Evolution of a Spherical Universe in a Short Range Collapse/Generation Interval

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Abstract. We study the final/initial behavior of a dust Universe with spatial spherical symmetry. This study is done in proximity of the collapse/generation times by an expansion in fractional Puiseux series. Even if the evolution of the universe has different behaviours depending on the initial data (in particular on the initial spatial curvature), we show that, in proximity of generation or collapse time, the Universe expands or collapses with the same behavior.

Keywords: Spherical Universe, Fractional Puiseux series, generation/ collapse time.

1 Introduction

In this paper we consider an Universe with spatial spherical symmetry around a physical point O and we analyze its behavior in proximity of the collapse/generation times. In this analysis we use an expansion of the exact solution of evolution equations in fractional power series (Puiseaux series).[1]

In particular we introduce the first principal curvature ω_1 of the initial spatial manifold V_3 into the evolution equations and we consider these equations in the three different cases of null, positive and negative principal curvature.

In other words, in a short range of times, it is impossible to distinguish the evolution of the Universe from the Euclidean case (where $\omega_1 = 0$).

Moreover this result allows the generalization of some of the results found in the previous papers [4], [8] in the spatially euclidean case (at least in an a suitable interval of time), also to the not euclidean case.

2 Evolution Equations

Since in the following we are going to consider dust universes with spatial spherical symmetry, we want to briefly summarize the previous main results in a form inspired by [3,4].

[1] A formal series of the form $\Sigma_{n=m}^{\infty} a_n z^{n/k}$ where m and k are integers such that $k \geq 1$ is called a Puiseux series or a fractional power series (see e. g. [5,6,7]).

Y. Shi et al. (Eds.): ICCS 2007, Part II, LNCS 4488, pp. 997–1003, 2007.

We will consider a dust system $\mathcal{C}$ which generates, during its evolution, a riemannian manifold, which has locally spatial spherical symmetry around a physical point O;[2] the metric can then be given the form [1,2]:[3]

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta = A^2(t,r)dr^2 + B^2(t,r)(d\theta^2 + \sin^2\theta d\varphi^2) - c^2\,dt^2 \ , \qquad (1)$$

where t is the proper time of each particle, r, θ, φ are co–moving spherical coordinates and we can interpret $B(t,r)$ as the intrinsic radius of the O–sphere $\mathcal{S}(r)$ at time t [1, Chap. XII, §11 p.411].[4]

We consider now the initial space-like hypersurface V_3 (with equation $t = 0$) and call r–shells the set of particles with co–moving radius r (i.e. the dust initially distributed on the surface of the geodesic sphere with center at O and radius r (O–sphere) $\mathcal{S}(r)$); in accordance with [3,4] we assign each particle of an r–shell the initial intrinsic radius $B(0,r)$ as radial co–moving coordinate r

$$B(0,r) = r \ . \qquad (2)$$

If we put $a(r) = A(0,r)$, the metric of the initial O–sphere V_3 takes the form:

$$d\sigma^2 = \gamma_{ij}\,dx^i\,dx^j = a^2(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \ , \qquad (3)$$

where $\gamma_{ij} \equiv g_{ij}$ is the metric tensor of V_3. If we introduce the first principal curvature of V_3, $\omega_1(r) = \frac{1}{r^2}\left(1 - \frac{1}{a(r)^2}\right)$ (see [1, Chap.VII §12 (43) p.205]), into the Tolmann–Bondi evolution equations [2,4], they become (see [9,10]):[5]

$$\begin{cases} A(t,r) &= \frac{B'(t,r)}{1-r^2\,\omega_1(r)} \\ \dot{B}(t,r)^2 &= -\omega_1(r)\,r^2\,c^2 + \frac{2\,G_N m(r)}{B(t,r)} \\ \mu(t,r) &= \frac{\mu_0(r)r^2}{B'(t,r)B^2(t,r)} \end{cases} \qquad (4)$$

where $\mu(t,r)$ is the mass density, $\mu_0 = \mu(0,r)$ is the initial mass density, and $m(r)$ is the so–called "*Euclidean mass*" [3,4]

$$m(r) = 4\pi \int_0^r \mu_0(s)\,s^2\,ds \ . \qquad (5)$$

The first principal curvature is very important for studying the geometrical property of V_3 in fact, as underlined in the paper [9,10], it completely determines its curvature properties.

[2] See [1, Chap. XII, §11 p.408] for a precise definition of spherical symmetry around a point O.

[3] In accordance with [1] (but differently from [3,4]) the latin indices will vary from 1 to 3, whereas the greek indices will vary from 1 to 4.

[4] At any point $\frac{1}{B^2}$ represents the gaussian curvature of the geodesic sphere with its centre at the centre of symmetry O and passing through the point [1, Chap. XII, §11 p.410].

[5] Hereafter a dot will denote differentiation with respect to t and a prime differentiation with respect to r.

Remark 1. In [4] it was demonstrated that given a spherical dust universe, for each material r–shell there exists a corresponding time $T(r)$ at which the dust distributed on the r–shell is collapsed into the symmetry center, so that we have a function $t = T(r)$ which satisfies $B(T(r), r) = 0$.[6]

3 Exact Solutions of Evolution Equations in Three Different Cases

Now we will focus our attention on a given single r-shell (that is we will consider r as a given fixed parameter), so we can regard the intrinsic radius $B = B(t; r)$ as a function of time only, and $r\,, \omega_1(r)\,, m(r)$ as constants. By introducing the new adimensional function $Y(t) = \frac{B(t;r)}{r}$ and the function $k(r) = \frac{G_N m(r)}{r^3}$ (which we will consider constant being r is a given fixed parameter), equation $(4)_2$ becomes

$$\dot{Y}^2(t) = -\omega_1\, \mathrm{c}^2 + \frac{2\,k}{Y(t)}\ . \tag{6}$$

In the following we will put $\varpi = |\omega_1|$ and will consider separately the three cases $\omega_1 = 0$, $\omega_1 = \varpi > 0$ and $\omega_1 = -\varpi < 0$, to get the corresponding exact solutions. The case $\omega_1 = 0$ corresponds to the Euclidean case $a^2(r) = 1$, already studied in [4]; it is the only case where it is possible to solve $(4)_2$ explicitly for B. Since $\omega_1(r) = 0$, equation $(4)_2$ becomes

$$\dot{B}^2 = \frac{2\,G_N m(r)}{B(t,r)} \quad \Rightarrow \quad \dot{Y}^2(t) = \frac{2\,k}{Y(t)}\ . \tag{7}$$

We can solve the previous equation by separating the variables:

$$\sqrt{2\,k}\,\mathrm{d}t = \pm\sqrt{Y}\,\mathrm{d}Y \quad \Rightarrow t - \tau(r) = \pm\frac{1}{3}\,\sqrt{\frac{2}{k}}\,Y^{\frac{3}{2}} \tag{8}$$

where we have to choose the plus sign if Universe is initially expanding ($\dot{B}(0; r) > 0$), the minus sign if Universe is initially contracting ($\dot{B}(0; r) < 0$) and $\tau(r)$ is an arbitrary function of the parameter r.

Remark 2. For $t = \tau(r) \Rightarrow Y = 0 \Rightarrow B = 0$ and we know, from remark 2, that for each r exists a unique instant $T(r)$ which satisfies $B(r, T(r)) = 0$, where $T(r)$ is the time at which the dust distributed on the r-shell is collapsed into the symmetry center, so we have $t = T(r)$ and consequently $\tau(r) \equiv T(r)$.

We can calculate the function $\tau(r) \equiv T(r)$ through the initial values $B(0; r) = r \to Y(0) = 1$

$$T(r) = \mp\frac{1}{3}\sqrt{\frac{2}{k(r)}} = \mp\frac{1}{3}\,\sqrt{\frac{2\,r^3}{G_N\, m(r)}}\ . \tag{9}$$

[6] If the initial mass density is constant, $T(r)$ is also constant.

It is possible to solve equation $(8)_2$ with respect to Y and we can write the solutions of equations (7) in the form [4]:

$$Y(t,r) = \left(1 - \frac{t}{T(r)}\right)^{\frac{2}{3}} \;\Rightarrow\; B(t,r) = r\left(1 - \frac{t}{T(r)}\right)^{\frac{2}{3}} . \tag{10}$$

If $\omega_1(r) = \varpi(r) > 0$, equation (4) becomes

$$\dot{Y}^2 = -\varpi\, c^2 + \frac{2\,k}{Y} . \tag{11}$$

Now we can write

$$\dot{Y}^2 = 2\,k\left(\frac{1}{Y} - \frac{1}{h}\right) \quad\Rightarrow\quad \frac{dY}{dt} = \pm\sqrt{\frac{2\,k}{h}}\sqrt{\frac{h-Y}{Y}} , \tag{12}$$

where $h(r) = \frac{2\,k(r)}{\varpi(r)c^2} > 0$. We can solve (12) by separating the variables:

$$\sqrt{\frac{2\,k}{h}}\,dt = \pm\sqrt{\frac{Y}{h\;-\;Y}}\,dY \quad\Rightarrow \tag{13}$$

$$\Rightarrow t - \tau(r) = \pm\sqrt{\frac{h}{2\,k}}\left[-\sqrt{(h-Y)Y} + h\;\arctan\left(\frac{\sqrt{Y}}{\sqrt{h-Y}}\right)\right] \tag{14}$$

where we have to choose the plus sign if Universe is initially expanding ($\dot{B}(0;r) > 0$), the minus sign if Universe is initially contracting ($\dot{B}(0;r) < 0$) and $\tau(r)$ is an arbitrary function of r.

Remark 3. Also in this case for $t = \tau(r) \Rightarrow Y = 0 \Rightarrow B = 0$ then, from Remark 2, $t = T(r)$ and consequently $\tau(r) \equiv T(r)$.

By substituting $B_{max} = h(r)\,r$ and $\sqrt{\frac{h}{2k}} = \frac{1}{\sqrt{c^2\,\varpi}}$ we find

$$t - T(r) = \pm\frac{1}{\sqrt{\varpi\,r^2\,c^2}}\left[-\sqrt{(B_{max}-B)B} + B_{max}\;\arctan\left(\sqrt{\frac{B}{B_{max}-B}}\right)\right] . \tag{15}$$

We can calculate the function $\tau(r) \equiv T(r)$ from the initial values $B(0;r) = r$

$$T(r) = \mp\frac{1}{\sqrt{\varpi\,r^2\,c^2}}\left[-\sqrt{(B_{max}-r)r} + B_{max}\;\arctan\left(\sqrt{\frac{r}{B_{max}-r}}\right)\right] . \tag{16}$$

Finally, when $\omega_1(r) = -\varpi(r) < 0$, equation (4) becomes

$$\dot{Y}^2 = \varpi\,c^2 + \frac{2\,k}{Y} \Rightarrow \dot{Y}^2 = \frac{\varpi\,c^2\,Y\;+\;2k}{Y} \tag{17}$$

$$\Rightarrow \frac{dY}{dt} = \pm\sqrt{2k}\sqrt{\frac{1+\frac{c^2\,\varpi}{2\,k}\,Y}{Y}} . \tag{18}$$

We can solve equation (18) separating the variables:

$$dt = \pm \frac{1}{\sqrt{2k}} \frac{\sqrt{Y}}{\sqrt{1 + \frac{c^2 \varpi}{2k} Y}} \Rightarrow \tag{19}$$

$$t - \tau(r) = \pm \frac{c\sqrt{k}\sqrt{Y\varpi}\sqrt{2k + c^2 Y \varpi} - 2k\, arcsinh(\frac{c\sqrt{Y\varpi}}{\sqrt{2k}})}{c^3 \varpi^{\frac{3}{2}}} \tag{20}$$

where we have to choose the plus sign if Universe is initially expanding ($\dot{B}(0;r) > 0$), the minus sign if Universe is initially contracting ($\dot{B}(0;r) < 0$) and $\tau(r)$ is an arbitrary function of r.

Remark 4. Also in this case for $t = \tau(r) \Rightarrow Y = 0 \Rightarrow B = 0$ then, from Remark 2, $t = T(r)$ and consequently $\tau(r) \equiv T(r)$.

So we can calculate the function $\tau(r)$ from the initial value $B(0;r) = r$

$$T(r) = \mp \frac{c\sqrt{k\,\varpi}\sqrt{2k + c^2 \varpi} - 2\;k\; arcsinh(\frac{c\sqrt{\varpi}}{\sqrt{2k}})}{c^3 \varpi^{\frac{3}{2}}} \tag{21}$$

4 Study of the Behaviour of the Universe in Proximity of the Collapse/Generation Times by an Expansion in Fractional Power Series

Now we want to study the behaviour of the universe in proximity of the collapse/expansion times by an expansion in fractional (Puiseux) series.[7]

Remark 5. In proximity of the times of generation or collapse the evolution has the same behaviour apart from its initial geometry. In addition the function $T(r)$ has approximately the same form in all of the three different cases $\omega_1 = 0$, $\omega_1 > 0$ and $\omega_1 < 0$.

4.1 Initial Principal Curvature ω_1 Positive

We already remarked that it is not possible to solve (14) explicitly with respect to B, but we can approximate the exact solution by an opportune fractional power series (or Puiseux series):[8]

$$\sqrt{\frac{h}{2k}}\left[-\sqrt{(h-Y)Y} + h \arctan\left(\frac{\sqrt{Y}}{\sqrt{h-Y}}\right)\right] = \tag{22}$$

[7] In [11] the approximate explicit solution was obtained through an expansion in power series of the parametric equations, therefore by a double expansion in power series.

[8] It is not possible to expand the second member of (14) in a simple power series with respect to Y, but we can develop it in Mac Lauren series with respect to $\sqrt{Y}$ thus obtaining a fractional power series. As it is known the fractional power series are particular cases of Puiseux series (see e.g. [5]).

$$= \frac{\sqrt{2}}{3\ \sqrt{k}} Y^{\frac{3}{2}} + \frac{1}{5\ h\sqrt{2\ k}} Y^{\frac{5}{2}} + \frac{3}{28\ \ h^2\ \sqrt{2\,k}} Y^{\frac{7}{2}} + \cdots \tag{23}$$

By truncating the fractional series to the first term (with precision 3/2), we find

$$t - \tau(r) = \pm\frac{1}{3}\sqrt{\frac{2}{k}}\ Y^{\frac{3}{2}}\ . \tag{24}$$

So in our approximation we found the same expression (8) that characterizes the case $\omega_1 = 0$: in proximity of the generation or collapse times, the r-shells expand or collapse with the same behaviour as in the case $\omega_1 = 0$ and the function $T(r)$ has, approximately, the form (9), in agreement with [11].

4.2 Initial Principal Curvature ω_1 Negative

Also in this case, being not possible to solve (20) explicitly with respect to B, we can approximate the exact solution by a Puiseux series:

$$\frac{\mathrm{c}\ \sqrt{Y\,\varpi}\,\sqrt{2\,k + c^2\,Y\,\varpi}\sqrt{k} - 2\ k\ arcsinh(\frac{c\,\sqrt{Y\,\varpi}}{\sqrt{2\,k}})}{c^3\,\varpi^{\frac{3}{2}}} = \tag{25}$$

$$= \frac{\sqrt{2}\,Y^{\frac{3}{2}}}{3\,\sqrt{k}} - \frac{c^2\,\omega\,Y^{\frac{5}{2}}}{10\,\sqrt{2}\,k^{\frac{3}{2}}} + \frac{3\,c^4\,\omega^2\,Y^{\frac{7}{2}}}{112\,\sqrt{2}\,k^{\frac{5}{2}}} + \cdots \tag{26}$$

By truncating the fractional series to the first term (with precision $\frac{3}{2}$), we find

$$t - \tau(r) = \pm\frac{1}{3}\sqrt{\frac{2}{k}}\ Y^{\frac{3}{2}} \tag{27}$$

So in our approximation we found again the same equation that characterizes the case $\omega_1 = 0$: in proximity of the generation or collapse times, the r-shells expand or collapse with the same behavior that in the case $\omega_1 = 0$. Moreover, also in this case, the function $T(r)$ has, approximately, the form (9) (see also [11]).

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