J. Phys. A: Math. Gen., Vol. 12, No. 5, 1979. Printed in Great Britain

Evolution of semiclassical quantum states in phase space

M V Berry† and N L Balazs‡

† H H Wills Physics Laboratory, Bristol University, Tyndall Avenue, Bristol, BS8 1TL, UK
‡ State University of New York at Stony Brook, NY 11794, USA

Received 14 September 1978

Abstract. We derive a semiclassical formula for the Wigner function $W(q, p, t)$ describing the evolution in the two-dimensional phase space qp of a nonstationary quantum state $\psi(q, t)$ for a system with one degree of freedom. The initial state $\psi(q, 0)$ corresponds to a family of classical orbits represented by a curve $\mathscr{C}_0$ in qp. Under the classical motion $\mathscr{C}_0$ evolves into a curve $\mathscr{C}_t$; we show that the region where W is large hugs $\mathscr{C}_t$ in an adiabatic fashion, and that W has semiclassical oscillations depending only on the geometry of $\mathscr{C}_t$ and neighbouring curves.

As $t \to \infty$, $\mathscr{C}_t$ can get very complicated, and we classify its convolutions as 'whorls' and 'tendrils', associated respectively with stable and unstable classical motion. In these circumstances the quantum function W cannot resolve the details of $\mathscr{C}_t$, and at time t_c there is a transition to new regimes, for which we make predictions about the morphology of ψ from the way $\mathscr{C}_t$ fills regions of phase space as $t \to \infty$. The regimes associated with whorls and tendrils are different. We expect $t_c = O(\hbar^{-2/3})$ for whorls and $t_c = O(\ln \hbar^{-1})$ for tendrils.

1. Introduction

Even in one dimension it is far from trivial to get a clear picture of the time development of quantum states, especially if the Hamiltonian is nonstationary. Here we show that under semiclassical conditions a natural framework for such problems is the phase space of the classical motion, in which an easily visualised, explicit expression can be derived for the quantum-mechanical Wigner function.

In § 2 we set up the initial state $\psi(q, 0; \mathscr{P}^*)$ corresponding to an initial curve $\mathscr{C}_0(\mathscr{P}^*)$ of points in the phase space q, p, labelled by the value $\mathscr{P}^*$ of a parameter $\mathscr{P}$. In other words, we are considering a *family* of quantum states, each of which corresponds to a *family* of classical orbits.

Then in § 3 we employ the time-dependent WKB method (Dirac 1947, Van Vleck 1928) to express the evolving wavefunction $\psi(q, t; \mathscr{P}^*)$ in terms of the curve $\mathscr{C}_t(\mathscr{P}^*)$ that develops in phase space from $\mathscr{C}_0(\mathscr{P}^*)$ under the classical dynamics. This approximation for ψ has the well known deficiency that it breaks down at caustics (turning points) of the classical motion, where $\mathscr{C}$ is perpendicular to the q direction in phase space.

This deficiency is remedied in § 4 by using the WKB expression for ψ to construct an approximation for Wigner's function $W(q, p, t; \mathscr{P}^*)$ in phase space, employing a method developed by Berry (1977a) for studying stationary states. The resulting formula for W has the advantages of being manifestly symmetric in q and p (and so not failing on caustics in the q or p spaces) and also of depending only on the geometry of the classical curves $\mathscr{C}_t(\mathscr{P})$. Our arguments here are similar in spirit to those developed by Heller (1977); he considers a wider class of cases, but we get more information about the quantum oscillations of W.

For a large class of Hamiltonians, almost all initial curves $\mathscr{C}_0(\mathscr{P})$ will develop as $t \to \infty$ into curves $\mathscr{C}_t(\mathscr{P})$ with infinitely complicated convolutions of two sorts, which we call 'whorls' and 'tendrils', related respectively to stable and unstable (stochastic) classical motion (reviewed by Berry 1978). Section 5 is devoted to a brief discussion of this asymptotic complication of $\mathscr{C}$.

When $\mathscr{C}$ gets sufficiently complicated, W can no longer follow its details, and the quantum state ψ must undergo a transition to a new regime whose nature depends on whether $\mathscr{C}$ grows whorls or tendrils as $t \to \infty$. Speculations about these transitions are presented in § 6. We conjecture:

(*a*) there will be precursors of the transition in the form of proliferating cusp catastrophes of ψ;

(*b*) the transition will occur at a time t_c, where $t_c = O(\hbar^{-2/3})$ for stable motion (whorls) and $t_c = O(\ln \hbar^{-1})$ for unstable motion (tendrils);

(*c*) after the transition the local average of the probability density $|\psi|^2$, and the spectrum of oscillations of ψ, will depend on how convolutions fill regions of phase space (Berry 1977b), and this is different for whorls and tendrils;

(*d*) where the classical motion is unbounded, Wigner's function W will eventually spread more slowly in phase space than $\mathscr{C}$ does, and this may explain some numerical results of Casati *et al* (1979) on the quantum pendulum.

2. The initial state

At $t = 0$ the quantum state $\psi(q, 0; \mathscr{P}^*)$ will be taken to correspond to a curve $\mathscr{C}_0(\mathscr{P}^*)$ in the phase space q, p. It will prove convenient to embed ψ and $\mathscr{C}$ in families parameterised by $\mathscr{P}$ (figure 1*a*). Then $\mathscr{C}_0(\mathscr{P})$ is defined by

$$\mathscr{C}_0(\mathscr{P}): F(q, p, 0; \mathscr{P}) = 0, \tag{1}$$

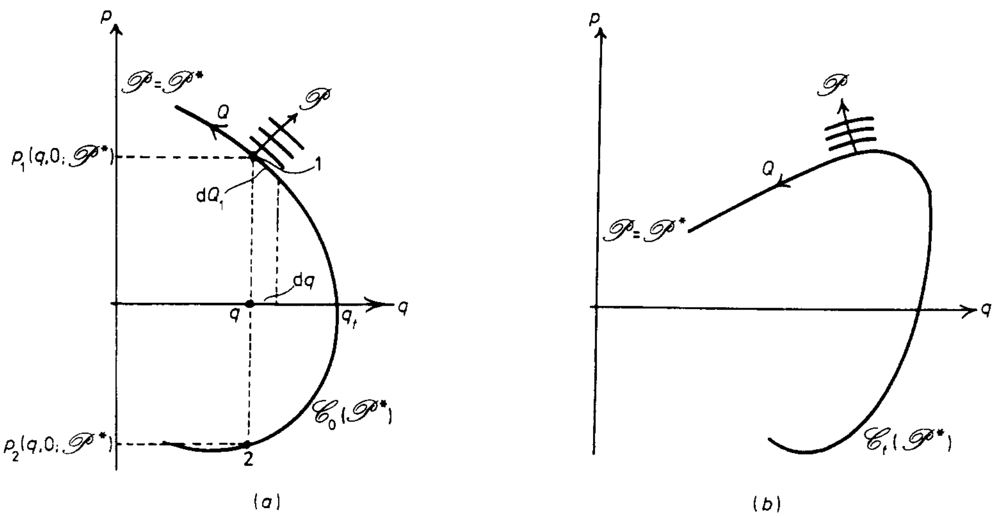

Figure 1. Geometry relating to family of curves parameterised by $\mathscr{P}$: (*a*) the initial curve $\mathscr{C}_0(\mathscr{P}^*)$; (*b*) the evolving curve $\mathscr{C}_t(\mathscr{P}^*)$.

where F is a family of functions of q and p. Obviously there are many ways to embed the curve of interest, $\mathscr{C}_0(\mathscr{P}^*)$, in a one-parameter family of neighbouring curves. The choice of the particular family (1) is determined by the assumed distribution of classical points along the curves $\mathscr{C}_0(\mathscr{P})$ as follows: Let Q be a coordinate along $\mathscr{C}$ (figure 1a), chosen so that the classical points on $\mathscr{C}_0(\mathscr{P})$ are distributed *uniformly* in Q; then Q will be considered as a new canonical coordinate in the phase space, and the parameter $\mathscr{P}$ will be taken as the conjugate momentum. The generator of this transformation is the action function $S_0(q;\mathscr{P})$, satisfying the relations (Synge 1960)

$$q, p \leftarrow S_0(q;\mathscr{P}) \rightarrow Q, \mathscr{P}$$

$$p = \partial S_0/\partial q, \qquad Q = \partial S_0/\partial \mathscr{P}. \tag{2}$$

Contributions to the semiclassical wave $\psi(q, 0; \mathscr{P}^*)$ arise where the line with constant q intersects $\mathscr{C}_0(\mathscr{P}^*)$; we label these intersections by an index (at the point q marked in figure 1a, for example, j can take the values 1 or 2), and denote by ψ_j the contribution of the jth intersection to ψ. Apart from a constant ϕ_j, the *phase* of ψ_j is simply $\hbar^{-1}$ times the action

$$S_{0j}(q;\mathscr{P}^*) = \int_{q_0}^{q} p_j(q', 0; \mathscr{P}^*)\, \mathrm{d}q', \tag{3}$$

where p_j $(q, 0; \mathscr{P}^*)$ denotes the jth branch (figure 1a) of the inverse function corresponding to (1), expressing p in terms of q along the curve $\mathscr{C}_0(\mathscr{P}^*)$, and q_0 is a constant. Of course S_{0j} in (3) is just one branch of the generating function $S_0(q;\mathscr{P})$ of the canonical transformation (2). The *modulus* $|\psi_j|$ is obtained in terms of the strength of the projections of segments $\mathrm{d}Q_j$ of $\mathscr{C}$ onto the segment $\mathrm{d}q$ (figure 1a), using the fact that the density of points along $\mathscr{C}$ is uniform in Q. Thus

$$I_0\, \mathrm{d}Q_j = |\psi_j(q, 0; \mathscr{P}^*)|^2\, \mathrm{d}q \tag{4}$$

or

$$|\psi_j|^2 = I_0\, \mathrm{d}Q_j/\mathrm{d}q = I_0(\partial^2 S_0(q;\mathscr{P})/\partial q\, \partial\mathscr{P})_{\mathscr{P}=\mathscr{P}^*}, \tag{5}$$

where I_0 is a constant, and the second equation in (2) has been used. Combining these results, the initial semiclassical wave is

$$\psi(q, 0; \mathscr{P}^*) = \sum_j \left| I_0 \frac{\partial^2 S_0(q;\mathscr{P}^*)}{\partial q\, \partial \mathscr{P}^*}\right|^{1/2} \exp\left[\mathrm{i}\left(\phi_j + \frac{S_0(q;\mathscr{P}^*)}{\hbar}\right)\right]. \tag{6}$$

This is a notoriously awkward expression, on account of the multivaluedness of the functions $p(q, 0; \mathscr{P})$ and $S_0(q;\mathscr{P})$ as embodied in the summations over j. The multivaluedness originates in the *caustics* or *turning points* q_t figure 1a) where $\mathscr{C}$ is perpendicular to the q direction and two branches j coalesce; typically q_t is a *fold catastrophe* of ψ (Poston and Stewart 1978). At q_t it is geometrically obvious that $\mathrm{d}Q/\mathrm{d}q$ diverges, and so the approximation (6) is no longer valid. Keller (1958) and Maslov (1972) have shown how these divergences connect the phases ϕ_j in (6). The time-dependent version (6), to be derived in the next section, will involve similar complications, but these will be eliminated in § 4 when the branches of the semiclassical wavefunction are synthesised into a single Wigner function valid throughout phase space.

The notation employed in this section can be illustrated by considering the particular case (Berry 1977a) of a stationary bound state of a time-independent Hamiltonian

H. The curves $\mathscr{C}$ (which do not change with time) are just the contours $H(q, p) = E$. The energy E could be the parameter $\mathscr{P}$ of equation (1); alternatively, that role could be played by the *action*

$$\mathscr{P}(E) = \frac{1}{2\pi} \oint p \, \mathrm{d}q = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}q \, \mathrm{d}p \Theta(E - H(q, p)), \tag{7}$$

where Θ denotes the unit step function. The action $\mathscr{P}(E)$ is $1/2\pi$ times the area enclosed by the curve $\mathscr{C}$ that corresponds to E. When $\mathscr{P}(E)$ and not E is the parameter, the coordinate Q conjugate to $\mathscr{P}$ is the *angle* variable, and points distributed round $\mathscr{C}$ uniformly in Q remain so as time proceeds (i.e. Q gives the invariant measure on $\mathscr{C}$—see Appendix 26 of Arnol'd and Avez 1968), and the corresponding semiclassical wave (6) will indeed represent a stationary state.

3. Evolution of the state

Each point q, p on the initial curve $\mathscr{C}_0(\mathscr{P}^*)$ moves according to dynamics governed by the system's Hamiltonian $H(q, p, t)$, so that at time t the curve has evolved into $\mathscr{C}_t(\mathscr{P}^*)$ (figure 1*b*), also embedded in the family parameterised by $\mathscr{P}$ and defined by

$$\mathscr{C}_t(\mathscr{P}): F(q, p, t; \mathscr{P}) = 0, \tag{8}$$

where $F(q, p, t; \mathscr{P})$ has evolved from $F(q, p, 0; \mathscr{P})$ by the action of $H(q, p, t)$. It is natural to ask: can the semiclassical wave $\psi(q, t; \mathscr{P}^*)$, which has evolved out of (6), be expressed in terms of the curves $\mathscr{C}_t(\mathscr{P})$?

The answer is affirmative. To show this, we begin by writing the Schrödinger equation that ψ must satisfy:

$$H(q, -\mathrm{i}\hbar \, \partial/\partial q, t)\psi = \mathrm{i}\hbar \, \partial\psi/\partial t. \tag{9}$$

A semiclassical solution of this equation, i.e. a solution correct to terms that do not vanish as $\hbar \to 0$ (Dirac 1947), shows that ψ is made up of contributions ψ_j whose phases are $\hbar^{-1}$ times solutions of the Hamilton–Jacobi equation. Again it is convenient to work with a family of solutions involving the parameter $\mathscr{P}$ and denoted by $S(q, t; \mathscr{P})$. Then the Hamilton–Jacobi equation is

$$H(q, \partial S(q, t; \mathscr{P})/\partial q, t) = -\partial S(q, t; \mathscr{P})/\partial t. \tag{10}$$

The moduli $|\psi_j|^2$ satisfy the continuity equation

$$\frac{\partial|\psi|^2}{\partial t} = -\frac{\partial}{\partial q}\left[|\psi|^2\left(\frac{\partial H}{\partial p}\right)_{p=\partial S/\partial q}\right] \tag{11}$$

whose solution (Van Vleck 1928) can be expressed in terms of S as

$$|\psi|^2 = \mathrm{const} \times \partial^2 S(q, t; \mathscr{P})/\partial q \, \partial\mathscr{P}. \tag{12}$$

The semiclassical wave evolving out of (6) is therefore

$$\psi(q, t; \mathscr{P}^*) = \sum_j \left| I_0 \frac{\partial^2 S_j(q, t; \mathscr{P}^*)}{\partial q \, \partial\mathscr{P}^*}\right|^{1/2} \exp\left[\mathrm{i}\left(\phi_j + \frac{S_j(q, t; \mathscr{P}^*)}{\hbar}\right)\right], \tag{13}$$

where S_j are branches of the solution of (10) whose initial value is (cf (3))

$$S_j(q, 0; \mathscr{P}) = S_{0j}(q; \mathscr{P}) = \int_{q_0}^{q} p_j(q', 0; \mathscr{P})\, \mathrm{d}q'. \tag{14}$$

It now remains to show how S thus defined depends on the curves $\mathscr{C}_t(\mathscr{P})$.

One technique *not* suitable for this purpose is the method of characteristics. This obscures the connection with the curves $\mathscr{C}_t(\mathscr{P})$ by expressing S in terms of the function $\bar{q}(t'; q, t, \mathscr{P})$. This gives that position $\bar{q}$ (at time t') which will arrive at time t at the position q with momentum determined by the curve $\mathscr{C}_t(\mathscr{P})$. The resulting solution of the Hamilton–Jacobi equation, satisfying the initial condition (14), is

$$S(q, t; \mathscr{P}) = S_0(\bar{q}(0; q, t, \mathscr{P}); \mathscr{P}) + \int_0^t \mathrm{d}t' L\left(\bar{q}(t'; q, t, \mathscr{P}), \frac{\partial \bar{q}}{\partial t'}(t', q, t, \mathscr{P}), t'\right), \tag{15}$$

where $L(\bar{q}, \partial\bar{q}/\partial t', t')$ denotes the Lagrangian.

However, this form of solution, involving the whole course of the trajectories over the time 0 to t, is unnecessarily complicated for our purposes. A much simpler expression for S exists, namely

$$S_j(q, t; \mathscr{P}) = \int_{q_0}^{q} p_j(q', t; \mathscr{P})\, \mathrm{d}q' - \int_0^t \mathrm{d}t' H(q_0, p_j(q_0, t'; \mathscr{P}), t'), \tag{16}$$

where $p_j(q, t; \mathscr{P})$ denotes the jth branch of the inverse function corresponding to (8), expressing p in terms of q along the curve $\mathscr{C}_t(\mathscr{P})$, and q_0 is the constant appearing in (3). There is no dependence in (16) on the details of the trajectories between times 0 and t; the first term depends only on the final curve $\mathscr{C}_t(\mathscr{P})$, and the second term involves only the time dependence of H at the fixed coordinate q_0 on the evolving curve $\mathscr{C}$.

Clearly the expression (16) satisfies the boundary condition (14). We now show that it satisfies the Hamilton–Jacobi equation (10). For simplicity we omit the subscripts j and the parameter $\mathscr{P}$. First we note that (16) gives the expected result

$$\partial S(q, t)/\partial q = p(q, t). \tag{17}$$

Next we calculate the time derivative in (10):

$$-\frac{\partial S(q, t)}{\partial t} = -\int_{q_0}^{q} \frac{\partial p(q', t)}{\partial t}\, \mathrm{d}q' + H(q_0, p(q_0, t), t). \tag{18}$$

Now

$$\frac{\partial p(q, t)}{\partial t} = \frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\partial p}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}t}, \tag{19}$$

where $\mathrm{d}q/\mathrm{d}t$ and $\mathrm{d}p/\mathrm{d}t$ denote the rates of change *along trajectories* passing through q, p at t, and $\partial p/\partial q$ denotes the slope of $\mathscr{C}_t(\mathscr{P})$ at q; the geometry of these derivatives is shown on figure 2, from which the truth of (19) is evident. Use of Hamilton's equations of motion now gives

$$\frac{\partial p(q, t)}{\partial t} = -\frac{\partial H(q, p(q(t)), t)}{\partial q} - \frac{\partial p}{\partial q}\frac{\partial H(q, p(q, t), t)}{\partial p} = -\frac{\mathrm{d}H(q, p(q, t), t)}{\mathrm{d}q}. \tag{20}$$

This makes the first term in (18) the integral of a derivative, so that

$$-\int_{q_0}^{q} \frac{\partial p(q', t)}{\partial t}\, \mathrm{d}q' = \int_{q_0}^{q} \frac{\mathrm{d}H(q', p(q', t), t)}{\mathrm{d}q'}\, \mathrm{d}q' = H\left(q, \frac{\partial S(q, t)}{\partial q}, t\right) - H(q_0, p(q_0, t), t), \tag{21}$$

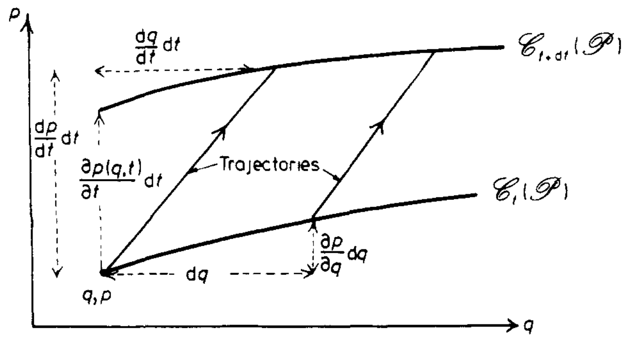

Figure 2. Geometry of derivatives describing slope and change of the curve $\mathscr{C}_t(\mathscr{P})$.

which on substitution into (18) shows that (16) does indeed satisfy the Hamilton–Jacobi equation. Dr J H Hannay (private communication) points out that an easy way to derive (16) is by integrating S in q, t space first from $t=0$ to t keeping q constant and equal to q_0, and then from q_0 to q keeping t constant.

Now that we have the semiclassical wavefunction given by equations (13) and (16), the next step is to use it to construct the Wigner function $W(q, p, t; \mathscr{P}^*)$ in phase space.

4. Wigner's function

From the semiclassical wavefunction ψ given by (13) we can construct the phase space distribution $W(q, p, t; \mathscr{P}^*)$ of Wigner (1932). This is defined as

$$W(q, p, t; \mathscr{P}^*) \equiv \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dX \exp\left(-\frac{2iX}{\hbar}\right)\psi(q+X, t; \mathscr{P}^*)\psi^*(q-X, t; \mathscr{P}^*). \tag{22}$$

W has the useful property of projecting along p to give the q space probability density:

$$\int_{-\infty}^{\infty} dp\, W(q, p, t; \mathscr{P}^*) = |\psi(q, t; \mathscr{P}^*)|^2. \tag{23}$$

Similarly, projection along q gives the momentum space probability density.

The obvious way to generate a semiclassical approximation to W is to use the method of stationary phase for the integration over X in (22), after substituting (13) and (16). At first sight it seems that the multivaluedness of p as a function of q on the classical curve $\mathscr{C}_t(\mathscr{P}^*)$ will cause great complications. However, these can be avoided by first evaluating (22) at points q, p close to $\mathscr{C}$ but not close to any caustics q_t. The resulting expression will then be rewritten so as to depend on the geometry of $\mathscr{C}$ in a manner manifestly symmetric in q and p, giving a W easily globalised to be valid near turning points and also far from $\mathscr{C}$.

Under the stated assumptions, the integral to be evaluated for W is

$$W(q, p, t; \mathscr{P}^*) = \frac{I_0}{\pi\hbar}\int_{-\infty}^{\infty} dX\left|\frac{\partial p(q+X, t; \mathscr{P}^*)}{\partial \mathscr{P}^*}\frac{\partial p(q-X, t; \mathscr{P}^*)}{\partial \mathscr{P}^*}\right|^{1/2}$$
$$\times \exp\left[\frac{i}{\hbar}\left(\int_{q-X}^{q+X} dq' p(q', t; \mathscr{P}^*) - 2pX\right)\right]. \tag{24}$$

Because q is far from any q_t, it is legitimate to use here the semiclassical approximation (13); and because q, p is close to $\mathscr{C}$, only one branch j of the inverse function $p(q, t; \mathscr{P})$ will contribute stationary points to the integrand, so that in writing (24) we have dropped the suffix j. Note that because Wigner's function (22) contains the product of a ψ and a ψ^* the following quantities do not appear in (24): the phases ϕ_j in (13), the constant q_0 in the action (16), and the explicitly time-dependent second term in (16).

The stationary-phase evaluation of (24) proceeds exactly as described in § 4 of Berry (1977a) for the special case of eigenstates of a time-independent Hamiltonian. Therefore we shall simply state the results of the analysis; readers can easily check the details. Assume first that q, p lies on the concave side of $\mathscr{C}$ (figure 3) in a region where $\mathscr{C}$ has no inflection points. Then the exponent in (24) is stationary for the two values $X = \pm X_0(q, p)$ corresponding to the end-points 1 and 2 (figure 3) of that chord of $\mathscr{C}$ whose midpoint is q, p. We employ the convention that 2 follows 1 moving clockwise around $\mathscr{C}$. The phase in (24) has the stationary value

$$\int_{q-X_0(q,p)}^{q+X_0(q,p)} \mathrm{d}p'\, p(q', t; \mathscr{P}^*) - 2pX_0 = A(q, p, t), \tag{25}$$

where $A(q, p, t)$ is the (positive) area (shaded in figure 3) between $\mathscr{C}_t(\mathscr{P}^*)$ and the chord 12.

As q, p moves onto $\mathscr{C}$ the two two stationary points 1 and 2 coalesce, thus invalidating the ordinary method of stationary phase. However, the more sophisticated method of Chester *et al* (1957) can be applied to yield an approximation for W in terms of the Airy function (Abramowitz and Stegun 1964). This is *uniformly valid*, and applies not only as q, p moves onto $\mathscr{C}$ but also when q, p lies on the convex side of $\mathscr{C}$, where the stationary values X_0 and the area $A(q, p, t)$ are imaginary. The resulting formula for W involves the geometry of the family of curves near $\mathscr{C}$ not only through the area $A(q, p, t)$ but also through the rates of change of the parameter $\mathscr{P}$, considered as a function $\mathscr{P}(q, p)$ as q and p are varied from the stationary points 1 and 2. To express this we shall use the abbreviated notation exemplified by

$$\mathscr{P}_q(1) \equiv (\partial \mathscr{P}(q', p')/\partial q')_{q'=q-X_0(q,p),\, p'=p(q-X_0(q,p),\, t;\, \mathscr{P}^*)}. \tag{26}$$

In terms of these quantities, W is given by

$$W(q, p, t; \mathscr{P}^*) = \frac{2\sqrt{2}I_0(3A(q, p, t)/2)^{1/6}\,\mathrm{Ai}[-(3A(q, p, t)/2\hbar)^{2/3}]}{\hbar^{2/3}(\mathscr{P}_q(2)\mathscr{P}_p(1) - \mathscr{P}_q(1)\mathscr{P}_p(2))^{1/2}}. \tag{27}$$

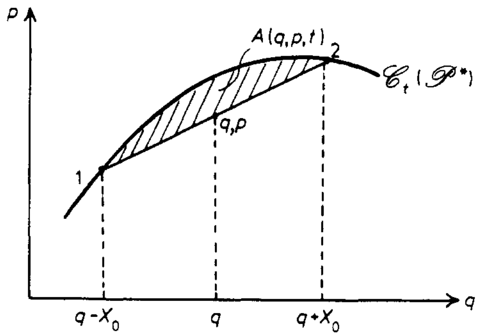

Figure 3. Chord construction for area A governing oscillations of Wigner's function at qp at time t.

This semiclassical limiting form of Wigner's function is our central result. It is rich with interesting properties which will now be summarised (for more detailed discussion see § 4 of Berry 1977a).

(*A*) W attains large positive values, of order $\hbar^{-2/3}$, close to the classical curve $\mathscr{C}_t(\mathscr{P}^*)$. (The actual maximum occurs a distance of order $\hbar^{2/3}$ away from $\mathscr{C}$ on its concave side.)

(*B*) Very close to $\mathscr{C}$, a tedious expansion yields the 'transitional approximation'

$$W(q,p,t;\mathscr{P}^*)=\frac{2I_0}{(\hbar^2B(q,p))^{2/3}}\,\mathrm{Ai}\left(\frac{2(\mathscr{P}(q,p)-\mathscr{P}^*)}{(\hbar^2B(q,p))^{1/3}}\right), \tag{28}$$

where

$$B(q,p)\equiv\mathscr{P}_q^2\mathscr{P}_{pp}+\mathscr{P}_p^2\mathscr{P}_{qq}-2\mathscr{P}_{pq}\mathscr{P}_p\mathscr{P}_q, \tag{29}$$

all derivatives being evaluated at q, p. B remains finite as q, p moves onto $\mathscr{C}$.

(*C*) The *classical limit* of W (as opposed to its semiclassical limiting form) can be obtained from (28) by letting $\hbar\to 0$ and using the result

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathrm{Ai}\left(\frac{x}{\epsilon}\right)=\delta(x). \tag{30}$$

This gives the simple expression (Heller 1977)

$$W_{\text{classical}}(q,p,t;\mathscr{P}^*)=I_0\delta(\mathscr{P}(q,p)-\mathscr{P}^*). \tag{31}$$

(In the special case where $\mathscr{P}$ represents action this formula was derived earlier by one of us (Balazs 1963, unpublished). On projection down p (cf. Berry 1977b), (31) recaptures the classical probability density $|\psi|^2$, which is a sum of terms of the form (5).

(*D*) On the concave side of $\mathscr{C}$ the Airy function in (27) has negative argument, so that W oscillates as $A(q,p,t)$ passes through multiples of $\hbar$. Thus $\mathscr{C}$ is decorated with *fringes* (first described with a formula valid for small p by Balazs and Zipfel 1973).

(*E*) On the convex side of $\mathscr{C}$ the Airy function has positive argument, and W decays exponentially away from $\mathscr{C}$ (Balazs and Zipfel 1973).

(*F*) The approximation (27) shares with the exact Wigner function the property of being formally symmetric in q and p, which is reassuring in view of the unsymmetrical manner of its derivation from (22) and (13). This symmetry means that the approximation for W is valid where its derivation is not, namely near turning points q_t. When projected down p, (27) gives not the original WKB approximation (13) for $|\psi(q,t;\mathscr{P}^*)|^2$ but the correct uniform approximation of Airy type (Langer 1937, Berry and Mount 1972) smoothly valid through q_t.

(*G*) The form of (27) shows that in this uniform approximation $W(q,p,t;\mathscr{P}^*)$ depends only on the form of the curves $\mathscr{C}_t(\mathscr{P})$ for $\mathscr{P}$ near $\mathscr{P}^*$, and not on the classical trajectory passing through q, p at t.

(*H*) If the quantum state ψ is normalised to unity, the approximation (27) satisfies to lowest order in $\hbar$ the following exact relations (Baker 1958):

$$\int_{-\infty}^{\infty}\mathrm{d}q\int_{-\infty}^{\infty}\mathrm{d}p\;W(q,p)=1, \qquad \int_{-\infty}^{\infty}\mathrm{d}q\int_{-\infty}^{\infty}\mathrm{d}p\;W^2(q,p)=h^{-1}. \tag{32}$$

The result (27) gives a pleasant picture of the evolution of the quantum state: Wigner's function hugs the evolving classical curve $\mathscr{C}_t(\mathscr{P}^*)$ in an 'adiabatic' fashion, and decorates it with Airy fringes. Nevertheless there are cases where it must be interpreted carefully. Consider the denominator in (27), as q, p varies (for fixed t). As was shown in § 5 of Berry (1977a), this vanishes on a catastrophe set $\mathscr{L}$ (figure 4) consisting of all qp at the mid-points of chords joining points on $\mathscr{C}$ with parallel tangents; for qp on $\mathscr{L}$, two or more stationary points X_0 of the integrand in (24) coalesce. Part of $\mathscr{L}$ is $\mathscr{C}$ itself, where the 'chords' have zero length, so that their ends are the same point and hence obviously have parallel tangents. In this case the factor $A^{1/6}$ in the numerator of (27) vanishes as well, in such a way as to keep W finite—this is just another way way of saying that the Airy approximation is uniform across $\mathscr{C}$. However, there are parts of $\mathscr{L}$ not on $\mathscr{C}$, and on these curves (27) breaks down.

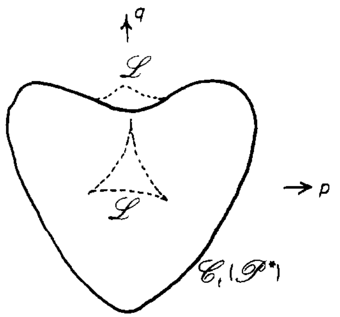

Figure 4. Catastrophe set $\mathscr{L}$, i.e. locus (broken curves) of mid-points of lines joining parallel tangents of $\mathscr{C}_t(\mathscr{P}^*)$ (full curve) on which the 'geometric' expression (27) for Wigner's function breaks down.

Such parts of $\mathscr{L}$ are of two sorts (both shown on figure 4): lines connecting inflections of $\mathscr{C}$, and lines that need not encounter $\mathscr{C}$. Further refinements of the method of stationary phase are required to evaluate W on or close to $\mathscr{L}$ (Berry 1977a), with the result that $|W|$ rises to large values (typically $O(\hbar^{-2/3})$). However, W oscillates along such $\mathscr{L}$ not on $\mathscr{C}$, so that when $\hbar = 0$ the limit is *zero* and not the delta-function (31) (which arises because of $\mathscr{C}$ the function W does not oscillate but remains positive). We emphasise that such complications concern only the evaluation of the integral (24) and do not indicate a breakdown of the basic WKB approximation of § 3. As time proceeds, however, it may happen that the spatial variation of H causes the WKB approximation to break down through a quite different cause, as will now be explained.

5. Whorls and tendrils

Over long times the classical motion may cause the curve $\mathscr{C}_t$ to convolute into very complicated shapes, and we expect such asymptotic complication to be the rule rather than the exception. In this section we shall argue that there are two principal forms of complexity, for which we shall introduce the terms 'whorl' and 'tendril', and in the next section we shall study the effects of whorls and tendrils on the evolution of the quantum state ψ.

In the case we shall consider, for which we can understand the main features of the evolution of $\mathscr{C}$ as $t \to \infty$, the Hamiltonian H describes a stationary system perturbed by a periodic force with period T (this includes the simple case where H is time-independent). Then snapshots of the phase plane at times 0, T, $2T$, ... correspond to successive *maps* $\mathcal{M}$ of *qp* onto itself. By Liouville's theorem $\mathcal{M}$ is area-preserving, and much is known about such maps; for reviews see Arnol'd and Avez (1968), Berry (1978), Ford (1975), Moser (1973). We intend to publish a detailed study (Berry *et al* 1979) of the quantisation of discrete-time maps of the *qp* plane ('quantum maps'), in which the evolution of curves will play a large part; therefore we confine ourselves here to summarising our principal conclusions and conjectures.

The evolution of $\mathscr{C}$ depends on the *fixed points* and *invariant curves* of $\mathcal{M}$. A fixed point *qp* maps onto itself after a finite number of iterations of $\mathcal{M}$, i.e. after a time nT, where n is an integer. Typical maps have infinite hierarchies of elliptic (stable) and hyperbolic (unstable) fixed points. An invariant curve maps onto itself under $\mathcal{M}$. Figure 5 shows some fixed points and invariant curves of a map with three commonly occurring regimes. The *first regime* is the neighbourhood of the origin, where $\mathcal{M}$ has an elliptic fixed point surrounded by smooth invariant curves. Then comes the *second regime*: a set of elliptic and hyperbolic fixed points (three of each in this case). The elliptic points are surrounded by 'islands' of smooth invariant curves. The hyperbolic points have chaotic 'area-filling' invariant curves which generate stochasticity (pseudo-randomess) in these deterministic systems. This set of fixed points is surrounded by more smooth invariant curves surrounding the origin. There is a largest such curve, beyond which lies the *third regime*, where no invariant curves or fixed points exist, and all points escape to infinity under $\mathcal{M}$.

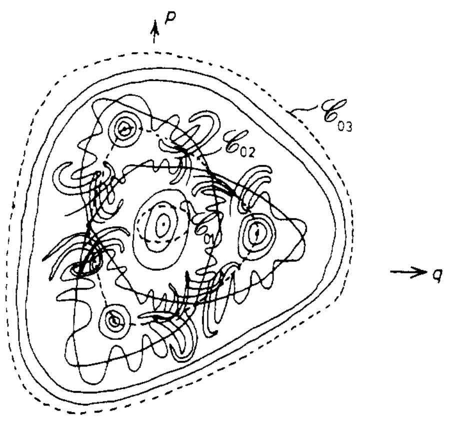

Figure 5. Invariant curves, and elliptic and hyperbolic fixed points, for a typical map. The evolution of the broken curves $\mathscr{C}_{01}$, $\mathscr{C}_{02}$ and $\mathscr{C}_{03}$ is shown on figure 6.

Now consider the fate of the three initial curves $\mathscr{C}_{01}$, $\mathscr{C}_{02}$ and $\mathscr{C}_{03}$ shown on figure 5. $\mathscr{C}_{01}$ lies in the first regime but does not coincide with any invariant curve. Each point on $\mathscr{C}$ maps round its own invariant curve, and since different invariant curves have different 'rotation numbers' the effect is to wrap $\mathscr{C}_{01}$ round the origin into the 'spiral galaxy' shown on figure 6(a). We call the result of this wrapping a 'whorl'. Whorls are not associated with classical stochasticity, and occur, for example, in the simple (integrable) case where H describes a time-independent, anharmonic oscillator and $\mathscr{C}_0$ corresponds to any initial quantum state that is not an eigenstate of H (if the oscillator is harmonic, $\mathscr{C}_0$ simply rotates and whorls do not develop).

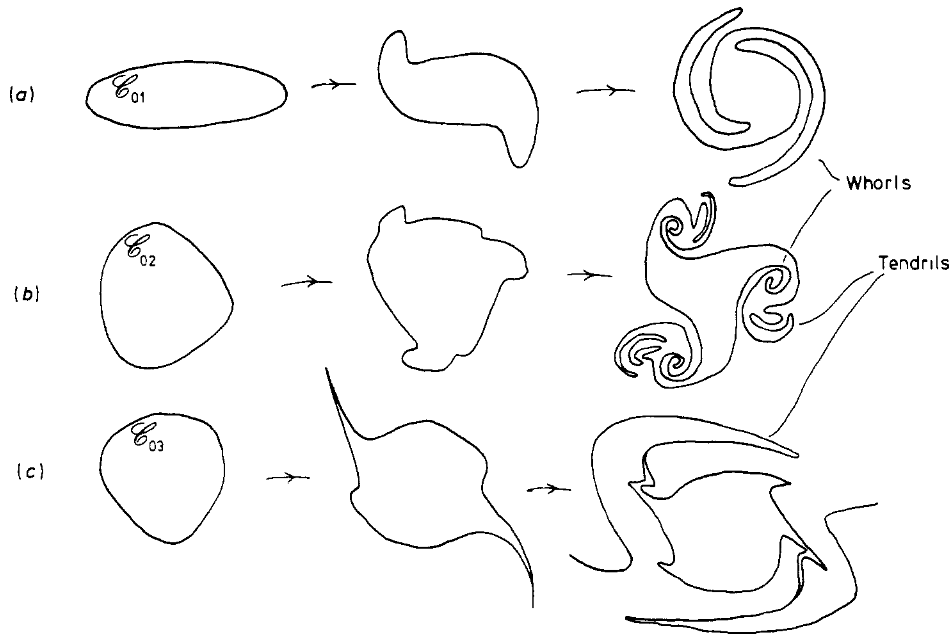

Figure 6. Evolution of curves $\mathscr{C}_{01}$, $\mathscr{C}_{02}$ and $\mathscr{C}_{03}$ on figure 5, showing development of whorls and tendrils.

$\mathscr{C}_{02}$ lies in the second regime and is drawn so that it passes through all six fixed points; these and only these points remain fixed as $\mathscr{C}$ evolves. Parts of $\mathscr{C}_{02}$ near the elliptic fixed points will develop whorls (figure 6*b*). Parts of $\mathscr{C}_{02}$ near the hyperbolic fixed points will swing back and forth like the nearby convoluted invariant curves of $\mathscr{M}$; we shall say that $\mathscr{C}$ grows 'tendrils' in such unstable regions of the phase plane. After many iterations $\mathscr{C}$ develops a fantastic complexity of whorls and tendrils as shown on figure 6(*b*).

$\mathscr{C}_{03}$ lies in the third regime. All its points are unstable so there will be no whorls; instead $\mathscr{C}$ grows tendrils that reach out towards infinity (figure 6*c*). The length of $\mathscr{C}$ grows rapidly while the area it encloses remains constant.

The difference between a whorl and a tendril can be described in terms of inflection points (figure 7): if $\mathscr{C}_0$ has no inflections, then as a whorl develops (figure 7*a*) only two inflections need form, while the development of a tendril (figure 7*b*) entails the formation of infinitely many inflections. Another difference is in the rate of growth of complexity, as measured say by the length L of $\mathscr{C}$. For whorls (associated with stable trajectories) we expect L to increase linearly with time, while for tendrils (associated

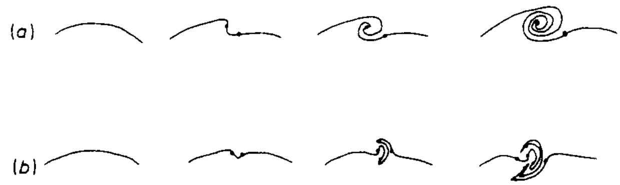

Figure 7. Development of (*a*) whorl, (*b*) tendrils. Inflections are indicated by dots.

with unstable trajectories) we expect L to increase exponentially fast. Finally, while whorls can be generated by a time-independent Hamiltonian, tendrils cannot; therefore the second and third regimes just discussed require a truly time-dependent Hamiltonian.

Preliminary computations by Dr M Tabor (see Berry *et al* 1979) confirm this picture of the evolution of curves under generic mappings. It seems likely that whorls and tendrils will also form in more general cases where $H(q, p, t)$ does not have an associated discrete mapping.

6. Quantum complexity

It seems obvious that Wigner's function $W(q, p, t)$ cannot follow the increasing complication of $\mathscr{C}$ as $t \to \infty$. The reason is that quantum functions on phase space can surely have no detail on areas smaller than $O(\hbar)$, whereas $\mathscr{C}$ develops structure down to arbitrarily fine scales. At some stage, therefore, there must be a transition to a new regime where the classical description is wrong, and W no longer follows the details of $\mathscr{C}$. This new regime will be different when $\mathscr{C}$ grows tendrils from when $\mathscr{C}$ grows whorls. In both cases, however, we expect the outcome as $t \to \infty$ to be a classical limit where W is positive in any region eventually filled by $\mathscr{C}$, with everchanging superimposed quantum oscillations that may bear no relation to $\mathscr{C}$.

A literal application of the geometric formula (27) to these new regimes predicts that W is the resultant of many superposed Airy functions, since most points qp lie at the midpoints of many chords, associated with different loops of $\mathscr{C}$. We have not made a detailed study of what this resultant could look like, and in any case it is not clear whether (27) would remain valid in these regimes. However, it is probably legitimate to use (27) to discuss the transition to the new regimes.

In the early stages of $\mathscr{C}$'s convolution its loops will be well separated in a quantum sense; by this we mean that there will be many Airy oscillations from one loop of $\mathscr{C}$ before the next loop is encountered. Equation (27) can be applied, and yields a W dominated by the windings of $\mathscr{C}$. The increasing complication of $\mathscr{C}$ will reveal itself in the wavefunction $\psi(q, t)$ through the increasing number of turning points q_t (where the tangent to $\mathscr{C}$ is perpendicular to the q axis). These turning points appear in pairs at the moments when an inflection of $\mathscr{C}$ turns so that its tangent lies parallel to the p axis. Such events are *cusp catastrophes* (Poston and Stewart 1978), and the corresponding behaviour of ψ is described by the function of Pearcey (1946), which shows how two Airy functions are born as the new turning points separate.

This proliferation of cusp catastrophes continues until the loops of $\mathscr{C}$ are no longer well separated, that is, until the Airy oscillations associated with different loops can no longer be distinguished. The scale of the oscillations is $\hbar^{2/3}$, as is clear from equations (27) and (28), and we can use this fact to estimate the 'transition time' t_c after which W no longer follows the details of $\mathscr{C}$. Let $D(t)$ be the distance between a typical pair of neighbouring loops of $\mathscr{C}$ at time t. Then t_c satisfies

$$D(t_c) = O(\hbar^{2/3}). \tag{33}$$

Now, in § 5, we claimed that the length of $\mathscr{C}$ should increase linearly or exponentially with t for whorls or tendrils respectively. Since the area enclosed by $\mathscr{C}$ remains constant, this suggests that $D(t) = O(t^{-1})$ for whorls and $D(t) = O(e^{-t})$ for tendrils.

Therefore we predict

$$\begin{aligned} t_c &= O(\hbar^{-2/3}) \text{ (whorls)} \\ t &= O(\ln \hbar^{-1}) \text{ (tendrils)} \end{aligned} \qquad (34)$$

for the time of transition to the new regimes of W.

What are the new regimes like? Consider first the case where $\mathscr{C}$ convolutes into *whorls*, and recall that whorls form when $\mathscr{C}_0$ passes near an elliptic fixed point (of the mapping associated with the Hamiltonian) which is surrounded by smooth invariant curves. To find the form of W we restrict ourselves for simplicity to an initial curve like $\mathscr{C}_{01}$ on figure 5, surrounding a single elliptic point, a situation shown in more detail on figure 8. Let the invariant curves be labelled by their *actions* I (defined as $1/2\pi$ times the phase-space area they enclose), and let positions on a given invariant curve be represented by the *angle* θ (defined as the coordinate canonically conjugate to I). For this particular family of curves, I and θ are respectively the momentum $\mathscr{P}$ and coordinate Q defined in § 2. θ is the invariant measure on the curve labelled I: points uniformly distributed in θ remain so as time proceeds.

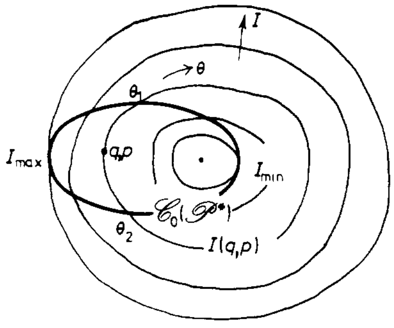

Figure 8. Initial curve $\mathscr{C}_0$ (thick curve) near an elliptic fixed point with invariant curves I (thin curves).

Let us use I, θ to label points in the phase space q, p. Then in the crudest classical approximation the initial form of Wigner's function is given by equation (31) as

$$W_{\text{classical}}(\theta, I, 0; \mathscr{P}^*) = I_0 \delta(\mathscr{P}(\theta, I) - \mathscr{P}^*) \qquad (35)$$

where $\mathscr{P}(\theta, I)$ is that curve in the family $\mathscr{C}_0(\mathscr{P})$ which passes through the point θ, I. Over long times each point of $\mathscr{C}_0$ rotates many times around its invariant curve I. This means that when the whorl is fully developed at $t = \infty$ the classical limit of Wigner's function depends only on I and can be obtained from (35) by averaging over θ and thus smoothing away the caustic singularities of the projection of $\mathscr{C}$, which get ever denser as $t \to \infty$. This gives

$$W_{\text{classical}}(I, t \to \infty; \mathscr{P}^*) = \frac{I_0}{2\pi} \int_0^{2\pi} d\theta\, \delta(\mathscr{P}(\theta, I) - \mathscr{P}^*). \qquad (36)$$

In the original coordinates $q, p, I = I(q, p)$ is just the label of the invariant curve passing through q, p.

The integration in (36) is easily performed: the delta function means that only those angles θ_l contribute which correspond to *intersections* of $\mathscr{C}_0(\mathscr{P}^*)$ (figure 8) with the invariant curve I (generally there is an even number of such intersections). Then

$$W_{\text{classical}}(I, t\to\infty; \mathscr{P}^*) = \frac{I_0}{2\pi}\sum_l \left|\frac{\partial\mathscr{P}(\theta_l, I)}{\partial\theta}\right|^{-1}_{\mathscr{P}=\mathscr{P}^*}. \tag{37}$$

As I approaches an invariant curve, labelled I_c, which *touches* $\mathscr{C}_0$, two intersections θ_l coalesce, $\partial\mathscr{P}/\partial\theta$ vanishes, and W as given by (37) diverges as $|I-I_c|^{-1/2}$. (In the simplest case, illustrated on figure 8, this happens only for the limiting actions $I_{\min}$ and $I_{\max}$ explored by $\mathscr{C}_0$.) Therefore, as $t\to\infty$, W does indeed fill the region occupied by the whorl, not uniformly, however, but with singularities for those actions I_c where $\mathscr{C}_0$ touches an invariant curve.

The result (37) is purely classical and does not exhibit the continual time variation that the quantum Wigner function must possess, even as $t\to\infty$, consequent upon the fact that the initial wavefunction is not an eigenstate of the Hamiltonian or any associated quantum map. In Appendix 1 we show that (37) can be derived on a simple model using quantum-mechanical arguments, provided the time variation of W is averaged away.

Nevertheless, by using arguments explained by Berry (1977b), equation (37) can be used to get a good picture of the wavefunction $\psi(q, t\to\infty; \mathscr{P}^*)$ corresponding to a fully developed whorl. Consider first the *locally averaged probability density* $|\psi|^2$. By equation (23) this is the projection of (37) onto the q axis, that is, the projection of the band of invariant curves between $I_{\min}$ and $I_{\max}$. The interesting q regions are near the singularities q_c of the projections of those invariant curves I_c where $W_{\text{classical}}$ is singular. It is easy to see that $|\psi|^2$ has step discontinuities at $q=q_c$. For consider the simple case of $I_c = I_{\max}$ where the invariant curves are circles; then the radius $(p^2+q^2)^{1/2}$ of an I curve is $\sqrt{2I}$, so q_c is simply $\sqrt{2I_{\max}}$, and for q near q_c

$$|\psi(q, t\to\infty)|^2 = \text{Re}\int_{-\infty}^{\infty}\frac{\text{d}p\times\text{const}}{(I_{\max}-I(q,p))^{1/2}} = \text{const}\times\text{Re}\int_{-(q_c^2-q^2)^{1/2}}^{(q_c^2-q^2)^{1/2}}\frac{\text{d}p}{(q_c^2-q^2-p^2)^{1/2}}$$
$$= \begin{cases}\text{const if } |q|<|q_c| \\ 0 \quad \text{if } |q|>|q_c|\end{cases}. \tag{38}$$

These step discontinuities are the analogues for a whorl as $t\to\infty$ of the caustic divergences of $|\psi|^2$ that originate from singularities of the projection of $\mathscr{C}_t$ when t is not large.

Now consider the *pattern of oscillations* of ψ near q as $t\to\infty$. This has a spectrum (Berry 1977b) consisting of wavelengths $\lambda = h/p$ for all p for which $W(q,p)$ exists at q. In the present case the spectrum is dominated by those p values lying on the critical invariant curves I_c for which W diverges. Therefore $\psi(q, t\to\infty)$ for a whorl is the superposition of a few (somewhat spectrally impure) oscillatory contributions.

Finally we enquire about W and ψ near *tendrils* when $t\to\infty$. This is the stochastic case, where the classical motion is unstable. If the tendrils are trapped between invariant curves (as is the case with $\mathscr{C}_{02}$ on figure 5), then it is likely that they will eventually cover some region of phase space uniformly (or nearly so) in an ergodic fashion, and we assume that this is in fact what happens. This behaves under projection very differently from a whorl; the boundaries of the tendril-filled region project onto *anticaustics* (Berry 1977b) where W vanishes. A simple model example is a circular patch $p^2+q^2<r^2$ of tendrilled curve, which projects to

$$|\psi(q,t\to\infty)|^2=\text{const}\times\int_{-\infty}^{\infty}\mathrm{d}p\,\Theta(r^2-p^2-q^2)=\begin{cases}\text{const}\times 2(r^2-q^2)^{1/2} & |q|<r\\ 0 & |q|>r,\end{cases} \tag{39}$$

where Θ denotes the unit step function. This result should be contrasted with (38).

The pattern of oscillations of ψ as $t\to\infty$ is also different for tendrils, since a uniformly filled region of phase space corresponds to a continuous spectrum of oscillations, corresponding to the range of p involved. Such a wave ψ, made up of contributions with different wavelengths, will resemble a *random function of* q.

When the tendrils are not trapped between invariant curves they escape to infinity, like $\mathscr{C}_{03}$ on figure 5. Then of course they can never fill any region in an ergodic fashion, and their convolutions diffuse outwards forever. The Wigner function must diffuse outwards too. As already explained, we expect W to follow $\mathscr{C}_t$ until the transition time t_c (equation 34). For $t>t_c$, however, the tendrils will be very thin on the scale of $\hbar$, and we no longer expect W to follow their details. Instead, the quantum W will diffuse outwards more slowly than than the classical probability.

We think that this simple observation explains the results of computations by Casati *et al* (1979) on the quantum-mechanical pendulum (free rotor with periodic coordinate q) subjected to a regular succession of sudden impulses. The system evolves from some given initial state $\psi(q,0)$, and Casati *et al* computed the quantum expectation value $E_{qu}(t)$ of the energy at time t, as well as the corresponding classical energy $E_{cl}(t)$. They found that E_{cl} and E_{qu} increased in very similar fashion from $0<t<\tau$, after which $E_{qu}(t)$ increased much more slowly than $E_{cl}(t)$. It is tempting to identify τ with our transition time t_c.

7. Conclusions

One reason for studying these time-dependent problems is of course to make mathematical models for naturally occurring processes. Our main reason here, though, is that (as realised by Casati *et al* 1979) one-dimensional non-stationary problems provide simple systems for which to study the quantum consequences of the transition from classically integrable to classically non-integrable motion. Previous studies (Percival 1973, Pomphrey 1974, Berry 1977a, b) have been restricted to bound states of stationary systems, for which the simplest case has two degrees of freedom so that phase space is four-dimensional and not so amenable to geometric intuition as the case we have treated here.

Our arguments about the way details of $\mathscr{C}_t$ get lost in the transition to new regimes as $t\to\infty$ provide a nice illustration of the different roles Planck's constant $\hbar$ can play in semiclassical mechanics (Berry 1977b, 1978). Before the transition, i.e. for $t<t_c$ (equation 34)), $\hbar$ imposes *oscillatory quantum detail* on a W and $|\psi|^2$ which otherwise jt merely the classical curve $\mathscr{C}_t$ and its projection. This is not surprising—after all $\hbar$ is an extra parameter not present in classical mechanics, and we expect it to describe a greater richness of behaviour. After the transition, however, $\hbar$ acts quite differently: *it smooths away the details of classical fine structure* (whorls and tendrils) which would otherwise develop as $t\to\infty$ down to arbitrarily fine scales.

It would be highly desirable to test the conjectures presented here, either numerically or, preferably, with the aid of exactly soluble models. A promising approach to these problems is based on the observation that the main features of time-dependent problems (evolution of curves $\mathscr{C}$, development of asymptotic complication in the form of whorls and tendrils, etc) are present in discrete-time, area-preserving maps of the plane. It turns out that these can often be quantised directly, and we are making the resulting 'quantum maps' the subject of a detailed study (Berry *et al* 1979).

Acknowledgments

We thank the US National Science Foundation for financial assistance, and Dr J H Hannay for a helpful discussion.

Appendix 1. Wigner's function for fully developed whorls

Here we give a quantum-mechanical derivation of the classical limit (37) of $W(q, p, t; \mathscr{P}^*)$ for a model system where an exact formal solution exists. This is the case of a time-independent Hamiltonian $\hat{H}$ corresponding to a classical anharmonic oscillator $H(q, p)$ with an elliptic fixed point at the origin (i.e. the contours of H surround $q = p = 0$). Let the eigenstates of H (assumed non-degenerate) be $|\phi_n\rangle$, i.e.

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle, \tag{A1.1}$$

and let the initial state $\psi(q, 0; \mathscr{P}^*)$ (which is not one of the $|\phi_n\rangle$) be now denoted by $|\psi_0\rangle$. Then the evolving state $\psi(q, t; \mathscr{P}^*)$ is given exactly by

$$|\psi_t\rangle = \sum_n \langle\phi_n|\psi_0\rangle \exp\left(-\frac{iE_n t}{\hbar}\right)|\phi_n\rangle. \tag{A1.2}$$

W is defined by (22), and for the state (A1.2) this gives

$$\begin{aligned} W(q, p, t; \mathscr{P}^*) &= \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dX \exp\left(-\frac{2ipX}{\hbar}\right) \sum_n \sum_m \langle\phi_n|\psi_0\rangle\langle\psi_0|\phi_m\rangle\phi_n(q+X) \\ &\quad \times \phi_m^*(q-X) \exp\left(-\frac{i(E_n - E_m)t}{\hbar}\right) \\ &= \sum_n |\langle\phi_n|\psi_0\rangle|^2 W_n(q, p) + \text{terms oscillating with } t, \end{aligned} \tag{A1.3}$$

where W_n denotes the (time-independent) Wigner function for the state $|\phi_n\rangle$. The slowest oscillations have period $h/(E_{n+1} - E_n)$, which by the correspondence principle is closely approximated for large n by the period of the associated classical motion round the contours of H. Whorls, being generated by *differences* in classical periods, develop over times much longer than this. Averaging away these 'classical' oscillations, as well as the faster quantum oscillations from levels E_m and E_n that are not neighbouring, gives a time-independent Wigner function whose classical limit we now examine.

By a well known semiclassical rule, the nth eigenstate $|\phi_n\rangle$ is associated with the contour (invariant curve) of H whose action ($1/2\pi$ times area in phase space) is

$$I_n = (n + \tfrac{1}{2})\hbar. \tag{A1.4}$$

By equation (31) with appropriate normalisation the classical Wigner function W_n of $|\phi_n\rangle$ is

$$W_n(q, p) = \delta(I(q, p) - I_n)/2\pi. \tag{A1.5}$$

Now we replace the summation over n in (A1.3) by an integration over I_n, and perform this using the delta function (A1.5). With an obvious change of notation for $|\phi_n\rangle$ this gives

$$W_{\text{classical}}(q, p, t\to\infty; \mathscr{P}^*) = \left|\int_{-\infty}^{\infty} \mathrm{d}q'\, \phi^*(q', I(q, p))\psi(q'm\ 0; \mathscr{P}^*)\right|^2 \Big/ 2\pi\hbar, \tag{A1.6}$$

so that the limiting Wigner function for the fully developed whorl is simply the square of the matrix element between $|\psi_0\rangle$ and a suitably chosen member of the set $|\phi_n\rangle$.

To evaluate the matrix element we employ WKB approximations for ψ and ϕ^*, and the method of stationary phase for the integration over q'. All discontinuous phases will be neglected since they only affect oscillatory contributions to (A1.6), which will subsequently be ignored. For ψ the WKB approximation is given by (6), and for ϕ by a similar expression with I_0 replaced by $1/2\pi$ (to ensure correct normalisation) and S_0 (given by equation 3) replaced by

$$F(q, I) = \int_{q_0}^{q} \pi(q'; I)\, \mathrm{d}q', \tag{A1.7}$$

where $\pi(q; I)$ is one of the momentum values on the curve I at q. Thus (A1.6) becomes

$$W_{\text{classical}}(q, p, t\to\infty; \mathscr{P}^*) = \frac{I_0}{2\pi\hbar}\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}q' \left(\frac{\partial p}{\partial \mathscr{P}^*}\frac{\partial \pi}{\partial I}\right)^{1/2}\right.$$
$$\left.\times \exp\left(\frac{i}{\hbar}\int_{q_0}^{q'} \mathrm{d}q''(p(q'', 0; \mathscr{P}^*) - \pi(q''; I(q, p)))\right)\right|^2. \tag{A1.8}$$

Stationary points of the exponent lie at $q' = q_l$, where

$$p(q_l, 0; \mathscr{P}^*) = \pi(q_l; I(q, p)). \tag{A1.9}$$

These are the *intersections* of $\mathscr{C}_0(\mathscr{P}^*)$ with the invariant curve I passing through the point q, p where W is being evaluated. According to the method of stationary phase the integral over q is the sum of oscillatory contributions from each q_l value. The non-oscillatory contribution to the square (A1.8) is

$$W_{\text{classical}}(q, p, t\to\infty; \mathscr{P}^*) = \frac{I_0}{2\pi}\sum_l \left|\frac{(\partial p/\partial \mathscr{P}^*)\,\partial\pi/\partial I}{\partial(p-\pi)/\partial q'}\right|_{q'=q_l}. \tag{A1.10}$$

That this is indeed the same as (37) can be seen from the following series of symbolic manipulations (which can be written in an explicit notation although this is extremely tedious):

$$\frac{\partial(p-\pi)/\partial q}{(\partial p/\partial\mathscr{P}^*)\,\partial\pi/\partial I} = \frac{\partial\mathscr{P}^*}{\partial q}\frac{\partial I}{\partial \pi} - \frac{\partial\mathscr{P}^*}{\partial p}\frac{\partial I}{\partial q} = \frac{1}{\partial H/\partial I}\left(\frac{\partial\mathscr{P}^*}{\partial q}\frac{\partial H}{\partial \pi} - \frac{\partial\mathscr{P}^*}{\partial p}\frac{\partial H}{\partial q}\right)$$
$$= \frac{1}{\omega(I)}\left(\frac{\partial\mathscr{P}^*}{\partial q}\dot{q} + \frac{\partial\mathscr{P}^*}{\partial p}\dot{p}\right) = \frac{1}{\omega(I)}\frac{\mathrm{d}\mathscr{P}^*}{\mathrm{d}t} = \frac{\partial\mathscr{P}^*(\theta, I)}{\partial\theta}. \tag{A1.11}$$

In these equations $\dot{q}$ and $\dot{p}$ denote the rates of change of q and p along the classical orbit, $\omega = \partial H/\partial I$ denotes the frequency of classical motion, and θ is the angle coordinate, varying as ωt + const.

References

Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Functions* (Washington: US National Bureau of Standards)
Arnol'd V I and Avez A 1968 *Ergodic Problems of Classical Mechanics* (New York: Benjamin)
Baker G A Jr 1958 *Phys. Rev.* **109** 2198–206
Balazs N L and Zipfel G G Jr 1973 *Ann. Phys. NY* **77** 139–56
Berry M V 1977a *Phil. Trans. R. Soc.* **287** 237–71
—— 1977b *J. Phys. A: Math. Gen.* **10** 2083–91
—— 1978 in *Topics in Nonlinear Dynamics* ed. S Jorna *AIP Conf. Series* **46** (New York: American Institute of Physics) chap. 2
Berry M V, Balazs N L, Tabor M and Voros A 1979 *Ann. Phys., NY* submitted for publication
Berry M V and Mount K E 1972 *Rep. Prog. Phys.* **35** 315–97
Casati G, Chirikov B V, Izraelev F M and Ford J 1979 in *Stochastic Behaviour in Classical and Quantum Hamiltonian Systems* ed G Casati and J Ford *Springer Notes in Physics* p 334
Chester G, Friedman B and Ursell F 1957 *Proc. Camb. Phil. Soc.* **53** 599–611
Dirac P A M 1947 *The Principles of Quantum Mechanics* (Oxford: Clarendon)
Ford J 1975 in *Fundamental Problems in Statistical Mechanics* Vol. III, ed M Cohen (Amsterdam: North-Holland) pp 215–55
Heller E J 1977 *J. Chem. Phys.* **67** 3339–51
Keller J B 1958 *Ann. Phys.* **4** 180–8
Langer R E 1937 *Phys. Rev.* **51** 669–76
Maslov V P 1972 *Théorie des Perturbations et des Methodes Asymptotiques* (Paris: Dunod)
Moser J 1973 *Stable and Random Motions in Dynamical Systems* (Princeton: University Press)
Pearcey T 1946 *Phil. Mag.* **37** 311–7
Percival I C 1973 *J. Phys. B: Atom. Molec. Phys.* **6** L229–32
Pomphrey N 1974 *J. Phys. B: Atom. Molec. Phys.* **7** 1909–15
Poston T and Stewart I 1978 *Catastrophe Theory and Its Applications* (London: Pitman)
Synge J L 1960 *Encyclopedia of Physics* ed. S Flügge (Berlin: Springer) **3/1** 1–225
Van Vleck J H 1928 *Proc. Math. Acad. Sci. USA* **14** 178–88
Wigner E P 1932 *Phys. Rev.* **40** 749–59