$B_{ik}\, u = -\frac{\partial(x^{2k})}{\partial x_i} \cdot \Delta^{N-k}\, u. \tag{66}$$

The first term on the right side of (64) is just

$$A\, B_2^N - A\, P_N\, u. \tag{67}$$

From the expressions (65) and (66) it is readily seen that each A_{ijk} is a linear combination of terms in the form $x^\alpha\, \partial^\beta\, u$ with $|\alpha+\beta| \leq 2N$, and each B_{ik} is such a sum with $|\alpha+\beta| \leq 2N-1$. Therefore, because of the assumption that $b_i = O(1+|x|)$ in (55), the absolute value of the second term in (64) is bounded by

$$M \sum_{|\alpha+\beta| \leq 2N} (1+|x|)^{|\alpha|}\; |\partial^\beta u|. \tag{68}$$

It is also easy to see that, under the assumptions (55) and (56) the third and fourth terms in (64) are bounded by an expression like (68). This applies equally well to $-A\, P_N\, u + P_N\, A\, u$. Thus we have finally the estimate

$$|\,[A, B_2^N]\, u\,| \leq M \sum_{|\alpha+\beta| \leq 2N} (1+|x|)^\alpha\; |\partial^\beta u|$$

from which follows (conf. [R-S], p. 141)

$$\|\,[A, B_2^N]\, u\,\| \leq M\, \|B_2^N\, u\|. \tag{69}$$

Having verified all the conditions (20) - (22) we arrive at the assertion in (i).

In view of the expression for A^*, the proof of the above assertion (i) and Theorems II.1.2 and II.1.3 imply assertion (ii).

Assertion (iii) is a direct consequence of (i) and (ii). □

In the above Theorems II.3.1 and II.3.2 the regularity and extendibility of the solutions to the equation (43) have been discussed in the Sobolev spaces H^ν and the modified Sobolev spaces $(LW)_2^\nu$ with ν even. Using the calculus of pseudo-differential operators we can treat the case H^ν for arbitrary index ν under the somewhat strong condition (54). We are wondering about the case $(LW)_2^\nu$ with odd index ν.

Theorem II.3.5. If the condition (54) is satisfied, then the semigroup e^{-tA} both restricts to a semigroup on each of the spaces $H^\nu, H^{\nu+}$ $(0 \leq \nu < \infty)$ and $H^{\nu-}$ $(0 \leq \nu \leq \infty)$, and extends to a c_0 semigroup on each of the spaces $H^{-\nu}$, $H^{(-\nu)-}$ $(0 \leq \nu < \infty)$ or $H^{(-\nu)+}$ $(0 < \nu \leq \infty)$.

Proof. Take $\Delta = S$. If is easily seen that $B_1^{-\nu/2}\, S \subset S \subset D(A)$ and $A\, B_1^{\nu/2}\, S \subset A\, S \subset S \subset D(B_1^{\nu/2})$. In the following, for the standard concepts, calculus and other results of pseudo-differential operators we refer to the monograph [Ku2] and the paper [Ku1].

The symbols corresponding to the pseudo-differential operators A and B are respectively

$$\sigma_A(x,\zeta) = -\sum_{j,k}^{n} a_{jk}\,\zeta_j\,\zeta_k + i\sum_{j=1}^{n} b_j(x)\,\zeta_j + c(x) \tag{70}$$

and

$$\sigma_{B_1^{\nu/2}}(x,\zeta) = (1+\zeta^2)^{\nu/2}. \tag{71}$$

Then $\sigma_A \in S^2_{1,0}$ and $\sigma_{B_1^{\nu/2}} \in S^{\nu}_{1,0}$. Furthermore

$$\sigma_{AB_1^{\nu/2}}(x,\zeta) = \sigma_A\,\sigma_{B_1^{\nu/2}} + \sum_{j=1}^{n}\frac{1}{i}\,\frac{\partial\sigma_A}{\partial\zeta_j}\,\frac{\partial\sigma_{B_1^{\nu/2}}}{\partial x_j} + R_2(x,\zeta)$$

$$= \sigma_A\,\sigma_{B_1^{\nu/2}} + R_2(x,\zeta),\quad R_2 \in S^{\nu+2-2}_{1,0} = S^{\nu}_{1,0} \tag{72}$$

and

$$\sigma_{B_1^{\nu/2}A}(x,\zeta) = \sigma_A\,\sigma_{B_1^{\nu/2}} + \sigma_A\,\sigma_{B_1^{\nu/2}} + \sum_{j=1}^{n}\frac{1}{i}\,\frac{\partial\sigma_{B_1^{\nu/2}}}{\partial\zeta_j}\cdot\frac{\partial\sigma_A}{\partial x_j} + \tilde{R}_2(x,\zeta)$$

$$= \sigma_A\,\sigma_{B_1^{\nu/2}} + \frac{\nu}{i}\,(\zeta^2+1)^{\nu/2-1}\,[i\sum_{j,k=1}^{n}\frac{\partial b_k}{\partial x_j}\,\zeta_j\,\zeta_k + \sum_{j=1}^{n}\frac{\partial c}{\partial x_j}\,\zeta_j]$$

$$+\tilde{R}_2(x,\zeta),\quad \tilde{R}_2(x,\zeta) \in S^{\nu}_{1,0}. \tag{73}$$

Thus

$$\sigma_{AB_1^{\nu/2}}(x,\zeta) - \sigma_{B_1^{\nu/2}A}(x,\zeta)$$

$$= \frac{\nu}{i}\,(\zeta^2+1)^{\nu/2-1}\,[i\sum_{j,k=1}^{n}\frac{\partial b_k}{\partial x_j}\,\zeta_j\,\zeta_k + \sum_{j=1}^{n}\frac{\partial c}{\partial x_j}\,\zeta_j] + R_2 - \tilde{R}_2 \in S^{\nu}_{1,0}. \tag{74}$$

Therefore the operator $(AB_1^{\nu/2} - B_1^{\nu/2}A)\,B_1^{\nu/2}$ is bounded on Δ in $L^2(\mathbb{R}^n)$. This is equally true for A^* instead of A. An invocation of the theorems in II.1 will complete the proof. □

III. Generation of Locally Equi-Bounded Semigroups in Inductive Limit of Banach Spaces

Since the end of sixties locally equi-continuous semigroup on locally convex spaces have been studied by several mathematicians. Cf. [Komu], [Ou], [Ba] and [De]. In those works necessary and sufficient conditions on a linear operator in a locally convex space are given so that it generates such a semigroup. The conditions are usually formulated in terms of the so called generalized resolvents or asymptotic resolvents involving the continuous semi-norms of the locally convex space. However, for the inductive limit space E^+ of a sequence $\{(E_n, \|\cdot\|_n) \mid n \in \mathbb{N}_0\}$ of Banach spaces very often it is hard even to imagin its continuous semi-norms. In this chapter we present a Hille-Yosida type theorem for the so called locally equi-bounded semigroups on such an inductive limit E^+, using directly the norms of the constituent spaces $(E_n, \|\cdot\|_n)$ $(n \in \mathbb{N}_0)$ instead of the continuous semi-norms of E^+. The concept of locally equi-bounded semigroups is in general stronger than that of locally equi-continuous semigroups. If, however, the inductive limit E^+ is regular and satisfies a certain interpolation type inequality as in Theorem I.1.6, then it turns out to be that the two concepts are equivalent. The results obtained together with Ouchi's theory can be readily applied to the spaces $X_B^{\sigma\pm}$ ($\sigma \in \mathbb{R}$ or $\pm\infty$) constructed in Chapter I. If the operator A generates a c_0 semigroup e^{tA} in X, then the results obtained here could be viewed as results of more flexible regularity and extendibility of the semigroup e^{tA}, Cf. Chapter II.

Let $\{(E_n, \|\cdot\|_n) \mid n \in \mathbb{N}_0\}$ be a sequence of Banach spaces such that each E_n is densely and continuously embedded in E_{n+1} for all $n \in \mathbb{N}_0$. Then, as in Chapter one, we can assume that the sequence $\{\|\cdot\|_n \mid n \in \mathbb{N}_0\}$ of norms is monotone decreasing. From the sequence of Banach spaces $\{E_n\}$ we form its inductive limit space $E^+ = \bigcup_{n=0}^{\infty} E_n$ with the corresponding topology denoted τ_{ind}. In this section we are concerned with the so called locally equi-bounded semigroups of linear operators on such inductive limits.

Definition III.1. A family $\{T_\alpha \mid \alpha \in I\}$ of linear operators defined on the inductive limit E^+ is said to be equi-bounded if and only if

$$\forall n \in \mathbb{N}_0 \, \exists k \in \mathbb{N}_0 \, \exists c_{n,k} \, \forall \alpha \in I \, \forall u \in E_n$$

$$T_\alpha u \in E_k \text{ and } \|T_\alpha u\|_k \leq c_{n,k} \|u\|_n. \tag{1}$$

In particular, a linear operator T on E^+ is said to be bounded if $\{T\}$ is equi-bounded. □

Definition III.2. A family of operators $\{T(t) \mid t \geq 0\}$ one E^+ is called a locally equi-bounded semigroup on E^+ iff it has the following properties:

(i) $T(0) = I$, the identity operator on E^+;

(ii) $T(t)\,T(s) = T(t+s)$ for all $s,t \geq 0$;

(iii) For any $\bar{t} > 0$, the subfamily $\{T(t) \mid 0 \leq t \leq \bar{t}\}$ is equi-bounded;

(iv) $\forall \bar{t} > 0\ \forall n \in \mathbb{N}_0\ \exists k \in \mathbb{N}_0\ \forall u \in E_n\ \forall t \in [0,\bar{t}]\ [T(t)u \in E_k$ and $T(\cdot)u : [0,\bar{t}] \to E_k$ is continuous$]$. □

Definition III.3. For a locally equi-bounded semigroup $\{T(t) \mid t \geq 0\}$ on E^+ we define its infinitesimal generator $\tilde{A}$ as follows: $D(\tilde{A}) = \bigcup_{n \in \mathbb{N}_0} D(\tilde{A}_n)$ and $\tilde{A}\,u = \tilde{A}_n\,u$ for $u \in D(\tilde{A}_n)$, where for all $n \in \mathbb{N}_0$.

$$D(\tilde{A}_n) = \{u \in E^+ \mid \exists \bar{t}_u > 0\ \forall t \in [0,\bar{t}_u],\ T(t)\,u \in E_n \text{ and } \lim_{t \to 0+} \frac{T(t)\,u - u}{t} \text{ exists in } E_n\}.$$

and

$$\tilde{A}_n\,u = \lim_{t \to 0+} \frac{T(t)\,u - u}{t}.$$

□

Before we can formulate a theorem of Hille-Yosida type still another definition is needed.

Definition III.4. Let V be a locally convex spaces and $A : D(A) \subset V \to V$ a densely defined linear operator in V. A pseudo-resolvent triple of A is a set $(P_{rs}(A), R, S)$ such that

(i) $P_{rs}(A) \subset \mathbb{C}$;

(ii) $R : P_{rs}(A) \to L(V) : \lambda \mapsto R(\lambda)$,
$S : P_{rs}(A) \to L(V) : \lambda \mapsto S(\lambda)$;

(iii) $\forall \lambda \in P_{rs}(A)\ R(\lambda)\,V \subset D(A)$;

(iv) $\forall \lambda \in P_{rs}(A)\ \forall u \in D(A)\ A\,R(\lambda)\,u = R(\lambda)\,A\,u$;

(v) $\forall \lambda,\mu \in P_{rs}(A)\ R(\lambda)\,R(\mu) = R(\mu)\,R(\lambda)$;

(vi) $\forall \lambda \in P_{rs}(A)\ \forall u \in V,\ (\lambda I - A)\,R(\lambda)\,u = u + S(\lambda)\,u$. □

Theorem III.5. Let $A : D(A) \subset E^+ \to E^+$ be a linear operator in E^+. Then, A is the infinitesimal generator for a locally equi-bounded semigroup $\{T(t) \mid t \geq 0\}$ on E^+ iff the following conditions are satisfied:

1) A_n, the part of A in E_n, is densely defined and closed in E_n, for each n large enough.

2) There exists a pseudo-resolvent triple $((\omega,\infty), R, S)$ $(\omega > 0)$ such that

   (i) $\forall n \in \mathbb{N}_0\ \exists m \in \mathbb{N}_0\ [R(\lambda)\,E_n \subset D(A_m)]$

(ii) $R(\lambda)$ and $S(\lambda)$ are infinitely strongly differentiable on (ω,∞), i.e. $\forall n \in \mathbb{N}_0 \, \exists m \in \mathbb{N}_0 \, \forall \lambda > \omega \, \forall u \in E_n \, [R(\cdot)u : (\omega,\infty) \to E_m$ and $S(\cdot)u : (\omega,\infty) \to E_m$ are infinitely differentiable].

(iii) The family of operators $\{\frac{\lambda^{k+1}}{k!} R^{(k)} \mid \lambda > \omega, \, k \in \mathbb{N}_0\}$ is equi-bounded on E^+.

(iv) There exists some $\bar{t} > 0$ such that the family $\{(\bar{t})^{-k} e^{\bar{t}\lambda} S^{(k)}(\lambda) \mid \lambda > \omega, \, k \in \mathbb{N}_0\}$ is equi-bounded on E^+.

□

The proof below is very much inspired by the work [Ou]. We first give the proof to the necessity of the conditions. In order to do so we need

Proposition III.6. Assume that $\{T(t) \mid t \geq 0\}$ is a locally equi-bounded semigroup on E^+. Then we have

(i) $\forall n \in \mathbb{N}_0 \, \exists m \in \mathbb{N}_0 \, \forall u \in E_n$

$$(E_m) \lim_{h \downarrow 0} \int_0^h T(s)\, u \, ds = u$$

Here and in the sequel the symbol (E_m) in front of an integral or another operation indicates that the respective operation is meaningful in the space E_m.

(ii) $\forall n \in \mathbb{N}_0 \, \forall \bar{t} > 0 \, \exists m \in \mathbb{N}_0 \, \forall u \in E_n \, \forall t \in [0,\bar{t}]$

$$(E_m) \int_0^t T(s)\, u \, ds \in D(\tilde{A}_m)$$

and

$$\tilde{A}_m\Big(\int_0^t T(s)\, u \, ds\Big) = T(t)\, u - u.$$

(iii) $\forall n \in \mathbb{N}_0 \, \forall \bar{t} > 0 \, \exists m \in \mathbb{N}_0 \, \forall u \in D(\tilde{A}_n) \, \forall t \in [0,\bar{t}]$

$$T(t)\, u \in D(\tilde{A}_m)$$

and

$$(E_m) \frac{d}{dt} (T(t)\, u) = \tilde{A}_m[T(t)\, u] = T(t) \tilde{A}_n \, u.$$

(iv) $\forall n \in \mathbb{N}_0 \, \forall \bar{t} > 0 \, \exists m \in \mathbb{N}_0 \, \forall u \in D(\tilde{A}_n) \, \forall t,s \in [0,\bar{t}]$

$$T(t)\,u - T(s)\,u = (E_m)\int_s^t T(r)\,\tilde{A}_n\,u\,dr$$

$$= (E_m)\int_s^t \tilde{A}_m\,[T(r)\,u]\,dr.$$

Proof. (i) follows readily from (iv) in Definition III.2.

(ii) Again by (iv) in Definition III.2 we have

$$\left.\begin{array}{l}\forall n \in \mathbb{N}_0\ \forall \bar{t} > 0\ \exists m \in \mathbb{N}_0\ \forall u \in E_n \\ T(\cdot)\,u : [0,\bar{t}+1] \to E_m \text{ is continuous.}\end{array}\right\} \tag{2}$$

Consequently, for $h \in (0,1)$ and $t \in [0,\bar{t}]$ we have

$$\frac{T(h)-I}{h}\int_0^t T(s)\,u\,ds = \frac{1}{h}\int_t^{t+h} T(s)\,u\,ds - \frac{1}{h}\int_0^h T(s)\,u\,ds.$$

Letting $h \to 0$ we conclude that $\int_0^t T(s)\,u\,ds \in D(\tilde{A}_m)$ and

$$\tilde{A}_m\,(\int_0^t T(s)\,u\,ds) = T(t)\,u - u.$$

(iii) As in (ii) above we have (2), which together with the equi-boundedness of the family $\{T(t) \mid t \in [0,\bar{t}]\}$ implies that $\forall t \in [0,\bar{t}]\ \forall u \in D(\tilde{A}_n)$

$$\frac{T(h)-I}{h}\,T(t)\,u = T(t)\,\frac{T(h)-I}{h}\,u \to T(t)\,\tilde{A}_n\,u \quad (h \to 0+) \text{ in } E_m.$$

Thus $T(t)\,u \in D(\tilde{A}_m)$ and

$$\frac{d^+}{dt}\,[T(t)\,u] = \tilde{A}_m\,T(t)\,u = T(t)\,\tilde{A}_n\,u. \tag{3}$$

However, from the equi-boundedness of $\{T(t) \mid 0 \le t \le \bar{t}\}$ it follows that

$$\frac{T(t)\,u - T(t-h)\,u}{h} - T(t)\,\tilde{A}_n\,u$$

$$= T(t-h)\left[\frac{T(h)\,u-u}{h} - \tilde{A}_n\,u\right] + [T(t-h)\,\tilde{A}_n\,u - T(t)\,\tilde{A}_n\,u]$$

$$\to 0 \quad (h \to 0+) \text{ in } E_m.$$

This together with (3) proves the existence of $\frac{d}{dt}\,[T(t)\,u]$ and hence the wanted assertion.

(iv) is obtained from (iii) by direct integration. □

Using the above proposition we now give

Proof to the necessity of the conditions: By (i) and (ii) in Proposition III.6 for $n=0, \bar{t}=1$ there exists $m \in \mathbb{N}_0$ such that

$$(E_m) \int_0^t T(s)\,u\,ds \in D(\tilde{A}_m), \quad \forall u \in E_0, \forall t \in [0,1] \tag{4}$$

and

$$\frac{1}{t}\int_0^t T(s)\,u\,ds \to u\,(t \to 0+) \text{ in } E_m, \; \forall u \in E_0. \tag{5}$$

Relation (4) implies that $\frac{1}{t}\int_0^t T(s)\,u\,ds \in D(\tilde{A}_m)$, which together with (5) shows that $D(\tilde{A}_m)$ is dense in E_0 in the norm of E_m. Since E_0 is dense in E_m, we arrive at the conclusion that $D(\tilde{A}_m)$ is dense in E_m.

By (iv) in Proposition III.6 we have

$$T(t)\,u - u = \int_0^t T(s)\,\tilde{A}\,u\,ds, \quad \forall u \in D(\tilde{A})$$

where the integral is meaningful in some E_n depending on u and t. Now let $u_j \in D(\tilde{A})$ $(j \in \mathbb{N}_0)$ be such that

$$\{u_j\} \subset E_m, \; \{\tilde{A}\,u_j\} \subset E_m \text{ and } u_j \to u \in E_m, \; \forall u_j \to \mathrm{v} \in E_m$$

for some $m \in E_m$. It holds true that

$$T(t)\,u_j - u_j = \int_0^t T(s)\,\tilde{A}\,u_j\,ds.$$

In view of the equi-boundedness of the family $\{T(s) \mid 0 \le s \le t\}$ letting $j \to \infty$ in the above equality we get

$$T(t)\,u - u = \int_0^t T(s)\,\mathrm{v}\,ds.$$

Dividing both sides of this equality by $t > 0$ then letting $t \to 0+$ we see that $u \in D(\tilde{A})$ and $\tilde{A}\,u = \mathrm{v}$.

Put $A = \tilde{A}$. In abuse of nolation we write A_n for the part of A (i.e. of $\tilde{A}$) in E_n $(n \in \mathbb{N}_0)$. Obviously $D(A_n) \supset D(\tilde{A}_n)$ for all $n \in \mathbb{N}_0$. Then, for m fixed above, $D(A_m)$ is dense in E_m. It is also closed in E_k. Indeed, for $\{u_j\} \subset D(A_m)$ such that $u_j \to u \in E_m$ and $A_m u \to v \in E_m$ it follows from last paragraph that $u \in D(A)$ and $A u = v$. So, $u \in D(A_m)$ and $A_m u = v$.

Let $\bar{t} > 0$ be fixed. By the equi-boundedness of the family $\{T(t) \mid 0 \le t \le \bar{t}+1\}$ we have

$$\forall n \in \mathbb{N}_0 \, \exists m \in \mathbb{N}_0 \, \exists C_{n,m} > 0 \, \forall u \in E_n \, \forall t \in [0, \bar{t}+1]$$

$$T(t)\, u \in E_m \text{ and } \| T(t)\, u \|_m \le C_{n,m} \| u \|_n . \tag{6}$$

Consequently a family of operators $\{R(\lambda) \mid \lambda > 0\}$ on E^+ is well defined by

$$R(\lambda)\, u = (E_m) \int_0^{\bar{t}} e^{-\lambda t}\, T(t)\, u\, dt \,,\; u \in E_n \,,\; \lambda > 0 \,,\; n \in \mathbb{N}_0 . \tag{7}$$

Therefore we have in E_m

$$R^{(k)}(\lambda)\, u \equiv \frac{d^k}{d\lambda^k} [R(\lambda)\, u] = (E_m) \int_0^{\bar{t}} (-t)^k e^{-\lambda t}\, T(t)\, u\, dt \,,\; k \in \mathbb{N}_0 .$$

So

$$\| R^{(k)}(\lambda)\, u \|_m \le \int_0^{\bar{t}} t^k e^{-\lambda t} \| T(t)\, u \|_m \, dt$$

$$\le C_{n,m} \int_0^{\infty} t^k e^{-\lambda t}\, dt\, \| u \|_n$$

$$= C_{n,m}\, k!\, \lambda^{-(k+1)} \| u \|_n \,,\; \forall u \in E_n \,,\; \forall \lambda > 0 \,,\; \forall k \in \mathbb{N}_0$$

which is precisely 2) (iii) in Theorem III.5.

For $u \in E_n$, $0 \le s \le 1$ and $\lambda > 0$ we have

$$\frac{T(s) - I}{s} R(\lambda)\, u$$

$$= \frac{1}{s} \left(\int_0^{\bar{t}} e^{-\lambda t}\, T(t+s)\, u\, dt - \int_0^{\bar{t}} e^{-\lambda t}\, T(t)\, u\, dt \right)$$

$$= \frac{e^{\lambda s}}{s} \int_{\bar{t}}^{s+\bar{t}} e^{-\lambda t}\, T(t)\, u\, dt - \frac{1}{s} \int_0^{s} e^{-\lambda t}\, T(t)\, u\, dt$$

$$+ \frac{e^{\lambda s} - 1}{s} \int_s^{\bar{t}} e^{-\lambda t}\, T(t)\, u\, dt$$

$$\to e^{-\lambda\bar{t}}\, T(\bar{t})\, u - u + \lambda R(\lambda)\, u \text{ in } E_m \text{ as } s \to 0.$$

Thus $R(\lambda)\, E_n \subset D(\tilde{A}_m) \subset D(A_m)$, i.e., condition 2) (i) is satisfied, and

$$A\, R(\lambda)\, u = e^{-\lambda\bar{t}}\, T(\bar{t})\, u - u + \lambda R(\lambda)\, u$$

or equivalently

$$(\lambda I - A)\, R(\lambda)\, u = u - e^{-\lambda\bar{t}}\, T(t)\, u.$$

Set $S(\lambda)\, u = -e^{-\lambda\bar{t}}\, T(\bar{t})\, u$ for $\lambda > 0$ and $u \in E^+$. Condition (iv) in Theorem III.5 is obviously verified. It is also easy to see that all the conditions in Definition III.4 are met, so $((0,\infty), R, S)$ is indeed a pseudo-resolvent triple. We thus have completed the proof to the necessity of the conditions in Theorem III.5. □

Assuming that an operator $A : D(A) \subset E^+ \to E^+$ satisfies all the conditions in Theorem III.5, we prove that it is the infinitesimal generator of a unique locally equi-bounded semigroup on E^+ by the following four lemmas.

Lemma III.1.7.

(i) $\forall n \in \mathbb{N}_0\, \exists m \in \mathbb{N}_0\, \forall u \in E_n$

$$\lim_{\lambda \to \infty} \lambda R(\lambda)\, u = u \text{ in } E_m.$$

(ii) Put $A(\lambda) = -\lambda I + \lambda^2 R(\lambda)$. Then $\forall n \in \mathbb{N}_0\, \exists m \in \mathbb{N}_0\, \forall u \in D(A_n)$

$$\lim_{\lambda \to \infty} A(\lambda)\, u = A\, u \text{ in } E_m.$$

(iii) The family of operators

$$\left\{ \frac{\lambda^{k+2}\, e^{\bar{t}\lambda}}{(\bar{t}\lambda)^{k+1} + (k+1)!} [R^{(k+1)} + (k+1)\, R(\lambda)\, R^{(k)}(\lambda)] \mid \lambda > \omega,\, k \in \mathbb{N}_0 \right\}$$

is equi-bounded on E^+.

Proof. (i) Conditions (ii), (iii) and (iv) in 2) of Theorem III.5 manifest themselves in the following way:

$$\forall n \in \mathbb{N}_0\, \exists m \in \mathbb{N}_0\, \exists C_{n,m} > 0\, \forall k \in \mathbb{N}_0\, \forall \lambda > \omega\, \forall u \in E_n$$

$$\left\| \frac{\lambda^{k+1}}{k!} R^{(k)}\, u \right\|_m \le C_{n,m} \| u \|_n \tag{8}$$

and

$$\| (\bar{t})^{-k} e^{\bar{t}\lambda} S^{(k)}(\lambda) u \| \leq C_{n,m} \| u \|_n. \tag{9}$$

In particular for all $u \in E_n$

$$\lim_{\lambda \to \infty} R(\lambda) u = \lim_{\lambda \to \infty} S(\lambda) u = 0 \text{ in } E_m. \tag{10}$$

On the other hand, conditions (iii), (iv) and (vi) in Definition III.4 imply that

$$\lambda R(\lambda) u = u + S(\lambda) u + R(\lambda) A u, \ \forall u \in D(A). \tag{11}$$

Relations (10) and (11) lead readily to the conclusion that $\lim_{\lambda \to \infty} \lambda R(\lambda) u = u$ in E_m for all $u \in D(A_n)$. This is actually true for all $u \in E_n$, for we can suppose $\overline{D(A_n)} = E_n$ and we have the following inequality

$$\| \lambda R(\lambda) v - v \|_m$$

$$\leq \| \lambda R(\lambda) (v - u) \|_m + \| u - v \|_m + \| \lambda R(\lambda) u - u \|_m$$

$$\leq C_{n,m} \| v - u \|_n + \| u - v \|_n + \| \lambda R(\lambda) u - u \|_m.$$

(ii) We have from (11)

$$A(\lambda) u = \lambda R(\lambda) A u + \lambda S(\lambda) u, \ \forall u \in D(A). \tag{12}$$

In view of (9) and the conclusion in (i) above we arrive at $(E_m) \lim_{\lambda \to \infty} A(\lambda) u = A u$ for all $u \in D(A_n)$.

(iii) Condition (vi) in Definition III.4 and condition 2) (i) in Theorem III.5 imply that

$$-A_m R(\lambda) u = u - \lambda R(\lambda) u + S(\lambda) u, \ \forall \lambda > \omega, \ \forall u \in E_n. \tag{13}$$

By condition 1) in Theorem III.1.5 we can consider A_m to be closed in E_m. Differentiating both sides of the above equality $(k+1)$ times, in view of the closedness of A_m we have in E_m

$$-A_m R^{(k+1)}(\lambda) u = -\lambda R^{(k+1)}(\lambda) - (k+1) R^{(k)}(\lambda) u + S^{(k+1)}(\lambda) u$$

$$\forall u \in E_n, \ \forall \lambda > \omega. \tag{14}$$

Multiplied on both sides by $R(\lambda)$ from the left the above equality becomes

$$(\lambda I - A_m) R(\lambda) R^{(k+1)} u = -(k+1) R(\lambda) R^{(k)}(\lambda) u + R(\lambda) S^{(k+1)}(\lambda) u$$

or

$$R^{(k+1)} u + (k+1) R(\lambda) R^{(k)}(\lambda) u = -S(\lambda) R^{(k+1)}(\lambda) u + R(\lambda) S^{(k+1)}(\lambda) u$$

$$\forall u \in E_n, \ \forall \lambda > \omega. \tag{15}$$

We finally arrive at the assertion in (iii) by combining (15) with (8) and (9). □

Lemma III.8. (i) A family of operators $\{T_\lambda(t) \mid \lambda > \omega, t \in [0,\bar{t}/4]\}$ on E^+ is well defined by

$$T_\lambda(t)\, u = e^{-\lambda t} \{u + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\lambda^2 t)^{k+1}}{(k+1)!} R^{(k)}(\lambda)\, u\}. \tag{16}$$

It is equi-bounded on E^+. Moreover

$$A(\lambda)\, T_\mu(t) = T_\mu(t)\, A(\lambda), \ \forall \lambda,\mu > \omega, \ \forall t \in [0,\bar{t}/4]. \tag{17}$$

(ii) $\forall n \in \mathbb{N}_0 \ \exists m \in \mathbb{N}_0 \ \forall u \in E_n$

$(E_m) \lim_{\lambda \to \infty} T_\lambda(t)\, u$ exists uniformly in $t \in [0,\bar{t}/4]$.

The limits, denoted by $T(t)$, $t \in [0,\bar{t}/4]$ constitute an equi-bounded family of operators on E^+.

Proof. (i) From (8) it readily follows that the family of operators $\{T_\lambda(t) \mid \lambda > \omega, t \in [0,\bar{t}/4]\}$ is well defined and equi-bounded on E^+. Indeed

$$\| T_\lambda(t)\, u \|_m \le C_{n,m} \| u \|_n, \ \forall u \in E_n, \ \forall \lambda > \omega, \ \forall t \in [0,\bar{t}/4]. \tag{18}$$

Moreover the relation $R(\lambda)\, R(\mu) = R(\mu)\, R(\lambda)$ in Definition III.4 (v) implies (17) directly.

(ii) Differentiating both sides of (16) we have for all $u \in E_n$

$$\frac{d}{dt} [T_\lambda(t)\, u] = -\lambda\, T_\lambda(t)\, u + e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\lambda^2 t)^k \lambda^2}{k!} R^{(k)}(\lambda)\, u$$

in E_m. This can be rewritten in the form

$$\frac{d}{dt} [T_\lambda(t)\, u] = A(\lambda)\, T_\lambda(t)\, u + P_{\lambda,t}\, u \tag{19}$$

where

$$P_{\lambda,t} = \lambda^2 e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(-\lambda^2 t)^{k+1}}{(k+1)!\,(k+1)!} [R^{(k+1)}(\lambda) + (k+1) R(\lambda) R^{(k)}(\lambda)]. \tag{20}$$

Lemma III.7. (iii) implies that for m large enough and $u \in E_n$, $\lambda > \omega$ and $t \in [0,\bar{t}/4]$

$$\| P_{\lambda,t}\, u \|_m \le \lambda^2 e^{-\lambda t} C_{n,m} \sum_{k=0}^{\infty} \frac{(\lambda^2 t)^{k+1}}{(k+1)!\,(k+1)!} \frac{(\bar{t}\lambda)^{k+1} + (k+1)!}{\lambda^{k+2}\, e^{\bar{t}\lambda}} \| u \|_n$$

$$= \lambda e^{-\lambda(t+\bar{t})} C_{n,m} \sum_{k=0}^{\infty} \left[\frac{(\lambda^2 t \bar{t})^{k+1}}{(k+1)!\,(k+1)!} + \frac{(\lambda t)^{k+1}}{(k+1)!} \right] \| u \|_n$$

$$\leq \lambda e^{-\lambda(t+\bar{t})} C_{n,m} (e^{\lambda\sqrt{t\bar{t}}} e^{\lambda\sqrt{t\bar{t}}} + e^{\lambda t}) \| u \|_n$$

$$= \lambda C_{n,m} (e^{-\lambda(\sqrt{\bar{t}}-\sqrt{t})^2} + e^{-\lambda\bar{t}}) \| u \|_n$$

$$\leq C_{n,m} \lambda e^{-\frac{1}{4}\bar{t}\lambda} \| u \|_n. \tag{21}$$

Since $T_\lambda(0) = I$ for all $\lambda > \omega$ and since the family of operators $\{T_\lambda(t) \mid \lambda > \omega, t \in [0,\bar{t}/4]\}$ is equi-bounded, we have for some sufficiently large m that in E_m

$$T_\lambda(t)u - T_\mu(t)u$$

$$= \int_0^t \frac{d}{ds} [T_\lambda(s) T_\mu(t-s) u] \, ds$$

$$= \int_0^t \{ \frac{d}{ds} [T_\lambda(s)] T_\mu(t-s) u + T_\lambda(s) \frac{d}{ds} [T_\mu(t-s) u] \} \, ds$$

$$\int_0^t \{A(\lambda) T_\lambda(s) T_\mu(t-s) u - T_\lambda(s) A(\mu) T_\mu(t-s) u$$

$$+ P_{\lambda,s} T_\mu(t-s) u - T_\lambda(s) P_{\mu,t-s} u \} \, ds, \ \forall u \in E_n.$$

Therefore, in view of (17) we have for m' sufficiently large

$$\| T_\lambda(t) u - T_\mu(t) u \|_{m'}$$

$$\leq \| \int_0^t T_\lambda(s) T_\mu(t-s) [A(\lambda) u - A(\mu) u] \, ds \|_{m'}$$

$$+ \| P_{\lambda,s} T_\mu(t-s) u - T_\lambda(s) P_{\mu,t-s} u \|_{m'}$$

$$\leq C_{m',m} \| A(\lambda) u - A(\mu) u \|_m + C_{m',n} (\lambda e^{-\frac{1}{4}\bar{t}\lambda} + \mu e^{-\frac{1}{4}\bar{t}\mu}) \| u \|_n, \ \forall u \in E_n. \tag{22}$$

Lemma III.7 (ii) and (22) then lead to the conclusion that for any $u \in D(A_n)$

$$\| T_\lambda(t) u - T_\mu(t) u \|_{m'} \to 0 \text{ as } \lambda, \mu \to \infty \text{ uniformly in } t \in [0,\bar{t}/4].$$

Hence uniformly in $t \in [0,\bar{t}/4]$ $\lim\limits_{\lambda \to \infty} T_\lambda(t) u$ exists, the limit of which is denoted by $T(t) u$. In fact this is true for all $u \in E_n$, for $\overline{D(A_n)} = E_n$ and

$$\| T_\lambda(t) v - T_\mu(t) v \|_{m'}$$

$$\le \| T_\lambda(t)v - T_\lambda(t)u \|_{m'} + \| T_\lambda(t)u - T_\mu(t)u \|_{m'} + \| T_\lambda(t)u - T_\mu(t)v \|_{m'}$$

$$\le C_{n,m'} \| u - v \|_n + \| T_\lambda(t)u - T_\mu(t)u \|_{m'} \ (u \in D(A_n),\ v \in E_n).$$

The equi-bounded of $\{T(t) \mid t \in [0,\bar{t}/4]\}$ follows from that of $\{T_\lambda(t) \mid \lambda > \omega,\ t \in [0,\bar{t}/4]\}$ and the uniform convergence in t of $\{T_\lambda(t)u\}$ as $\lambda \to \infty$. □

Lemma III.9. The family of operators $\{T(t) \mid t \in [0,\bar{t}/4]\}$ has the following properties:

(i) $T(0) = I$;

(ii) $\forall n \in \mathbb{N}_0\ \exists m \in \mathbb{N}_0\ \forall u \in E_n$

$$T(\cdot)u : [0,\bar{t}/4] \to E_m \text{ is continuous};$$

(iii) $\forall n \in \mathbb{N}_0\ \exists m \in \mathbb{N}_0\ \forall u \in D(A_n)$

$T(\cdot)u : [0,\bar{t}/4] \to E_m$ is continuously differentiable

$T(t)\,D(A_n) \subset D(A_m)$ for all $t \in [0,\bar{t}/4]$ and

$$\frac{d\,T(t)u}{dt} = A_m\,T(t)u = T(t)\,A_n\,u;$$

(iv) $T(t)\,T(s) = T(t+s)$ for all $t,s \ge 0$ such that $s + t \le \bar{t}/4$.

Proof. Statement (i), $T(0) = I$, is obvious from the definition. Statement (ii) follows from the continuity of $T_\lambda(\cdot)u : [0,\bar{t}/4] \to E_m$ and the uniform convergence of $T_\lambda(\cdot)u$ to $T(\cdot)u$ on $[0,\bar{t}/4]$ (Lemma III.8 (ii)).

(iii) For $n \in \mathbb{N}_0$ let m be chosen such that Lemma III.7 (ii) and Lemma III.8 (ii) hold. Since the family $\{T_\lambda(t) \mid \lambda > \omega,\ t \in [0,\bar{t}/4]\}$ is equi-bounded (Lemma III.8 (i)) there exists an $m' \ge m$ such that for any $u \in D(A_n)$

$$\| T_\lambda(t)\,A(\lambda)u - T(t)\,A\,u \|_{m'}$$

$$\le \| T_\lambda(t)\,(A(\lambda)u - A\,u) \|_{m'} + \| (T_\lambda(t) - T(t))A\,u \|_{m'}$$

$$\le C_{n,m'} \| A(\lambda)u - A\,u \|_m + \| (T_\lambda(t) - T(t))A\,u \|_m.$$

Thus, Lemma III.7 (ii) and Lemma III.8 (ii) lead to the assertion that $T_\lambda(\cdot)A(\lambda)u$ converges to $T(\cdot)A\,u$ in $E_{m'}$ uniformly in $t \in [0,\bar{t}/4]$. This combined with (19) and (21) implies that $\frac{d}{dt}[T_\lambda(t)u]$ converges to $T(t)\,A\,u$ in $E_{m'}$, uniformly in $t \in [0,\bar{t}/4]$. Therefore $\frac{d}{dt}[T(t)u]$ exists in $E_{m'}$ and

$$\frac{d}{dt}[T(t)u] = \lim_{\lambda \to \infty} \frac{d}{dt}[T_\lambda(t)u] = T(t)A\,u,\ u \in D(A_n). \tag{22'}$$

On the other hand for $u \in D(A_n)$

$$A_m R(\lambda) u = R(\lambda) A_n u. \tag{23}$$

Differentiating the above equality for k times, by virtue of the closedness of A_m we get

$$R^{(k)}(\lambda) D(A_n) \subset D(A_m), \ \forall k \in \mathbb{N}_0, \ \forall \lambda > \omega \tag{24}$$

$$A_m R^{(k)}(\lambda) u = R^{(k)}(\lambda) A_n u, \ \forall k \in \mathbb{N}_0, \ \forall \lambda > \omega, \ \forall u \in D(A_n). \tag{25}$$

From (24) and (25) by the same reason as above we have

$$T_\lambda(t) D(A_n) \subset D(A_m), \ \forall \lambda > \omega, \ \forall t \in [0, \bar{t}/4] \tag{26}$$

$$A_m T_\lambda(t) u = T_\lambda(t) A_n u, \ \forall \lambda > \omega, \ \forall t \in [0, \bar{t}/4], \ \forall u \in D(A_n). \tag{27}$$

Letting $\lambda \to \infty$ in (26) and (27), in view of the closedness of A_m again and Lemma III.8 (ii). We obtain finally

$$T(t) D(A_n) \subset D(A_m), \ \forall t \in [0, \bar{t}/4] \tag{28}$$

$$A_m T(t) u = T(t) A_n u, \ \forall t \in [0, \bar{t}/4], \ \forall u \in D(A_n). \tag{29}$$

(22)' and (29) together provide the wanted result.

(iv) Let $u \in D(A_n)$. From the equi-boundedness of the family $\{T(t) \mid 0 \le t \le \bar{t}/4\}$ and what we have proved in (iii) above it follows that for sufficiently large m' the function $T(t - \cdot) T(s + \cdot) u : [0, t] \to E_{m'}$ is continuously differentiable and

$$\begin{aligned} &T(t+s)u - T(t)T(s)u \\ &= \int_0^t \frac{d}{dr} [T(t-r) T(s+r) u] \, dr \\ &= \int_0^t [-T(t-r) A T(s+r) u + T(t-r) T(s+r) A u] \, dr \\ &= \int_0^t T(t-r) T(s+r) \ (Au - Au) \, dr = 0. \end{aligned}$$

Since $\overline{D(A_n)} = E_n$ and each of the operators $T(t)$ $(0 \le t \le \bar{t}/4)$ is continuous, $T(t) T(s) u = T(t+s) u$ for all $u \in E_n$, hence for all $u \in E_+$. □

Lemma III.10 (i) If we extend the family $\{T(t) \mid t \in [0, \bar{t}/4]\}$ of operators on E^+ to the family $\{T(t) \mid t \in [0, \infty)\}$ of operators on E^+ defined by

$$T(t) = [T(\bar{t}/4)]^k \, T(t'), \ \text{for } t = k\,\bar{t}/4 + t', \ 0 \le t' < \bar{t}/4 \tag{30}$$

then the later is a locally equi-bounded semigroup of operators on E^+.

(ii) The infinitesimal generator $\tilde{A}$ of $\{T(t) \mid t \in [0, \infty)\}$ is equal to the operator A.

(iii) If $\{S(t) \mid t \geq 0\}$ is a locally-bounded semigroup of operators on E^+ which has A as its infinitesimal generator, then $S(t) = T(t)$ for all $t \geq 0$.

Proof. The assertion in (i) follows readily from Lemma III.9.

(ii) Since $D(A) = \bigcup_{n \in \mathbb{N}_0} D(A_n)$, Lemma III.9 (iii) implies that $A \subset \bar{A}$. Let us show the converse. We have

$$R(\mu)\, T_\lambda(t)\, u = T_\lambda(t)\, R(\mu)\, u\,, \quad \forall \lambda,\mu > \omega\,, \quad \forall t \in [0,\bar{t}/4]\,, \quad \forall u \in E^+$$

and

$$R(\mu)\, T(t)\, u = T(t)\, u = T(t)\, R(\mu)\, u\,, \quad \forall \mu > \omega\,, \quad \forall t \in [0,\bar{t}/4]\,, \quad \forall u \in E^+.$$

Let $u \in D(\bar{A}_n)$. Choose $m \in \mathbb{N}_0$ such that $R(\lambda)\, E_n \subset D(A_m)$

$$\| R(\lambda)\, u \|_m \leq C_{n,m} \| u \|_n\,, \quad \forall u \in E_n$$

and such that Lemma III.7 (i) is verified. Letting $t \to 0+$ in

$$\frac{T(t)-I}{t}\, R(\lambda)\, u = R(\lambda)\, \frac{T(t)-I}{t}\, u$$

we get

$$A_m\, R(\lambda)\, u = \bar{A}_m\, R(\lambda)\, u = R(\lambda)\, \bar{A}_n\, u.$$

Multiplying both sides of the above equality by λ and then letting $\lambda \to \infty$, by the closedness of A_m we have $u \in D(A_m)$ and $A_m u = \bar{A}_n u$. In conclusion we have proved that $A = \bar{A}$.

(iii) Let $u \in D(A_n)$. In E_m with sufficiently large m we have

$$\begin{aligned} & T(t)\, u - S(t)\, u \\ &= \int_0^t \frac{d}{ds}\, \{S(t-s)\, T(s)\, u\}\, ds \\ &= \int_0^t \{-S(t-s)\, A\, T(s)\, u + S(t-s)\, T(s) A\, u\}\, ds \\ &= \int_0^t \{-S(t-s)\, T(s)\, A\, u + S(t-s)\, T(s)\, A\, u\}\, ds = 0. \end{aligned}$$

Since $\overline{D(A_n)} = E_n$ this is true for all $u \in E_n$, hence for all $u \in E^+$. □

Example III.11. Let $X = L^2(\mathbb{R})$, $B = e^{x^2 - \frac{d^2}{dx^2}}$ and $A = i\,a\, \frac{d^2}{dx^2} + b\, \frac{d}{dx} + c$ with $a \in \mathbb{R}$, $b \in \mathbb{C}$, $c \in \mathbb{C}$. Then, in the notation of Chapter I, $X_B^{0+} = S_{1/2}^{1/2}$, one of the spaces of type S

introduced by Gelfand and Shilov, cf. [Zh]. There are several characterizations of the space $S^{1/2}_{1/2}$. Here we use the one given by Van Eijndhoven and Meyers ([E-M])

$X^\tau_B = \{u \mid u$ extends to an entire analytic function $u(x+iy)$ such that $\| u \|^2_\tau = \iint_{\mathbb{R}^2} | u(x+iy) |^2 \exp[\tanh(\frac{\tau}{2}) x^2 - \frac{1}{\tanh(\frac{\tau}{2})} y^2\, dxdy < \infty\}$.

We can consider operator A as acting on the space S of Schwartz rapidly decreasing functions on $\mathbb{R}$. Moreover

$$A\, u = F^{-1}\, (-a\, i\, \zeta^2 + b\, i\, \zeta + c)\, F\, u\,,\ \ u \in S$$

where F and F^{-1} are the Fourier transform and its inverse. Recall that each of the spaces X^τ_B is F and F^{-1} invariant. Then, it is easy to see that the operator A maps each space X^τ_B into X^σ_B for any $\sigma \in (0,\tau)$. Therefore A maps $X^{0+}_B = S^{1/2}_{1/2}$ into itself and for any $\tau > 0$, A_τ, the part of A in X^τ_B is densely defined and closed. Let $\overline{t} > 0$ be fixed. Put

$$R(\lambda)\, u = \int_0^{\overline{t}} e^{-t\lambda}\, F^{-1}\, e^{t(-ai\zeta^2+bi\zeta+c)}\, F\, u\; dt \ \ (\lambda > 0\,, u \in S^{1/2}_{1/2})$$

and

$$S(\lambda)\, u = -F^{-1}\, e^{-\lambda\overline{t}-ai\zeta^2+bi\zeta+c}\, F\, u.$$

Note that

$$| \exp 2t\, [-ia(x+iy)^2 + i\, b(x+iy) + c] \,| \le$$

$$\le \exp 2t\, [\, | a |\, (\varepsilon x^2 + \frac{1}{\varepsilon} y^2) + \tfrac{1}{2}\, | \operatorname{Re} b \,|\, (1+y^2) + \tfrac{1}{2}\, (| I_m b \,|\, (\varepsilon x^2 + \frac{1}{\varepsilon}) + | \operatorname{Re} c \,|\,].$$

Then clearly $R(\lambda)$ and $S(\lambda)$ $(\lambda > 0)$ are well defined linear operators on $S^{1/2}_{1/2}$. It is simple to check that $((0,\infty), R(\lambda), S(\lambda))$ is a pseudo-resolvent triple on $S^{1/2}_{1/2}$. Also, for each $\tau > 0$ there exists a $\sigma > 0$ such that $R(\lambda)\, X^\tau_B \subset X^\sigma_B$; thus $R(\lambda) X^\tau_B \subset D(A_{\sigma'})$ for $\sigma' \in (0,\sigma)$, i.e. condition 2) (i) in Theorem III.5 is satisfied. Moreover, for any $\tau > 0$ there exists $\sigma \in (0,\tau)$ such that $R(\cdot)\, u : (0,\infty) \to X^\sigma_B$ and $S(\cdot)\, u : (0,\infty) \to X^\sigma_B$ are infinitely differentiable for all $u \in X^\tau_B$. Explicitly, for all $k \in \mathbb{N}_0$ any $u \in X^\tau_B$

$$R^{(k)}(\lambda)\, u = \int_0^{\overline{t}} (-t)^k\, e^{-\lambda t}\, F^{-1}\, e^{t(-ai\zeta^2+bi\zeta+c)}\, F\, u\; dt$$

and

$$S^{(k)}(\lambda)\, u = -(-t)^k\, F^{-1}\, e^{-\lambda\overline{t}-ai\zeta^2+bi\zeta+c}\, F\, u.$$

It then readily follows that

$$\| R^{(k)}(\lambda) u \|_\sigma \leq C_{\tau,\sigma} \, k! \, \lambda^{-(k+1)} \| u \|_\tau$$

and $(\lambda > 0, k \in I\!N_0, u \in X_B^\tau)$

$$\| S^{(k)}(\lambda) u \|_\sigma \leq C_{\tau,\sigma}(\overline{t})^k \, e^{-\lambda \overline{t}} \| u \|_\tau .$$

So conditions 2) (ii), (iii) and (iv) in Theorem III.5 hold true

In summary we have checked all the conditions of Theorem III.5, which then garantees that the operator A in $S_{1/2}^{1/2}$ is the infinitesimal generator of a locally equi-bounded semigroup $\{T(t) \mid t \geq 0\}$ on $S_{1/2}^{1/2}$. Indeed we have the explicit expression

$$T(t)\, u = F^{-1} \, e^{t(-ai\zeta^2 + bi\zeta + c)} \, F u.$$

A family of operators $\{T(t) \mid t \geq 0\}$ on E^+ is called a locally equi-continuous semigroup on E^+ if in Definition III.2 of locally equi-bounded semigroups conditions (iii) and (iv) are replaced by the following ones respectively.

(iii)' For any $\overline{t} > 0$, the family $\{T(t) \mid 0 \leq t \leq \overline{t}\}$ is equi-continuous.

(iv)' The mapping $T(\cdot)\, u : [0,\infty) \to E^+$ is continuous for all $u \in E^+$.

Obviously a locally equi-bounded semigroup $\{T(t) \mid t \geq 0\}$ on E^+ is a locally equi-continuous semigroup on E^+. We are going to prove

Theorem III.12. Assume that the inductive limit E^+ is regular and satisfies an interpolation type inequality as in Theorem I.1.6. Then a locally equi-continuous semigroup $\{T(t) \mid t \geq 0\}$ on E^+ is in fact a locally equi-bounded semigroup on E^+. □

Before the proof of this theorem we formulate a more general result of an arbitrary family of operators.

Theorem III.13. Suppose that the assumptions on the inductive limit E^+ in Theorem III.12 are satisfied. Then for a family of operators $\{T_\alpha \mid \alpha \in I\}$ on E^+ the following three statements are mutually equivalent:

(i) $\{T_\alpha \mid \alpha \in I\}$ is equi-continuous on E^+.

(ii) For any sequence $\{u_k\}$ converging (to zero) in E_n there exists $m \in I\!N$ such that $\{T_\alpha u_k\} \subset E_m$ for all $k \in I\!N_0$ and $\alpha \in I$ and the family $\{\{T_\alpha u_k\}_{k \in I\!N_0} \mid \alpha \in I\}$ of sequences converge (to zero) uniformly in E_m.

(iii) For any $n \in \mathbb{N}_0$ there exists $m \in \mathbb{N}_0$ and constant $M_{n,m}$ such that

$$\| T_\alpha u \|_m \le M_{n,m} \| u \|_n , \quad \forall u \in E_n , \quad \forall \alpha \in I.$$

Proof. The same arguments in the proof of Theorem I.1.6 apply to a family of operators here. We omit the details by only pointing out that the equi-continuity of a family of operator $\{T_\alpha \mid \alpha \in I\}$ implies the boundedness of the set $\{T_\alpha u \mid \alpha \in I\}$ for any fixed u. ▯

Now we give

Proof of Theorem III.12. According to Theorem III.13 condition (iii)' above implies condition (iii) in Definition III.2. It remains to check condition (iv). Let $n \in \mathbb{N}_0$ and $\bar{t} > 0$ be given. By condition (iii) just verified there exist $m \in \mathbb{N}_0$ and constant $M_{n,m} > 0$ such that

$$\| T(t) u \|_m \le M_{n,m} \| u \|_n , \quad \forall u \in E_n , \quad \forall t \in [0,\bar{t}]. \tag{31}$$

On the other hand, for m so fixed, by our assumption there exists $m' \in \mathbb{N}_0$ such that

$$\| v \|_{m'} \le \phi_{m,m'} (\| v \|_m , \| v \|), \quad \forall v \in E_m. \tag{32}$$

Here $\phi_{m,m'} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function monotone increasing in each of its arguments and such that $\phi_{m,m'}(M,s) \to 0$ as $s \to 0$ for each fixed $M > 0$; $\| \cdot \|$ is the norm of a normed space E such that $E^+ \hookrightarrow E$.

For $u \in E_n$ relations (31) and (32) lead to that

$$\| T(t) u - T(t_0) u \|_{m'} \le \phi_{m,m'} (2M_{n,m} \| u \|_n , \| T(t) u - T(t_0) u \|)$$

from which follows readily the continuity of $T(\cdot) u : [0,\bar{t}] \to E_{m'}$. So condition (iv) in Definition III.2 is satisfied. ▯

To conclude we remark that the spaces $X_B^{\sigma +}$ $(\sigma \in \mathbb{R})$ satisfy the assumptions in Theorem III.12 above so the concepts of locally equi-bounded semigroups and of locally equi-continuous semigroups on them are equivalent.

IV. Hilbert Spaces of Harmonic Functions and Linear Operators

In Section 1 we give a brief review of the classical theory of harmonic functions and spherical harmonics, especially the results which will be used in the subsequent sections.

In Section 2 we give conditions on a weight function $\mu : (0,\infty) \to (0,\infty)$ and a nonnegative sequence $\{\lambda_m\}_{m\in \mathbb{N}_0}$ such that the (pre-) Hilbert space $HA^q(\mu) = \{u \mid u$ harmonic in $\mathbb{R}^q$ and $\| u \|_\mu = (\int_{\mathbb{R}^q} |u(x)|^2 \mu(|x|)\, dx)^{1/2} < \infty\}$ can be identified with $(D(\Lambda), \|\cdot\|_\Lambda)$ where $\Lambda : D(\Lambda) \subset L^2(S^{q-1}) \to L^2(S^{q-1})$ is a nonnegative self-adjoint operator in $L^2(S^{q-1})$ defined by

$$A\, u = \sum_{m=0}^{\infty} \lambda_m P_m u$$

for

$$u \in D(A) = \{u \in L^2(S^{q-1}) \mid \| u \|_\Lambda^2 = \sum_{m=0}^{\infty} \lambda_m^2 \| P_m u \|^2 < \infty\}.$$

Here $P_m : L^2(S^{q-1}) \to \mathrm{H}_m^q$ is the projection operator onto the subspace of spherical harmonics of degree m in q dimensions.

In Section 3 we give a characterization of the ranges of the propagation operators for the fractional spherical reaction-diffusion equation

$$\frac{\partial u}{\partial t} = -(-\Delta_{LB})^{\nu/2}\, u$$

with Δ_{LB} the spherical Laplace-Beltrami operator on S^{q-1}.

Finally in Section 4 we are concerned with several naturally arising linear operators in spaces of harmonic functions on $\mathbb{R}^q$. Namely, the differentiation operators ∂_k, the multiplication operators M_k, the "general linear" operators L_A and the harmonic product $\odot$. In particular we present weighted Hilbert spaces of harmonic functions wherein the differentiation operators are continuous or even compact.

IV.1. Preliminaries

Let $q \in \mathbb{N}$. As usual $\mathbb{R}^q$ stands for the Euclidean space of dimension q. The inner product in $\mathbb{R}^q$ is denoted by $x \cdot y$ for $x,y \in \mathbb{R}^q$ with norm $|x|$. For $x \in \mathbb{R}^q$ and $r > 0$ we set $B^q(x,r) = \{y \in \mathbb{R}^q \mid |y - x| < r\}$, the ball in $\mathbb{R}^q$ with center x and radius r; $\overline{B}^q(x,r)$ stands for the closure of $B^q(x,r)$. We adopt the abbreviations that $B^q(0,r) \equiv B^q(r)$, $\overline{B}^q(0,r) \equiv \overline{B}^q(r)$.

For a twice continuously differentiable function $u(x)$ on Ω, an open set in $\mathbb{R}^q$, we define the Laplace operator Δ and the orbital angular momentum operator L^2 as follows

$$(\Delta u)(x) = \sum_{k=1}^{q} \frac{\partial^2 u(x)}{\partial x_k^2}, \quad x \in \Omega \tag{1}$$

$$(L^2 u)(x) = \frac{1}{2} \sum_{1 \le j,k \le q} (x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j})^2 u(x), \quad x \in \Omega. \tag{2}$$

A direct while somewhat tedious calculation gives rise to

Proposition IV.1.1. For a C^2 function on Ω there always holds the identity

$$|x|^2 \Delta u = L^2 u + \partial_n^2 u + (q-2)\, \partial_n u \tag{3}$$

where $\partial_n u = \sum_{k=1}^{q} x_k \frac{\partial u}{\partial x_k}$. □

If spherical coordinates are introduced, viz.

$$x_1 = r \cos\theta_1$$

$$x_k = r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{k-1} \cos\theta_k, \; 2 \le k \le q-1$$

$$x_q = r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{q-2} \sin\theta_{q-1} \tag{4}$$

then clearly $\partial_n = r \frac{\partial}{\partial r}$. Similarly

$$L^2 u = \frac{\partial^2 u}{\partial \theta_1^2} + \sum_{k=2}^{q-1} \frac{1}{\sin^2\theta_1 \cdots \sin^2\theta_{k-1}} \frac{\partial^2 u}{\partial \theta_k^2} + \sum_{k=1}^{q-2} (q-k-1) \frac{\operatorname{ctan}\theta_k}{\sin^2\theta_1 \cdots \sin^2\theta_{k-1}} \frac{\partial u}{\partial \theta_k}. \tag{5}$$

Thus we have the expression for Δ in polar coordinates

$$r^2 \Delta u = \Delta_{LB}\, u + (r \frac{\partial}{\partial r})^2 u + (q-2)(r \frac{\partial}{\partial r}) u. \tag{6}$$

Here Δ_{LB} stands for the right hand side in (5), namely, the Laplace-Beltrami operator.

Let $S^{q-1} = \{x \in \mathbb{R}^q \mid |x| = 1\}$ be the unit sphere in $\mathbb{R}^q$. The points on S^{q-1} will be denoted by Greek letters ζ, η et al. If not mentioned otherwise we always adopt the custom of relating x and ζ such that $x = r\zeta$ with $r = |x|$. Similarly, if a function u is defined on a domain Ω containing S^{q-1}, $u(x)$ represent the function defined on Ω while $u(\zeta)$ stands for its restriction on S^{q-1}.

A function $u(x)$ in $C^2(\Omega)$ is said to be harmonic iff $\Delta u = 0$. For a harmonic function $u(x)$ on $\mathbb{R}^q$ which is homogeneous of degree m, one has, since $(r \frac{\partial}{\partial r})\, u(x) = (r \frac{\partial}{\partial r})(r^m u(\zeta)) = m\, r^m u(\zeta) = m\, u(x)$, that

$$L^2 u(x) = -m\,(m+q-2)\, u(x) \tag{7}$$

or

$$\Delta_{LB}\, u(\zeta) = -m\,(m+q-2)\, u(\zeta). \tag{8}$$

Let H_m^q denote the set of all homogeneous harmonic polynomials of degree m in q variables, and let $\mathbf{H}_m^q$ be the set of the restrictions $p(\zeta)$ of $p(x)$ in H_m^q. The entries in H_m^q and $\mathbf{H}_m^q$ are respectively called solid harmonics and spherical harmonics of degree m and dimension q. Both H_m^q and $\mathbf{H}_m^q$ are vector spaces with the usual linear operations and they are isomorphic to each other in the natural way. Let $N(q,m)$ denote their dimension. Then we have via a clever counting (cf. [Mü])

$$\begin{aligned} &N(q,0) = 1 \\ &N(q,m) = (2m+q-2)\, \frac{\Gamma(m+q-2)}{\Gamma(m+1)\,\Gamma(q-1)}\,, \quad m \geq 1. \end{aligned} \tag{9}$$

In particular $N(2,m) = 2$ and $N(3,m) = 2m+1$. From (9) readily follows the estimate that

$$N(q,m) \leq K_q\, m^{q-2} \quad (K_q \text{ constant}). \tag{10}$$

Let $L^2(S^{q-1})$ be the L^2-space of functions on S^{q-1} with inner product defined by

$$(u,v) = \int_{S^{q-1}} u(\zeta)\, \overline{v(\zeta)}\, d\,\sigma_{q-1}(\zeta) \tag{11}$$

for $u, v \in L^2(S^{q-1})$, and corresponding norm

$$\| u \| = \Big(\int_{S^{q-1}} |\, u(\zeta) \,|^2\, d\,\sigma_{q-1}(\zeta) \Big)^{1/2}. \tag{12}$$

Here $d\,\sigma_{q-1}$ stands for the area element on S^{q-1}. Each of the spaces $\mathbf{H}_m^q$ is a finite dimensional subspace of $L^2(S^{q-1})$. For $p_m \in H_m^q$ and $q_n \in H_n^q$, Green's theorem implies that

$$\int_{S^{q-1}} [p_m(\zeta)\, \frac{\partial \bar{q}_n}{\partial r}(\zeta) - \bar{q}_n(\zeta)\, \frac{\partial p_m}{\partial r}(\zeta)]\, d\,\sigma_{n-1} = 0.$$

Since $\dfrac{\partial u}{\partial r} = m\, u$ for a homogeneous function of degree m we have $(n-m)\,(p_m, q_n) = 0$. Thus, if $n \neq m$ then $(p_m, q_n) = 0$. This shows that the subspaces $\mathbf{H}_m^q (m \in \mathbb{N}_0)$ are mutually orthogonal. Their linear span is actually dense in $L^2(S^{q-1})$. In order to show this let us first prove that any polynomial p_m in P_m^q (the set of homogeneous polynomials of degree m in q variables) can be uniquely written as $p_m(x) = \tilde{p}_m(x) + |\, x\,|^2\, p_{m-2}(x)$ where $\tilde{p}_m \in H_m^q$ and $p_{m-2} \in P_{m-2}^q$. In fact, if we define

$$< p_m, q_m > = p_m(\partial)\, q_m\,, \quad p_m, q_m \in P_m^q \tag{13}$$

then it is easy to see that $< \cdot\,, \cdot >$ is a inner product in P_m^q. For any $\tilde{p}_m \in H_m^q$ and $p_{m-2} \in P_{m-2}^q$ we have

$$< \tilde{p}_m,\, |\, x\,|^2\, p_{m-2} > = p_{m-2}(\partial)\, \Delta \tilde{p}_m = 0$$

which implies that H_m^q is orthogonal to $|\, x\,|^2\, P_{m-2}^q$ in P_m^q. They are in fact orthogonal duals of each other. For, if $< p_m,\, |\, x\,|^2\, p_{m-2} > = 0$ for all $p_{m-2} \in P_{m-2}^q$ and some $p_m \in P_m^q$ then

$<\Delta p_m, p_{m-2}> = 0$ with $\Delta p_m \in P^q_{m-2}$ and hence $\Delta p_m = 0$ by setting $p_{m-2} = \Delta p_m$. Therefore the decomposition $p_m(x) = \tilde{p}_m(x) + |x|^2 p_{m-2}(x)$ holds true. Continuing in this way we derive the decomposition

$$p_m(x) = \begin{cases} \tilde{p}_m(x) + |x|^2 \tilde{p}_{m-2}(x) + |x|^4 \tilde{p}_{m-4}(x) + \cdots + |x|^m \tilde{p}_0(x) \\ \tilde{p}_m(x) + |x|^2 \tilde{p}_{m-2}(x) + |x|^4 \tilde{p}_{m-4}(x) + \cdots + |x|^{m-1} \tilde{p}_1(x) \end{cases} \tag{14}$$

for m even or odd respectively. In particular we have shown that $p_m(\zeta) = \sum_{j=0}^{m} \tilde{p}_j(\zeta)$. On the other hand the space $C(S^{q-1})$, functions continuous on S^{q-1}, is obviously dense in $L^2(S^{q-1})$ and the Stone-Weierstrass theorem in its turn implies that Span $\{p_m(\zeta) \mid p_m \in P^q_m, m \in \mathbb{N}_0\}$ is dense in $C(S^{q-1})$. In summary we arrive at the assertion that

$$L^2(S^{q-1}) = \bigoplus_{m=0}^{\infty} \mathrm{H}^q_m. \tag{15}$$

In the sequel $P_m : L^2(S^{q-1}) \to \mathrm{H}^q_m$ will be the projection mapping onto H^q_m.

Using the invariance property of H^q_m with respect to the group $SO(q)$ we have the following crucial

Proposition IV.1.2. Given $m \in \mathbb{N}_0$ and any orthonormal basis $\{e^q_{m,j} \mid m \in \mathbb{N}_0\}$ of H^q_m. Then

$$\sum_{j=1}^{N(q,m)} e^q_{m,j}(\zeta)\, \overline{e^q_{m,j}(\eta)} = \frac{N(q,m)}{\sigma_{q-1}} P^q_m(\zeta \cdot \eta),\ \zeta,\eta \in S^{q-1} \tag{16}$$

where $\sigma_{q-1} = \int_{S^{q-1}} d\sigma_{q-1} = \dfrac{2\pi^{q/2}}{\Gamma(q/2)}$ is the total area of the unit sphere. Here $P^q_m(t)$ is a Gegenbauer polynomial given by

$$P^q_m(t) = (-1/2)^m \frac{\Gamma(\frac{q-1}{2})}{\Gamma(m+\frac{q-1}{2})} (1-t)^{-\frac{q-3}{2}} \left[\frac{d}{dt}\right]^m (1-t^2)^{m+\frac{q-3}{2}}. \tag{17}$$

□

For the proof we refer to [Mü]. Multiplying both sides of (16) by $u(\eta)$ and integrating with respect to η we arrive at

Proposition IV.1.3. For $u(\zeta) \in L^2(S^{q-1})$ and $m \in \mathbb{N}_0$ it holds true that

$$(P_m u)(\zeta) = \frac{N(q,m)}{\sigma_{q-1}} \int_{S^{q-1}} P_m^q(\zeta \cdot \eta)\, u(\eta)\, d\sigma_{q-1}(\eta). \tag{18}$$

□

The Gegenbauer polynomials enjoy the following properties.

Proposition IV.1.4.

1) P_m^q is a polynomial of degree m

2) $\int_{-1}^{1} P_n^q(t)\, P_m^q(t)\, (1-t^2)^{\frac{q-3}{2}}\, dt = 0$ for $n \neq m$ (19)

3) $P_m^q(1) = 1$, $m \in \mathbb{N}_0$

4) For $t, \tau \in \mathbb{R}$ with $|t| < 1$ and $|\tau| \leq 1$

$$\sum_{m=0}^{\infty} N(q,m)\, t^m\, P_m^q(\tau) = \frac{1-t^2}{(1+t^2-2t\tau)^{q/2}} \tag{20}$$

where for each fixed t the series converges uniformly in $\tau \in [-1,1]$. □

Setting $\zeta = \eta$ in (16) and using 3) in Proposition IV.1.4 we have

Proposition IV.1.5. For $u \in \mathrm{H}_m^q$

$$|u(\zeta)| \leq \left[\frac{N(q,m)}{\sigma_{q-1}}\right]^{½} \|u\|, \quad \zeta \in S^{q-1}. \tag{21}$$

The following characteristic property of harmonic functions, the mean value property, is proved by a clever use of Green's theorem.

Theorem IV.1.6.

1) If $u : \Omega \subset \mathbb{R}^q \to \mathbb{C}$ is a harmonic function in the region Ω, then, for any $x^0 \in \Omega$ and $r > 0$ such that $B(x^0, r) \subset \Omega$, we have

$$u(x^0, r) \equiv \frac{1}{\sigma_{q-1}} \int_{S^{q-1}} u(x^0 + r\zeta)\, d\sigma_{q-1}(\zeta) = u(x^0). \tag{22}$$

2) Conversely, if a function $u : \Omega \subset \mathbb{R}^q \to \mathbb{C}$ is continuous and for any $x^0 \in \Omega$ there exists an $r^0 > 0$ such that $B(x^0, r^0) \subset \Omega$ and whenever $r \subset r^0$ equality (22) holds true, then u is infinitely differentiable and harmonic in Ω. □

The above Theorem IV.1.6 has the following two corollaries.

Theorem IV.1.7. For a harmonic function $u : \Omega \subset \mathbb{R}^q \to \mathbb{R}$, on the region Ω if $A = \sup_{x \in \Omega} u(x) < \infty$ and u is not a constant on Ω, then $u(x) < A$ for all $x \in \Omega$. Thus, if $u : \overline{\Omega} \subset \mathbb{R}^q \to \mathbb{R}$ is harmonic in Ω and continuous in $\overline{\Omega}$, then either u is a constant on Ω or u attains its maximum only on the boundary. $\square$

Theorem IV.1.8. Let $\{u_n(x)\}$ be a sequence of harmonic functions on the region Ω. If for any $x^0 \in \Omega$ there exists an $r^0 > 0$ such that $B(x^0, r^0) \subset \Omega$ and $\{u_n(x)\}$ converges uniformly on $B(x^0, r^0)$, then it converges to a harmonic function on Ω. $\square$

Applying the above properties of harmonic functions we can solve the Dirichlet problem for the unit ball.

Theorem IV.1.9.

(i) Assume that $u(\zeta) \in L^2(S^{q-1})$ and let $P_m u$ $(m \in \mathbb{N}_0)$ be its projection onto the subspace $\mathbf{H}^q_m$. Then the series

$$\sum_{m=0}^{\infty} (P_m u)(x) = \sum_{m=0}^{\infty} r^m (P_m u)(\zeta), \quad x = r\zeta, \ r < 1 \tag{23}$$

converges uniformly on the ball $B^q(\rho)$ for each $\rho < 1$, the sum of which, denoted by $u(x)$, is harmonic in $B^q(1)$ and is equivalently given by the Poisson formula

$$u(x) = \frac{1}{\sigma_{q-1}} \int_{S^{q-1}} \frac{1 - |x|^2}{|x - \zeta|^2} u(\zeta)\, d\sigma_{q-1}(\zeta). \tag{24}$$

Moreover, setting $u_r(\zeta) = u(r\zeta)$ for $r \in (0,1)$ and $\zeta \in S^{q-1}$, we have $\| u_r - u \| \to 0$ as $r \uparrow 1$.

(ii) If $u(\zeta) \in C(S^{q-1})$, then there exists a unique function $u(x)$ which is harmonic in $B^q(1)$ and continuous on $\overline{B}^q(1)$ with the given boundary values $u(\zeta)$ on S^{q-1}. It is given by (23) or equivalently by the Poisson formula (24). $\square$

The following theorem on the central expansion of harmonic functions is important in our further work.

Theorem IV.1.10. (Cf. [Gr])

(i) $u(\zeta) \in L^2(S^{q-1})$ can be extended to a harmonic function $u(x)$ on $B^q(R)$ $(R > 1)$, iff

$$\sum_{m=0}^{\infty} r^{2m} \| P_m u \|^2 < \infty \quad \text{for any } r \in (0,R). \tag{25}$$

(ii) If $u(x)$ is a harmonic function on $B^q(R)$, then the series

$$\sum_{m=0}^{\infty} r^m (P_m u)(\zeta) = \sum_{m=0}^{\infty} (P_m u)(x) \tag{26}$$

converges to $u(x)$ uniformly on each ball $B^q(r)$, $r \in (0,R)$.

Proof. i) Assume that the condition (25) is satisfied. Then for $0 < r < \rho < R$ we have

$$\begin{aligned} |(P_m u)(x)| &= | r^m (P_m u)(\zeta) | \\ &\le \left[\frac{K_q}{\sigma_{q-1}} \right]^{1/2} r^m m^{\frac{q-2}{2}} \| P_m u \| \quad \text{(by (10) and (21))} \\ &\le \frac{1}{2} \left[\frac{K_q}{\sigma_{q-1}} (r/\rho)^{2m} m^{q-2} + \rho^{2m} \| P_m u \|^2 \right]. \end{aligned} \tag{27}$$

Therefore, the series in (26) converges uniformly on each ball $B^q(r)$ $(r \in (0,R))$. Theorem IV.1.8 above then implies that the sum, denoted by $u(x)$, is harmonic on $B^q(R)$. Especially, $\sum_{m=0}^{\infty} (P_m u)(\zeta)$ converges uniformly in $\zeta \in S^{q-1}$ and hence in $L^2(S^{q-1})$. This shows that $u(x)$ is indeed an extension of $u(\zeta)$.

Conversely, suppose that $u(\zeta)$ is extendible to a harmonic function $u(x)$ on the ball $B^q(R)$ $(R > 1)$. For fixed $r \in (1,R)$ set u_r by $u_r(\zeta) = u(r\zeta)$, $\zeta \in S^{q-1}$. Theorem IV.1.9 (ii) above then leads to the assertion that the function $v_1(x)$ on $\bar{B}^q(1)$ defined by

$$\begin{cases} v_1(x) = \sum_{m=0}^{\infty} \rho^m (P_m u_r)(\zeta), \quad x = \rho\zeta, \rho < 1 \\ v_1(\zeta) = u_r(\zeta) = u(r\zeta), \quad \zeta \in S^{q-1} \end{cases}$$

is harmonic in $B^q(1)$ and continuous on $\bar{B}^q(1)$. On the other hand it is easy to verify that the function $v_2(x) = u(r\rho\zeta)$ $(x = \rho\zeta)$ is harmonic in $B^q(1)$ and continuous on $\bar{B}^q(1)$ with the boundary values $v_2(\zeta) = u(r\zeta)$ $(\zeta \in S^{q-1})$. By the uniqueness of the solution to the Dirichlet problem (Theorem IV.1.9 (ii)) we then have $v_1 \equiv v_2$. In particular $v_1(\frac{1}{r}\zeta) = v_2(\frac{1}{r}\zeta)$, namely

$$u(\zeta) = \sum_{m=0}^{\infty} r^{-m} (P_m u_r)(\zeta) \text{ uniformly in } \zeta \in S^{q-1}.$$

Thus we conclude that

$$P_m u = r^{-m} P_m u_r \tag{28}$$

from which it follows that

$$\sum_{m=0}^{\infty} r^{2m} \| P_m u \|^2 = \sum_{m=0}^{\infty} \| P_m u_r \|^2 < \infty.$$

This completes the proof for (i).

The assertion in (ii) readily follow from the proof for (i). □

Finally we give the following theorem which provides estimates for the values of the derivatives of a homogeneous harmonic polynomial in terms of the L^2-norm on the unit sphere of the polynomial itself. This will play an essential role in our investigation of differential operators in spaces of harmonic functions (cf. Section IV.4 below).

Theorem IV.1.11. For $p_m \in H_m^q$ and $\alpha \in \mathbb{N}_0^q$ there exists a constant C_α such that

$$\left| \frac{\partial^\alpha p_m}{\partial x^\alpha} (\zeta) \right| \le C_\alpha \, m^{\frac{q-1}{2}+|\alpha|} \, \| p_m \| , \ \zeta \in S^{q-1}. \tag{29}$$

For the proof of the above theorem we need

Lemma IV.1.12. For a harmonic function u in $B^q(\bar{x},r)$ and $\alpha \in \mathbb{N}_0^q$

$$\left| \frac{\partial^\alpha u}{\partial x^\alpha} (\bar{x}) \right| \le A_\alpha \, r^{-\frac{q}{2}-|\alpha|} \, \Big(\int_{B^q(\bar{x},r)} | u(y) |^2 \, dy \Big)^{1/2} \tag{30}$$

where A_α is a constant depending only on α.

Proof. Take a radial C^∞ function $\phi(x)$ on $\mathbb{R}^q$ (i.e., $\phi(x) = \tilde{\phi}(|x|)$ for a function $\tilde{\phi} : \mathbb{R}^+ \to \mathbb{R}$), which has its support in $B^q(1)$ and which is normalized in the sense that $\int_{\mathbb{R}^q} \phi(x)\, dx = 1$. Then

$$\int_0^1 \rho^{q-1} \sigma_{q-1} \tilde{\phi}(\rho)\, d\rho = \int_0^1 \rho^{q-1} d\rho \int_{S^{q-1}} \phi(\rho\zeta)\, d\sigma_{q-1}(\zeta) = \int_{\mathbb{R}^q} \phi(y)\, dy = 1.$$

From this and the mean value theorem, for $\phi_r(y) = r^{-q} \phi(y/r)$ we have

$$\int_{\mathbb{R}^q} u(\bar{x}-y)\, \phi_r(y)\, dy$$

$$= \int_0^r \rho^{q-1} d\rho \int_{S^{q-1}} u(\bar{x}-\rho\zeta)\, r^{-q} \phi(\frac{\rho}{r}\zeta)\, d\sigma_{q-1}(\zeta)$$

$$= \int_0^r \rho^{q-1} d\rho\, r^{-q} \tilde{\phi} \left[\frac{\rho}{r} \right] \sigma_{q-1}\, u(\bar{x})\, d\rho$$

$$= (\int_0^1 \rho^{q-1}\, \sigma_{q-1}\, \bar{\phi}(\rho)\, d\rho)\; u(\bar{x}) = u(\bar{x}).$$

This can be rewritten as

$$u(\bar{x}) = \int_{\mathbb{R}^q} u(y)\, \phi_r(\bar{x}-y)\, dy.$$

Hence

$$\frac{\partial^\alpha u}{\partial x^\alpha}(\bar{x}) = \int_{\mathbb{R}^q} u(y)\, \frac{\partial^\alpha \phi_r}{\partial x^\alpha}(\bar{x}-y)\, dy.$$

Noticing that $\frac{\partial^\alpha \phi_r}{\partial x^\alpha}(x) = r^{-q-|\alpha|}\, \frac{\partial^\alpha \phi}{\partial x^\alpha}\left[\frac{x}{r}\right]$ we have by Schwartz inequality

$$\left|\frac{\partial^\alpha u}{\partial x^\alpha}(\bar{x})\right| \le (\int_{B^q(\bar{x},r)} |\, u(y)\,|^2\, dy)^{1/2}\, (\int_{B^q(\bar{x},r)} r^{-2q-2|\alpha|} \left|\frac{\partial^\alpha \phi}{\partial x^\alpha}\left[\frac{\bar{x}-y}{r}\right]\right|^2 dy)^{1/2}$$

$$\le A_\alpha\, r^{-\frac{q}{2}-|\alpha|}\, (\int_{B^q(\bar{x},r)} |\, u(y)\,|^2\, dy)^{1/2}$$

where

$$A_\alpha = (\int_{\mathbb{R}^q} \left|\frac{\partial^\alpha \phi}{\partial x^\alpha}(x)\right|^2 dx)^{1/2}.$$

□

Proof of Theorem IV.1.11. For $\varepsilon > 0$ we have

$$\int_{B^q(1+\varepsilon)} |\, p_m(x)\,|^2\, dx = \int_0^{1+\varepsilon} r^{q-1}\, dq \int_{S^{q-1}} |\, p_m(r\zeta)\,|^2\, d\,\sigma_{q-1}(\zeta)$$

$$= \int_0^{1+\varepsilon} r^{2m+q-1}\, dr \int_{S^{q-1}} |\, p_m(\zeta)\,|^2\, d\,\sigma_{q-1}(\zeta)$$

$$= \frac{(1+\varepsilon)^{2m+q}}{2m+q}\, \|\, p_m\,\|^2.$$

For $\bar{x} \in S^{q-1}$ by the above lemma we obtain

$$\left|\frac{\partial^\alpha p_m}{\partial x^\alpha}(\bar{x})\right| \le A_\alpha\, \varepsilon^{-\frac{q}{2}-|\alpha|}\, (\int_{B^q(\bar{x},\varepsilon)} |\, p_m(x)\,|^2\, dx)^{1/2}$$

$$\leq A_\alpha\, \varepsilon^{-\frac{q}{2}-|\alpha|} \left(\int_{B^q(1+\varepsilon)} |p_m(x)|^2\, dx \right)^{\frac{1}{2}}$$

$$\leq A_\alpha \frac{(1+\varepsilon)^{m+\frac{q}{2}}\, \varepsilon^{-\frac{q}{2}-|\alpha|}}{(2m+q)^{\frac{1}{2}}} \| p_m \|.$$

Choosing $\varepsilon = 1/m$ we arrive at

$$\left| \frac{\partial^\alpha p_m}{\partial x^\alpha}(\bar{x}) \right| \leq A_\alpha \frac{(1+1/m)^{m+q/2}\, m^{\frac{q}{2}+|\alpha|}}{(2m+q)^{\frac{1}{2}}} \| p_m \|$$

$$\leq C_\alpha\, m^{\frac{q-1}{2}+|\alpha|} \| p_m \|.$$

□

The above Theorem IV.1.11 can be found in [St], Appendix C. Here we have a little more precise estimate on the power of m in (29), which is important to us.

IV.2. Identification of Weighted Hilbert spaces of Harmonic Functions on $\mathbb{R}^q$ with Domains of Positive Self-adjoint Operators in $L^2(S^{q-1})$

As is seen in the last section we have an identity decomposition for the Hilbert space $L^2(S^{q-1})$ into spherical harmonics: $L^2(S^{q-1}) = \bigoplus_{m=0}^{\infty} \mathrm{H}^q_m$. For a sequence $\{\lambda_m\}_{m\in \mathbb{N}_0}$ of positive numbers, then, we have a well defined positive self-adjoint operator Λ on $L^2(S^{q-1})$, i.e.

$$\Lambda u = \sum_{m=0}^{\infty} \lambda_m P_m u$$
$$D(\Lambda) = \{ u \in L^2(S^{q-1}) \mid \sum_{m=0}^{\infty} \lambda_m^2 \| P_m u \|^2 < \infty \}. \tag{31}$$

The domain $D(\Lambda)$, equipped with the inner product

$$(u,v)_\Lambda = \sum_{m=0}^{\infty} \lambda_m^2 (P_m u, P_m v) \tag{32}$$

and corresponding norm

$$\| u \|_\Lambda = \left(\sum_{m=0}^{\infty} \lambda_m^2 \| P_m u \|^2 \right)^{\frac{1}{2}} \tag{33}$$

is a Hilbert space by itself. If the sequence $\{\lambda_m\}$ is such that $R^m / \lambda_m = O(1)$ for any $R > 0$, then Theorem IV.1.10 implies that each u in $D(\Lambda)$ extends uniquely to a harmonic function on $\mathbb{R}^q$.

On the other hand, it is easy to see that the space

$$HA^q(\mu) = \{u(x) \text{ harmonic on } \mathbb{R}^q \mid \|u\|_\mu = \{\int_{\mathbb{R}^q} |u(x)|^2 \mu(|x|)\,dx\}^{1/2} < \infty\} \tag{34}$$

is a pre-Hilbert space with the inner product

$$(u, v)_\mu = \int_{\mathbb{R}^q} u(x)\,\overline{v(x)}\,\mu(|x|)\,dx. \tag{35}$$

Here $\mu : (0,\infty) \to (0,\infty)$ is a Lebesgue measurable weight function.

The following fundamental result is on the identification of a space $HA^q(\mu)$ with a space $D(\Lambda)$ for an appropriate pair of weight function μ and sequence $\{\lambda_k\}$.

Theorem IV.2.1. Assume that a weight function $\mu : (0,\infty) \to (0,\infty)$ and a positive sequence $\{\lambda_m\}_{m\in \mathbb{N}_0}$ satisfy the conditions below:

1) $\int_0^\infty r^{2m+q-1}\,\mu(r)\,dr < \infty, \quad \forall m \in \mathbb{N}_0$

2) $R^m / \lambda_m = O(1), \quad \forall R > 0$

3) $\{\lambda_m^2\} \sim \int_0^\infty r^{2m+q-1}\,\mu(r)\,dr.$ Here and afterwards, for two sequences of positive numbers $\{a_m\}$ and $\{b_m\}$ we write $\{a_m\} \sim \{b_m\}$ iff $0 < \liminf (a_m b_m^{-1}) \le \limsup (a_m b_m^{-1}) < \infty$.

Then the space $HA^q(\mu)$ is isomorphic to the space $D(\Lambda)$ as normed spaces. The isomorphism is exactly the restriction-extension mapping. Furthermore, if, instead of condition 3) above, $\lambda_m^2 = \int_0^\infty r^{2m+q-1}\,\mu(r)\,dr$, then the isomorphism is actually an isometry.

Proof. Let $u(\zeta) \in D(\Lambda)$. Because of the condition 2) above Theorem IV.1.10 ensures that $u(\zeta)$ extends uniquely to a harmonic function $u(x)$ on $\mathbb{R}^q$, namely

$$u(x) = u(r\zeta) = \sum_{m=0}^{\infty} r^m\,(P_m u)\,(\zeta). \tag{36}$$

It is clear that

$$\int_{S^{q-1}} |u(r\zeta)|^2\, d\sigma_{q-1}(\zeta) = \sum_{m=0}^{\infty} r^{2m}\,\|P_m u\|^2. \tag{37}$$

Therefore

$$\int_{\mathbb{R}^q} \mu(|x|)\,|u(x)|^2\,dx$$

$$= \int_0^\infty r^{q-1}\,\mu(r)\,dr \int_{S^{q-1}} |u(r\zeta)|^2\,d\sigma_{q-1}(\zeta)$$

$$= \int_0^\infty r^{q-1}\,\mu(r)\,dr \sum_{m=0}^{\infty} r^{2m}\,\|P_m u\|^2$$

$$= \sum_{m=0}^{\infty} \left(\int_0^\infty r^{2m+q-1}\,\mu(r)\,dr\right)\|P_m u\|^2. \tag{38}$$

This, in view of the condition 3) above, leads readily to the conclusion that $u(x) \in HA^q(\mu)$.

Conversely, if $u(x) \in HA^q(\mu)$, then Theorem IV.1.10. ii) implies that Equation (36) is valid, so are (37) and (38). From the latter and the condition 3) immediately follows that $u(\zeta) \in D(\Lambda)$.

Finally the relation (38) and the condition 3) lead readily to the conclusion that the restriction-extension mapping is an isomorphism or isometry between the spaces $HA^q(\mu)$ and $D(\Lambda)$ under the respective conditions. □

Corollary IV.2.2. Under the conditions in Theorem IV.2.1 above the space $HA^q(\mu)$ is complete. So it is a Hilbert space. □

Corollary IV.2.3. Under the conditions in Theorem IV.2.1 above we have $HA^q(\mu) = \bigoplus_{m=0}^{\infty} H_m^q$. Moreover, if $\{e_{m,j}^q(\zeta) \mid 1 \le j \le N(q,m)\}$ is an orthonormal basis in $\mathbf{H}_m^q$, then $\{\frac{1}{\mu_m} e_{m,j}^q(x) \mid 1 \le j \le N(q,m)\}$ is an orthonormal basis in H_m^q considered as a subspace of $HA^q(\mu)$. Here $\mu_m = (\int_0^\infty r^{2m+q-1}\, m(r)\,dr)^{1/2}$.

Proof. For $p_m \in H_m^q$ and $q_n \in H_n^q$ we have

$$(p_m, q_n)_\mu = \int_{\mathbb{R}^q} \mu(|x|)\,p_m(x)\,\overline{q_n(x)}\,dx = \left(\int_0^\infty r^{m+n+q-1}\,\mu(r)\,dr\right)(p_m, q_n). \tag{34}$$

Hence $H_m^q \perp H_n^q$ in $HA^q(\mu)$ if $m \ne n$, for $\mathbf{H}_m^q \perp \mathbf{H}_n^q$ if $m \ne n$.

Let $u \in HA^q(\mu)$. Since the last expression in (38) is obviously $\sum_{m=0}^{\infty} \|P_m u\|_\mu^2$, we have

$$\| u \|_\mu^2 = \sum_{m=0}^{\infty} \| P_m u \|_\mu^2. \tag{35}$$

Therefore

$$u = \sum_{m=0}^{\infty} P_m u \text{ in } HA^q(\mu). \tag{36}$$

Thus $HA^q(\mu) = \bigoplus_{m=0}^{\infty} H_m^q$. The last assertion in the present corollary follows directly from (34). □

Corollary IV.2.4. Under the conditions in Theorem IV.2.1 above the space $HA^q(\mu)$ has a reproducing kernel $K_\mu^q(x,y)$. Explicitly $K_\mu^q(x,y) = \sum_{m=0}^{\infty} K_{\mu,m}^q(x,y)$, where for each m $K_{\mu,m}^q(x,y)$ is the reproducing kernel for the subspace H_m^q, namely

$$K_{\mu,m}^q(x,y) = \frac{N(q,m)}{\sigma_{q-1}\,\mu_m^2} \, |x|^m \, |y|^m \, P_m^q\left[\frac{x}{|x|} \cdot \frac{y}{|y|}\right] \tag{37}$$

where P_m^q is the Gegenbauer polynomial (see (17)).

Proof. For the theory of reproducing kernel we refer to [Ar], [Yo], [Ma1] and [Ma2].

Let $u \in HA^q(\mu)$. Then $u(x) = \sum_{m=0}^{\infty} r^m (P_m u)(\zeta)$. In view of the estimate $N(q,m) \le K_q\, m^{q-2}$ and condition 2) in Theorem IV.2.1 we have

$$\begin{aligned} |u(x)| &\le \sum_{m=0}^{\infty} r^m \, |(P_m u)(\zeta)| \\ &\le \sum_{m=0}^{\infty} r^m \left[\frac{N(q,m)}{\sigma_{q-1}}\right]^{1/2} \| P_m u \| \quad \text{(Prop. IV.1.5)} \\ &\le \left[\sum_{m=0}^{\infty} \frac{N(q,m)\, r^{2m}}{\sigma_{q-1}\, \lambda_m^2}\right]^{1/2} \left(\sum_{m=0}^{\infty} \lambda_m^2 \| P_m u \|^2\right)^{1/2} \\ &\le C \| u \|_\mu \end{aligned}$$

where C is a constant depending only on q, μ and $\{\lambda_m\}$. The above inequality implies that for each fixed $x \in \mathbb{R}^q$, the functional $u \mapsto u(x)$ is continuous on $HA^q(\mu)$. Hence $HA^q(\mu)$ is a Hilbert space with a reproducing kernel.

Let $\{e_{m,j}^q \mid 1 \le j \le N(q,m)\}$ be an orthonormal basis in $\mathbf{H}_m^q$. Corollary IV.2.3 above then shows that $\{\frac{1}{\mu_m} e_{m,j}^q(x) \mid 1 \le j \le N(q,m)\}$ is an orthonormal basis in H_m^q. So its reproducing kernel is give by

$$K^q_{\mu,m}(x,y) = \sum_{m=0}^{N(q,m)} \frac{1}{\mu_m} e^q_{m,j}(x) \cdot \overline{\frac{1}{\mu_m} e^q_{m,j}(y)}$$

$$= \frac{|x|^m |y|^m}{\mu_m^2} \sum_{m=0}^{N(q,m)} e^q_{m,j}\left[\frac{x}{|x|}\right] \overline{e^q_{m,j}\left[\frac{y}{|y|}\right]}$$

$$= \frac{N(q,m)}{\sigma_{q-1}\mu_m^2} |x|^m |y|^m P^q_m\left[\frac{x}{|x|} \cdot \frac{y}{|y|}\right]. \tag{38}$$

Of course the reproducing kernel of $HA^q(\mu)$ is given by $K^q_\mu(x,y) = \sum_{m=0}^{\infty} K^q_{\mu,m}(x,y)$, for $HA^q(\mu) = \bigoplus_{m=0}^{\infty} H^q_m$. □

Corollary IV.2.5.

(i) For $u \in H^q_m$

$$|u(x)| \le \left[\frac{N(q,m)}{\sigma_{q-1}}\right]^{1/2} \frac{|x|^m}{\mu_m} \|u\|_\mu. \tag{39}$$

(ii) For $u \in HA^q(\mu)$

$$|u(x)| \le \left[\sum_{m=0}^{\infty} \frac{N(q,m)\ |x|^{2m}}{\sigma_{q-1}\ \mu_m^2}\right]^{1/2} \|u\|_\mu. \tag{40}$$

(iii) For $u \in HA^q(\mu)$ and $m \in \mathbb{N}_0$

$$(P_m u)(x) = \frac{N(q,m)\ |x|^m}{\sigma_{q-1}\ \mu_m^2} \int_{\mathbb{R}^q} |y|^m P^q_m\left[\frac{x}{|x|} \cdot \frac{y}{|y|}\right] u(y)\, dy. \tag{41}$$

Proof. (i) and (ii) follow from the last corollary. Multiplying both sides of (38) by $u(y)$ and then integrating over $\mathbb{R}^q$ we obtain (40) readily. □

The above considerations on reproducing kernels is motivated by [Ma1] and [Ma2] where Martens treated in effect the space $HA^q(e^{-\frac{1}{2}|x|^2})$.

Example IV.2.6. For $\mu(r) = e^{-ar^b}$ $(a,b>0)$ and $\lambda_m^2 = \left[\frac{2m}{eab}\right]^{\frac{2}{b}m} m^{\frac{q}{b}-\frac{1}{2}}$ $(m \in \mathbb{N}_0)$ all the conditions in Theorem IV.2.1 above are satisfied. Thus, $HA^q(e^{-ar^b})$ is a Hilbert space isomorphic to the space $D(\Lambda)$ and all the assertions in the above corollaries hold true. In fact we have

Lemma IV.2.7.

$$\int_0^\infty e^{-ar^b}\, r^{2m+q-1}\, dr \simeq (2\pi)^{½}\, b^{-½}\, (eab)^{-(2m+q)/b}\, (2m+q)^{\frac{2m+q}{b}-\frac{1}{2}}. \tag{41}$$

Here and in the sequel for two sequences $\{a_m\}$ and $\{b_m\}$ of positive numbers we write $\{a_m\} \simeq \{b_m\}$ iff $\lim_{m\to\infty} a_m / b_m = 1$.

Proof. Substituting s for $a\,r^b$ we have

$$\int_0^\infty e^{-ar^b}\, r^{2m+q-1}\, dr$$

$$= \int_0^\infty e^{-s}\, a^{-\frac{2m+q}{b}}\, b^{-1}\, s^{\frac{2m+q}{b}-1}\, ds$$

$$= a^{-\frac{2m+q}{b}}\, b^{-1}\, \Gamma\left[\frac{2m+q}{b}\right].$$

Then an invocation to Stirrings formula

$$\Gamma(\theta) \simeq (2\pi)^{½}\, \theta^{\theta-½}\, e^{-\theta} \quad (\theta \to \infty)$$

yields (41). □

IV.3. The Range of the Propagation Operator for the Spherical Reaction-Diffusion Equation

We proceed to discuss the spherical reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \Delta_{LB}\, u. \tag{42}$$

Here $u = u(\zeta, t)$ $(\zeta \in S^{q-1}, t > 0)$ is the unknown function and Δ_{LB} is the Laplace-Beltrami operator for S^{q-1} (cf. (2)-(6) in IV.1). As is observed in IV.1 we have

$$\Delta_{LB}\, u = -m(m+q-2)\, u\,, \quad \forall u \in \mathbf{H}_m^q. \tag{43}$$

Thus the operator Δ_{LB}, as defined on the algebraic direct sum of $\mathbf{H}_m^q$ $(m \in \mathbb{N}_0)$, is essentially self-adjoint in $L^2(S^{q-1})$. Its self-adjoint closure, still denoted Δ_{LB}, is given by

$$\begin{aligned} &\Delta_{LB}\, u = -\sum_{m=0}^\infty m(m+q-2)\, P_m\, u \\ &u \in D(\Delta_{LB}) = \{u \in L^2(S^{q-1}),\ \sum_{m=0}^\infty [m(m+q-2)]^2\, \| P_m\, u \|^2 < \infty\}. \end{aligned} \tag{44}$$

The solution $u = u(\zeta, t)$ of (42) for the initial value $u(0) = \mathrm{v} \in L^2(S^{q-1})$ is given by

$$u = e^{t\Delta_{LB}}\,\mathrm{v} = \sum_{m=0}^{\infty} e^{-tm(m+q-2)}\, P_m \mathrm{v}. \tag{45}$$

The range of the propagation operator at time t, $R(e^{t\Delta_{LB}})$, i.e., the possible states of the system at time t starting from the initial states in $L^2(S^{q-1})$, is given by

$$D(e^{-t\Delta_{LB}}) = \{u \in L^2(S^{q-1}) \mid \sum_{m=0}^{\infty} e^{2tm(m+q-2)} \, \| P_m u \|^2 < \infty\}. \tag{46}$$

If instead of Equation (42) we consider the more general fractional spherical reaction-diffusion equation

$$\frac{\partial u}{\partial t} = -(-\Delta_{LB})^{\nu/2} u \qquad (\nu > 1) \tag{42'}$$

then in place of (44)-(46) we have

$$\begin{aligned} -(-\Delta_{LB})^{\nu/2} u &= -\sum_{m=0}^{\infty} \{m(m+q-2)\}^{\nu/2} P_m u \\ u \in D(-(-\Delta_{LB})^{\nu/2}) &= \{u \in L^2(S^{q-1}) \mid \sum_{m=0}^{\infty} \{m(m+q-2)\}^{\nu} \, \| P_m u \|^2 < \infty\} \end{aligned} \tag{44'}$$

$$u = e^{-t(-\Delta_{LB})^{\nu/2}}\,\mathrm{v} = \sum_{m=0}^{\infty} e^{-t\{m(m+q-2)\}^{\nu/2}}\, P_m \mathrm{v} \tag{45'}$$

and

$$D(e^{t(-\Delta_{LB})^{\nu/2}}) = \{u \in L^2(S^{q-1}) \mid \sum_{m=0}^{\infty} e^{2t\{m(m+q-2)\}^{\nu/2}} \, \| P_m u \|^2 < \infty\}. \tag{46'}$$

The purpose of this section is to characterize $D(e^{t(-\Delta_{LB})^{\nu/2}})$, more precisely, to identify it with a weighted space of harmonic functions on $\mathbb{R}^q$. Note that $D(e^{t(-\Delta_{LB})^{\nu/2}})$ equipped with the norm

$$\| u \|_{\nu,t} = \Big(\sum_{m=0}^{\infty} e^{2t\{m(m+q-2)\}^{\nu/2}} \, \| P_m u \|^2\Big)^{1/2} \tag{47}$$

is just the space X_B^t for $X = L^2(S^{q-1})$ and $B = e^{(-\Delta_{LB})^{\nu/2}}$, in the notation of Chapter I.

Theorem IV.3.1. Given $1 < \nu \leq 2$ and $t > 0$. The space $X_B^t = D(e^{t(-\Delta_{LB})^{\nu/2}})$ is isomorphic to the space $HA^q(\mu)$ as normed spaces under the restriction-extension mapping. Here $\mu : (0,\infty) \to (0,\infty)$ is the weight function given by

$$\mu(r) = r^{-2} \, |\log r|^{\frac{2-\nu}{2(\nu-1)}} \, e^{\frac{2(\nu-1)}{\nu^{\frac{\nu}{\nu-1}}\, t^{\frac{1}{\nu-1}}}} \, |\log r|^{\frac{\nu}{\nu-1}}, \quad r > 0. \tag{48}$$

The Hilbert space $HA^q(\mu)$ has all the properties stated in Theorem IV.2.1 and Corollaries IV.2.2-

5.

Proof. Obviously conditions 1) and 2) in Theorem IV.2.1 hold true. The assertions in the present theorem are proved provided condition 3) in Theorem IV.2.1 is verified, which is done in the next lemma. □

Lemma IV.3.2. For $1 < \nu \leq 2$ and $t > 0$ fixed and μ described as in Theorem IV.3.1 above we have

$$\int_0^\infty r^{2m+q-1}\, \mu(r)\, dr \sim e^{2t\{m(m+q-2)\}^{\nu/2}}. \tag{49}$$

□

In order to prove this lemma we need the following delicate result of Brands ([Br]) on the asymptotic behaviours of integrals.

Theorem IV.3.3. Assume that a function $M\ \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the following conditions:

(i) $M \in C([0,\infty, R)$, $M \in C^2([a,\infty, R)$ for some $a \geq 0$, $M''(x) > 0\ (x \geq a)$ and $M'(x) \to \infty\ (x \to \infty)$.

(ii) There exists a positive function α on $[b,\infty)$ for some $b \geq 0$ such that

$$\alpha(x)\,(M''(x))^{1/2} \to \infty\ (x \to \infty);$$

$$\alpha(x) \leq x\ (x \geq b);$$

$$\forall \varepsilon > 0\, \exists A > 0\, \forall x \geq A\ \forall y \geq 0\, [\,|y-x| \leq \alpha(x) \Rightarrow |C''(y) - C''(x)| \leq \varepsilon C''(x)].$$

Then the integral

$$I(M,t) = \int_{-\infty}^{+\infty} e^{tx - M(|x|)}\, dx$$

has the following asymptotic behaviour as $t \to \infty$:

$$I(M,t) = \left[\frac{2\pi}{m'(m^{\leftarrow}(t))}\right]^{1/2} e^{M^{x}(t)}\,(1+0(1))$$

$$= [2\pi (m^{\leftarrow})'(t)]^{1/2}\, e^{M^{x}(t)}\,(1+0(1)) \tag{50}$$

where $m(t) = M'(t)$, $m^{\leftarrow}(t)$ is the inverse function of $m(t)$ and $M^{x}(t) = \int_0^t m^{\leftarrow}(\tau)\, d\tau$. □

Corollary IV.3.4. For $M : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the conditions in the theorem above and

$$M(x) = \frac{A}{\alpha+1}\, x^{\alpha+1} + c \log x \text{ for } x \text{ sufficiently large } (A > 0,\ \alpha > 0,\ c \in I\!R) \tag{51}$$

we have

$$I(M,t) \sim t^{\frac{1}{2}(\frac{1}{\alpha}-1)-\frac{c}{\alpha}}\, e^{\frac{\alpha}{1+\alpha} A^{-1/\alpha} t^{1+1/\alpha}} \qquad (t \to \infty). \tag{52}$$

Proof. We need to know the asymptotic behaviour of $m^{\leftarrow}(t)$ as $t \to \infty$. We have

$$y = m(x) = M'(x) = A\, x^{\alpha} + \frac{c}{x} \qquad (x \text{ large}). \tag{53}$$

Since

$$m'(x) = M''(x) = A\, \alpha x^{\alpha-1} - \frac{c}{x^2} > 0 \text{ for } x \text{ large enough}$$

the function $m(x)$ is strictly increasing in a neighbourhood of ∞. Therefore the inverse function $m^{\leftarrow}(y)$ exists in a neighbourhood of ∞. Substituting $\frac{1}{\sigma^{\frac{\alpha}{\alpha+1}}}$ for y and $\frac{1}{\sigma^{\frac{1}{\alpha+1}}\,\omega}$ for x in (53) we have

$$F(\sigma,\omega) = \omega^{\alpha} - A - c\,\sigma\,\omega^{\alpha+1} = 0. \tag{54}$$

Clearly $F(0,A^{1/\alpha}) = 0$ and $F(\sigma,\omega)$ is analytic in (σ,ω) in the neighbourhood of $(0,A^{1/\alpha})$. According to Weierstrass' theorem we have the following expansion in a neighbourhood of $\sigma = 0$:

$$A^{-1/\alpha}\,\omega = 1 + w_1\,\sigma + \frac{w_2}{2}\,\sigma^2 + \cdots \tag{55}$$

So is true the expansion

$$\frac{1}{A^{-1/\alpha}\,\omega} = 1 - w_1\,\sigma + \cdots \tag{56}$$

Inserting (55) in (54) and equating the coefficient of σ on both sides we obtain $w_1 = \frac{c\,A^{1/\alpha}}{\alpha}$. This together with (56) implies that

$$x = \frac{1}{\sigma^{\frac{1}{\alpha+1}}\,\omega}$$

$$= \frac{A^{-1/\alpha}}{\sigma^{\frac{1}{\alpha+1}}}\,(1 - w_1\,\sigma + \cdots)$$

$$= A^{-1/\alpha} y^{1/\alpha} - \frac{c}{\alpha} y^{-1} + c' y^{-(2+\frac{1}{\alpha})} + \cdots \quad \text{(for } y \text{ large enough)} \tag{57}$$

where c' is a constant. The series converges uniformly in a neighbourhood of $y = \infty$, so does its derivative. Thus we have

$$(m^{\leftarrow})' \sim t^{1/\alpha - 1} \tag{58}$$

$$e^{M^*(t)} \sim e^{\int_a^t m^{\leftarrow}(\tau)\,d\tau}$$

$$\sim e^{\frac{\alpha}{1+\alpha} A^{-1/\alpha} t^{1+1/\alpha}} - \frac{C}{\alpha} \log t. \tag{59}$$

Relation (52) then follows from relations (50), (58) and (59). □

Proof of Lemma IV.3.2. For $1 < \nu \le 2$ we have

$$e^{2t\{m(m+q-2)\}^{\nu/2}} \sim e^{2tm^{\nu}[1+\frac{\nu}{2}\frac{q-2}{m}]}.$$

For $M(x)$ in Corollary IV.3.4 we have

$$I(M, 2m+q-2)$$

$$\sim (2m+q-2)^{\frac{1}{2}(\frac{1}{\alpha}-1)-\frac{c}{\alpha}} \, e^{\frac{\alpha}{1+\alpha} A^{-1/\alpha} (2m)^{1+1/\alpha} [1+(1+1/\alpha)\frac{q-2}{2m}]} \quad (\alpha \ge 1). \tag{61}$$

Consequently if we could take α, c and A such that

$$\begin{cases} \frac{1}{2}(\frac{1}{\alpha} - 1) - \frac{c}{\alpha} = 0 \\ 1 + \frac{1}{\alpha} = \nu \\ \frac{\alpha}{1+\alpha} A^{-1/\alpha} 2^{1+\frac{1}{\alpha}} = 2t \end{cases} \tag{62}$$

then

$$I(M, 2m+q-2) \sim e^{2t\{m(m+q-2)\}^{\nu/2}}. \tag{63}$$

It turns out that the system of equations (62) indeed has a set of solutions:

$$\begin{cases} \alpha = \dfrac{1}{\nu - 1} \\ c = \dfrac{\nu - 2}{2(\nu - 1)} \\ A = \dfrac{2}{(t\nu)^{\frac{1}{\nu - 1}}}. \end{cases} \tag{64}$$

On the other hand

$$\int_0^\infty r^{2m+q-1}\,\mu(r)\,dr = \int_{-\infty}^\infty e^{(2m+q-2)^x}\,\mu(e^x)\,e^{2x}\,dx. \tag{65}$$

Relation (49) then follows from the relations (63), (64), (65), (48) and the fact that for any continuous function μ on $(0,\infty)$ and $\delta < 1$ there exists a constant C_μ such that

$$| \int_{1-\delta}^{1+\delta^{-1}} r^{2m+q-1}\,\mu(r)\,dr \,|$$

$$\leq C_\mu \frac{1}{2m+q}\,[(1+\delta^{-1})^{2m+q} - (1-\delta)^{2m+q}]$$

$$= o(e^{2tm(m+q-2))^{\nu/2}}) \quad (m \to \infty).$$

Thus Lemma IV.3.2 is proved, so is Theorem IV.3.1. $\square$

Finally we remark that Theorem IV.3.1 is still valid for $\nu > 2$ in case $q = 2$.

IV.4. Linear Operators in Spaces of Harmonic Functions

In the last sections we introduced some Hilbert spaces of harmonic functions on $\mathbb{R}^q$. We are now in a position to discuss several naturally arising linear operators in these spaces. In order to do so it seems appropriate to introduce the space $HA(\mathbb{R}^q)$ of all harmonic functions in $\mathbb{R}^q$.

Proposition IV.4.1. The space $HA(\mathbb{R}^q)$ equipped with the norms $\{\|\cdot\|_r \mid r > 1\}$

$$\| u \|_r = (\sum_{m=0}^\infty r^{2m} \, \| P_m u \|^2)^{½} \tag{66}$$

is a Frechét space. For any $R > 0$, a sequence of functions $\{u_n\}$ in $HA(\mathbb{R}^q)$ converges to zero uniformly on each ball $B^q(r)$ with $r < R$ iff $\| u_n \|_r \to 0$ for all $r < R$. So, convergence to zero of a sequence $\{u_n\}$ in $HA^q(\mathbb{R}^q)$ means its uniform convergence to zero on each compact set in $\mathbb{R}^q$.

Proof. The inequality (cf. (27) in IV.1)

$$
\begin{aligned}
|P_m u(x)| &= |r^m (P_m u)(\zeta)| \\
&\le \left[\frac{K_q}{\sigma_{q-1}}\right]^{1/2} r^m \, m^{\frac{q-2}{2}} \, \| P_m u \| \\
&\le \tfrac{1}{2} \, [\varepsilon \frac{K_q}{\sigma_{q-1}} \left[\frac{r}{\rho}\right]^{2m} m^{q-2} + \varepsilon^{-1} \rho^{2m} \| P_m u \|^2]
\end{aligned} \tag{67}
$$

$\varepsilon \in (0,1)$, $\rho \in (r,R)$

leads to the assertion that if $\| u_n \|_r \to 0$ for all $r < R$ then $\{u_n\}$ converges to zero uniformly on each ball $B^q(r)$ with $r < R$.

Set $u_r(x) = u(rx)$. Then the equality (cf. (28) in IV.1)

$$r^m P_m u = P_m u_r$$

implies the converse, i.e., that if $\{u_n\}$ converges to zero uniformly on each ball $B^q(r)$ with $r < R$ then $\| u_n \|_r \to 0$ for all $r < R$.

As a consequence it follows that $HA(\mathbb{R}^q)$ is complete with respect to the norms $\{\| \cdot \|_r\}$ (cf. Theorem IV.1.8). $\square$

In the following for a pair $(\mu, \{\lambda_m\}_{m \in \mathbb{N}_0})$ of weight function μ and positive sequence $\{\lambda_m\}$ we always assume that the conditions in Theorem IV.2.1 are satisfied so that the space $HA^q(\mu)$ is isomorphic to the space $D(\Lambda)$ under the restriction-extension correspondence. Corollary IV.2.5 (ii) then implies that the space $HA^q(\mu)$ for each μ is continuously and densely embedded in the space $HA(\mathbb{R}^q)$.

For convenience of reference here we settle down a few special notations for the spaces we are most interested in. $HA^q(e^{-ar^b}) \equiv HA^q(a,b)$ $(a,b > 0)$ (cf. Example IV.2.6). The symbol $HA^q_{\Delta_{LB}}(\nu,t)$ $(1 < \nu \le 2, t > 0)$ stands for the space $HA^q(\mu)$ with μ given in (48), i.e., the space corresponding to the range of the propagator for the fractional spherical reaction-diffusion equation (cf. IV.3). We also form inductive limits and projective limits of these spaces:

$$HA^q_e(a+, b) = \operatorname*{indlim}_{a' \uparrow a} HA^q_e(a', b) \quad (0 < a \le \infty)$$

$$HA^q_e(a-, b) = \operatorname*{projlim}_{a' \downarrow a} HA^q_e(a', b) \quad (0 \le a < \infty)$$

$$HA^q_{\Delta_{LB}}(\nu, t+) = \operatorname*{indlim}_{t' \downarrow t} HA^q_{\Delta_{LB}}(\nu, t') \quad (0 \le t < \infty)$$

$$HA^q_{\Delta_{LB}}(\nu, t-) = \operatorname*{projlim}_{t' \uparrow t} HA^q_{\Delta_{LB}}(\nu, t') \quad (0 < t \le \infty).$$

Let us begin with the differentiation operators $\dfrac{\partial}{\partial x_k} \equiv \partial_k$ $(1 \le k \le q)$.

Proposition IV.4.2. (i) The differentiation operator $\partial_k (1 \le k \le q)$ is continuous in $HA(\mathbb{R}^q)$ and $P_m \partial_k = \partial_k P_{m+1}$ $(1 \le k \le q\,,\, m \in \mathbb{N}_0)$ on $HA(\mathbb{R}^q)$.

(ii) If $\lambda_m^2(2)/\lambda_{m+1}^2(1) = 0(m^{-(q+1)})$ then the differentiation operator ∂_k is continuous from $HA^q(\mu(1))$ to $HA^q(\mu(2))$.

(iii) If $\lambda_m^2(2)/\lambda_{m+1}^2(1) = o(m^{-(q+1)}$ then ∂_k $(1 \le k \le q)$ is compact from $HA^q(\mu(1))$ to $HA^q(\mu(2))$.

Proof. (i) Let $u \in HA(\mathbb{R}^q)$. We have $u(x) = \sum_{m=0}^{\infty} (P_m u)(x)$ with uniform convergence on compacta of $\mathbb{R}^q$. For $m \in \mathbb{N}_0$ clearly $\dfrac{\partial(P_{m+1} u)(x)}{\partial x_k} \in H_m^q$ and by Theorem IV.1.11 we have

$$\left| \frac{\partial(P_{m+1} u)(x)}{\partial x_k} \right| \le C(m+1)^{\frac{q+1}{2}} r^m \, \| P_{m+1} u \|$$

$$= C R^{-1} (m+1)^{\frac{q+1}{2}} \left[\frac{r}{R} \right]^m R^{m+1} \, \| P_{m+1} u \| \;\; (0 < r < R). \tag{68}$$

Therefore the series $\sum_{m=0}^{\infty} \dfrac{\partial(P_m u)(x)}{\partial x_k}$ converges uniformly on compacta in $\mathbb{R}^q$ and hence holds true the equality

$$\frac{\partial u(x)}{\partial x_k} = \sum_{m=0}^{\infty} \frac{\partial(P_m u)(x)}{\partial x_k}.$$

So $P_m \partial_k u = \partial_k P_{m+1} u$. From (68) and Proposition IV.4.1 follows also the continuity of ∂_k in $HA(\mathbb{R}^q)$.

(ii), (iii) For $u \in HA^q(\mu(1))$ we have

$$\| \partial_k u \|^2_{\Lambda(2)} = \sum_{m=0}^{\infty} \lambda_m^2(2) \, \| P_m(\partial_k u) \|^2$$

$$= \sum_{m=0}^{\infty} \lambda_m^2(2) \, \| \partial_k P_{m+1} u \|^2$$

$$\le \sum_{m=0}^{\infty} \lambda_m^2(2) \, C(m+1)^{q+1} \, \| P_{m+1} u \|^2$$

$$= C \sum_{m=0}^{\infty} \frac{\lambda_m^2(2)}{\lambda_{m+1}^2(1)} m^{q+1} \cdot \lambda_{m+1}^2(1) \, \| P_{m+1} u \|^2. \tag{69}$$

The assertions in (ii) and (iii) then follow easily from the above estimate. $\square$

Corollary IV.4.3.

(i) The differential operator ∂_k is continuous in the Hilbert spaces $HA^q_{\Delta_{LB}}(\nu,t)$ $(1<\nu\leq 2, t>0)$ and $HA^q_e(a,b)$ with $a>0$ and $0<b\leq \dfrac{2}{q+1}$. It is in fact compact in those spaces except for $b=\dfrac{2}{q+1}$ for which we do not know whether it is compact.

(ii) The differential operator ∂_k is compact from $HA^q_e(a_1,b)$ to $HA^q_e(a_2,b)$ for all $a_1>0$, $a_2>0$ and $b>0$ such that $a_1<a_2$.

(iii) The differential operator ∂_k is compact on each of the spaces $HA^q_e(a+,b)$ and $HA^q_e(a-,b)$.

Proof. The assertions in the present corollary follow from Proposition IV.4.2 above and the next lemma. □

Lemma IV.4.4. (i) For $\nu>1$

$$\{(m+1)\,(m+1+q-2)\}^{\nu/2}-\{m(m+q-2)\}^{\nu/2}\sim m^{\nu-1}$$

(ii) For $0<a_1\leq a_2$ and $b>0$

$$\frac{\left[\dfrac{2m}{ea_2\,b}\right]^{\frac{2}{b}m} m^{\frac{q}{b}-\frac{1}{2}}}{\left[\dfrac{2(m+1)}{ea_1\,b}\right]^{\frac{2}{b}(m+1)} (m+1)^{\frac{q}{b}-\frac{1}{2}}}\sim\left[\frac{a_1}{a_2}\right]^{\frac{2}{b}m} m^{-\frac{2}{b}}.$$

□

We continue to study the multiplication operators. For $1\leq k\leq q$ define $M_k:L^2(S^{q-1})\rightarrow L^2(S^{q-1})$ by $(M_k u)(\zeta)=\zeta_k u(\zeta)$. It is obvious that M_k is continuous and $\|M_k\|=1$. For $u\in HA(\mathbb{R}^q)$ we first define $(M_k u)(\zeta)=\zeta_k u(\zeta)$ on S^{q-1} as above, then extend it to a harmonic function on $\mathbb{R}^q$ provided this is possible at all. Such an operation is still denoted by M_k.

Proposition IV.4.5. (i) For $m\in\mathbb{N}_0$ and $p_m(\zeta)\in\mathbf{H}^q_m$ we have $(\mathbf{H}^q_{-1}\equiv\{0\})$

$$(M_k p_m)(\zeta)=(P_{m-1}M_k p_m)(\zeta)+(P_{m+1}M_k p_m)(\zeta) \tag{70}$$

where

$$(P_{m-1}M_k p_m)(\zeta)=\frac{1}{2m+q-2}\,(\partial_k p_m)(\zeta) \tag{71}$$

and

$$(P_{m+1}M_k p_m)(\zeta)=(M_k p_m)(\zeta)-\frac{1}{2m+q-2}\,(\partial_k p_m)(\zeta). \tag{72}$$

Corresponding

$$(P_{m-1} M_k p_m)(x) = \frac{1}{2m+q-2} (\partial_k p_m)(x) \tag{73}$$

$$(P_{m+1} M_k p_m)(x) = x_k p_m(x) - \frac{|x|^2}{2m+q-2} (\partial_k p_m)(x) \tag{74}$$

and

$$(M_k p_m)(x) = x_k p_m(x) + \frac{1-|x|^2}{2m+q-2} (\partial_k p_m)(x). \tag{75}$$

(ii) $M_k[HA(\mathbb{R}^q)] \subset HA(\mathbb{R}^q)$ and $M_k : HA(\mathbb{R}^q) \to HA(\mathbb{R}^q)$ is continuous. Moreover

$$(P_m M_k u)(\zeta) = \zeta_k (P_{m-1} u)(\zeta) - \frac{1}{2m+q-4} (\partial_k P_{m-1} u)(\zeta)$$

$$+ \frac{1}{2m+q} (\partial_k P_{m+1} u)(\zeta) \tag{76}$$

$$(P_m M_k u)(x) = x_k (P_{m-1} u)(x) - \frac{|x|^2}{2m+q-4} (\partial_k P_{m-1} u)(x)$$

$$+ \frac{1}{2m+q} (\partial_k P_{m+1} u)(x) \tag{77}$$

and

$$(M_k u)(x) = x_k u(x) + (1-|x|^2) \sum_{m=0}^{\infty} \frac{1}{2m+q} (\partial_k P_{m+1} u)(x) \tag{78}$$

$$= x_k u(x) + (1-|x|^2)\ (2\partial_n + q)^{-1}\, \partial_k u. \tag{78'}$$

(iii) If

$$\frac{\lambda_m^2(2)}{\lambda_{m+1}^2(1)} = 0(1), \quad \frac{\lambda_m^2(2)}{\lambda_{m-1}^2(1)} = 0(1) \tag{79}$$

then $M_k [HA^q(\mu(1))] \subset HA^q(\mu(2))$ and $M_k : HA^q(\mu(1)) \to HA^q(\mu(2))$ is continuous.

If instead of (79) we have

$$\frac{\lambda_m^2(2)}{\lambda_{m+1}^2(1)} = o(1), \quad \frac{\lambda_m^2(2)}{\lambda_{m-1}^2(1)} = o(1) \tag{80}$$

then M_k is compact.

Proof. (i) For $p_m(x) \in H_m^q$ we have by calculation

$$\Delta(|x|^n p_m(x)) = n(2m+n+q-2)\, |x|^{n-2} p_m(x). \tag{81}$$

In particular $n = 2$ gives rise to the identity

$$\Delta(|x|^2 p_m(x)) = 2(2m+q)\, p_m(x). \tag{82}$$

On the other hand we have

$$\Delta(x_k p_m(x)) = 2(\partial_k p_m)(x). \tag{83}$$

The identities (82) and (83) together suggest that

$$\Delta\,[x_k p_m(x) - \frac{|x|^2}{2(m-1)+q}\,(\partial_k p_m(x)] = 0. \tag{84}$$

This together with $\Delta\,[(\partial_k p_m(x))] = 0$ leads readily to all the assertions in (i).

(ii) For $u(x) \in HA(\mathbb{R}^q)$ we have $u(\zeta) = \sum_{m=0}^{\infty} (P_m u)(\zeta)$ and $(M_k u)(\zeta) = \sum_{m=0}^{\infty} \zeta_k (P_m u)(\zeta)$ where the series converges uniformly and absolutely in $\zeta \in S^{q-1}$. From (70), (71) and (72) follow, therefore, (76) and (77). In particular

$$P_m M_k u = P_m M_k P_{m-1} u + P_m M_k P_{m+1} u. \tag{85}$$

Since $\| M_k \| = \| P_m \| = 1$ we have for $r > 1$

$$\| M_k u \|_r^2 = \sum_{m=0}^{\infty} r^{2m} \| P_m M_k u \|^2 \le \sum_{m=0}^{\infty} r^{2m} (\| P_{m-1} u \| + \| P_{m+1} u \|)^2$$

$$\le 2 \sum_{m=0}^{\infty} r^{2m} (\| P_{m-1} u \|^2 + \| P_{m+1} u \|^2).$$

This together with Theorem IV.1.10 and Proposition IV.4.1 ensures that $M_k u \in HA(\mathbb{R}^q)$ and that $M_k : HA(\mathbb{R}^q) \to HA(\mathbb{R}^q)$ is continuous. (78) then follows directly from (77).

(iii) The respective assertions follow easily from the following estimate

$$\| M_k u \|_{\Lambda(2)}^2 = \sum_{m=0}^{\infty} \lambda_m^2(2) \| P_m M_k u \|^2$$

$$\le 2 \sum_{m=0}^{\infty} \lambda_m^2(2) (\| P_{m-1} u \|^2 + \| P_{m+1} u \|^2).$$

□

Corollary IV.4.6. The multiplication operator M_k is a compact operator from $HA_e^q(a_1, b)$ to $HA_e^q(a_2, b)$ $(0 < a_1 < a_2,\, b > 0)$ or from $HA_{\Delta_{LB}}^2(\nu, t_1)$ to $HA_{\Delta_{LB}}^2(\nu, t_2)$ $(1 < \nu \le 2,\, 0 < t_2 < t_1)$. Hence it is compact operator on the inductive limits $HA_e^q(a+, b)$ and $HA_{\Delta_{LB}}^2(\nu, t+)$ or on the projective limits $HA_e^q(a-, b)$ and $HA_{\Delta_{LB}}^q(\nu, t-)$.

Proof. The assertions in the present corollary are immediate consequences of the above Proposition IV.4.5 and Lemma IV.4.4. □

We remark that the possibility of the decomposition (70) was known before, see [S-W] Chapter VI, Lemma 3.4, and [Mar1], Lemma 2.12. In (i) of Proposition IV.4.5 above we gave an explicit formula for the components $P_{m-1} M_k p_m$ and $P_{m+1} M_k p_m$.

Recall that by definition $-\Delta_{LB}$ is the nonnegative self-adjoint operator in $L^2(S^{q-1})$ given by

$$(-\Delta_{LB}\, u)\,(\zeta) = \sum_{m=0}^{\infty} m(m+q-2)\,(P_m u)\,(\zeta)$$

for

$$u = \sum_{m=0}^{\infty} P_m u \in D(\Delta_{LB}) = \{u \in L^2(S^{q-1}) \mid \sum_{m=0}^{\infty} \{m(m+q-2)\}^2 \, \| P_m u \|^2 < \infty\}.$$

Now for $u \in HA(\mathbb{R}^q)$ we first define $(\Delta_{LB}\, u)\,(\zeta)$ as above (note that indeed $u(\zeta) \in D(\Delta_{LB})$) then extend it to a harmonic function in $\mathbb{R}^q$. Theorem IV.1.10 ensures that this is allowable. Furthermore Proposition IV.4.1 implies that such an operation (still denoted Δ_{LB}) is continuous on $HA(\mathbb{R}^q)$. In fact it will be shown that it coincides with the arbital angular momentum operator L^2 defined in (2) in IV.1. Equation (78) demonstrates that M_k differs from $x_k\cdot$ by a term

$$(1-|x|^2) \sum_{m=0}^{\infty} \frac{1}{2m+q}\,(\partial_k P_{m+1}\, u)\,(x).$$

For some special expressions involving the ∂_k's and M_k's, we will show, however, that it does not matter if the M_k's are replaced by the ordinary multiplications $x_k\cdot$. We will also show that the multiplication operators M_k are not continuous in any Hilbert spaces of harmonic functions $HA^q(\mu)$ where the differential operators ∂_k are continuous.

Proposition IV.4.7. (i) For $u \in HA(\mathbb{R}^q)$ we have

$$\sum_{k=1}^{q} M_k \partial_k u = \partial_n u = \sum_{k=1}^{q} x_k \partial_k u \tag{86}$$

and

$$\sum_{k=1}^{q} \partial_k M_k u = \partial_n u + (q-1)\,u + (q-2) \sum_{m=0}^{\infty} \frac{1}{2m+q-2}\, P_m u. \tag{87}$$

(ii) $\Delta_{LB} : HA(\mathbb{R}^q) \to HA(\mathbb{R}^q)$ is continuous and for $u \in HA(\mathbb{R}^q)$

$$\Delta_{LB}\, u = -\sum_{m=0}^{\infty} m(m+q-2) P_m u = -\partial_n^2 u - (q-2)\,\partial_n u = L^2 u. \tag{88}$$

(iii) For $u \in HA(\mathbb{R}^q)$ and $1 \le j,\, k \le q$

$$(M_k \partial_j - M_j \partial_k)\, u = (x_k \partial_j - x_j \partial_k)\, u. \tag{89}$$

So

$$\frac{1}{2} \sum_{1 \le k,j \le q} (M_k \partial_j - M_j \partial_k)^2 u = L^2 u = \Delta_{LB} u. \tag{90}$$

Proof. (i) For $u = \sum_{m=0}^{\infty} P_m u \in HA(\mathbb{R}^q)$ we have by Propositions IV.4.2 and IV.4.5 that

$$\partial_k u = \sum_{m=0}^{\infty} \partial_k P_{m+1} u$$

$$M_k u = x_k u + (1 - |x|^2) \sum_{m=0}^{\infty} \frac{1}{2m+q} (\partial_k P_{m+1} u).$$

Therefore

$$M_k \partial_k u = x_k \partial_k u + (1 - |x|^2) \sum_{m=0}^{\infty} \frac{1}{2m+q} (\partial_k^2 P_{m+2})$$

and

$$\partial_k M_k u = x_k \partial_k u + u - 2 x_k \sum_{m=0}^{\infty} \frac{1}{2m+q} (\partial_k P_{m+1} u)$$

$$+ (1 - |x|^2) \sum_{m=0}^{\infty} \frac{1}{2m+q} (\partial_k^2 P_{m+1} u).$$

From these follow readily (86) and (87) (note that $\Delta P_m u = 0$).

(ii) The continuity of $\Delta_{LB} : HA(\mathbb{R}^q) \to HA(\mathbb{R}^q)$ was already observed in the remarks prior to the proposition. Furthermore

$$\Delta_{LB} u = - \sum_{m=0}^{\infty} m(m+q-2) P_m u$$

$$= - \sum_{m=0}^{\infty} [\partial_n^2 + (q-2)\partial_n] P_m u$$

$$= - \sum_{m=0}^{\infty} [(\sum_{k=0}^{q} \partial_k M_k)^2 + (q-2)(\sum_{k=0}^{q} \partial_k M_k)] P_m u$$

$$= - [(\sum_{k=0}^{q} \partial_k M_k)^2 + (q-2)(\sum_{k=0} \partial_k M_k)] \sum_{m=0}^{\infty} P_m u$$

(∂_k's and M_k's are continuous in $HA(\mathbb{R}^q)$)

$$= - [\partial_n^2 + (q-2)\partial_n] u \quad \text{(by (i) again)}$$

$$= L^2 u.$$

Statement (iii) is proved similarly by noticing that

$$M_j \partial_k u = x_j \partial_k u + (1 - |x|^2) \sum_{m=0}^{\infty} \frac{1}{2m+q} \partial_j \partial_k P_{m+2}.$$

□

Corollary IV.4.8.

(i) $\sum_{k=1}^{q} M_k \partial_k$ is continuous in no Hilbert spaces of harmonic functions $HA^q(\mu)$.

(ii) M_k is not continuous in any Hilbert spaces of harmonic functions $HA^q(\mu)$ in which the differentiation operator is continuous.

Proof. By (i) of the above proposition we have

$$(\sum_{k=1}^{q} M_k \partial_k) p_m = \partial_n p_m = m\, p_m \text{ for } p_m \in H_m^q \quad (m \in \mathbb{N}_0).$$

□

Now for any given real $q \times q$ square matrix A we define an operator L_A on $HA(\mathbb{R}^q)$ as follows. For $u \in HA(\mathbb{R}^q)$ first define $(L_A u)(\zeta) = u(A\zeta)$ on $\zeta \in S^{q-1}$, then extend it to a harmonic function on $\mathbb{R}^q$. The possibility of such an extension will be seen below.

Proposition IV.4.9.

(i) The operator L_A is a well defined continuous operator on $HA(\mathbb{R}^q)$.

(ii) If $A^T A = I$, then L_A is unitary on any of the Hilbert space $HA^q(\mu)$.

(iii) If $|A| = \sup\{|Ax| \mid x \in \mathbb{R}^q, |x| = 1\} < 1$ then the operator L_A is compact on any Hilbert space $HA^q(\mu)$.

(iv) If

$$\frac{\lambda_m^2(2)}{\lambda_m^2(1)} |A|^{2m} m^q \in l^1 \tag{91}$$

then L_A is compact from $HA^q(\mu(1))$ to $HA^q(\mu(2))$.

Proof. (i) Let $u \in HA(\mathbb{R}^q)$. By definition

$$(L_A u)(\zeta) = \sum_{k=0}^{\infty} (P_k u)(A\zeta)$$

where the series converges uniformly in $\zeta \in S^{q-1}$ and hence also in $L^2(S^{q-1})$. Therefore

$$(P_m L_A u)(\zeta) = \sum_{k=0}^{\infty} P_m [(P_k u)(A\zeta)]$$

$$= \sum_{k\geq m}^{\infty} P_m [(P_k u)(A\zeta)].$$

Here we have used the fact that $(P_k u)(Ax)$ is a homogeneous polynomial of degree k and hence $P_m [(P_k u)(A\zeta)] = 0$ for $m > k$. Further, we have

$$\| P_m L_A u \| \leq \sum_{k\geq m}^{\infty} \| P_m[(P_k u)(A\zeta)\|$$

$$\leq \sum_{k\geq m}^{\infty} \| (P_k u)(A\zeta)\|$$

$$\leq \sum_{k\geq m}^{\infty} |A\zeta|^k \left[\frac{N(q,k)}{\sigma_{q-1}}\right]^{1/2} \| P_k u \|$$

$$\leq \sum_{k\geq m}^{\infty} |A|^k \left[\frac{N(q,k)}{\sigma_{q-1}}\right]^{1/2} \| P_k u \|.$$

Therefore

$$(\sum_{m=0}^{\infty} r^{2m} \| P_m L_A u \|^2)^{1/2} \quad (r > 1)$$

$$\leq \sum_{m=0}^{\infty} \sum_{k\geq m}^{\infty} r^m |A|^k \left[\frac{N(q,k)}{\sigma_{q-1}}\right]^{1/2} \| P_k u \|$$

$$\leq \sum_{m=0}^{\infty} (k+1) r^k |A|^k \left[\frac{N(q,k)}{\sigma_{q-1}}\right]^{1/2} \| P_k u \|$$

$$\leq \left[\frac{K_q}{\sigma_{q-1}}\right]^{1/2} \sum_{k=0}^{\infty} (k+1) |A|^k k^{(q-2)/2} r^k \| P_k u \| \tag{92}$$

$$\leq \left[\frac{K_q}{\sigma_{q-1}}\right]^{1/2} (\sum_{k=0}^{\infty} (k+1)^q (|A| / (|A|+1))^{2k})^{1/2} (\sum_{k=0}^{\infty} [(|A|+1) r]^{2k} \| P_k u \|^2)^{1/2}.$$

Theorem IV.1.10 and Proposition IV.4.1 imply then that $L_A u \in HA(\mathbb{R}^q)$ and L_A is continuous in $HA(\mathbb{R}^q)$.

(ii) is readily seen by

$$\| L_A u \|_\mu^2 = \int_{\mathbb{R}^q} \mu(|x|) \, |u(A\zeta)|^2 \, d\sigma_{q-1}(\zeta)$$

$$= \int_{\mathbb{R}^q} \mu(|x|)\, |u(\zeta)|^2\, d\sigma_{q-1}(\zeta) = \|u\|_\mu^2.$$

(iii) follows readily from

$$(\sum_{k=K}^{\infty} \lambda_k^2 \|P_k L_A u\|^2)^{1/2} \le \left[\frac{K_q}{\sigma_{q-1}}\right]^{1/2} (\sum_{k=K}^{\infty} (k+1)^q\, |A|^{2k})^{1/2} (\sum_{k=K}^{\infty} \lambda_k^2 \|P_k u\|^2)^{1/2}$$

(cf. (92)) and the fact that

$$\sum_{k=0}^{\infty} (k+1)^q\, |A|^{2k} < \infty \text{ for } |A| < 1.$$

(iv) is shown similarly by noticing the estimate

$$\sum_{k=K}^{\infty} \lambda_k^2(2) \|P_k L_A u\|^2 \le \left[\frac{K_q}{\sigma_{q-1}}\right]^{1/2} (\sum_{k=K}^{\infty} (k+1)^q\, |A|^{2k} \left[\frac{\lambda_k(2)}{\lambda_k(1)}\right]^2)^{1/2} (\sum_{k=K}^{\infty} \lambda_k^2(1) \|P_k u\|^2)^{1/2}.$$

□

Corollary IV.4.10.

(i) For any matrix A, L_A is a compact operator from $HA^q_{\lambda_{LB}}(\nu, t_1)$ to $HA^q_{\lambda_{LB}}(\nu, t_2)$ $(0 < t_2 < t_1)$ or from $HA^q_e(a_1, b)$ to $HA^q_e(a_2, b)$ provided either $\left[\frac{a_1}{a_2}\right]^b |A| < 1$ or $\left[\frac{a_1}{a_2}\right]^b |A| = 1$ but $b < \frac{2}{q-1}$.

(ii) L_A is compact on each of the spaces $HA^q_{\lambda_{LB}}(\nu, t+)$, $HA^q_{\lambda_{LB}}(\nu, t-)$, $HA^q_e(\infty+, b)$ and $HA^q_e(0-, b)$.

Proof. The claims in the present corollary are direct consequences of the above Proposition IV.4.9 and Lemma IV.4.4. □

Our next concern is the so called harmonic product. For $u(x) \in HA(\mathbb{R}^q)$ and $\nu(x) \in HA(\mathbb{R}^q)$ we will show below that their point wise product on the sphere, $w(\zeta) = u(\zeta)\, \nu(\zeta)$ for $\zeta \in S^{q-1}$, can be extended to a harmonic function $w(x)$ on the whole space $\mathbb{R}^q$. It is called the harmonic product of u and ν and is denoted $w = u \odot \nu$.

Proposition IV.4.11.

(i) The mapping $\odot : HA(\mathbb{R}^q) \times HA(\mathbb{R}^q) \to HA(\mathbb{R}^q)$ is well defined and continuous.

(ii) Assume that $\mu_m(1)\, \mu_n(2) \ge c^{-1} m_l$ for a positive constant c and all $m, n, l \in \mathbb{N}_0$ with $m + n = l$. Then, $\odot : HA^q(\mu_1) \times HA^q(\mu_2) \to HA^q(\mu_3)$ is compact if $\{\mu_l(3)\, m_l\, l^{q-2}\} \in l^1$.

Proof. (i) Let $u, \mathrm{v} \in HA(\mathbb{R}^q)$. By definition we have

$$w(\zeta) = u(\zeta)\, \mathrm{v}(\zeta) = \sum_{l=0}^{\infty} \sum_{m+n=l} (P_m u)(\zeta)\,(P_n \mathrm{v})(\zeta) \tag{93}$$

where the series converges absolutely and uniformly in $\zeta \in S^{q-1}$. We note that

$$P_k [\sum_{m+n=l} (P_m u)(\zeta)\,(P_n \mathrm{v})(\zeta)] = 0 \text{ for } k > l. \tag{94}$$

On the other hand

$$| \sum_{m+n=l} (P_m u)(\zeta)\,(P_n \mathrm{v})(\zeta) |$$

$$\leq \frac{K_q}{\sigma_{q-1}}\, l^{q-2} \sum_{m+n=l} \| P_m u \| \, \| P_n \mathrm{v} \|. \tag{95}$$

The above (93), (94) and (95) lead to that

$$\| w \|_r = (\sum_{k=0}^{\infty} r^{2k} \| P_k w \|^2)^{1/2}$$

$$\leq \sum_{k=0}^{\infty} r^k \| P_k w \|$$

$$\leq \sum_{k=0}^{\infty} r^k \sum_{l=0}^{\infty} \| P_k [\sum_{m+n=l} (P_m u)(\zeta)\,(P_n \mathrm{v})(\zeta)] \|$$

$$\leq \sum_{k=0}^{\infty} r^k \sum_{l \geq k}^{\infty} \frac{K_q}{\sigma_{q-1}^{1/2}}\, l^{q-2} \sum_{m+n=l} \| P_m u \| \, \| P_n \mathrm{v} \|$$

$$\leq \sum_{l=0}^{\infty} \frac{K_q}{\sigma_{q-1}^{1/2}}\, l^{q-2}\, r^l \sum_{m+n=l} \| P_m u \| \, \| P_n \mathrm{v} \| \quad (r > 1)$$

$$= \sum_{l=0}^{\infty} \frac{K_q}{\sigma_{q-1}^{1/2}}\, l^{q-2}\, (r/R)^l \sum_{m+n=l} R^m \| P_m u \| \cdot R^n \| P_n \mathrm{v} \| \quad (R > r)$$

$$\leq \frac{K_q}{\sigma_{q-1}^{1/2}} (\sum_{l=0}^{\infty} l^{q-2}\, (r/R)^l)\, \| u \|_R \, \| \mathrm{v} \|_R. \tag{96}$$

From this estimate and Proposition IV.4.1 it readily follows that the mapping $\odot : HA(\mathbb{R}^q) \times HA(\mathbb{R}^q)$ is well defined and continuous.

(ii) The assertion follows from the next estimate which is derived entirely similarly to (96) above

$$\| w \|_{\Lambda_3} \leq \frac{K_q\, c}{\sigma_{q-1}^{1/2}} (\sum_{l=0}^{\infty} \mu(3)\, m_l\, l^{q-2})\, \| u \|_{\Lambda_1} \, \| \mathrm{v} \|_{\Lambda_2}. \tag{97}$$

Corollary IV.4.12.

(i) The mapping $\odot$ acts from $HA^q_{\Delta_{LB}}(\nu,t) \times HA^q_{\Delta_{LB}}(\nu,s)$ to $HA^q_{\Delta_{LB}}(\nu,p)$ continuously for $1 < \nu \leq 2$, $t > 0$, $s > 0$ and

$$p < p_0(t,s) = \frac{ts}{(t^{\frac{1}{\nu-1}} + s^{\frac{1}{\nu-1}})^{\nu-1}}. \tag{98}$$

(ii) Given $g \in HA^q_{\Delta_{LB}}(\nu,s)$, the operator of multiplication by g, Mg, defined by $Mg = g \odot \cdot$, is a compact operator from $HA^q_{\Delta_{LB}}(\nu,t)$ to $HA^q_{\Delta_{LB}}(\nu,p)$ where p meets the condition (98) above. Consequently Mg is compact on the inductive limit $HA^q_{\Delta_{LB}}(\nu,0+)$. If, moreover, $g \in HA^q_{\Delta_{LB}}(\nu,\infty)$, then Mg is compact on each of the inductive or projective limits $HA^q_{\Delta_{LB}}(\nu,t\pm)$.

(iii) $HA_{\Delta_{LB}}suq(\nu,0+)$ and $HA^q_{\Delta_{LB}}(\nu,\infty)$ are topological algebras, i.e., the operations of addition, scalar multiplication and harmonic product are well defined and are continuous in each of the two spaces.

Proof. We have for $t,s \geq 0$ and $m,n,l \in \mathbb{N}_0$ with $m + n = l$ the estimate

$$\begin{aligned} & t\{m(m+q-2)\}^{\nu/2} + s\{n(n+q-2)\}^{\nu/2} \\ & \geq t\,m^\nu + s\,n^\nu \\ & \geq t\left[\frac{l}{(t/s)^{\frac{1}{\nu-1}}+1}\right]^\nu + s\left[l - \frac{l}{(t/s)^{\frac{1}{\nu-1}}+1}\right]^\nu \\ & = p_0(t,s)\,l^\nu. \end{aligned} \tag{99}$$

Now it remains to apply Proposition IV.4.11 above. ▯

At the end of this section let us examin the translation operator τ_a defined by $(\tau_a u)(x) = u(x-a)$.

Proposition IV.4.13.

(i) The translation operator τ_a ($a \in \mathbb{R}^q$) is well defined and continuous on the space $HA(\mathbb{R}^q)$.

(ii) The operator $\tau_a - I$ is compact in each of the spaces $HA^q_{\Delta_{LB}}(\nu,t)$ $(1 < \nu \leq 2,\ t > 0)$ and $HA^q_\ell(a,b)$ with $0 < b < \frac{2}{q+1}$ and $a > 0$. It is continuous in $HA^q_\ell(a,b)$ for all $a > 0$ and $0 < b < 1$.

(iii) The Taylor series converges properly in each of the spaces $HA^q_{\Delta_{LB}}(\nu,t)$ $(1 < \nu \leq 2,\ t > 0)$ and $HA^q_\ell(a,b)$ $(0 < a,\ 0 < b \leq \frac{2}{q+1})$

$$\tau_a = e^{-(a,\nabla)}. \tag{100}$$

Namely

$$u(x-a) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} [(a,\nabla)^m u](x)$$

$$= \sum_{m=0}^{\infty} (-1)^m \sum_{|\alpha|=m} \frac{a^\alpha}{\alpha!} (\partial^\alpha u)(x). \tag{101}$$

Proof. (i) Proposition IV.4.1 implies the assertions here directly.
(ii) and (iii). Under the conditions in (iii) above the operator (a,∇) is continuous in the respective spaces. Hence the series

$$e^{-(a,\nabla)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (a,\nabla)^m \tag{102}$$

converges to a continuous operator. On the other hand Taylor's theorem reads

$$(\tau_a u)(x) = u(x-a) = \sum_{m=0}^{M} \frac{(-1)^m}{m!} [(a,\nabla)^m u](x)$$

$$+ \frac{(-1)^{M+1}}{(M+1)!} [(a,\nabla)^{M+1} u](x-\theta a) \quad (0<\theta<1). \tag{103}$$

For u in the respective spaces the convergence of the series in (102) implies that the series in (101) converges uniformly on compacta of $\mathbb{R}^q$. Therefore

$$\frac{(-1)^{M+1}}{(M+1)!} [(a,\nabla)^{M+1} u](x) \to 0 \text{ as } M \to \infty \tag{104}$$

uniformly on compacta of $\mathbb{R}^q$. Now (104) and (103) together yields the equality (101) or (100). From this immediately follow the first assertion in (ii) and the second assertion for $b \le \frac{2}{q+1}$. However the latter is true also for $b \in (\frac{2}{q+1}, 1)$. This can be observed from the relation

$$e^{-a|x|^b} \sim e^{-a|x+a|^b} \text{ as } |x| \to \infty \ (b<1). \tag{105}$$

□

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Summary

This thesis is concerned with constructions of special scales of Banach spaces and with the study of linear operators and of evolution equations in such scales.

In Chapter I we deal with the construction of so called regular spaces and hyperspaces as well as linear operators therein. For a Banach space $(X, \|\cdot\|)$ and an invertible operator B of positive type in X a scale of Banach spaces $\{X_B^\sigma \mid \sigma \in \mathbb{R}\}$ is defined. More precisely, for $\sigma \geq 0, X_B^\sigma = (D(B^\sigma), \|\cdot\|_\sigma)$ with $\|u\|_\sigma = \|B^\sigma u\|$ for $u \in D(B^\sigma)$, and $X_B^{-\sigma}$ = completion of X with respect to the norm $\|u\|_{-\sigma} = \|B^\sigma u\|$ for $u \in X$. From these Banach spaces we form their natural inductive limits $X_B^{\sigma+} = \bigcup_{\tau>\sigma} X_B^\tau$ $(\sigma \in [-\infty, +\infty))$ and projective limits $X_B^{\sigma-} = \bigcap_{\tau<\sigma} X_B^\tau$ $(\sigma \in (-\infty, +\infty])$. Thus we have the scheme

$$X_B^{(-\sigma)-} \supset X_B^{-\sigma} \supset X_B^{(-\sigma)+} \supset X \supset X_B^{\sigma-} \supset X_B^\sigma \supset X_B^{\sigma+}.$$

In the above diagram the spaces to the right of X are called regular spaces and those to the left of X hyper-spaces. We are thus led to the study of the inductive limits and projective limits of a sequence of Banach spaces in general, as well as linear operators therein. In doing so, the topological structures of the regular spaces and hyper-spaces are clarified and several types of continuous linear mappings on them are characterized. By choosing different space X and different operator B various classical spaces of functions or generalized functions are realized functional-analytically as regular spaces or hyper-spaces in the above sense.

In Chapter II we discuss the regularity and extendibility of a C_0 semigroup e^{-tA} on X with respect to a scale of Banach spaces $\{X_B^\sigma \mid \sigma \in \overline{\mathbb{R}}\}$ constructed in Chapter I. More precisely, conditions between the operators A and B^σ (or A^* and $(B^*)^\sigma$) are given so that the semigroup e^{-tA} on X restricts to a C_0 semigroup on X_B^σ (or extends to a C_0 semigroup on $X_B^{-\sigma}$). Applications to matrix operators in l^2 and to second order partial differential operators on $L^2(\mathbb{R}^n)$ are presented. We also set two criteria for an infinite matrix (a_{jk}) to generate a C_0 semigroup on l^2.

In Chapter III we formulate and prove a Hille-Yosida type theorem for locally equi-continuous semigroups on the inductive limit space of a sequence of Banach spaces. We emphasize that instead of semi-norms of the inductive limit, which are hard to find and to deal with, we use the norms of the constituents of the inductive limit. The result together with that of Ouchi applies readily to the spaces $X_B^{\sigma\pm}$ defined in Chapter I.

In Chapter IV weighted L^2 spaces of harmonic functions on $\mathbb{R}^q (q \geq 2)$ and several naturally arising linear operators in them are studied. The central idea is an identification of a weighted L^2 space of harmonic functions on $\mathbb{R}^q$ with the domain of a suitable positive self-adjoint operator in $L^2(S^{q-1})$ (S^{q-1} the unit sphere in $\mathbb{R}^q$); the identification is the natural restriction-extension procedure. In particular, we have natural weighted L^2 spaces of harmonic functions on $\mathbb{R}^q$ wherein the differentiation operators are continuous or even compact. Also, working in the opposite

direction we arrive at a complete characterization of the ranges of the propagator of the fractional spherical diffusion equation $\frac{\partial u}{\partial t} = -(-\Delta_{LB})^{\nu/2} u$, where Δ_{LB} is the Laplace-Beltrami operator on S^{q-1}.

Curriculum Vitae

The author of the present thesis was born on September 3, 1962 at Zhi Dan County, Shaanxi Province, China. He passed the annual national examinations for entrance into higher educations and was enrolled into Mathematics Department of Xi'an Jiaotong University at Xi'an in the fall of 1978. In the same department he got a Bachelor of Science degree and a Master of Science degree in July 1982 and Deceber 1984 successively. His master thesis was supervised by Prof. You Zhao-Yong. From September 1985 onwards he has been working as a research assistant in the Department of Mathematics and Computing Science, Eindhoven University of Technology, under supervision of Prof. Dr. Ir. Jan de Graaf.

简　　历

本学位论文作者于一九六二年九月三日（夏历任寅年八月初五）出生在中国陕西省志丹县。七八年秋通过全国统考入西安交通大学数学系学习。在该校该系分别于八二年七月和八五年十二月获理科学士和理科硕士学位。硕士论文由游兆永教授指导下完成。从八五年九月起作为研究助理在荷兰埃因侯温大学数学与计算科学系分析研究组杨·德·格拉夫（Jan de Graaf）教授指导下工作。

STELLINGEN

behorende bij het proefschrift

EVOLUTION EQUATIONS
and
SCALES OF BANACH SPACES

door

LIU GUI-ZHONG

1.

For a homogeneous harmonic polynomial h of degree m in $I\!R^q$ the following estimate holds true:

$$\left|\frac{\partial h}{\partial x}(\zeta)\right| \leq m\left[\frac{N(q,m)}{\sigma_{q-1}}\right]^{½} \|h\|_{L^2(S^{q-1})}.$$

Here S^{q-1} is the unit sphere in $I\!R^q$ with center at the origin and with total Lebesque measure σ_{q-1}, and $N(q,m)$ is the number of linear independent homogeneous harmonic polynomials of degree m in $I\!R^q$. See [L-M].

2.

The operator $L^2+\sum_{k=1}^{q} a_k\partial_k$ $(a_k\in \mathbb{C})$ generates a C_0 semigroup in $HA^q_{\Delta_{LB}}(\nu)$ $(1<\nu\leq 2)$ and $HA^q_e(a,b)$ $(a>0,\, b>0$ if $q\leq 4$; $a>0,\, 0<b<\frac{2}{q-4}$ if $q>4)$. Here $L^2=\frac{1}{2}\sum_{k,j=1}^{q}(x_k\partial_i-x_j\partial_k)^2$, and $HA^q_{\Delta_{LB}}(\nu)$ and $HA^q_e(a,b)$ are the weighted Hilbert spaces of harmonic functions defined in Chapter IV Sections 2 and 3 of this thesis.

3.

The operator $x_k\partial^\alpha$ $(1\leq k\leq q,\ |\alpha|=2)$ is compact in the spaces $HA^q_{\Delta_{LB}}(\nu)$ $(1<\nu\leq 2)$ and $HA^q_e(a,b)$ $(a>0,\, 0<b<\frac{2}{q+3})$, and continuous in $HA^q_e(a,b)$ $(a>0,\, b=\frac{2}{q+3})$.

4.

Since $\sum_{k=1}^{q} a_k M_k$ $(a_k\in\mathbb{C})$ is a bounded perturbation of the positive self-adjoint operator Δ_{LB} in $L^2(S^{q-1})$, the operator $\Delta_{LB}+\sum_{k=1}^{q} a_k M_k$ is the infinitesimal generator of a C_0 semigroup $T(t)$ in $L^2(S^{q-1})$. Actually $T(t)$ restricts to a C_0 semigroup on each of the spaces $HA^q_{\Delta_{LB}}(\nu)$ or $HA^q_e(a,b)$.

5.

Let $A: D(A)\subset X\to X$ be an ω-accretive nonlinear operator in a Banach space $(X, \|\cdot\|)$ with X^* Fréchet differentiable. $\bar{A}$ and J_λ denote the closure and the resolvent $(\lambda I+A)^{-1}$ of A respectively. Assume furthermore that $R(I+\lambda A)\supset \overline{D(A)}$ for sufficiently small positive value of λ. Then the semigroup $\{S(t)\}_{t\geq 0}$ on $\overline{D(A)}$ generated by A is differentiable in the following sense:

(i) For each $x\in \hat{D}(A)=\{x\in\overline{D(A)};\ \lim_{t\to 0+} t^{-1}\|x-J_t x\|<\infty\}$ both $\lim_{t\to 0+} t^{-1}(x-S(t)x)$ and $\lim_{t\to 0+} t^{-1}(x-J_t x)$ exist and are equal.
Defining the common limit to be $A^* x$ for each $x\in\hat{D}(A)$, A^* is precisely the infinitesimal generator of $\{S(t)\}_{t\geq 0}$.

(ii) $(\bar{A})^0$ is single valued, $D((\bar{A})^0) = D(\bar{A}) = \hat{D}(A)$ and $(\bar{A})^0 = A^*$. Here for an operator B in X, $B^0 x = \{y \mid y \in Bx, \|y\| = |Bx|\}$ with $|Bx| = \inf \{\|y\| \mid y \in Bx\}$.
Thus

$$\frac{d^+}{dt}[S(t)x] + (\bar{A})^0 S(t)x = 0, \; \forall x \in D(A), \; \forall t \geq 0.$$

See [L1].

6.

Let $(E, \|\cdot\|, K)$ be a regular and strongly minimal ordered Banach space and let (V,d) be a complete quasimetric space with $d : V \times V \to E$. Assume that $F : V \to E$ is a lower semi-continuous mapping bounded from below. For arbitrary $\varepsilon > 0$, $e > \theta$ (zero in E), choose $u \in V$ such that

$$F(u) \not\geq \inf_V F + \varepsilon e.$$

Then, there exists some $v \in V$ (an approximate Pareto minimum) satisfying the following relations:

$$e \not\leq d(u, v)$$

$$F(v) \leq F(u)$$

$$F(w) \not\ll F(v) - \varepsilon d(v,w), \; \forall w \in V$$

$$F(w) \not\leq F(v) - \varepsilon d(v,w), \; \forall w \in V \text{ such that } d(v,w) \gg \theta.$$

See [L2].

7.

Let $\{T_k \mid 1 \leq k \leq K\}$ be a finite family of bounded linear operators in a Banach space $(X, \|\cdot\|)$ with spectral radii $\{\gamma(T_k) \mid 1 \leq k \leq K\}$. If they are mutually commutative, i.e., $T_k T_j = T_j T_k$ for all $1 \leq j, k \leq K$, then, for any given $\varepsilon > 0$ there exists an equivalent new norm $|\cdot|$ on X such that

$$|T_k| \leq \|T_k\|, \; 1 \leq k \leq K$$

and

$$\gamma(T_k) \leq |T_k| \leq \gamma(T_k) + \varepsilon.$$

Here $\|T_k\|$ and $|T_k|$ stand for the operator norms of T_k with respect to the old norm $\|\cdot\|$ and new norm $|\cdot|$ of X respectively. See [Y-L].

8.

World peace can only be realized by its partition into a large number of small states, not by grouping together into a few ones.

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