Mon. Not. R. Astron. Soc. **332,** 676–688 (2002)

Evolution equations for the perturbations of slowly rotating relativistic stars

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Accepted 2002 January 11. Received 2001 January 4; in original form 2001 August 22

ABSTRACT

We present a new derivation of the equations governing the oscillations of slowly rotating relativistic stars. Previous investigations have been mostly carried out in the Regge–Wheeler gauge. However, in this gauge the process of linearizing the Einstein field equations leads to perturbation equations in a form that cannot be used to perform numerical time evolutions. It is only through the tedious process of combining and rearranging the perturbation variables in a clever way that the system can be cast into a set of hyperbolic first-order equations, which is then well suited for the numerical integration. The equations remain quite lengthy, and we therefore rederive them in a different gauge. Using the ADM formalism, one immediately obtains a first-order hyperbolic evolution system, which is remarkably simple and can be integrated numerically without many further manipulations. Moreover, the symmetry between the polar and axial equations becomes directly apparent.

Key words: relativity – stars: neutron – stars: oscillations – stars: rotation.

1 INTRODUCTION

The theory of non-radial perturbations of relativistic stars has been a field of intensive study for more than three decades, beginning with the pioneering paper of Thorne & Campolattaro in 1967. These authors focused on perturbations of non-rotating stars, while Hartle (1967) laid the foundations for computing rotating relativistic stellar models. He also devised a way of modelling slowly rotating stars. This was widely used in following works since the problem becomes one dimensional and therefore much simpler than the two-dimensional case of rapidly-rotating and strongly-deformed stars. It was well known from the very early days that rotating relativistic bodies can become unstable with respect to gravitational radiation. The most interesting instability mechanism in the context of *r*-mode oscillations is the Chandrasekhar–Friedman–Schutz (CFS) mechanism (Chandrasekhar 1970; Friedman & Schutz 1978), where the gravitational radiation continuously removes angular momentum from a backwards moving mode, thus reducing the total angular momentum of the star and slowing it down.

As the full set of perturbation equations for rotating relativistic stars is quite complicated, it was only in the last decade that people started to compute their oscillation modes. In most studies the slow-rotation approximation is still used to tackle the problem. Investigating the axisymmetric perturbations, Chandrasekhar & Ferrari (1991) showed how rotation induces coupling of the polar and axial modes, which are decoupled in the non-rotating case. (Polar or even parity modes are characterized by a sign change under parity transformation according to $(-1)^l$, while the axial ones change as $(-1)^{l+1}$.) Soon after, Kojima (1992) presented the first complete derivation of the coupled polar and axial perturbation equations.

For more than 40 yr, it was quite common to work in the Regge–Wheeler gauge (Regge & Wheeler 1957), although some groups have used different gauges or the gauge-invariant formulation of Moncrief (1974). In a series of papers devoted to the study of the stability properties of non-radial oscillations in relativistic non-rotating stars, Battiston, Cazzola & Lucaroni introduced in 1971 a different gauge, which, however, has not received much attention since (Battiston, Cazzola & Lucaroni 1971; Cazzola & Lucaroni 1972, 1974, 1978; Cazzola, Lucaroni & Semenzato 1978a,b).

Since the perturbation equations of non-rotating stars are fairly simple, there is no real advantage of one gauge over the other. For rotating stars, however, the equations become much more complicated and choosing the 'right' gauge can make life much simpler. In

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particular, when one is interested in the time-dependent problem, the perturbation equations have to be brought into a form suitable for the numerical evolution. In this case, the effort that has to be spent manipulating the equations can depend considerably on the chosen gauge.

Time evolutions of the perturbations of non-rotating stars have been carried out, first for the axial equations (Andersson & Kokkotas 1996) and then for the polar equations using the Regge–Wheeler gauge by Allen et al. (1998) and Ruoff (2001). Allen et al. (1998) managed to write down the evolution equations as two relatively simple wave equations for the metric perturbations and one wave equation for the fluid enthalpy perturbation inside the star. Ruoff (2001) rederived these equations using the Arnowitt–Deser–Misner (ADM) formalism (Arnowitt et al. 1962). They were used to evolve and study initial data representing the late stage of a binary neutron star head-on collision (Allen et al. 1999).

Using the ADM formalism, Ruoff & Kokkotas (2001, 2002) derived the evolution equations for the axial perturbations of slowly rotating stars as a first-order system both in space and time, which could be used for the numerical evolution without many further manipulations. In the non-rotating case, it is an easy task to transform the first-order system into a single wave equation for a metric variable. In the rotating case, however, this is not possible because of the rotational correction terms.

When looking at the set of polar equations derived by Kojima (1992) it is apparent that the presence of mixed spatial and time derivatives makes them unsuitable for the numerical time integration. Nevertheless, using a number of successive manipulations and the introduction of many additional variables, we were able to cast the equations into a hyperbolic first-order form.

A more natural way to obtain a first-order-in-time set of equations automatically is to use the ADM formalism. However, as we shall explain, even in that case the polar equations in the Regge–Wheeler gauge need to be manipulated further before they are suitable for a numerical integration. In general the ADM formalism yields a set of partial differential equations that are first order in time, but second order in space. For the numerical evolution, this is not ideal and one would rather have a pure first-order system, or if possible a complete second-order system, thus representing generalized wave equations. As we mentioned above, in the non-rotating case, it is easy to convert the perturbation equations into a set of wave equations. However, in the rotating case, this is no longer possible, even in the simple case of purely axial perturbations. To illustrate the problems associated with the Regge–Wheeler gauge, let us recall Einstein's (unperturbed) evolution equations written in the ADM formalism:

$$(\partial_t - L_\beta)\gamma_{ij} = -2\alpha K_{ij}, \tag{1}$$

$$(\partial_t - L_\beta)K_{ij} = -\alpha_{;ij} + \alpha[R_{ij} + K^k_k K_{ij} - 2K_{ik}K^k_j - 4\pi(2T_{ij} - T^\nu_\nu \gamma_{ij})], \tag{2}$$

with α denoting the lapse function, β^i the shift vector, L_β the Lie-derivative with respect to β^i, γ_{ij} the metric of a space-like three-dimensional hypersurface with Ricci tensor R_{ij}, and K_{ij} its extrinsic curvature. It is obvious that the only second-order spatial derivatives are $\partial_i\partial_j\alpha$ and $\partial_i\partial_j\gamma_{kl}$ with the latter originating from the Ricci tensor R_{ij}. This is still true for the linearized version of equations (1) and (2).

In the Regge–Wheeler gauge, we have a non-vanishing perturbation of the lapse α and the diagonal components of the spatial perturbations h_{ij}. Using the notation of Ruoff (2001), the polar perturbations can be written as

$$\alpha \sim \sum_{l,m} S_1^{lm}(t,r) Y_{lm}(\theta,\phi), \tag{3}$$

$$h_{ij} \sim \sum_{l,m} \begin{pmatrix} S_3^{lm}(t,r) & 0 & 0 \\ 0 & T_2^{lm}(t,r) & 0 \\ 0 & 0 & \sin^2\theta T_2^{lm}(t,r) \end{pmatrix} Y_{lm}(\theta,\phi). \tag{4}$$

The perturbation equations obtained from equation (2) contain second r-derivatives of S_1 and T_2. Note that they do not contain second derivatives of S_3, because only the angular components of the metric are differentiated twice with respect to r. In the axial case there is only one perturbation function for the angular metric components, but it is set to zero in the Regge–Wheeler gauge. Therefore the ADM formalism immediately yields a first-order system.

The polar equations, in contrast, can be cast into a fully first-order system only through the introduction of auxiliary variables. In the non-rotating case, this is a fairly easy task, but for the rotating case it turns out to be considerably more complicated. One of the main reasons is that, in the non-rotating case, we have a simple proportionality between S_1 and S_3, which we can use to eliminate S_1 in all the equations. This is crucial since S_1 represents the perturbation of the lapse, for which the ADM formalism does not provide an evolution equation. In the rotating case, however, the relation between S_1 and S_3 contains various rotational correction terms and the replacement of S_1 by S_3 would lead to a considerable inflation of the equations.

Instead of manipulating the perturbation equations in the Regge–Wheeler gauge, we therefore look for a gauge in which the perturbation equations, by construction, do not show any second-order spatial derivative. We have seen that the second derivatives originate from the angular terms of the spatial metric perturbations and the perturbation of the lapse function. It seems natural therefore to set these to zero. For the axial case this is already realized in the Regge–Wheeler gauge. It is only for the polar perturbations that we need a different gauge. From the seven polar components of the metric, Regge & Wheeler (1957) chose to set the components V_1, V_3 and T_1 to zero, which, in the notation of Ruoff (2001), represent the polar angular vector perturbations and one of the angular tensor perturbations. We now proceed differently: we keep the vector perturbations V_1 and V_3, while we set the tensor perturbations T_1, T_2, together with the perturbation of the lapse S_1, to zero. With this choice we expect the ADM formalism to provide us with an evolution system without any second r-derivatives.

We note again that this gauge was actually introduced thirty years ago by Battiston et al. (1971) to derive the perturbation equations for non-radial oscillations of non-rotating neutron stars and to investigate their stability properties in a subsequent series of papers (Cazzola & Lucaroni 1972, 1974, 1978; Cazzola, Lucaroni & Semenzato 1978a,b). The first paper is of particular relevance, as they show that this gauge does not contain any further gauge freedom and establish the relation with the Regge–Wheeler gauge. From now on, we will call this gauge the BCL gauge.

This paper is structured as follows. In Section 2, we will use the ADM formalism to derive the time-dependent perturbation equations for slowly rotating relativistic stars in the BCL gauge. Section 3 contains a brief discussion of the non-rotating limit, and conclusions will be drawn in Section 4. In the appendix, we present the perturbation equations as they follow directly from Einstein's equations in a form similar to the equations in the Regge–Wheeler gauge given by Kojima (1992). Throughout the paper, we adopt the metric signature $(-+++)$, and we use geometrical units with $c = G = 1$. Derivatives with respect to the radial coordinate r are denoted by a prime, while derivatives with respect to time t are denoted by a dot. Greek indices run from 0 to 3, Latin indices from 1 to 3.

2 THE PERTURBATION EQUATIONS IN THE ADM FORMALISM

The metric describing a slowly rotating neutron star in spherical coordinates (t, r, θ, ϕ) is

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\nu} & 0 & 0 & -\omega r^2 \sin^2\theta \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -\omega r^2 \sin^2\theta & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{5}$$

where ν, λ and the 'frame dragging' ω are functions of the radial coordinate r only. With the neutron star matter described by a perfect fluid with pressure p, energy density ϵ, and 4-velocity

$$U^\mu = (e^{-\nu}, 0, 0, \Omega e^{-\nu}), \tag{6}$$

the Einstein equations together with an equation of state $p = p(\epsilon)$ yield the well-known TOV equations plus an extra equation for the frame dragging, which to linear order is given by

$$\varpi'' - \left[4\pi r e^{2\lambda}(p+\epsilon) - \frac{4}{r}\right]\varpi' - 16\pi e^{2\lambda}(p+\epsilon)\varpi = 0, \tag{7}$$

where

$$\varpi := \Omega - \omega \tag{8}$$

represents the angular velocity of the fluid relative to the local inertial frame. In the language of the ADM formalism, we have to express the background metric (5) in terms of lapse, (covariant) shift and the 3-metric, which we denote by A, B_i and γ_{ij}, respectively. Explicitly, we have

$$A = \sqrt{B^i B_i - g_{00}} = e^{\nu} + O(\omega^2), \tag{9}$$

$$B_i = (0, 0, -\omega r^2 \sin^2\theta), \tag{10}$$

$$\gamma_{ij} = \begin{pmatrix} e^{2\lambda} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{11}$$

The extrinsic curvature of the space-like hypersurface described by γ_{ij} can be computed using

$$K_{ij} = \frac{1}{2A}(B^k \partial_k \gamma_{ij} + \gamma_{ki}\partial_j B^k + \gamma_{kj}\partial_i B^k), \tag{12}$$

yielding the only non-vanishing components

$$K_{13} = K_{31} = -\frac{1}{2}\omega' e^{-\nu} r^2 \sin^2\theta. \tag{13}$$

The perturbations of the background lapse A, shift B_i, 3-metric γ_{ij}, extrinsic curvature K_{ij}, 4-velocity U_i, energy density ϵ and pressure p will be denoted by α, β_i, h_{ij}, k_{ij}, u_i, $\delta\epsilon$ and δp, respectively. The twelve evolution equations for h_{ij} and k_{ij} are obtained by linearizing the non-linear ADM equations (1) and (2). Working in the slow-rotation approximation, we keep only terms up to order Ω (or ω). The background quantities B^k and K_{ij} are first order in Ω, hence we can neglect any products thereof. The trace of the background extrinsic curvature $\gamma^{ij}K_{ij}$ vanishes as well and the perturbation equations reduce to:

$$\partial_t h_{ij} = \partial_i\beta_j + \partial_j\beta_i - 2(Ak_{ij} + K_{ij}\alpha + \Gamma^k_{ij}\beta_k + B_k\delta\Gamma^k_{ij}), \tag{14}$$

$$\partial_t k_{ij} = \alpha[R_{ij} + 4\pi(p-\epsilon)\gamma_{ij}] - \partial_i\partial_j\alpha + \Gamma^k_{ij}\partial_k\alpha + \delta\Gamma^k_{ij}\partial_k A + A\{\delta R_{ij} + K_{ij}k - 2(K^k_i k_{jk} + K^k_j k_{ik}) + 4\pi[(p-\epsilon)h_{ij} + \gamma_{ij}(\delta p - \delta\epsilon) - 2(p+\epsilon)(u_i\delta u_j + u_j\delta u_i)]\} + B^k\partial_k k_{ij} + (\partial_k K_{ij} - K^l_i\partial_j\gamma_{kl} - K^l_j\partial_i\gamma_{kl})\beta^k + k_{ik}\partial_j B^k + k_{jk}\partial_i B^k + K^k_i\partial_j\beta_k + K^k_j\partial_i\beta_k \quad (15)$$

where

$$k := \gamma^{ij}k_{ij}, \quad (16)$$

$$\delta\Gamma^k_{ij} := \frac{1}{2}\gamma^{km}(\partial_i h_{mj} + \partial_j h_{mi} - \partial_m h_{ij} - 2\Gamma^l_{ij}h_{lm}), \quad (17)$$

$$\delta R_{ij} := \partial_k\delta\Gamma^k_{ij} - \partial_j\delta\Gamma^k_{ik} + \Gamma^l_{ij}\delta\Gamma^k_{lk} + \Gamma^k_{lk}\delta\Gamma^l_{ij} - \Gamma^l_{ik}\delta\Gamma^k_{lj} - \Gamma^k_{lj}\delta\Gamma^l_{ik}. \quad (18)$$

To obtain a closed set of evolution equations, we will also use the four evolution equations for the fluid perturbations that follow from the linearized conservation law $\delta T^{\mu\nu}_{;\mu} = 0$. Finally we need the four linearized constraint equations, which allow us to construct physically valid initial data and to monitor the accuracy of the numerical evolution:

$$\gamma^{ij}\delta R_{ij} - h^{ij}R_{ij} - 2K^{ij}k_{ij} = 16\pi[\delta\epsilon + 2e^{-\nu}(p+\epsilon)(\Omega - \omega)\delta u_3], \quad (19)$$

$$-8\pi[(p+\epsilon)\delta u_i + u_i(\delta p + \delta\epsilon)] = \gamma^{jk}(\partial_i k_{jk} - \partial_j k_{ik} - \Gamma^l_{ik}k_{jl} + \Gamma^l_{jk}k_{il} - \delta\Gamma^l_{ik}K_{jl} + \delta\Gamma^l_{jk}K_{il}) - h^{jk}(\partial_i K_{jk} - \partial_j K_{ik} - \Gamma^l_{ik}K_{jl} + \Gamma^l_{jk}K_{il}). \quad (20)$$

We assume the oscillations to be adiabatic, so that the relation between the Eulerian pressure perturbation δp and energy density perturbation $\delta\epsilon$ is given by

$$\delta p = \frac{\Gamma_1 p}{p+\epsilon}\delta\epsilon + p'\xi^r\left(\frac{\Gamma_1}{\Gamma} - 1\right), \quad (21)$$

where Γ_1 represents the adiabatic index of the perturbed configuration, Γ is the background adiabatic index

$$\Gamma = \frac{p+\epsilon}{p}\frac{\mathrm{d}p}{\mathrm{d}\epsilon}, \quad (22)$$

and ξ^r is the radial component of the fluid displacement vector ξ^μ. The latter is related to the (covariant) 4-velocity perturbations δu_μ through

$$\delta u_\mu = (g_{\mu\nu} + u_\mu u_\nu)L_u\xi^\nu + \frac{1}{2}u_\mu u^\kappa u^\lambda h_{\kappa\lambda} + u^\nu h_{\mu\nu}, \quad (23)$$

where L_u denotes the Lie derivative along u^μ. For the r component, this gives us

$$(\partial_t + \Omega\partial_\phi)\xi^r = e^{-2\lambda}(e^\nu\delta u_r - \beta_r - \Omega h_{r\phi}). \quad (24)$$

Next, we expand the complete set of perturbation variables into spherical harmonics $Y_{lm} = Y_{lm}(\theta,\phi)$. This will enable us to eliminate the angular dependence and obtain a set of equations for the coefficients, which only depend on t and r. It is only then that we can finally choose our gauge. In principle, choosing the gauge amounts to providing prescriptions for the lapse function and shift vector. In perturbation theory, the gauge can be used to set some of the ten metric perturbations to zero. We could, for instance, set $\alpha = \beta_i = 0$, and we would be left with only the six components h_{ij}. Note that setting α to zero is possible, since this quantity is only the perturbation of the background lapse A, and the latter does not vanish.

However, our actual goal is to set some of the spatial perturbation components h_{ij} to zero, namely the angular components h_{ab} with $a,b = \{\theta,\phi\}$. In principle we need prescribe the values of h_{ij} only once for the initial data, and not during the evolution. The only way to keep h_{ab} zero throughout the evolution is to choose our gauge such that the evolution equations for h_{ab} become trivial, i.e. we have to enforce

$$\partial_t h_{ab} = 0, \quad a,b = \{\theta,\phi\}. \quad (25)$$

We will see that this requirement leads to a unique algebraic condition for the shift vector β_i. With the definitions

$$X_{lm} := 2(\partial_\theta - \cot\theta)\partial_\phi Y_{lm}, \quad (26)$$

$$W_{lm} := \left(\partial^2_{\theta\theta} - \cot\theta\partial_\theta - \frac{\partial^2_{\phi\phi}}{\sin^2\theta}\right)Y_{lm} = [2\partial^2_{\theta\theta} + l(l+1)]Y_{lm}, \quad (27)$$

we expand the metric as follows. For the polar part we choose

$$\alpha = 0, \tag{28}$$

$$\beta_i^{\text{polar}} = \sum_{l,m} (e^{2\lambda} S_2^{lm}, V_1^{lm} \partial_\theta, V_1^{lm} \partial_\phi) Y_{lm}, \tag{29}$$

$$h_{ij}^{\text{polar}} = \sum_{l,m} \begin{pmatrix} e^{2\lambda} S_3^{lm} & V_3^{lm} \partial_\theta & V_3^{lm} \partial_\phi \\ \star & 0 & 0 \\ \star & 0 & 0 \end{pmatrix} Y_{lm}, \tag{30}$$

and the axial part is

$$\beta_i^{\text{axial}} = \sum_{l,m} \left(0, -V_2^{lm} \frac{\partial_\phi}{\sin\theta}, V_2^{lm} \sin\theta \partial_\theta \right) Y_{lm}, \tag{31}$$

$$h_{ij}^{\text{axial}} = \sum_{l,m} \begin{pmatrix} 0 & -V_4^{lm} \dfrac{\partial_\phi}{\sin\theta} & V_4^{lm} \sin\theta \partial_\theta \\ \star & 0 & 0 \\ \star & 0 & 0 \end{pmatrix} Y_{lm}. \tag{32}$$

The asterisks stand for symmetric components. For the extrinsic curvature we have no vanishing components

$$k_{ij}^{\text{polar}} = \frac{1}{2} e^{-\nu} \sum_{l,m} \begin{pmatrix} e^{2\lambda} K_1^{lm} Y_{lm} & e^{2\lambda} K_2^{lm} \partial_\theta Y_{lm} & e^{2\lambda} K_2^{lm} \partial_\phi Y_{lm} \\ \star & (rK_4^{lm} - \Lambda K_5^{lm}) Y_{lm} + K_5^{lm} W_{lm} & K_5^{lm} X_{lm} \\ \star & K_5^{lm} X_{lm} & \sin^2\theta [(rK_4^{lm} - \Lambda K_5^{lm}) Y_{lm} - K_5^{lm} W_{lm}] \end{pmatrix}, \tag{33}$$

$$k_{ij}^{axial} = \frac{1}{2} e^{-\nu} \sum_{l,m} \begin{pmatrix} 0 & -e^{2\lambda} K_3^{lm} \dfrac{\partial_\phi Y_{lm}}{\sin\theta} & e^{2\lambda} K_3^{lm} \sin\theta \partial_\theta Y_{lm} \\ \star & -K_6^{lm} \dfrac{X_{lm}}{\sin\theta} & K_6^{lm} \sin\theta W_{lm} \\ \star & K_6^{lm} \sin\theta W_{lm} & K_6^{lm} \sin\theta X_{lm} \end{pmatrix}. \tag{34}$$

Here and throughout the whole paper, we use the shorthand notation

$$\Lambda := l(l+1). \tag{35}$$

We note that the somewhat peculiar expansions for the coefficient K_5^{lm} can actually be written as

$$W_{lm} - \Lambda Y_{lm} = 2\partial_{\theta\theta}^2 Y_{lm}, \tag{36}$$

$$-\sin^2\theta (W_{lm} + \Lambda Y_{lm}) = 2(\cos\theta \sin\theta \partial_\theta + \partial_{\phi\phi}^2) Y_{lm}, \tag{37}$$

which are essentially the diagonal terms of the Regge–Wheeler tensor harmonic $\Psi_{\alpha\beta}^{lm}$ (cf. equation 20 of Ruoff 2001). However, we prefer to write them in terms of W_{lm} and Y_{lm} because it is only for these quantities that simple orthogonality relations apply. Furthermore, we note that in the definition of the polar components of the extrinsic curvature, we differ from the notation of Ruoff (2001), where the meaning of K_4 and K_5 is reversed (cf. equation 24). Also, the expansion for the axial perturbations slightly differs from that of Ruoff & Kokkotas (2001, 2002).

In their original paper, Battiston et al. (1971) did not use the ADM formalism to fix the gauge but instead defined their gauge by directly setting h_{tt}, $h_{\theta\theta}$, $h_{\theta\phi}$ and $h_{\phi\phi}$ to zero. The relation between h_{tt} and the lapse α is given by

$$h_{tt} = 2A\alpha + 2B^i \beta_i = 2e^\nu \alpha - 2\omega h_{t\phi}, \tag{38}$$

so that in the rotating case $h_{tt} \neq 0$ although the lapse α vanishes. In the non-rotating case $\beta_i = 0$ and both α and h_{tt} vanish. If we insisted on keeping a vanishing h_{tt} also in the rotating case, we would obtain a non-vanishing lapse, leading to the undesired second r derivatives in the perturbation equations.

Finally, the fluid perturbations are decomposed as

$$\delta u_i^{\mathrm{polar}} = -e^{\nu}\sum_{l,m}(u_1^{lm}, u_2^{lm}\partial_\theta, u_2^{lm}\partial_\phi)Y_{lm}, \tag{39}$$

$$\delta u_i^{\mathrm{axial}} = -e^{\nu}\sum_{l,m}\left(0, -u_3^{lm}\frac{\partial_\phi}{\sin\theta}, u_3^{lm}\sin\theta\partial_\theta\right)Y_{lm}, \tag{40}$$

$$\delta\epsilon = \sum_{l,m}\rho^{lm}Y_{lm}, \tag{41}$$

$$\delta p = (p+\epsilon)\sum_{l,m}H^{lm}Y_{lm}, \tag{42}$$

$$\xi^r = \left[\nu'\left(1-\frac{\Gamma_1}{\Gamma}\right)\right]^{-1}\sum_{l,m}\xi^{lm}Y_{lm}. \tag{43}$$

From equation (21), we obtain the relation

$$\rho^{lm} = \frac{(p+\epsilon)^2}{\Gamma_1 p}(H^{lm}-\xi^{lm}). \tag{44}$$

With the above expansion, the evolution equations for h_{ij} read

$$(\partial_t + \mathrm{i}m\omega)S_3^{lm}Y_{lm} = [2(S_2^{lm})' + 2\lambda' S_2^{lm} - K_1^{lm}]Y_{lm} + 2\omega e^{-2\lambda}[(V_3^{lm})' - \lambda' V_3^{lm}]\partial_\phi Y_{lm} + 2\omega e^{-2\lambda}[(V_4^{lm})' - \lambda' V_4^{lm}]\sin\theta\partial_\theta Y_{lm}, \tag{45}$$

$$\partial_t\left(V_3^{lm}\partial_\theta - V_4^{lm}\frac{\partial_\phi}{\sin\theta}\right)Y_{lm} = \left[(V_1^{lm})' - \frac{2}{r}V_1^{lm} + e^{2\lambda}(S_2^{lm}-K_2^{lm})\right]\partial_\theta Y_{lm} - \left[(V_2^{lm})' - \frac{2}{r}V_2^{lm} - e^{2\lambda}K_3^{lm}\right]\frac{\partial_\phi Y_{lm}}{\sin\theta} - \omega\Lambda V_4^{lm}\sin\theta Y_{lm}, \tag{46}$$

$$\partial_t(V_3^{lm}\partial_\phi + V_4^{lm}\sin\theta\partial_\theta)Y_{lm} = \left[(V_1^{lm})' - \frac{2}{r}V_1^{lm} + e^{2\lambda}(S_2^{lm}-K_2^{lm})\right]\partial_\phi Y_{lm} + \left[(V_2^{lm})' - \frac{2}{r}V_2^{lm} - e^{2\lambda}K_3^{lm}\right]\sin\theta\partial_\theta Y_{lm}, \tag{47}$$

$$0 = \left(2S_2^{lm} - \frac{\Lambda}{r}V_1^{lm} - K_4^{lm} + \frac{\Lambda}{r}K_5^{lm}\right)Y_{lm} + 2\omega e^{-2\lambda}(V_3^{lm}\partial_\phi Y_{lm} + V_4^{lm}\sin\theta\partial_\theta Y_{lm}), \tag{48}$$

$$0 = (V_1^{lm} - K_5^{lm})W_{lm} + (V_2^{lm} - K_6^{lm})\frac{X_{lm}}{\sin\theta}, \tag{49}$$

$$0 = (V_1^{lm} - K_5^{lm})X_{lm} - (V_2^{lm} - K_6^{lm})\sin\theta W_{lm}. \tag{50}$$

In every equation a sum over all l and m is still implied. From equations (49) and (50) we immediately obtain our condition for the shift components:

$$V_1^{lm} = K_5^{lm}, \tag{51}$$

$$V_2^{lm} = K_6^{lm}, \tag{52}$$

and from equation (48) it follows, after multiplication with Y_{lm}^* and integration over the 2-sphere, that

$$S_2^{lm} = \frac{1}{2}K_4^{lm} - \omega e^{-2\lambda}(\mathrm{i}mV_3^{lm} + \mathcal{L}_1^{\pm1}V_4^{lm}), \tag{53}$$

where we have defined the operator $\mathcal{L}_1^{\pm1}$, which couples the equations of order l to the equations of order $l+1$ and $l-1$, according to

$$\mathcal{L}_1^{\pm1}A^{lm} := \sum_{l'm'}A^{l'm'}\int_{S_2}Y_{lm}^*\sin\theta\partial_\theta Y_{l'm'}\,\mathrm{d}\Omega = (l-1)Q_{lm}A^{l-1m} - (l+2)Q_{l+1m}A^{l+1m}, \tag{54}$$

with

$$Q_{lm} := \sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}}. \tag{55}$$

Below we will also need

$$\mathcal{L}_2^{\pm1}A^{lm} := \sum_{l'm'}A^{l'm'}\int_{S_2}\partial_\theta Y_{lm}^*\sin\theta Y_{l'm'}\,\mathrm{d}\Omega = -(l+1)Q_{lm}A^{l-1m} + lQ_{l+1m}A^{l+1m} \tag{56}$$

and

$$\mathcal{L}_3^{\pm 1} A^{lm} := \sum_{l'm'} A^{l'm'} \left[l'(l'+1) \int_{S_2} Y_{lm}^* \cos\theta Y_{l'm'} \, d\Omega + \int_{S_2} Y_{lm}^* \sin\theta \partial_\theta Y_{l'm'} \, d\Omega \right] = (l-1)(l+1) Q_{lm} A^{l-1m} + l(l+2) Q_{l+1m} A^{l+1m}. \tag{57}$$

The operator $\mathcal{L}_3^{\pm 1}$ can actually be expressed in terms of $\mathcal{L}_1^{\pm 1}$ and $\mathcal{L}_2^{\pm 1}$:

$$\mathcal{L}_3^{\pm 1} = -\frac{1}{2}[\mathcal{L}_1^{\pm 1}(\Lambda - 2) + \mathcal{L}_2^{\pm 1}\Lambda]. \tag{58}$$

For notational simplicity, we will from now on omit the indices l and m for the perturbation variables. By making use of the above relations we can eliminate the spherical harmonics and obtain the following simple set of evolution equations for the metric perturbations:

$$(\partial_t + \mathrm{i}m\omega) S_3 = K_4' - K_1 + \lambda' K_4 - 2\omega' e^{-2\lambda}(\mathrm{i}m V_3 + \mathcal{L}_1^{\pm 1} V_4), \tag{59}$$

$$(\partial_t + \mathrm{i}m\omega) V_3 = K_5' - e^{2\lambda} K_2 + \frac{1}{2} e^{2\lambda} K_4 - \frac{2}{r} K_5, \tag{60}$$

$$(\partial_t + \mathrm{i}m\omega) V_4 = K_6' - e^{2\lambda} K_3 - \frac{2}{r} K_6. \tag{61}$$

In a similar way, we obtain the evolution equations for the six extrinsic curvature components:

$$\begin{aligned}(\partial_t + \mathrm{i}m\omega) K_1 = {} & e^{2\nu - 2\lambda} \left[\left(\nu' + \frac{2}{r} \right) S_3' - 2\frac{\Lambda}{r^2} V_3' + 2\lambda' \frac{\Lambda}{r^2} V_3 + 2\left(\frac{\nu'}{r} - \frac{\lambda'}{r} - \frac{e^{2\lambda} - 1}{r^2} + e^{2\lambda} \frac{\Lambda}{2r^2} \right) S_3 \right] \\ & + 8\pi e^{2\nu}(p + \epsilon) C_\mathrm{s}^{-2}[(C_\mathrm{s}^2 - 1)H + \xi] - 2e^{-2\lambda}\omega' \left[\mathrm{i}m \left(K_5' - \frac{2}{r} K_5 \right) + \mathcal{L}_1^{\pm 1} \left(K_6' - \frac{2}{r} K_6 \right) \right],\end{aligned} \tag{62}$$

$$\begin{aligned}(\partial_t + \mathrm{i}m\omega) K_2 = {} & e^{2\nu - 2\lambda} \left[\left(\nu' + \frac{1}{r} \right) S_3 - \frac{2}{r^2} V_3 \right] \\ & + \frac{\mathrm{i}m r^2}{2\Lambda} e^{-2\lambda} \left[\omega' \left(K_4' - K_1 + \lambda' K_4 - 4\frac{\Lambda - 1}{r^2} K_5 \right) - 16\pi\varpi(p + \epsilon)(e^{2\lambda} K_4 + 2e^{2\nu} u_1) \right] - \frac{\omega' e^{-2\lambda}}{\Lambda} \mathcal{L}_1^{\pm 1}[(\Lambda - 2) K_6],\end{aligned} \tag{63}$$

$$\begin{aligned}(\partial_t + \mathrm{i}m\omega) K_3 = {} & e^{2\nu - 2\lambda} \frac{\Lambda - 2}{r^2} V_4 + e^{-2\lambda} \frac{\omega'}{\Lambda}[2\mathrm{i}m K_6 + (\Lambda - 2)\mathcal{L}_2^{\pm 1} K_5] \\ & - \frac{r^2}{2\Lambda} e^{-2\lambda} \mathcal{L}_2^{\pm 1}[\omega'(K_4' - K_1 + \lambda' K_4) - 16\pi\varpi(p + \epsilon)(e^{2\lambda} K_4 + 2e^{2\nu} u_1)],\end{aligned} \tag{64}$$

$$\begin{aligned}(\partial_t + \mathrm{i}m\omega) K_4 = {} & e^{2\nu - 2\lambda} \left[S_3' + 2\left(\nu' - \lambda' + \frac{1}{r} \right) S_3 - \frac{2\Lambda}{r^2} V_3 \right] + 8\pi r e^{2\nu}(p + \epsilon) C_\mathrm{s}^{-2}[(C_\mathrm{s}^2 - 1)H + \xi] \\ & + r(\mathcal{L}_1^{\pm 1} - \mathcal{L}_2^{\pm 1})[\omega' K_3 + 16\pi e^{2\nu}\varpi(p + \epsilon) u_3],\end{aligned} \tag{65}$$

$$\begin{aligned}(\partial_t + \mathrm{i}m\omega) K_5 = {} & e^{2\nu - 2\lambda} \left[V_3' + (\nu' - \lambda') V_3 - \frac{1}{2} e^{2\lambda} S_3 \right] \\ & + \frac{r^2}{\Lambda} \left\{ \mathrm{i}m \left[\omega' \left(\frac{1}{2} K_4 - K_2 \right) - 16\pi e^{2\nu}\varpi(p + \epsilon) u_2 \right] - \mathcal{L}_2^{\pm 1}[\omega' K_3 + 16\pi e^{2\nu}\varpi(p + \epsilon) u_3] \right\},\end{aligned} \tag{66}$$

$$(\partial_t + \mathrm{i}m\omega) K_6 = e^{2\nu - 2\lambda}[V_4' + (\nu' - \lambda') V_4] - \frac{r^2}{\Lambda} \left\{ \mathrm{i}m[\omega' K_3 + 16\pi e^{2\nu}\varpi(p + \epsilon) u_3] + \mathcal{L}_2^{\pm 1} \left[\omega' \left(\frac{1}{2} K_4 - K_2 \right) - 16\pi e^{2\nu}\varpi(p + \epsilon) u_2 \right] \right\}. \tag{67}$$

It is worth pointing out the symmetry between the polar and axial equations. Each pair V_3 and V_4, K_2 and K_3, and K_5 and K_6 represents the polar and axial counterparts of a metric or extrinsic curvature perturbation. Thus, each associated pair of equations (60) and (61), (63) and (64), and (66) and (67) has basically the same structure, with only the polar equations containing additional terms as there are more polar variables than axial ones.

The final set of evolution equations is that for the fluid quantities, resulting from $\delta(T^{\mu\nu}_{\mu}) = 0$ and equation (24):

$$(\partial_t + \mathrm{i}m\Omega)H = C_s^2\left[e^{2\nu-2\lambda}\left[u_1' + \left(2\nu' - \lambda' + \frac{2}{r}\right)u_1 - e^{2\lambda}\frac{\Lambda}{r^2}u_2\right] + \frac{1}{2}K_1 + \frac{1}{r}K_4 - \frac{\Lambda}{r^2}K_5\right.$$
$$\left. + \varpi e^{-2\lambda}\left\{\mathrm{i}m\left[V_3' + \left(\frac{2}{r} - \lambda'\right)V_3 + e^{2\lambda}\left(H - \frac{1}{2}S_3\right)\right] + \mathcal{L}_1^{\pm1}\left[V_4' + \left(\frac{2}{r} - \lambda'\right)V_4\right]\right\}\right]$$
$$- \nu'\left[e^{2\nu-2\lambda}u_1 + \frac{1}{2}K_4 + \varpi e^{-2\lambda}(\mathrm{i}mV_3 + \mathcal{L}_1^{\pm1}V_4)\right], \tag{68}$$

$$(\partial_t + \mathrm{i}m\Omega)u_1 = H' + \frac{p'}{\Gamma_1 p}\left[\left(\frac{\Gamma_1}{\Gamma} - 1\right)H + \xi\right] - \mathrm{i}m\left\{e^{-2\nu}\varpi\left(K_5' - \frac{2}{r}K_5\right) + \left[\omega' + 2\varpi\left(\nu' - \frac{1}{r}\right)\right]u_2\right\}$$
$$- \mathcal{L}_1^{\pm1}\left\{e^{-2\nu}\varpi\left(K_6' - \frac{2}{r}K_6\right) + \left[\omega' + 2\varpi\left(\nu' - \frac{1}{r}\right)\right]u_3\right\}, \tag{69}$$

$$(\partial_t + \mathrm{i}m\Omega)u_2 = H + \frac{\varpi}{\Lambda}\{\mathrm{i}m[2u_2 - e^{-2\nu}(\Lambda - 2)K_5] + 2\mathcal{L}_3^{\pm1}u_3 - e^{-2\nu}\mathcal{L}_1^{\pm1}[(\Lambda - 2)K_6]\} - \frac{\mathrm{i}mr^2}{\Lambda}A, \tag{70}$$

$$(\partial_t + \mathrm{i}m\Omega)u_3 = 2\frac{\varpi}{\Lambda}[\mathrm{i}m(u_3 + e^{-2\nu}K_6) - \mathcal{L}_3^{\pm1}(u_2 + e^{-2\nu}K_5)] + \frac{r^2}{\Lambda}\mathcal{L}_2^{\pm1}A, \tag{71}$$

$$(\partial_t + \mathrm{i}m\Omega)\xi = \nu'\left(\frac{\Gamma_1}{\Gamma} - 1\right)\left[e^{2\nu-2\lambda}u_1 + \frac{1}{2}K_4 + \varpi e^{-2\lambda}(\mathrm{i}mV_3 + \mathcal{L}_1^{\pm1}V_4)\right], \tag{72}$$

where

$$A = \varpi C_s^2\left\{e^{-2\lambda}\left[u_1' + \left(2\nu' - \lambda' + \frac{2}{r}\right)u_1 - e^{2\lambda}\frac{\Lambda}{r^2}u_2\right] + e^{-2\nu}\left(\frac{1}{2}K_1 + \frac{1}{r}K_4 - \frac{\Lambda}{r^2}K_5\right)\right\} + \left[\varpi\left(\nu' - \frac{2}{r}\right) + \omega'\right]\left(e^{-2\lambda}u_1 + \frac{1}{2}e^{-2\nu}K_4\right), \tag{73}$$

and the sound speed C_s is defined by

$$C_s^2 = \frac{\Gamma_1}{\Gamma}\frac{\mathrm{d}p}{\mathrm{d}\epsilon}. \tag{74}$$

The evolution equations comprise fourteen equations in total: four axial and ten polar. In the non-rotating case, they can be reduced to four wave equations, one for the axial and two for the polar metric perturbations plus one wave equation for the fluid variable H. The fluid equation for the axial velocity perturbation u_3 vanishes in the non-rotating case, whereas equation (72) for the displacement variable ξ does so in the barotropic case.

Finally we have our constraint equations: the Hamiltonian constraint

$$8\pi r^2 e^{2\lambda}\rho = rS_3' - \Lambda V_3' + \left(1 - 2r\lambda' + \frac{1}{2}e^{2\lambda}\Lambda\right)S_3 + \Lambda\left(\lambda' - \frac{1}{r}\right)V_3$$
$$+ r^2e^{2\lambda}\left\{\mathrm{i}m\left[\frac{1}{2}\omega' e^{-2\nu}K_2 + 16\pi\varpi(p+\epsilon)u_2\right] + \mathcal{L}_1^{\pm1}\left[\frac{1}{2}\omega' e^{-2\nu}K_3 + 16\pi\varpi(p+\epsilon)u_3\right]\right\}, \tag{75}$$

and the three momentum constraints

$$8\pi re^{2\nu}(p+\epsilon)u_1 = K_4' - \frac{\Lambda}{r}K_5' - K_1 + e^{2\lambda}\frac{\Lambda}{2r}K_2 - \nu'K_4 + \frac{\Lambda}{r^2}(1 + r\nu')K_5 + \frac{\mathrm{i}m}{4}r\omega' S_3 - [8\pi r(p+\epsilon)\varpi + 2e^{-2\lambda}\omega'](\mathrm{i}mV_3 + \mathcal{L}_1^{\pm1}V_4), \tag{76}$$

$$16\pi re^{2\nu}(p+\epsilon)u_2 = -rK_2' + rK_1 + (r\nu' - r\lambda' - 2)K_2 + K_4 - \frac{2}{r}K_5 + e^{-2\lambda}\frac{r\omega'}{\Lambda}[2\mathrm{i}mV_3 - (\Lambda - 2)\mathcal{L}_2^{\pm1}V_4]$$
$$+ \frac{\mathrm{i}mr^3}{\Lambda}\left[\frac{1}{2}e^{-2\lambda}\omega' S_3' - 16\pi\varpi(p+\epsilon)\{S_3 + C_s^{-2}[(C_s^2 + 1)H - \xi]\}\right], \tag{77}$$

$$16\pi re^{2\nu}(p+\epsilon)u_3 = -rK_3' + (r\nu' - r\lambda' - 2)K_3 + \frac{\Lambda - 2}{r}K_6 + e^{-2\lambda}\frac{r\omega'}{\Lambda}[2\mathrm{i}mV_4 + (\Lambda - 2)\mathcal{L}_2^{\pm1}V_3]$$
$$- \frac{r^3}{\Lambda}\mathcal{L}_2^{\pm1}\left[\frac{1}{2}e^{-2\lambda}\omega' S_3' - 16\pi\varpi(p+\epsilon)\{S_3 + C_s^{-2}[(C_s^2 + 1)H - \xi]\}\right]. \tag{78}$$

Without the coupling terms to the polar perturbations, the axial equations (61), (64), (67), (71) and (78) are equivalent to equations (7)–(10) and (12) of Ruoff & Kokkotas (2002).

3 THE NON-ROTATING LIMIT

The non-rotating limit is obtained by letting $\Omega \to 0$. As is well known, in this case the polar and axial parts of the equations completely decouple and the equations can be cast into a set of wave equations (Allen et al. 1998; Ruoff 2001). Nevertheless, it is instructive to consider it in the BCL gauge. For barotropic perturbations ($\Gamma_1 = \Gamma$), the polar evolution equations can then be easily transformed into three wave equations for the rescaled metric variables $S = e^{\nu-\lambda}S_3$ and $V = e^{\nu-\lambda}V_3/r$ and the rescaled fluid variable $\tilde{H} = e^{-\nu-\lambda}H/r$:

$$\frac{\partial^2 S}{\partial t^2} = \frac{\partial^2 S}{\partial r_*{}^2} + e^{2\nu-2\lambda}\left\{\left[\nu'(\nu'-\lambda') + 3\frac{\nu'}{r} + \frac{\lambda'}{r} - \frac{3}{r^2} - e^{2\lambda}\frac{\Lambda-1}{r^2} - \lambda''\right]S + \frac{4\Lambda}{r^2}(1-r\nu')V\right\}$$
$$+ 8\pi e^{2\nu}\left\{(C_s^2-1)\left[\tilde{\rho}' + \left(\nu' - \frac{1}{r}\right)\tilde{\rho}\right] + (C_s^2)'\tilde{\rho}\right\}, \tag{79}$$

$$\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 V}{\partial r_*{}^2} + e^{2\nu-2\lambda}\left[\left(\frac{\nu'}{r} - \frac{\lambda'}{r} + 2\frac{e^{2\lambda}-1}{r^2} - e^{2\lambda}\frac{\Lambda}{r^2}\right)V - e^{2\lambda}\left(\frac{\nu'}{r} + \frac{\lambda'}{r} - \frac{1}{r^2}\right)S\right] + 4\pi e^{2\nu}(C_s^2-1)\tilde{\rho}, \tag{80}$$

$$\frac{\partial^2 \tilde{H}}{\partial t^2} = e^{2\nu-2\lambda}\left[C_s^2\frac{\partial^2\tilde{H}}{\partial r^2} - [C_s^2\lambda' + \nu']\frac{\partial\tilde{H}}{\partial r} + \left\{C_s^2\left[\lambda'\left(\frac{3}{r}+\lambda'\right) + \frac{e^{2\lambda}-1}{r^2} - \lambda'' - e^{2\lambda}\frac{\Lambda}{r^2}\right] + \frac{\lambda'}{r} + 2\frac{\nu'}{r} + \frac{\nu'}{C_s^2}[\nu'+\lambda']\right\}\tilde{H}\right]$$
$$+ e^{2\nu}\left[\frac{r\nu'}{2}[C_s^2-1]\frac{\partial S}{\partial r} + \left\{C_s^2\left[\frac{\nu'}{2}(r\lambda' - r\nu' + 6) + \lambda' - \frac{e^{2\lambda}-1}{r}\right] + \frac{\nu'}{2}[r\lambda' - r\nu' - 2]\right\}S - \nu'\Lambda[C_s^2-1]V\right]. \tag{81}$$

In equations (79) and (80), r_* is the well-known tortoise coordinate, which is related to r through

$$\frac{\mathrm{d}}{\mathrm{d}r_*} = e^{\nu-\lambda}\frac{\mathrm{d}}{\mathrm{d}r}. \tag{82}$$

Furthermore, one can express the energy density $\tilde{\rho}$ in terms of $\tilde{H}$, which in the barotropic case reduces to

$$\tilde{\rho} = \frac{p+\epsilon}{C_s^2}\tilde{H}. \tag{83}$$

Although the equations in the first-order form are quite simple, the above set of wave equations is more complicated than the equivalent set in the Regge–Wheeler gauge (equations 14, 15 and 16 of Allen et al. 1998). However, there is a clear advantage if one is interested in computing the gauge-invariant Zerilli function Z in the exterior. Following Moncrief (1974), the definition of the Zerilli function is

$$Z = \frac{r^2(\Lambda k_1 + 4e^{-4\lambda}k_2)}{r(\Lambda-2)+2M} \tag{84}$$

with

$$k_1 = -2e^{-\lambda-\nu}V \tag{85}$$

and

$$k_2 = \frac{1}{2}e^{3\lambda-\nu}S. \tag{86}$$

In terms of S and V this gives us

$$Z = \frac{2r^2e^{-\lambda-\nu}}{r(\Lambda-2)+2M}(S-\Lambda V), \tag{87}$$

which is a simple algebraic relation in contrast with the equations in the Regge–Wheeler gauge, which includes a spatial derivative of one of the metric perturbations (see equation 20 of Allen et al. 1998, or equation 60 of Ruoff 2001). In the Regge–Wheeler gauge, the two metric variables (S and F in the notation of Allen et al. 1998, and S and T in the notation of Ruoff 2001) have different asymptotic behaviour at infinity, in particular one (F or T) is linearly growing with r. It is only through the delicate cancellation of the growing terms that the Zerilli function remains finite at infinity. However, this cancellation can only occur if both metric variables exactly satisfy the Hamiltonian constraint. Any (numerical) violation leads to an incomplete cancellation, and the Zerilli function starts to grow at large radii. As a result it could be rather difficult in the numerical time evolution to extract the correct amount of gravitational radiation emitted from the neutron star. With the above relation (87), we do not expect such difficulties to occur.

4 CONCLUSIONS

We have presented the derivation of the perturbation equations for slowly rotating relativistic stars using the BCL gauge, which was first used by Battiston et al. in 1971. This gauge is defined by setting the metric perturbations α, $h_{\theta\theta}$, $h_{\theta\phi}$ and $h_{\phi\phi}$ to zero. In the non-rotating case, the condition of zero lapse leads to a complete vanishing of h_{tt}. However, in the rotating case h_{tt} becomes non-zero (see also the appendix). For the axial perturbations the BCL gauge coincides with the Regge–Wheeler gauge; it is only for the polar perturbations that the two gauges differ. The advantage of the BCL gauge over the Regge–Wheeler gauge is that in the ADM formalism, the evolution equations do not a priori contain any second-order spatial derivatives. Instead, one is immediately lead to a hyperbolic set of first-order evolution equations, which can be used for the numerical time evolution without major modifications. Even though it is also possible to derive a hyperbolic set in the Regge–Wheeler gauge, the procedure is rather tedious and requires the introduction of carefully-chosen new variables in order to replace the second-order derivatives.

The perturbation equations for slowly rotating relativistic stars form a set of fourteen evolution equations plus four constraints. In the non-rotating barotropic case, it is possible to cast the polar equations into a system of three wave equations – as it is with the Regge–Wheeler gauge. Although these wave equations are not simpler than the corresponding ones in the Regge–Wheeler gauge, the first-order system actually is. Moreover, we have a simple algebraic relation between the metric variables and the Zerilli function. It was demonstrated by Ruoff (2001) that the accurate numerical evaluation of the Zerilli function in the Regge–Wheeler gauge is somewhat difficult and requires high resolution because a small numerical violation of the Hamiltonian constraint can lead to very large errors in the Zerilli function. This should not be the case in the BCL gauge as relation (87) does not involve any derivatives.

A further advantage of these evolution equations is that the inclusion of the source terms describing a particle orbiting the star can be accomplished in a straightforward way. This is not the case for the Regge–Wheeler gauge, as, even for the non-rotating case, one is forced to include second-order derivatives of the polar source terms (Ruoff, Laguna & Pullin 2001). Since the source terms contain δ-functions, one has to deal with second-order derivatives. In the axial case, no derivatives appear, and the perturbation equations with the source terms are quite simple (Ruoff et al. 2001). We expect the same to be the case for the polar equations in the BCL gauge, where it should be possible to plug the source terms into the equations for the extrinsic curvature without generating any derivatives.

In subsequent papers, we will present results from the numerical evolution of the perturbation equations of slowly rotating relativistic stars in the BCL gauge, with a particular focus on oscillation modes that are unstable with respect to gravitational radiation. We also plan to include the contribution of a test particle acting as a source of excitation for the stellar oscillations.

ACKNOWLEDGMENTS

We thank Nils Andersson, Luciano Rezzolla and Nikolaos Stergioulas for many helpful comments. JR is supported by Marie Curie Fellowship No. HPMF-CT-1999-00364. AS is supported by the Greek National Scholarship foundation (I. K. Y.). This work has been supported by the EU Programme 'Improving the Human Research Potential and the Socio-Economic Knowledge Base' (Research Training Network Contract HPRN-CT-2000-00137).

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APPENDIX A: THE PERTURBATION EQUATIONS FOLLOWING FROM EINSTEIN'S EQUATIONS

Kojima (1992) derived the perturbation equations in the Regge–Wheeler gauge directly from the linearized Einstein equations without resorting to the ADM formalism. In this section we repeat this calculation using the BCL gauge. In order to facilitate the comparison with

Kojima's equations, which use the more familiar notation of Regge & Wheeler (1957), we switch to a similar notation. In the Regge–Wheeler gauge, the quantities h_0 and h_1 denote the axial perturbations of $h_{t\{\phi,\theta\}}$ and $h_{r\{\phi,\theta\}}$, respectively, whereas the corresponding polar perturbations are set to zero. Since in the BCL gauge the latter do not vanish, we denote them by $h_{0,p}$ and $h_{1,p}$, respectively, and, in order to avoid confusion, we denote the axial ones by $h_{0,a}$ and $h_{1,a}$. The remaining non-zero polar perturbations are then H_1 and H_2. Thus, the expansion of the metric in the BCL gauge reads:

$$h_{\mu\nu} = \sum_{lm} \begin{pmatrix} -2\omega(h_{0,p}^{lm}\partial_\phi + h_{0,a}^{lm}\sin\theta\partial_\theta) & H_1^{lm} & h_{0,p}^{lm}\partial_\theta - h_{0,a}^{lm}/\sin\theta\partial_\phi & h_{0,p}^{lm}\partial_\phi + h_{0,a}^{lm}\sin\theta\partial_\theta \\ \star & e^{2\lambda}H_2^{lm} & h_{1,p}^{lm}\partial_\theta - h_{1,a}^{lm}/\sin\theta\partial_\phi & h_{1,p}^{lm}\partial_\phi + h_{1,a}^{lm}\sin\theta\partial_\theta \\ \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \end{pmatrix} Y_{lm}. \tag{A1}$$

Note that the component h_{tt} is not zero, which is a consequence of the relation between the perturbation of the lapse α and h_{tt} given by equation (38). The relation between the above variables and the ones used in the previous sections is as follows (we again omit the indices l and m):

$$H_1 = e^{2\lambda}K_4 - \omega(\mathrm{i}mV_3 + \mathcal{L}_1^{\pm1}V_4), \tag{A2}$$

$$H_2 = S_3, \tag{A3}$$

$$h_{0,p} = K_5, \tag{A4}$$

$$h_{0,a} = K_6, \tag{A5}$$

$$h_{1,p} = V_3, \tag{A6}$$

$$h_{1,a} = V_4, \tag{A7}$$

$$R = -u_1, \tag{A8}$$

$$V = -u_2, \tag{A9}$$

$$U = -u_3. \tag{A10}$$

The extrinsic curvature components can be expressed as

$$K_1 = 2e^{-2\lambda}\{H_1' - \lambda' H_1 + \omega[\mathrm{i}m(h_{1,p}' - \lambda' h_{1,p}) + \mathcal{L}_1^{\pm1}(h_{1,a}' - \lambda' h_{1,a})] - \dot{H}_2 - \mathrm{i}m\omega H_2, \tag{A11}$$

$$K_2 = e^{-2\lambda}\left[h_{0,p}' - \frac{2}{r}h_{0,p} + H_1 - \dot{h}_{1,p} + \omega\mathcal{L}_1^{\pm1}h_{1,a}\right], \tag{A12}$$

$$K_3 = e^{-2\lambda}\left[h_{0,a}' - \frac{2}{r}h_{0,a} + \dot{h}_{1,a} - \mathrm{i}m\omega h_{1,a}\right]. \tag{A13}$$

A frequently occurring combination of variables in the perturbation equations is $h_0' - \dot{h}_1$ for both the axial and polar cases, which we abbreviate with the following functions:

$$Z_a = h_{0,a}' - \dot{h}_{1,a}, \tag{A14}$$

$$Z_p = h_{0,p}' - \dot{h}_{1,p}. \tag{A15}$$

The equations belonging to the (tt), (tr), (rr) and the addition of the ($\theta\theta$) and ($\phi\phi$) components can be written as

$$A_{lm}^{(I)} + \mathrm{i}mC_{lm}^{(I)} + \mathcal{L}_2^{\pm1}B_{lm}^{(I)} + \mathcal{L}_4^{\pm1}\tilde{A}_{lm}^{(I)} = 0, \tag{A16}$$

with

$$\mathcal{L}_4^{\pm1}A_{lm} := -\frac{1}{2}(\mathcal{L}_1^{\pm1} + \mathcal{L}_2^{\pm1})A_{lm} = Q_{lm}A_{l-1m} + Q_{l+1m}A_{l+1m} \tag{A17}$$

and

$$A^{(tt)} = \frac{2e^{2\nu}}{r^2}\left[rH_2' - \Lambda h_{1,p}' - 16\pi r^2 e^{2\lambda} C_{\rm s}^{-2}(H-\xi) + \Lambda\left(\lambda' - \frac{1}{r}\right)h_{1,p} + \left(1 - 2r\lambda' + \frac{\Lambda e^{2\lambda}}{2}\right)H_2\right], \tag{A18}$$

$$\tilde{A}^{(tt)} = 0, \tag{A19}$$

$$B^{(tt)} = 2\omega Z_a' + \left[\omega' - 2\omega\left(\lambda' + \nu' - \frac{2}{r}\right)\right]Z_a \frac{4\omega}{r} h_{0,a}' - 32\pi\Omega(p+\epsilon)e^{2\nu+2\lambda}U + \frac{2}{r}\left[-\omega' + \omega\left(2\nu' + 2\lambda' - \frac{2}{r} - e^{2\lambda}\frac{\Lambda - 2}{r}\right)\right]h_{0,a}, \tag{A20}$$

$$C^{(tt)} = 2\omega(Z_p' - H_1' + e^{2\lambda}\dot{H}_2) + [\omega' - 2\omega(\lambda' + \nu')]h_{0,p}' - \frac{2}{r}\left[\omega' - 2\omega\left(\nu' + \lambda' - \frac{e^{2\lambda} - 1}{r}\right)\right]h_{0,p} - \left[\omega' - 2\omega\left(\nu' + \lambda' - \frac{2}{r}\right)\right]\dot{h}_{1,p}$$
$$+ [\omega' + 2\omega(\lambda' - \nu')]H_1 - 32\pi\Omega e^{2\nu+2\lambda}(p+\epsilon)V, \tag{A21}$$

$$A^{(tr)} = \frac{2}{r}\dot{H}_2 + \frac{\Lambda}{r^2}(Z_p + H_1 - 2h_{0,p}' + 2\nu' h_{0,p}) + 16\pi(p+\epsilon)(e^{2\nu}R - H_1), \tag{A22}$$

$$\tilde{A}^{(tr)} = \frac{2\Lambda\omega}{r^2}h_{1,a}, \tag{A23}$$

$$B^{(tr)} = \left[\frac{\Lambda\omega}{r^2} - 16\pi\Omega(p+\epsilon)\right]h_{1,a}, \tag{A24}$$

$$C^{(tr)} = \left(\frac{2\omega}{r} + \frac{\omega'}{2}\right)H_2 - 16\pi\Omega(p+\epsilon)h_{1,p}, \tag{A25}$$

$$A^{(rr)} = \dot{H}_1 + e^{2\nu}\frac{\Lambda}{2r}\left(\lambda' + \frac{1}{r}\right)h_{1,p} - 4\pi r e^{2\nu+2\lambda}(p+\epsilon)H - \frac{e^{2\nu}}{2r}\left[(2r\nu' + 1) - \frac{\Lambda}{2}e^{2\lambda}\right]H_2 - e^{2\nu}\frac{\Lambda}{2r}h_{1,p}', \tag{A26}$$

$$\tilde{A}^{(rr)} = 0, \tag{A27}$$

$$B^{(rr)} = \omega h_{0,a}' + \frac{\omega'}{2}h_{0,a} - \left(\omega + \frac{r\omega'}{4}\right)Z_a, \tag{A28}$$

$$C^{(rr)} = \omega h_{0,p}' - \left(\omega + \frac{r\omega'}{4}\right)Z_p + \left(\frac{\omega'}{2} - e^{2\lambda}\frac{\Lambda\omega}{r}\right)h_{0,p} + \left(\omega + \frac{r\omega'}{4}\right)H_1, \tag{A29}$$

$$A^{(\theta\theta+\phi\phi)} = -\ddot{H}_2 + 2e^{-2\lambda}\left[\dot{H}_1' + \left(\frac{1}{r} - \lambda'\right)\dot{H}_1\right] - e^{2\nu-2\lambda}\left(\nu' + \frac{1}{r}\right)H_2' - \frac{\Lambda}{r^2}(\dot{h}_{0,p} - e^{2\nu-2\lambda}h_{1,p}')$$
$$- 16\pi e^{2\nu}(p+\epsilon)H - e^{2\nu}\left(\frac{\Lambda}{2r^2} + 16\pi p\right)H_2 + \frac{\Lambda}{r^2}e^{2\nu-2\lambda}(\nu' - \lambda')h_{1,p}, \tag{A30}$$

$$\tilde{A}^{(\theta\theta+\phi\phi)} = 0, \tag{A31}$$

$$B^{(\theta\theta+\phi\phi)} = 2\omega e^{-2\lambda}\left[h_{0,a}'{}' - Z_a' + \left(\frac{1}{r} - \lambda'\right)(h_{0,a} - Z_a)\right] + 2\omega' e^{-2\lambda}\left(h_{0,a}' - \frac{2}{r}h_{0,a}\right) - 16\pi e^{2\nu}\varpi(p+\epsilon)U, \tag{A32}$$

$$C^{(\theta\theta+\phi\phi)} = 2\omega e^{-2\lambda}\left[h_{0,p}'{}' - Z_p' + H_1' - e^{2\lambda}\left(\dot{H}_2 + \frac{\Lambda}{r^2}h_{0,p}\right) + \left(\frac{1}{r} - \lambda'\right)(\dot{h}_{1,p} + H_1)\right] + e^{-2\lambda}\omega'\left(H_1 + 2h_{0,p}' - \frac{4}{r}h_{0,p}\right) - 16\pi\varpi e^{2\nu}(p+\epsilon)V. \tag{A33}$$

The $(t\theta)$ and $(r\theta)$ components are

$$\Lambda a_{lm}^{(I)} + {\rm i}md_{lm}^{(I)} + \mathcal{L}_3^{\pm 1}\tilde{a}_{lm}^{(I)} + \mathcal{L}_2^{\pm 1}\eta_{lm}^{(I)} = 0, \tag{A34}$$

with

$$a^{(t\theta)} = -\dot{H}_2 + e^{-2\lambda}\left[H_1' - Z_p' + \frac{2}{r}h_{0,p}' + \left(\lambda' + \nu' - \frac{2}{r}\right)Z_p + (\nu' - \lambda')H_1 - \frac{2}{r^2}(r\lambda' - r\nu' + e^{2\lambda} - 1)h_{0,p}\right] + 16\pi e^{2\nu}(p+\epsilon)V, \tag{A35}$$

$$d^{(t\theta)} = e^{-2\lambda}\left\{2\Lambda\omega\left[h_{1,p}' + \left(\frac{1}{r} - \nu'\right)h_{1,p}\right] + \omega'\left[\frac{r^2}{2}H_2' + 2h_{1,p}\right]\right\} - 16\pi r^2\varpi(p+\epsilon)[H_2 + (1 + C_{\rm s}^{-2})H + C_{\rm s}^{-2}\xi], \tag{A36}$$

$$\tilde{a}^{(t\theta)} = 2\omega' e^{-2\lambda}h_{1,a}, \tag{A37}$$

$$\eta^{(t\theta)} = -\Lambda\omega e^{-2\lambda}[h_{1,a}' + (\nu' - \lambda')h_{1,a}], \tag{A38}$$

$$a^{(r\theta)} = -\dot{Z}_p + \frac{1}{r}\left(2e^{2\nu-2\lambda} - \frac{\Lambda}{2}\right)h'_{1,p}\left[8\pi e^{2\nu}(p+\epsilon) + \frac{\Lambda}{2r^2}(1+r\lambda') - \frac{2}{r^2}e^{2\nu}\right]h_{1,p} + \left[\nu'(e^{2\nu}-1) + \frac{1}{2r}\left(\frac{\Lambda}{2}e^{2\lambda}-1\right)\right]H_2 - 4\pi r e^{2\lambda}(p+\epsilon)H, \tag{A39}$$

$$d^{(r\theta)} = 16\pi r^2 \varpi(p+\epsilon)(H_1 + e^{2\nu}R) - \omega\Lambda(H_1 + Z_p - h_{0,p}) + \omega'\left(\frac{r^2}{2}\dot{H}_2 - 2(\Lambda+2)h_{0,p}\right), \tag{A40}$$

$$\tilde{a}^{(r\theta)} = 2\omega' h_{0,a}, \tag{A41}$$

$$\eta^{(r\theta)} = \Lambda[\omega(h'_{0,a} - Z_a) + \omega' h_{0,a}]. \tag{A42}$$

From the $(t\phi)$ and $(r\phi)$ components we get

$$\Lambda b^{(I)}_{lm} + \mathrm{i}mc^{(I)}_{lm} + \mathcal{L}_3^{\pm1}\tilde{b}^{(I)}_{lm} + \mathcal{L}_2^{\pm1}\zeta^{(I)}_{lm} = 0, \tag{A43}$$

with

$$b^{(t\phi)} = -Z'_a + \left(\nu' + \lambda' - \frac{2}{r}\right)Z_a + \frac{2}{r}h'_{0,a} + 16\pi e^{2\nu+2\lambda}(p+\epsilon)U - \left[\frac{2}{r}\left(\nu' + \lambda' - \frac{1}{r}\right) - e^{2\lambda}\frac{\Lambda-2}{r^2}\right]h_{0,a}, \tag{A44}$$

$$c^{(t\phi)} = -3\Lambda\omega h'_{1,a} + \left[\Lambda\omega\left(3\lambda' - \nu' - \frac{2}{r}\right) - (\Lambda-2)\omega'\right]h_{1,a}, \tag{A45}$$

$$\tilde{b}^{(t\phi)} = -2e^{-2\lambda}\omega' h_{1,p}, \tag{A46}$$

$$\zeta^{(t\phi)} = 2\omega\Lambda[e^{2\lambda}H_2 + (\nu' - \lambda')h_{1,p}] + \omega'\left(\Lambda h_{1,p} - \frac{r^2}{2}H'_2\right) + 16\pi r^2 e^{2\lambda}\varpi(p+\epsilon)[H_2 + (1 + C_{\mathrm{s}}^{-2})H + C_{\mathrm{s}}^{-2}\xi], \tag{A47}$$

$$b^{(r\phi)} = \dot{Z}_a - \frac{2}{r}e^{2\nu-2\lambda}h'_{1,a} - e^{2\nu}\left[\frac{\Lambda-2}{r^2} + \frac{2}{r}e^{-2\lambda}(\nu' - \lambda')\right]h_{1,a}, \tag{A48}$$

$$c^{(r\phi)} = \Lambda\omega h'_{0,a} + 2\left[(\Lambda+1)\omega' - \frac{\Lambda\omega}{r}\right]h_{0,a}, \tag{A49}$$

$$\tilde{b}^{(r\phi)} = 2\omega' h_{0,p}, \tag{A50}$$

$$\zeta^{(r\phi)} = \omega'\left(\Lambda h_{0,p} - \frac{r^2}{2}\dot{H}_2\right) - 16\pi r^2\varpi(p+\epsilon)(e^{2\nu}R + H_1). \tag{A51}$$

From the $(\theta\phi)$ and the subtraction of $(\theta\theta)$ and $(\phi\phi)$ components we get

$$\Lambda s_{lm} - \mathrm{i}mf_{lm} + \mathcal{L}_2^{\pm1}g_{lm} = 0, \tag{A52}$$

$$\Lambda t_{lm} + \mathrm{i}mg_{lm} + \mathcal{L}_2^{\pm1}f_{lm} = 0, \tag{A53}$$

with

$$f = \omega' r^2 e^{-2\lambda}\left(Z_p - \frac{2}{r}h_{0,p}\right) - 16\pi r^2\varpi e^{2\nu}(p+\epsilon)V, \tag{A54}$$

$$g = -\omega' r^2 e^{-2\lambda}\left(Z_a - \frac{2}{r}h_{0,a}\right) + 16\pi r^2\varpi e^{2\nu}(p+\epsilon)U, \tag{A55}$$

$$s = -\dot{h}_{0,p} + e^{2\nu-2\lambda}[h'_{1,p} + (\nu' - \lambda')h_{1,p}] - \frac{e^{2\nu}}{2}H_2 - \mathrm{i}m\omega h_{0,p}, \tag{A56}$$

$$t = -\dot{h}_{0,a} + e^{2\nu-2\lambda}[h'_{1,a} + (\nu' - \lambda')h_{1,a}] - \mathrm{i}m\omega h_{0,a}. \tag{A57}$$

These equations are fully equivalent to those derived within the ADM formalism. Although they still contain some second-order derivatives, they can be easily converted into first-order or even characteristic form by introducing a few auxiliary variables.

This paper has been typeset from a TEX/LATEX file prepared by the author.