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Evolution of Social networks

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Abstract

Modeling the evolution of networks is central to our understanding of modern large communication systems, such as the World-Wide-Web, as well as economic and social networks. The research on social and economic networks is truly interdisciplinary and the number of modeling strategies and concepts is enormous. In this survey we present some modeling approaches, covering classical random graph models and game-theoretic models, which may be used to provide a unified framework to model and analyze the evolution of networks.

1. Introduction

The importance of network structure in social and economic systems is by now very well understood. In sociology and applied statistics the study of social ties among actors is a classical field, which has been established as the subject of social network analysis (a classical reference is Wasserman and Faust, 1994). More recently the networks perspective has been discovered by game theorists, economists, as well as computer scientists and physicists attempting to model the *evolution of networks*. Of course all these subjects put different emphasis on what is considered to be a "good" model of network formation. Traditionally economists are used to interpret observed social structures (e.g. a collaboration network between firms in an industrial cluster) as *equilibrium phenomena* which can be rationalized by the preferences of the agents. Game theoretic reasoning based on optimizing behavior is the obvious tool used in this literature. Computer scientists, on the other hand, prefer to think of network formation in terms of dynamic network formation algorithms. Of course these algorithms can be often given behavioral foundations or

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interpretations. Physicists tend to think as networks as an outgrowth of complex system analysis. In this field the main interest is to understand and characterize the statistical regularities of large networks, using a reduced-form description of the dynamical system (the so-called “mean-field” model). The paradigmatic example for a large network where this approach has turned out to be quite successful in reproducing the measured stylized facts is the world wide web (a nice overview on this literature is given in Dorogovtsev and Mendes, 2003). By now the number of publications on the evolution of social and economic networks has exploded, and it would be impossible to provide a survey covering all the models developed in the above mentioned disciplines. For this very reason, we have decided to focus in this survey on two particular promising approaches to model the evolution of social and economic networks. Before describing these models, however, let us provide some motivation why we think that this survey provides a good contribution to the literature. There are many excellent textbooks and surveys already available, so any new survey needs some words of justification. Recent textbooks discussing models of dynamic stochastic network formation are Chung and Lu (2006) and Durrett (2007). From an economic perspective the textbooks by Vega-Redondo (2007), Goyal (2007) and Jackson (2008) provide concise introductions to the fields. There are also various surveys discussing this interdisciplinary topic from different perspectives.[1] What distinguishes this survey from existing ones is that it tries to give the reader a brief overview on recent attempts to model network formation with a particular focus on the evolution of networks either in the language of stochastic processes or game theory. Moreover, we try to highlight the potential connections between these two seemingly separate modeling strategies and we try to give some suggestions for further research in this field.

1.1 Overview

The first part of this review article (Chapter 2) discusses random graph models. These models are the cornerstone for the statistical analysis of networks and have had a large impact on theoretical models of network evolution. Moreover this approach has a long tradition in social network analysis, and provides a natural bridge to the more recent models of network evolution used in computer science, mathematics and physics. Following the terminology of Chung and Lu (2006), we focus on ”off-line” models. Hence, we consider network formation models in which the number of nodes, or the *population*

[1] See among others, Jackson (2003), Jackson (2005), Van den Nouweland (2005), Goyal (2005), Snijders et al. (2006), Robins et al. (2007), Goldenberg et al. (2009).

size, is a given parameter.[2] Our focus is on the formulation of network evolution as edge-based stochastic processes. We provide concrete examples in which we relate this model to classical models in mathematical sociology, as well as to more recent models from mathematics and economics. It is shown that the model we present is equivalent to various *inhomogeneous random graph models* (Bollobás et al., 2007). This model family is rather rich. It contains well-known statistical block-models, as well as the classical Bernoulli random graph model due to Erdös and Rényi (1959) as special cases.
Section 3 presents an alternative approach of network modeling which has been advanced by economic theorists. It uses game theoretical concepts to interpret network structures as *equilibrium phenomena* of strategically acting players who create and destroy links according to their incentives. Two particular approaches have turned out to be useful in this domain: The semi-cooperative solution concept of *pairwise-stability* (Jackson and Wolinsky, 1996), and various modifications of *Nash equilibrium*. While this approach is inherently static and evolution of networks is modeled implicitly, we present also a dynamic model of network formation that gives rise to these stability concepts. Beside this game-theoretic approach, there have been also some dynamic "learning" models, which combine elements from the statistical literature on network evolution, surveyed in Chapter 2 with the just mentioned game theoretic concepts. Such models have been further developed in a model family which we call *co-evolutionary processes of networks and play* (Staudigl, 2010). The final part of this survey (Chapter 4) presents a modest attempt to synthesize the strategic approach of network formation with the random graph approach of section 2. Finally, in section 5 we make some suggestions for future research.

1.2 Notation

We use both a traditional graph theoretic definition of networks, as well as its (equivalent) algebraic definition. We treat networks and graphs as synonymous objects. A *graph* is a pair $G = ([N], E)$, where $[N] := \{1, 2, \dots, N\}$ is the set of *vertices* (or *nodes*), and $E \equiv E(G) \subset [N] \times [N]$ is the set of *edges* (or *links*). The entries $\{i, j\} \in E(G)$ represent the *bilateral connections (links)* in network G. In this survey our main focus will be on the evolution of *undirected* networks, meaning that the edge $\{i, j\}$ is equivalent to the edge $\{j, i\}$ for any pair $i, j \in [N]$.[3] For ease of notation, we denote by $ij = ji \equiv \{i, j\} \in E(G)$ a

[2] "On-line" models are models in which the population is growing over time. This important class of models contains the very popular preferential attachment models (Barabási and Albert, 1999), which we are not touching in this survey. Excellent summaries of these fascinating models can be found in Newman (2003) and, in more rigorous manner, in Chung and Lu (2006) and Durrett (2007).

[3] Our focus on undirected networks does not mean that we think directed networks are less important. However many of the game-theoretic concepts, which we are going to introduce in Section 3, have a more

link between player i and player j in network $E(G)$. We denote by $\mathcal{G}[N]$ the set of simple graphs on the vertex set $[N]$.

Given a network $G \in \mathcal{G}[N]$, the *neighbors of player i* are represented by the set $N_i(G) := \{j \in [N] \mid ij \in E(G)\}$. Similarly, $E_i(G) := \{ij \in E(G) \mid j \in [N]\}$ denotes the *set of player i's links* in G and $E_{-i}(G) := E(G) \setminus E_i(G)$ denotes the set of links in G in which player i is not involved. We denote by $\eta_i(G) := |E_i(G)|$ the *degree* of player i. For two networks $G, G' \in \mathcal{G}[N]$ let $G \oplus G'$ be the network obtained by adding the links of both networks, i.e. $G \oplus G' := ([N], E(G) \cup E(G'))$ and let $G \ominus G' := ([N], E(G) \setminus E(G'))$ denote the network obtained by *deleting* the set of links $E \cap E'$ from network G. Abusing notation, we will also use $G \oplus l$ to denote the addition of links $l \subseteq E(G^c \ominus G)$ to G and $G \ominus l'$ to denote the deletion of links $l' \subseteq E(G)$ from networks G. We say that there exists a *walk* in a network G between two players i and j if there exists a sequence of players $i_1, ..., i_K$ such that $i_1 = i$ and $i_K = j$ and $i_k i_{k+1} \in E(G)$ for all $k = 1, ..., K-1$. A *path* is a walk using mutually distinct edges. The *distance* between two nodes i and j in network G, denoted by $d_{ij}(G)$ is then the length of the shortest path between these nodes.
An equivalent algebraic definition of a graph is given by introducing a function $A_{ij} : \mathcal{G}[N] \to \{0, 1\}$ defined by

$$A_{ij}(G) = \begin{cases} 1 & \text{if } (i, j) \in E(G), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The matrix-valued function $A : \mathcal{G}[N] \to \{0, 1\}^{N \times N}$ defined by $A(G) = \left[A_{ij}(G); 1 \leq i, j \leq N\right]$ is called the *adjacency matrix* of the graph G.

2. Stochastic models of network evolution

Social behavior is complex, and stochastic models allow us to capture both the regularities in the processes giving rise to network ties while at the same time recognizing that there is variability that we are unlikely to be able to model in detail. To capture all the variability in social network modeling there is not much hope for a canonical network formation model which is able to capture all details which one might think are important in a concrete study. Nevertheless, one may attempt to start thinking about families of models which are parsimonious enough to get pointed predictions, and on the other hand

natural interpretation in terms of undirected graphs. Many of the models presented in this survey can be adapted to allow for directed networks, and, in particular, the random graph models which are presented in Section 2 can be used to model the evolution of directed as well as undirected networks.

rich enough in order to be able to reproduce as many stylized facts the researcher is aiming to model (cf. Section 2.1). Random graph models are powerful tools in this respect. In this section we introduce a rather general model of a random graph process, which will turn out to be useful our definition of a co-evolutionary process of networks and play, to be defined in Section 4.

2.1 Random graphs

Technically speaking a *random graph model* is a probability space $(\mathcal{G}[N], 2^{\mathcal{G}[N]}, \mathsf{P})$, where P is a probability measure defined on the power set $2^{\mathcal{G}[N]}$. The probability measure P assigns to each graph a weight, which should reflect the likelihood that a certain graph structure appears in our model. The question is now what a "natural" random graph model should be. Let us start with a classical example. An historically very important random graph model is the *Bernoulli graph*, often simply called the *Erdös-Rényi* graph.[4] The Bernoulli random graph is built on the assumption that edges are formed independently with constant probability $p \in [0,1]$. This implies that the random graph model $(\mathcal{G}[N], 2^{\mathcal{G}[N]}, \mathsf{P})$ is determined by the probability measure

$$\mathsf{P}(\{G\}) = \prod_{i=1}^{N} \prod_{j>i} p^{A_{ij}(G)}(1-p)^{1-A_{ij}(G)}.$$

The advantage of the Bernoulli graph model is its simplicity. In fact, the complete random graph model is described by two parameters: the *population size* (N), and the *edge-success probability* (p). Hence, this model is by now rather well understood, and we refer the reader to Bollobás (2008) for an in-depth study of this model. It comes with little surprise that such a simple model is rarely a good description of a real-world network. However it can serve as a benchmark to compare real world characteristics with the predictions of Bernoulli graphs which presume independence of link formation. Compared to random graphs, real world networks are often observed to have smaller average path length (an effect coined small worlds phenomenon),[5] higher clustering (friends of friends are more likely to be friends),[6] exhibit homophily (connections between nodes of similar kind are more likely),[7] and often exhibit a power law degree distribution (more nodes with very

[4]Erdös and Rényi (1959, 1960) introduced the slightly different model in which the number of vertices and edges are given parameters. The Bernoulli graph model is due to Gilbert (1959).

[5]See the famous letter experiment of Milgram (1967). Other studies include Garfield (1979), Watts (1999), and Dodds et al. (2003).

[6]See e.g. Watts and Strogatz (1998), Watts (1999) and Newman (2003, 2004).

[7]See Lazarsfeld and Merton (1954), Blau (1977), Marsden (1988), McPherson et al. (2001), Golub and Jackson (2012), among others.

high and very low degree compared to Bernoulli graphs).[8] In particular, the lack of *correlation across links* is a well-known deficit of the Bernoulli graph. For a more detailed discussion we refer the reader to Jackson (2008) and Vega-Redondo (2007). Moreover, the notion of a random graph is inherent *static*. In the next section we are going to discuss models which are able to generate more realistic network structures, and are dynamic, hence "evolutionary" models.

2.2 Network formation as a stochastic process

A classical approach in social network theory is to view the evolution of a network as a stochastic process. This approach has been strongly influenced by the Markov graph model of Frank and Strauss (1986), and laid the foundation for the important model class of *exponential random graphs*.[9] In the following we will describe a fairly general dynamic network formation model in terms of a continuous-time Markov jump processes.

In a dynamic model of network formation we would like to capture two things: First, the network should be viscous: Links are deleted and formed over time. Second, the likelihood that a link is formed or destroyed should be made dependent on some characteristics of the vertices in the graph. The following network formation algorithm captures both these requirements.

We are given some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The sample space Ω might be larger than the set of graphs.[10] We call a Markov jump process $\{\gamma(t)\}_{t\in\mathbb{R}_+}$ a random graph process if each $\gamma(t)$ is a $\mathcal{G}[N]$-valued random variable, measurable with respect to the σ-field $\sigma(\{\gamma(s); s \leq t\})$. The dynamic evolution of the random graph process consists of the following steps:

Link creation: With a constant rate $\lambda \geq 0$ the network is allowed to expand. Let $W : \mathcal{G}[N] \to \mathbb{R}_+^{N\times N}$ be a bounded matrix-valued function, whose components $w_{ij}(G)$ define the *intensities* of link formation between vertex i and j. The function W will be called the *attachment mechanism* of the process.

Link destruction: Let $\xi \geq 0$ denote the constant rate of link destruction. Let $V : \mathcal{G}[N] \to \mathbb{R}_+^{N\times N}$ be a bounded matrix-valued function, whose components $v_{ij}(G)$ define the

[8]First stated by Price (1965). Other studies include Kochen et al. (1989), Seglen (1992), Albert et al. (1999), Amaral et al. (2000)

[9]See Snijders et al. (2006) for a recent survey, and Chatterjee and Diaconis (2011) who clarify some mathematical problems associated with this model.

[10]This will be necessary in order to model the co-evolution of networks and play in section 4.

intensities of link destruction between vertex i and j. The function V is called the *volatility mechanism* of the process.

The generator describing this Markov process will be described in more detail in Section 4.

Remark 2.1.

- In some models of network formation the rate of link destruction ξ has been interpreted as environmental volatility (See in particular the papers Marsili et al., 2004; Ehrhardt et al., 2006, 2008a,b). This is also the motivation behind our definition of a volatility mechanism.
- In principle the intensities of link creation and destruction can be asymmetric, i.e. we do not require in the construction that $w_{ij}(G) = w_{ji}(G)$, or $v_{ij}(G) \equiv v_{ji}(G)$, respectively. Hence, in principle the network formation process can be used to model the formation of directed as well as undirected networks.

Let us now illustrate some simple examples which can be modeled using our network formation algorithm.

2.2.1 A popularity model

Wasserman (1980) proposes the following model of the evolution of a directed network. The only difference between a directed and an undirected network is that the adjacency relationship between two vertices is not necessarily symmetric. Hence, a link between vertices i and j can exist without the need that there is a link between j and i. Directed networks are very frequently used in social network analysis (see e.g. Snijders, 2001, and the references therein), and are also of big importance in models of growing networks (the "on-line" models), modeling the evolution technological networks such as the world-wide-web. An "off-line" version of the preferential attachment model can be obtained by assuming that the intensities of link creation and destruction are positively correlated with the "popularity" of a node. There are several measures of popularity, or centrality (cf. Freeman (1979) or Bonacich (1987), see also Section 3.2). In the context of a directed graph, a natural and simple measure of popularity of a node is its *indegree*, denoted by $\eta_i^+(G)$. Mathematically, the indegree of a node is given by $\eta_i^+(G) = \sum_{j=1}^N A_{ji}(G)$. i.e. the number of vertices $j \in [N]$ which choose to be connected to i. A simple model of popularity is obtained by assuming the link formation intensities are increasing functions of the in-degree of vertex j, i.e.

$$w_{ij}(G) = \alpha_1 + \beta_1 \eta_j^+(G).$$

Similarly, it might be reasonable to assume that the rate of link destruction is an increasing function of the in-degree of j, i.e.

$$v_{ij}(G) = \alpha_2 + \beta_2 \eta_j^+(G).$$

The coefficients $\alpha_i, \beta_i, i = 1, 2$, are given constants, which can be estimated from a given data set of networks. This gives rise to a simple model of network formation based on popularity. Wasserman (1980) provides a detailed study of this model.

2.2.2 Inhomogeneous random graphs

A straightforward extension of the classical Erdös-Rényi model is the inhomogeneous random graph model. It is constructed as follows. Let $\mathcal{G}[N]$ the set of undirected graphs. Recall that we can represent every such graph with its (symmetric) adjacency matrix $A(G) = [A_{ij}(G); 1 \leq i, j \neq N]$. Suppose that the intensities of link creation and link destruction are respectively given by the functions

$$w_{ij}(G) = (1 - A_{ij}(G))\kappa_{ij}, \text{ and } v_{ij}(G) = A_{ij}(G)\delta_{ij}. \tag{2}$$

The scalars κ_{ij}, δ_{ij} are, for simplicity, assumed to be positive and symmetric, meaning that $\kappa_{ij} \equiv \kappa_{ji}$ and $\delta_{ij} \equiv \delta_{ji}$ for all $i, j \in [N]$. Additionally we assume that $\lambda = \xi = 1$. This essentially says that the processes of link creation and link destruction run on the same time scale. Then the following general picture emerges.

Theorem 2.2 (Staudigl (2012))**.** *Consider the random graph process $\{\gamma(t)\}_{t\in\mathbb{R}_+}$ with attachment and volatility mechanism W and V given by the functions (2). Then the graph process is ergodic with unique invariant measure*

$$\mathsf{P}(\{G\}) = \prod_{i=1}^{N} \prod_{j>i} (p_{ij})^{A_{ij}(G)} (1 - p_{ij})^{1-A_{ij}(G)}, \tag{3}$$

where $p_{ij} = \frac{\kappa_{ij}}{\kappa_{ij}+\delta_{ij}}$ is the edge-success probability of vertex i and j.

The random graph measure (3) describes the probability space of an *inhomogeneous* *random graph*. The wonderful work of Bollobás et al. (2007) studies this model in detail.[11] It contains the Erdös-Rényi model as a special case by setting $\kappa_{ij} \equiv \kappa$ and $\delta_{ij} \equiv \delta$. Moreover, it generalizes certain networks based on clustering nodes according to some notion of "similarity", as explained in the next subsection.

2.2.3 Multi-Type random networks

In general networks are complex objects and therefore difficult to analyze. However, it is often the case that vertices in a network can be classified to belong to certain *groups*. In a social network a natural classification of the vertices can be made according to criteria such as gender, income and age. In an industrial network it might be natural to group the vertices (the firms in the industry) according to their field of specialization or size. Indeed, a prevalent fact in social networks is the phenomenon of *homophily*, meaning that vertices of similar characteristics are more likely to be connected. Fienberg et al. (1985) introduced such *blockmodels* into the statistical literature of social networks. Recently, this type of networks has also been used in economic theory (Golub and Jackson, 2012), where it has been called a multi-type random network. The general network formation model can be used in a very simple way to construct multi-type random networks, as we would like to illustrate now. Suppose that the set of vertices can be partitioned into finitely many types $k \in \{1, \ldots, m\}$ of respective sizes N_k. The vector $\vec{N} = (N_1, \ldots, N_m)$ defines the partition of the population of nodes into its types. The number $N_k \in \{0, 1, \ldots, N\}$ is either deterministically given, or random. To exploit the group classification of the society, assume that the intensities of link formation and link destruction can be modeled by functions

$$w_{ij}(G) = (1 - A_{ij}(G))\kappa_{rl}, \text{ and } v_{ij}(G) = A_{ij}(G)\delta_{rl}$$

whenever vertex i is a member of group r, and vertex j is a member of group l for $1 \leq r, l \leq m$. This is readily seen to be a special case of the inhomogeneous random graph studied above. The edge-success probabilities between members of group r and group l are given by $p_{rl} = \frac{\kappa_{rl}}{\kappa_{rl}+\delta_{rl}}$. The nice feature of the multi-type random network is that it reduces the complexity of the random graph model tremendously. Compared to the inhomogeneous random graph model the multi-type random graph has the advantage that instead of computing (or estimating) an $N \times N$ matrix of edge-success probabilities, it suffices to find an appropriate partitioning of the vertices and then compute (or estimate) the edge-success probabilities across the various groups.

[11] See also Söderberg (2002) and Park and Newman (2004).

2.3 An outlook

Our proposed strategy to model dynamic network formation is rather parsimonious. The stochastic process is entirely specified by the attachment and the volatility mechanism. Particularly appealing variations are the inhomogeneous random graph model, and as a special case the multi-type random network. All these models rely on a particular choice of the intensities of link creation and link destruction, which leads us directly to the question how these intensities should be chosen. If we want a model-driven approach to dynamic network formation processes, then these intensities should be *derived* from an underlying model of network formation. This requires that we set-up a specific model specifying the *incentives* of the vertices in the graph to connect to each other. Economic and Game theoretic reasoning is natural for this task, which leads us directly to the next section, where some recent approaches to *strategic network formation* are discussed.

3. Game theoretic models of network evolution

The approach taken by the statistical models presented in the previous chapter describe *how* networks form and evolve from an observer's point of view. Thus, on the macro level these models give a good approximation of the likelihood with which a given network is observed. While answering the question of *how* network formation takes place, they do not explain *why* networks form and evolve. To access the *why* we have to go to the micro level and understand the forces that drive the nodes to connect to each other. In the context of economics we think of networks representing connections between economic agents. Economic agents are driven by incentives, hence they connect to each other because of payoff or utility resulting from these connections. Some examples of economic models where the payoff results from or is affected by the network itself are presented in Section 3.2. If payoff then depends on connections which result from all agents decisions, i.e. the whole network, then game theory seems to be an appropriate tool to model network formation. This chapter introduces central concepts of the game theoretic approach to model network formation.

3.1 Networks and Utilities

Suppose that agents have preferences solely over the set of possible networks. For each player, this preference ordering can be presented by a *utility function* $u_i : \mathcal{G}[N] \to \mathbb{R}$, with the standard assumptions on preference orderings.[12] By $u = (u_1, ..., u_N)$, we denote the

[12] In particular, completeness and transitivity.

profile of utility functions. Decisions to form or to sever links typically depend on marginal utility of adding/deleting links. Player i's *marginal utility of deleting* a set of links $l \subseteq E(G)$ *currently* in network G is thus given by $mu_i(G, l) := u_i(G) - u_i(G \ominus l)$. Analogously, we denote by $mu_i(G \oplus l, l) := u_i(G \oplus l) - u_i(G)$ the *marginal utility of adding* the set of *new* links $l \subseteq E(G^c \ominus G)$ to network G. A rather natural behavioral assumption is that if the marginal utility to player i of a given link is positive, then a player would like to form respectively keep the link and if negative, she would delete respectively not form the link. When studying network formation, externalities of own and other players' links on marginal utility play a central role. If externalities from own links are always positive (negative), a utility function is called *convex (concave) in own links*, i.e. $\forall G \in \mathcal{G}[N]$, $\forall l_i \subseteq E_i(G^c \ominus G)$, and $\forall ij \in E_i(G^c \ominus (G \oplus l_i)) : mu_i(G \oplus ij, ij) \leq (\geq) mu_i(G \oplus l_i \oplus ij, ij)$.[13] A weaker version is given by *ordinal convexity (concavity) in own links* which is satisfied by a utility function u_i if $\forall G \in \mathcal{G}[N]$, $\forall l_i \subseteq E_i(G^c \ominus G)$, and $\forall ij \in E_i(G^c \ominus (G \oplus l_i)) : mu_i(G \oplus ij, ij) \geq 0 \;\Rightarrow (\Leftarrow) mu_i(G \oplus l_i \oplus ij, ij) \geq 0$.[14] Similarly a utility function satisfies *strategic complements (substitutes)* if $\forall G \in \mathcal{G}[N]$, $\forall l_{-i} \subseteq E_{-i}(G^c \ominus G)$, and $\forall ij \in E_i(G^c \ominus G) : mu_i(G \oplus ij, ij) \leq (\geq) mu_i(G \oplus l_{-i} \oplus ij, ij)$. In a similar way as above this notion can be weakened to hold only in ordinal terms, i.e. a utility function satisfies *ordinal strategic complements (substitutes)* if $\forall G \in \mathcal{G}[N]$, $\forall l_{-i} \subseteq E_{-i}(G^c \ominus G)$, and $\forall ij \in E_i(G^c \ominus G) : mu_i(G \oplus ij, ij) \geq 0 \;\Rightarrow (\Leftarrow)\; mu_i(G \oplus l_{-i} \oplus ij, ij) \geq 0$.

3.2 Examples

We present here three basic examples of utility functions that depend only on the network itself. These may arise naturally (e.g. because players want to be as central as possible) or as a result from a multistage game which gives rise to such a utility function by backward induction.

3.2.1 The Connections Model

When agents derive utility solely from the network, one might think that agents want to be as central as possible. There are several prominent centrality measures, mainly introduced in the sociology literature.[15] Among those is closeness centrality which is a rather intuitive definition of centrality considering the distances from a given node in

[13] The definition is taken from Hellmann (2012). Other notions are introduced in Bloch and Jackson (2007) and Calvó-Armengol and Ilkiliç (2009). However, it is shown in Hellmann (2012) that all definitions are equivalent.

[14] The definition is again taken from Hellmann (2012). Calvó-Armengol and Ilkiliç (2009) introduce the concept of α−submodularity which is equivalent to ordinal convexity if it holds for all $\alpha \geq 0$.

[15] See, e.g. Freeman (1979) or Bonacich (1987) for an overview and introduction of some centrality measures in sociology.

the network to all the other nodes. Here, the distance between two nodes is simply the length of the shortest path connecting both nodes which is set to infinity if there exists no path. Centrality should then be decreasing in the distances. Jackson and Wolinsky (1996) introduce a model were players strive for closeness centrality but have to pay costs for each link they maintain. The notion of closeness that they use is often referred to as decay centrality, $Cl_i(G) = \sum_{j\neq i} \delta^{d_{ij}(G)}$ such that the distance between two nodes, $d_{ij}(G)$, is discounted by a $\delta \in (0, 1)$. The utility function for a player $i \in [N]$ when she strives for closeness but incurs cost of link formation c for each link is then given by

$$u_i^{Co}(G) = Cl_i(G) - |N_i(G)|c = \sum_{j\neq i} \delta^{d_{ij}(G)} - c\eta_i(G) \tag{4}$$

In Jackson and Wolinsky (1996) this is called the homogeneous connections model.[16] It can be shown that the homogeneous connections model satisfies concavity, but neither strategic complements nor strategic substitutes are satisfied (see e.g.Calvó-Armengol and Ilkiliç, 2009; Buechel and Hellmann, 2012).

3.2.2 Local Complementarities and Bonacich centrality

In a similar sense as the connections model of Jackson and Wolinsky (1996), one could also think about players striving for centrality with respect to other notions. Another prominent centrality notion is due to Bonacich (1987). It is similar to the closeness (decay) centrality, but instead of counting only paths between nodes all possible walks are considered while also discounted for length. Recalling that the ij-th entry of k-th power of the adjacency matrix, A^k_{ij}, represents the number of walks between i and j of length k, this so called Bonacich centrality is hence given by $b_i(G, \delta) = \left(\sum_{k\in\mathbb{N}} \delta^k A^k\right)\vec{1}$. The sum $\sum_{k\in\mathbb{N}} \delta^k A^k$ converges to $(I - \delta A)^{-1}$ if $0 < \delta < \lambda_1(A)^{-1}$, where $\lambda_1(A)$ is the Perron-Frobenius eigenvalue of A.[17] Thus, we can define a utility function were players strive for Bonacich centrality similarly to Jackson and Wolinsky (1996) by

$$u_i^{BC}(G) = b_i(G, \delta) - |N_i(G)|c = \left[(I - \delta A)^{-1}\vec{1}\right]_i - c\eta_i(G) \tag{5}$$

Another motivation for considering network formation according to Bonacich centrality is given in Ballester et al. (2006). Extending their approach to include network

[16] A more general functional form is presented in Jackson and Wolinsky (1996) such that the discount factor δ and the cost of linking c can be made player specific.

[17] Note that for infinitesimally $\delta > 0$ both the Bonacich centrality and the closeness centrality give infinitely more weight to nodes of smaller distance and is hence proportional to the (in-)degree centrality used in Section 2.2.1.

formation, König et al. (2012) consider a two stage game. In the first stage the network forms and in the second stage effort $x_i \in \mathbb{R}_+$ is chosen in a game of local complementarities where payoff is given by $\pi_i(x, G) := x_i + \delta \sum_{j=1}^{n} A_{ij} x_i x_j - \frac{1}{2} x_i^2$. Here the second stage payoff can be interpreted as benefits from own production and from cooperation with neighbors minus cost of exerting the effort. Ballester et al. (2006) show that the unique equilibrium in the second stage is given by $x_i^* = b_i(G, \delta)$. Thus, solving the first stage under assumption of play of the unique Nash equilibrium in the second stage and considering cost of link formation c obtains the utility function (5). One can easily derive that u^{BC} satisfies convexity and strategic complements.[18]

3.2.3 R&D Collaborations between Firms

Another example where payoff is not solely dependent on the network structure but can be reduced to that is presented in Goyal and Joshi (2003). In their setup, the economic agents are firms which produce a homogeneous product and compete in quantities on a single market. However, firms are also able to form bilateral R&D collaborations to lower their marginal cost of producing the output $mc_i(G) = \gamma_0 - \gamma \eta_i(G)$ with parameters $\gamma, \gamma_0 \in \mathbb{R}_+$ such that $\gamma < \frac{\gamma_0}{N-1}$. The model is a two stage game where in the first stage network formation takes place and in the second stage firms compete in the market. Assuming linear demand $P(q) = \max[0, \alpha - \sum_{j \in [N]} q_j]$ with $q \geq 0$ being total quantity and market size $\alpha > 0$, the payoff in the second stage is given by $\tilde{\pi}_i(q, G) = (\alpha - \sum_{j \in [N]} q_j) q_i - q_i mc_i(G)$. Play of Cournot equilibrium in the second stage implies that the problem in the first stage can be reduced to the following payoff function.

$$(6) \qquad u_i^{R\&D}(G) := \frac{\left((\alpha - \gamma_0) + N\gamma\eta_i(G) - \gamma \sum_{j \neq i} \eta_j(G)\right)^2}{(N+1)^2} - \eta_i(G)c.$$

Thus, for the first stage optimization problem of the firms, the payoff can be reduced to only depend on the network itself. It can be shown that this payoff function satisfies convexity and strategic complements (see e.g. Dawid and Hellmann (2012), where the network evolution of this R&D model is studied, presented in Section 3.6). Goyal and Moraga-Gonzalez (2001) extend this setup to a three stage game which includes choice of efforts devoted to R&D.

[18] Defining $\Delta(ij)$ to be the $n \times n$ matrix with $\Delta(ij)_{kl} = 1$ if $k, l \in \{i, j\}$, $k \neq l$ and $\Delta(ij)_{kl} = 0$ else, it is straightforward to see that u^{BC} satisfies convexity and strategic complements, since $[A + \Delta(ij)]^k - A^k \leq [A + \Delta(lm) + \Delta(ij)]^k - [A + \Delta(lm)]^k$ for all mutually distinct $i, j, l, m \in [N]$ and for all $k \in [N]$.

3.3 Network Formation Modeled as Non-cooperative Games

The first attempt to model network formation as a non-cooperative game is due to Aumann and Myerson (1988). The objective is to model network formation where payoff is given by the Myerson value. Aumann and Myerson (1988) propose an extensive form game of perfect information where pairs of players are given link formation opportunities according to an exogenously given order, called the *rule of order*. In order to form a link both players have to agree and once formed, a link can never be destroyed. The procedure is repeated until all remaining pairs reject the link formation opportunity. Each outcome is associated with a graph G which is evaluated according to the network utility function (see Section 3.1). It is straightforward to see that subgame perfect equilibria exist in this game. This game has found only few applications in the literature since the game lacks a behavioral motivation and subgame perfect equilibria are often hard to obtain analytically.[19]

A more natural approach is the Consent Game or Myerson game, introduced in Myerson (1991). In this model, network formation is formalized in terms of a simultaneous move game where players announce their desired links in a network. A link between two players is formed if and only if both players announce each other in their sets of desired links. Hence, a link requires the *consent* of both involved players to be formed. Formally, the game in normal form is given by $\Gamma^C = (N, S, \tilde{u})$ such that $S = S_1 \times \ldots S_n$ where $S_i = \{0, 1\}^{N \setminus \{i\}}$.[20] A link is formed if both involved players announce that they want to form that link, i.e. $s_{ij} = s_{ji} = 1$. This defines an outcome rule $\tilde{G}$ which maps strategies into networks, $\tilde{G} : S \to \mathcal{G}$, such that $ij \in E(\tilde{G}(s))$ if and only if $s_{ij} = s_{ji} = 1$. The *game form* $(N, S, \tilde{G})$ then gives rise to the *Consent Game* $\Gamma^C = (N, S, \tilde{u})$ since players have a preference ordering over the set of networks represented by a utility function (see Section 3.1) which makes it straightforward to define payoffs of the Consent Game by $\tilde{u}_i(s) := u_i(\tilde{G}(s))$.

Since this game is a game in normal form, the most natural equilibrium concept is that of Nash equilibrium. A network G^* is defined to be *Nash stable*, denoted by $NS(u)$, if it is supported by a Nash equilibrium in the Consent game, i.e. if there exists a strategy profile $s^* \in S$ such that $\tilde{G}(s^*) = G^*$ and $\tilde{u}_i(s_i, s^*_{-i}) \leq \tilde{u}_i(s^*)$ for all $i \in [N]$ and $s_i \in S_i$.

However, the concept of Nash stability has some drawbacks. It is straightforward to show that a network G is Nash stable if and only if

$$(7) \qquad u_i(G) \geq u_i(G \ominus l_i) \quad \text{for all } i \in N \text{ and } l_i \subset E_i(G).$$

[19] See also Van den Nouweland (2005).

[20] A strategy $s_i \in S_i$ can be interpreted in the following way: if $s_{ij} = 1$, then player i announces that she wants to have a link with player j, otherwise if $s_{ij} = 0$, then player j announces that she does not want to have a connection with j.

That is, each network where no player has an incentive to delete any subset of her links is Nash stable. In particular, the empty network is always Nash stable independently of the profile of utility functions.[21] Thus simple non–cooperative solution concepts cannot account for the bilateral nature of network formation. Therefore, in some works (see e.g. Dutta and Mutuswami, 1997, Dutta et al., 1998), equilibrium concepts are used that involve coalitional deviations such as *strong Nash equilibrium* (Aumann, 1959), *coalition-proof Nash equilibrium* (Bernheim et al., 1987), and also *undominated Nash equilibrium*. These concepts, however, involve high computational and analytical efforts since all possible deviations of sub–coalitions have to be computed.

3.4 Pairwise Stability and Refinements

The Consent Game resembles that players are in control of their links, i.e. each player can delete any set of links without consent of others, but to form a link any two involved players must agree. However, the nature of the non-cooperative solution concepts leads to unsatisfying stability concepts. To account for this cooperative feature of network formation Jackson and Wolinsky (1996) adapt a solution concept from the well established theory of matching.[22] Instead of modeling the game explicitly they rather define desired properties directly on the set of networks.

Definition 3.1. *[Jackson and Wolinsky (1996)] A network $G \in \mathcal{G}[N]$ is pairwise stable (PS) if*

i) for all $ij \in E(G) : mu_i(G, ij) \geq 0$ and

ii) for all $ij \in E(G^c) \setminus E(G) : mu_i(G \oplus ij, ij) > 0 \;\Rightarrow\; mu_j(G \oplus ij, ij) < 0$.

A network is thus pairwise stable, if *i*) no player has an incentive to delete *one* of her links, and *ii*) there does not exists a pair of players who want to form a mutually beneficial link. The definition of pairwise stability implicitly makes assumptions on the underlying rules of network formation. Let us define the sets of links which can be added, respectively deleted, according to this link formation rules. For $G \in \mathcal{G}[N]$ define

$$ADD(G) := \left\{ij \in E(G^c \ominus G) \mid mu_i(G \oplus ij, ij) > 0, mu_j(G \oplus ij, ij) \geq 0\right\}$$

[21] The reason is that the strategy profile $s^{\emptyset} \in S$, defined by $s^{\emptyset}_{ij} = 0$ for all $i, j \in N$ and $i \neq j$, always leads to the empty network even with one player deviating, i.e. $\tilde{G}(s_i, s^{\emptyset}_{-i}) = G^{\emptyset}$ for all $s_i \in S_i$ and $s^{\emptyset}_{-i} = (s^{\emptyset}_j)_{j \neq i}$. Same considerations hold for any network that satisfies (7).

[22] In matching theory known as stable matching, see e.g. Roth and Sotomayor (1990)

as the set of links that can be added to G, and define

$$DEL(G) := \left\{ ij \in E(G) \mid \exists k \in \{i, j\} : \; mu_k(G, ij) < 0 \right\}$$

as the set of links that can be deleted from G.

Although no actual game is modeled, the solution concept does reflect the intuition from the Consent Game: players are in control of their links (one at a time) and form a link only if both players benefit. The solution concept is static but can be motivated by evolutionary models of network formation (see Section 3.6). Pairwise stability is the most commonly used notion in the literature of network formation since it reflects naturally the bilateral nature of network formation and is easy to use, i.e. it does not require a high computational effort to calculate pairwise stable networks. However, it also has some limitations since it only considers one link at a time. In the definition of Nash stability, however, it is ruled out that players have an incentive to cut *a set* of own links. A natural refinement of both stability concepts is thus to consider the formation of mutually beneficial links as well as the deletion of more than one link at a time. This is expressed in the following notion of stability.

Definition 3.2. *[Jackson and Wolinsky (1996)] A network $G \in \mathcal{G}[N]$ is pairwise Nash stable (PNS) if*

i) G is Nash stable and

ii) for all $ij \in E(G^c) \setminus E(G) : mu_i(G \oplus ij, ij) > 0 \;\Rightarrow\; mu_j(G \oplus ij, ij) < 0$.

By (7), condition i) is simply equivalent to $u_i(G) \geq u_i(G \ominus l_i)$ for all $l_i \subset E(G)$. Thus, a network that is pairwise Nash stable is also robust against a deletion of a *set of links* by any player, while pairwise stability only considers one link at a time.

Let us denote by $NS(u)$ $PS(u)$, $PNS(u)$ the set of stable networks depending on the profile of utility functions u. From the definition of PNS it is clear that

$$(8) \qquad PNS(u) = NS(u) \cap PS(u),$$

which particularly implies that any PNS network is also NS and PS. While equivalence of NS and PNS can only happen in very special cases,[23] Calvó-Armengol and Ilkiliç (2009) present conditions such that the concepts PS and PNS coincide.

Proposition 3.3. *[Calvó-Armengol and Ilkiliç (2009)] $PS(u) = PNE(u)$ if u is ordinally concave.*

[23]For $NS(u) = PNS(u)$ we need to have that every network where no links can be deleted does not contain a link which is beneficial to both involved players. This is satisfied if e.g. only the empty network is PNS.

Intuitively, this can be obtained since each single link in $PS(u)$ is beneficial and by concavity stays beneficial when links are deleted. Thus, no set of links can be deleted implying that (7) holds and hence $PS(u) \subset NS(u)$ from which the statement follows by (8).

These introduced stability notions are very basic and most commonly notions of stability. Other refinements of the concepts include

- strong and weak stability (Dutta and Mutuswami, 1997),
- bilateral stability (Goyal and Vega-Redondo, 2007),
- pairwise stability with transfers (Bloch and Jackson, 2007),
- strict pairwise stability (Chakrabarti and Gilles, 2007),
- unilateral stability (Buskens and van de Rijt, 2008), and
- strict Nash stability (supported by strict Nash equilibrium).

3.5 Existence and Uniqueness of Stable Networks

Since the stability notions presented in Section 3.4 are widely used notions, general properties of stable networks like existence and uniqueness are of interest. These rather static properties are shown by means of network evolution in the form of improving paths (cf. Definition 3.5) which are possible paths of a best-response process (see Section 3.6).

One approach to prove existence of stable networks (in this case PS networks) is taken by Jackson and Watts (2001) by imposing a potential function and in a similar way Chakrabarti and Gilles (2007) show existence of a stronger stability notion. Potentials in games are introduced by Monderer and Shapley (1996).

Definition 3.4. *[Monderer and Shapley (1996)] Let $\Gamma = (N, S, \pi)$ be a game in strategic form. A function $\rho : S \to \mathbb{R}$ is an ordinal [exact] potential for Γ if for every $i \in N$, for every $s_{-i} \in S_{-i}$ and for all $s_i, \tilde{s}_i \in S_i$ it holds that:*

$$\pi_i(s_i, s_{-i}) - \pi_i(\tilde{s}_i, s_{-i}) > 0 \Leftrightarrow \rho(s_i, s_{-i}) - \rho(\tilde{s}_i, s_{-i}) > 0 \quad [\pi_i(s_i, s_{-i}) - \pi_i(\tilde{s}_i, s_{-i}) = \rho(s_i, s_{-i}) - \rho(\tilde{s}_i, s_{-i})]$$

Existence of a potential function in non-cooperative games rule out best response cycles. In a similar way cycling behavior in network formation can be ruled out. To define what we mean by cycling behavior consider the following definition due to Jackson and Watts (2001).

Definition 3.5. *An improving path from network G to network G' is a finite sequence of networks $(G_1, ..., G_K)$ such that $G_k \in \mathcal{G}[N]$ for all $k = 1, ..., K$, $G_1 = G$, $G_K = G'$, and for all $k = 1, ..., K-1$ it holds that either*

- $G_{k+1} = G_k \ominus ij$ *and* $ij \in DEL(G_k)$, *or*
- $G_{k+1} = G_k \oplus ij$ *and* $ij \in ADD(G_k)$.

Note that the definition of *improving paths* rests on the underlying rules of network formation that are assumed for pairwise stability. A link is destructed if there exists a player who wants to delete that link and a link is constructed if there exists a pair of players who want to form that link. Therefore it is clear that the set of improving paths emanating from a PS network is empty. An improving path can thus be viewed as network evolution since these are the possible paths of a best-response dynamics (see Section 3.6). Given the notion of improving paths, Jackson and Watts (2001) define an (improving) *cycle* C as an improving path $(G_1, ..., G_K)$ such that $G_1 = G_K$. Furthermore, a cycle C is called a *closed cycle*, if for all networks $G \in C$ there does not exists an improving path leading to a network $G' \notin C$. Concerning the existence of pairwise stable networks and closed cycles, Jackson and Watts (2001) get the following result.

Lemma 3.6. *[Jackson and Watts, 2001] For any profile of utility functions u, there exists at least one pairwise stable network or a closed cycle of networks.*

The idea to prove existence of a PS network is now to exclude the existence of cycles. For the result the following definition of no indifference is needed.

Definition 3.7. *[Jackson and Wolinsky (1996)] The utility function u_i of player i exhibits no indifference if for all $G \in \mathcal{G}[N]$ and for any link $ij \in E_i(G^c \ominus G)$ the following holds: $u_i(G) \neq u_i(G \oplus ij)$.*

As in non-cooperative games the existence of a potential for networks is sufficient to rule out exactly cycles, thereby giving a sufficient condition for the existence of PS networks. Jackson and Wolinsky (1996) impose the existence of a function w which is similar to that of a potential.

Proposition 3.8. *[Jackson and Watts (2001)] If there exists a function $w : \mathcal{G}[N] \mapsto \mathbb{R}$ such that $G' \in ADD(G) \cup DEL(G) \Leftrightarrow w(G') > w(G)\ \forall G, G'$ that are separated by one link, then there exist no cycles. If u exhibits no indifference then there exist no cycles only if such a function exists.*

The function w in Proposition 3.8 is similar to a potential function since Chakrabarti and Gilles (2007) show that existence of a function is implied by existence of what Chakrabarti and Gilles (2007) define as *ordinal network potential* which in turn is implied by existence of an ordinal potential of the Consent Game (cf. Definition 3.4). As an example, the connections model introduced in Section 3.2.1 admits such a function w only for certain values of c and δ. If e.g. $N(\delta - \delta^{N-1}) < c < \delta$ (which only holds for high values of δ when N is large) then it is easy to check that the function $w^{Co}(G) = \sum_{i \in [N]} u_i^{Co}(G)$ satisfies the condition of Proposition 3.8.[24]

Existence of an ordinal potential for the Consent Game particularly implies the existence of a network $G \in \mathcal{G}[N]$ which satisfies even stronger conditions, namely which is both PNS and such that there exists no link $ij \in G^c \ominus G$ with $mu_i(G \oplus ij, ij) + mu_j(G \oplus ij, ij)$. Chakrabarti and Gilles (2007) define these networks as *strictly pairwise stable*.

Proposition 3.9. *[Chakrabarti and Gilles (2007)] If the profile of utility functions u is such that the Consent Game admits an ordinal potential, then there exists a* strictly pairwise stable *network.*

The condition of existence of a function w (resp. of an ordinal potential) to show existence of PS networks is stronger than necessary since cycling behavior is ruled out completely. Moreover, it is not easy to check whether a utility function delivers such a function. Externality conditions on marginal utilities are, however, easy to check and seem to be very natural assumptions (cf. Section 3.2 for examples). The implications of these are studied in Hellmann (2012). Existence is guarenteed if the externalities are always positive and a kind of uniqueness is implied if externalities are negative.

Proposition 3.10. *[Hellmann (2012)] If a profile of utility functions $u = (u_1, ..., u_n)$ satisfies ordinal convexity in own links and the ordinal strategic complements property, then:*

(1) There does not exist a closed improving cycle.

(2) There exists a PNS (and hence also a PS) network.

The intuition for this result is that any link that is beneficial stays beneficial when other links are added. Thus, starting from the empty network, a link once improvingly added will never be deleted improvingly. Moreover, if every player has the same utility function, one can easily conclude that if the empty network is not P(N)S then the complete network is uniquely P(N)S (since then every link is beneficial and stays beneficial) and if the complete network is not P(N)S then the empty network is uniquely P(N)S. An example, where the conditions of Proposition 3.10 are satisfied, is given by u^{BC}, presented in Section 3.2.2, where players strive for Bonacich centrality. In that example, every player has the same

[24] For very low values of c such that any link is always desirable and for very high values of c, the function $w^{Co}(G)$ also satisfied the condition of Proposition 3.8.

utility function, and hence, either the empty network or the complete network is uniquely P(N)S or both are P(N)S.

When imposing instead negative externalities from links a more general kind of uniqueness result can be established.

Proposition 3.11. *[Hellmann (2012)] Let the profile of utility functions u satisfy ordinal concavity in own links, the ordinal strategic substitutes property, and no indifference. Then:*

(1) If G is PS, then for all $G' \in \mathcal{G}[N]$ such that $E(G') \subsetneq E(G)$ or $E(G') \supsetneq E(G)$ it holds that G' is not PS (and hence not PNS).

(2) If G^c or $G^\emptyset$ is PS, then there exists no other PS network.

The intuition here is that a link, if not liked by one player, will not be added when adding other links. Moreover, it is shown in Hellmann (2012) that if one can construct particular improving paths starting from the empty network then there exists a unique PS network.

Definition 3.12. *[Hellmann (2012)] An iterated elimination of dominated links addition path is an improving path $(G_1^+ ... G_K^+)$ with $G_1^+ = G^\emptyset$ (starting from the empty network) such that for all $k = 1, ..., K-1 : G_{k+1}^+ := G_k^+ \oplus ij$ if $ij \in ADD(G_k^+)$ and $mu_i(ADD(G_k^+) \oplus G_k^+, ij) \geq 0$ and $mu_j(ADD(G_k^+) \oplus G_k^+, ij) \geq 0$.*

An iterated elimination of dominated links deletion path is an improving path $(G_1^- ... G_L^-)$ with $G_1^- = G^c$ (starting from the complete network) such that for all $l = 1, ..., L-1 : G_{l+1}^- := G_l^- \ominus ij$ if $ij \in DEL(G_l^-)$ and there exists $\xi \in \{i, j\}$ such that $mu_\xi((G_l^- \ominus DEL(g_l^-)) \oplus ij, ij) \leq 0$.

An iterated elimination of dominated links addition path is hence a sequence of networks starting from the empty network, where links are added such that they stay beneficial to both involved players, even after all other possible candidates (the set $ADD(G)$) are added. These links must be contained in any PS network, thus implying the following result.

Proposition 3.13. *[Hellmann (2012)] If the profile of utility functions u satisfies ordinal concavity in own links, the ordinal strategic substitutes property and no indifference and if there exists an iterated elimination of dominated links addition path and an iterated elimination of dominated links deletion path terminating at the same network $G^* \in \mathcal{G}[N]$, then G^* is uniquely PS.*

The result also holds for PNS by Proposition 3.3. Examples of utility functions satisfying ordinal concavity and ordinal strategic substitutes are the models of Patent Races and Friendship by Goyal and Joshi (2006b), and the Free-Trade-Agreements-Model by Goyal and Joshi (2006a).

3.6 Evolution

The static approach in the definitions of stability implicitly assumes some underlying rules of network formation as argued above. Moreover by the construction of improving paths some evolution of networks is already modeled. These could be seen as a pure best response dynamic with discrete time $t \in \mathbb{N}$, where at each time step one link is selected randomly (according to some probability distribution with full support on $E(G^c)$) to be altered and that decision is made by the two adjacent players. If decisions are made myopically and optimally then the trajectories of such a dynamic process are exactly given by the improving paths in Definition 3.5. In that sense, the construction of the pairwise stable networks by improving paths in the existence results already implicitly model evolution of networks. In particular, the pairwise stable networks are then absorbing states of such a Markov process. Together with the closed improvement cycles they form the set of recurrent classes. Lemma 3.6 then follows easily since any finite Markov process must have at least one recurrent class.

Such a pure best–response process is explicitly studied in Watts (2001) for the connections model (cf. Section 3.2.1). While any improving path starting from the empty network terminates at trivial stable networks (empty network for high costs, complete for low costs), it is shown for intermediate costs that the likelihood of emergence the star network, which is among the PS networks for that cost range and efficient, decreases with the number of players N and goes to 0 as $N \to \infty$. The reason is that such a dynamics is path dependent and once two distinct pairs form a link (which is beneficial) the improving path will never lead to the star network.

To avoid path dependency Jackson and Watts (2002a) introduce perturbation in the decisions of players in the sense of Foster and Young (1990); Kandori et al. (1993); Young (1993). This is interpreted as making mistakes. Formally the timing of the process can be described as follows (where the first two are exactly the same as in Watts (2001) and in the definition of improving paths)[25]:

1. at each point in time $t \in \mathbb{N}$ a network $G_t \in \mathcal{G}[N]$ is given and one link is selected according to a probability distribution with full support on $E(G^c)$
2. the link is added (not deleted) if $ij \in ADD(G_t) \oplus (G^c \ominus DEL(G_t))$ while it is not added (respectively deleted) if $ij \in (G^c \ominus ADD(G_t)) \oplus DEL(G_t)$
3. with low probability $\varepsilon > 0$ the decision is reversed

[25] A continuous-time version of this process is presented in section 4.

For every $0 < \varepsilon < 1$ the such defined process defines an irreducible Markov chain on the set of networks $\mathcal{G}[N]$. It therefore has a unique invariant distribution μ^{ε}, which describes the probability with which a given network can be observed. Path dependency (in particular the starting network) does not influence this probability. The networks $G \in \mathcal{G}[N]$ such that $\lim_{\varepsilon \to 0} \mu^{\epsilon}(G) > 0$ are called *stochastically stable*. These are the networks which will be observed most of the time when the noise limit goes to 0. Naturally only networks contained in recurrent classes of the Markov process are candidates for stochastic stability. The concept of stochastic stability hence also presents an evolutionary selection mechanism among the stable networks (and closed cycles). Jackson and Watts (2002a) show that stochastically stable networks can be derived by counting mistakes involved in the transition from one recurrent class to another in a similar way as established already in game theory (see Young (1998); Sandholm (2010)). However, the construction of these trees for every recurrence class and calculation of minimal number of mistakes can be quite complex. Therefore Tercieux and Vannetelbosch (2006) present conditions for pairwise stable networks (or sets of networks) to be stochastically stable that are easy to check, called p-pairwise stability.

Dawid and Hellmann (2012) study the evolution of R&D collaborations in the model setting of Goyal and Joshi (2003)(see Section 3.2.3), using the dynamics of Jackson and Watts (2002a) to provide a selection of pairwise stable networks. It can be shown that any PS and PNS network is of dominant group architecture such that there is one completely connected group and all other players isolated. The sizes of the dominant group are sensitive to the cost of link formation, but there is no unique prediction with respect to the networks which will be observed. Moreover, the minimal size of the component in a non-empty network is increasing in the cost of link formation for a certain cost range, which is somehow counter–intuitive (cf. Goyal and Joshi (2003) and Dawid and Hellmann (2012)). When introducing the evolutionary process above it can be shown that the size of the dominant group in stochastically stable networks is generically unique and monotonically decreasing in cost of link formation. Further, there exists a lower bound on the group size of connected firms such that a non-empty network can be stochastically stable. Thus, introducing stochastic perturbations into the best–response dynamics leads to a selection of pairwise stable networks.

4. Co-evolutionary processes

We have now seen two distinct approaches to model network formation. Section 2 discussed the statistical/mathematical approach to network formation using random graph models. As emphasized in that section, the main focus in this literature is in defining dynamic processes of network formation in order to generate networks which display some desired stylized features observed in real-world graphs. Section 3 provided a different approach to network formation using game theoretic concepts and equilibrium reasoning to understand and describe observed networks. In the last section of this review article we describe a class of models which is able to combine these two elements. Following Staudigl (2010), we call this model class *co-evolutionary processes of networks and play*. To construct such a model we first have to define the class of strategic interactions we are interested in.

4.1 Games with local interaction structure

Recall that a normal form game is a tuple $\langle [N], (u_i)_{i\in[N]}, (S_i)_{i\in[N]}\rangle$, where

- $[N] = \{1, 2, \dots, N\}$ is the set of players,
- $u_i : \times_{i\in[N]} S_i \to \mathbb{R}$ is the (von Neumann-Morgenstern) utility function of the players, assigning a utility index $u_i(s_1, \dots, s_N)$ to each action profile $s := (s_1, \dots, s_N) \in S = \times_{i\in[N]} S_i$.

In this definition only implicitly the interaction structure of the players is captured. Actually it is hidden in the definition of the utility function u_i, by specifying the effect the action of player j has on player i. In order to make this dependency structure more explicit it is useful to separate these effects and define a preference relation directly on the product set of action profiles and interaction structures (i.e. networks). To do so, let us redefine the strategic interaction as a game with *local interaction structure*, following Morris (1997). We call a game with local interaction structure a tuple $\langle [N], (u_i)_{i\in[N]}, (S_i)_{i\in[N]}, \mathcal{G}[N]\rangle$. The difference between this definition and the classical definition given above is that we define the utility functions of players as mappings $u_i : S \times \mathcal{G}[N] \to \mathbb{R}$. This is a natural extension of the preference structure used in strategic network formation models, where the first factor of the product set (hence, the action profile) is essentially a dummy variable. In game theory, models with local interaction structure have a quite long tradition (early contributions are Blume, 1993, 1995; Ellison, 1993). In a co-evolutionary process of networks and play we think of a game with local interaction structure as a ”stage game”, and the dynamic interaction is repeated over time, allowing the players to change their actions, as well as to influence the local interaction structure. With this motivation in mind, let us now make the construction of a co-evolutionary process precise.

4.2 Co-evolution of networks and play

Starting from a given local interaction game $\langle [N], (u_i)_{i\in[N]}, (S_i)_{i\in[N]}, \mathcal{G}[N]\rangle$ we now construct an explicit dynamic model which includes a dynamic model of network formation as well as a dynamic model of action revision of the player. The model class we are presenting in this survey build on the following hypothesis:

(i) Agents are myopic in their linking and action decision (cf. Section 3.6);

(ii) The dynamic process is stationary.

Hypothesis (i) is a huge simplification and essentially boils down to the hypothesis that players base their decision's today only on the current information they have. They do not attempt to forecast the impact of their behavior on the future evolution of the state and its consequence on their own payoff. Such an assumption is standard practice in evolutionary game theory (Weibull, 1995; Hofbauer and Sigmund, 1998). Together with hypothesis (ii), it allows us to model a co-evolutionary process as a time-homogeneous Markov processes living on the product set $S\times\mathcal{G}[N]$. More formally, a *co-evolutionary model with noise* a family of continuous-time Markov jump processes $\{\{X^{\varepsilon}(t)\}_{t\in R_+}\}_{\varepsilon\in(0,\bar{\varepsilon}]}$, defined on the finite state space $\mathcal{X} = S \times \mathcal{G}[N]$, the set of pairs of *action profiles* and *networks*. A realization $\{X^{\varepsilon}(t) = x\}$ defines an action profile $\sigma(x) = (\sigma_i(x))_{i\in[N]} \in S$, and a network $\gamma(x) \in \mathcal{G}[N]$.[26] In concrete examples it is often more convenient to encode the network via its adjacency matrix. With a slight abuse of notation we denote the adjacency matrix of the network $\gamma(x)$ by $A(x)$.

Remark 4.1. The stochastic process X^{ε} is indexed by the parameter $\varepsilon > 0$. In the evolutionary literature this parameter often has the interpretation of behavioral noise parameter, which models some elements of deviations from behavior based on pure optimization (cf. section 3.6). For more details on stochastic models in evolutionary game theory see Young (1998) and Sandholm (2010).

We now turn the specification of the random process. Concrete examples are presented below. The Markov jump process is constructed from the following components:

Action adjustments: With constant rate 1 the action profile $\sigma(x)$ is allowed to change. Conditional on this event, player $i \in [N]$ is chosen uniformly at random to update his current action. If i gets a revision opportunity, he draws an action from the distribution $b_i^{\varepsilon}(\cdot|x) \in \Delta(S_i)$. The mapping $b_i^{\varepsilon} : \mathcal{X} \to \Delta(S_i)$ is called the *choice function* Hofbauer and Sandholm (2002) of player i. Choice functions are basic objects in game

[26] Hence, we think of the mappings σ and γ as the projection mappings onto their relevant factors.

theory, and we refer the reader to the book Sandholm (2010) for many examples of choice functions studied in the literature.

Link creation: With constant rate $\lambda \geq 0$ the process allows the network to expand. As in Section 2 we model this process by a matrix-valued function $W^{\varepsilon} : \mathcal{X} \to \mathbb{R}^{N\times N}$, whose elements are bounded non-negative functions $w^{\varepsilon}_{ij}(x)$, which specify the *intensity* of link creation given the current state $x \in \mathcal{X}$. For each $x \in \mathcal{X}$ and $\varepsilon > 0$ the matrix $W^{\varepsilon}(x)$ is symmetric, and called the *attachment mechanism* of the process.

Link destruction: With constant rate $\xi \geq 0$ a link becomes destroyed. The *intensity* with which the link between player i and j is destroyed is modeled by matrix-valued function $V^{\varepsilon} : \mathcal{X} \to \mathbb{R}^{N\times N}$, whose elements are bounded non-negative functions $v^{\varepsilon}_{ij}(x)$. For each $x \in \mathcal{X}$ and $\varepsilon > 0$ the matrix $V^{\varepsilon}(x)$ is symmetric, and called the *volatility mechanism* of the process.

This description can be summarized in terms of the generator of the process X^{ε}, which is a linear operator Q^{ε}, acting on bounded functions $f \in \mathbb{R}^{\mathcal{X}}$, specified as

$$Q^{\varepsilon}(x,x') = \begin{cases} b^{\varepsilon}_i(s_i|x) & \text{if } x' = ((s_i,\sigma_{-i}(x)),G(x)),\ s_i \neq \sigma_i(x), \\ \lambda w^{\varepsilon}_{ij}(x) & \text{if } x' = (\sigma(x),G(x)\oplus ij),\ j\neq i, \\ \xi v^{\varepsilon}_{ij}(x) & \text{if } x' = (\sigma(x),G(x)\ominus ij),\ j\neq i. \end{cases}$$

Here we have used the classical game theoretic notation for an action profile as $s = (s_i, s_{-i})$. Let us end the description of the process with some remarks.

Remark 4.2.

- A co-evolutionary process is general enough to cover most evolutionary models which have been studied in the literature. In particular, evolutionary learning models on fixed interaction structures are obtained by specializing the setting to $\lambda = \xi = 0$. In this case, we envision a finite set of players who interact with a subset of the player population and have to stick to one action in each individual interaction. Such models are very prominent in the local interaction literature (see e.g. Alós-Ferrer and Weidenholzer, 2007, and the literature mentioned in Section 4.1).

- From its very construction it is obvious that the process X^{ε} is indeed an extension of the random graph process of Section 2. The added feature is that the attachment and the volatility mechanism are now functions defined on the domain $\mathcal{X}$.

4.3 A micro-founded model for inhomogeneous random graphs.

As mentioned in the introductory section of this survey, the theory of random graphs provides in essence 2 classes of model: the "On-line" models frequently motivated by the preferential attachment model of Barabási and Albert (1999), and generalized random graphs (Newman, 2003, and Section 2.2 of this survey). An interesting and important question is now which random graph models are expected to appear in a co-evolutionary process. To give a characterization of the class of networks generated by co-evolutionary processes we would like to rely on our knowledge about general random graph models. In particular we would like to use the general characterization Theorem 2.2. To apply this theorem to the co-evolutionary process X^{ε}, we will need to make some assumptions, and introduce some more notation. First we have to develop notation to define a random graph process *conditional* on a fixed profile of actions $s \in S$. For a given profile $s \in S$, let us define the s-section of the state space $\mathcal{X}$ as the set $\mathcal{X}_s := \{s\}\times\mathcal{G}[N]$. An s-conditional random graph process is a continuous-time Markov process $\{\gamma^{\varepsilon}(t)\}_{t\geq 0}$ in $\mathcal{X}_s$ with attachment mechanism $W^{\varepsilon}|_{\mathcal{X}_s}$ and $V^{\varepsilon}|_{\mathcal{X}_s}$. Observe that the intensities can only vary with the network on the s-section. Our goal is to derive a *conditional random graph measure* $\mathsf{P}(\cdot|s)$, which is a random graph measure in the sense of random graph theory, but where we condition on the action profile used by the players.

We impose the following set of assumptions on the network formation mechanism:

Assumption 4.3. *The co-evolutionary process X^{ε} satisfies the following assumptions:*

(i) $\lambda, \xi > 0$;

(ii) If $A_{ij}(x) = 0$ and $\varepsilon > 0$, then $w^{\varepsilon}_{ij}(x) > 0$. If $A_{ij}(x) = 1$ or $i = j$, then $w^{\varepsilon}_{ij}(x) = 0$ for all ε;

(iii) For all pairs of players $i, j \in [N]$ and states $x \in \mathcal{X}$ we have $w^{\varepsilon}_{ij}(x) = \kappa^{\varepsilon}_{ij}(\sigma(x))(1 - A_{ij}(x))$, and $v^{\varepsilon}_{ij}(x) = \delta^{\varepsilon}_{ij}(\sigma(x))A_{ij}(x)$, where $\kappa^{\varepsilon}_{ij}$ and $\delta^{\varepsilon}_{ij}$ are positive functions.

Remark 4.4.

- The items listed under Assumption 4.3 are not minimal in order to be able to apply Theorem 2.2 to characterize the conditional random graph measure. Item (ii) is more restrictive than actually needed. The only requirement we need is that the generator of the conditional random graph process satisfies the necessary irreducibility assumptions in order to guarantee the existence of a unique invariant measure. However, item (iii) is necessary as will be seen in below.
- Our definition of a conditional random graph process is closely related to a random process with latent space variables (Hoff et al., 2002).

Item (iii) of Assumption 4.3 is arguably the most restrictive one. It requires that the intensities of link creation and destruction are functions only of the given action profile. One can imagine examples where this assumption makes sense, but it is clear that many examples will not fit this description. Nevertheless we have the following characterization result, which is a straightforward application of Theorem 2.2.

Proposition 4.5. *Consider a co-evolutionary process X^{ε} as defined above whose attachment and volatility mechanism satisfies assumption 4.3. Conditional on an action profile $a \in \mathcal{A}$, the conditioned random graph process $\{\gamma^{\varepsilon}(t)\}_{t\geq 0}$ has a unique invariant graph measure agreeing with the probability measure of an inhomogeneous random graph*

$$\mathsf{P}^{\varepsilon}(\{G\}|s) = \prod_{i=1}^{N} \prod_{j>i} (p_{ij}^{\varepsilon}(s))^{A_{ij}(G)} (1 - p_{ij}^{\varepsilon}(s))^{1-A_{ij}(G)},$$

where $p_{ij}^{\varepsilon}(s) := \frac{\kappa_{ij}^{\varepsilon}(s)}{\kappa_{ij}^{\varepsilon}(s)+\delta_{ij}^{\varepsilon}(s)}$ is the edge-success probability of vertex i and j.

4.4 Examples of co-evolutionary processes

In this section we are about to present some simple examples of co-evolutionary processes of networks and play. In these examples network formation is naturally coupled to an underlying interaction game. In particular, link creation and destruction are defined in such a way so that they reflect the incentives of the agents in a simple way. The examples we present differ in some important facts. The first model, which is a variation of the model presented in Jackson and Watts (2002b), uses the idea of pairwise stability (see 3.4) to construct a network formation model. The second example, which is based on Staudigl (2011, 2012), presents an evolutionary model of network formation which is more in the spirit of matching.

4.4.1 A co-evolutionary model based on Pairwise stability

Jackson and Watts (2002b) combines the network formation model of Jackson and Wolinsky (1996) (see Chapter 3.4) with the dynamic network formation model due to Watts (2001), presented in section 3.6. This paper takes the best-response with mutations model of Kandori et al. (1993) and Young (1993) as choice function. Let us introduce the model by Jackson and Watts (2002b) briefly. The local interaction game is a symmetric 2×2 coordination game. Hence $S_1 = S_2 = \{1, 2\}$. Assume that for each link a player has to pay a constant marginal cost $\phi > 0$ for each link. The utility function of player i is given by

$$u_i(s,G) = \sum_{j\neq i} \pi(s_i,s_j)A_{ij}(G) - \phi\eta_i(G),$$

where

$$\pi(1,s) = \begin{cases} a & \text{if } s = 1, \\ b & \text{if } s = 2 \end{cases} \quad \text{and } \pi(2,s) = \begin{cases} c & \text{if } s = 1, \\ d & \text{if } s = 2. \end{cases}$$

Note that, for a fixed profile of action a, the marginal utility of the link (i, j) for player i is given exactly by $mu_i(G(x) \oplus ij, ij) = \pi(\sigma_i(x), \sigma_j(x))$.
The co-evolutionary process is set up as follows:

Action adjustment: Assume that each player receives with uniform probability $1/N$ the opportunity to change his action. Conditional on this event he selects action $a \in \mathcal{A}$ with probability

$$b_i^{\varepsilon}(a|x) = \begin{cases} 1 - \frac{\varepsilon}{2} & \text{if } \sigma_i(x) \neq a \text{ and } \{a\} = \arg\max_{s'\in\{1,2\}} u_i((s', \sigma_{-i}(x)), G(x)), \\ 1 - \frac{\varepsilon}{2} & \text{if } \sigma_i(x) = a \text{ and } \{\sigma_i(x)\} = \arg\max_{s'\in\{1,2\}} u_i((s', \sigma_{-i}(x)), G(x)), \\ \frac{\varepsilon}{2} & \text{otherwise.} \end{cases}$$

This choice function says that a player abandons his currently used action with relatively high probability, if there exists a strictly better action. Otherwise he sticks to his action and switches only with the relatively small probability ε.

Link creation: With rate $\lambda > 0$ a link becomes created. Jackson and Watts (2002b) model network formation in the flavor of pairwise stability as discussed in 3. Using the notation of Section 3.4 let

$$ADD(x) = \{ij \in E(G^c \ominus G(x)) | \pi(\sigma_i(x), \sigma_j(x)) > \phi \text{ and } \pi(\sigma_j(x), \sigma_i(x)) \geq \phi\}$$

the set of links that are mutually profitable. Similarly we define

$$DEL(x) = \{ij \in E(G(x)) | \pi(\sigma_i(x), \sigma_j(x)) < \phi \text{ or } \pi(\sigma_j(x), \sigma_i(x)) < \phi\}.$$

Let $m(x) := |ADD(x)|$ the number of mutually profitable links and $d(x) := |E(G^c)| - |E(G(x))|$ the number of links that can be formed at $x \in \mathcal{X}$. Jackson and Watts (2002b) assume that a previously non-existing link becomes active with probability $1 - \varepsilon$ if both players mutually agree. With the small probability ε all links have a chance to be formed. The rate that a currently non-existing link ij will be added is

$$w^{\varepsilon}_{ij}(x) := \begin{cases} \frac{1-\varepsilon}{m(x)} + \frac{\varepsilon}{d(x)} & \text{if } ij \in ADD(x), \\ \frac{\varepsilon}{d(x)} & \text{otherwise.} \end{cases}$$

Link destruction: With rate $\xi > 0$ links become destroyed. Conditional on this event, pick one edge $ij \in E(G(x))$ uniformly at random and allow the incident players to re-evaluate the benefits arising from this connection. Denote by $\bar{m}(\omega) = |DEL(x)|$ the number of active links where at least one player benefits from the deletion of the link. If (i, j) is a link where at least one player is better off after its deletion it is assumed that with large probability $1 - \varepsilon$ it will be destroyed. With the small probability ε every link can be destroyed once it has been selected. This leads to the following version of the volatility mechanism:

$$\forall (i, j) \in E(G(x)) : v^{\varepsilon}_{i,j}(x) = \begin{cases} \frac{1-\varepsilon}{\bar{m}(x)} + \frac{\varepsilon}{|E(G((x))|} & \text{if } ij \in DEL(x), \\ \frac{\varepsilon}{|E(G((x))|} & \text{otherwise.} \end{cases}$$

Clearly this model is a version of a co-evolutionary process. The attachment and volatility mechanism depends however in a non-trivial way on the current network and the action chosen by the players, so that we cannot apply Proposition 4.5 to this model. However, this model can still be quite well understood in the extreme case where $\varepsilon \to 0$. Jackson and Watts (2002b) show that in this particular limit the random graph measure peaks out at the complete network, and a single strongly symmetric action profile, which depends on the parameters of the function π.

4.4.2 An analytically tractable model

In this section we present a co-evolutionary process of networks and play which is analytically tractable. This approach is due to Staudigl (2011) and Staudigl (2012). We sketch the model presented in Staudigl (2012), as it is closely related to the multi-type random network presented in Section 2.2.3.

Consider a society of N players, playing a game with local interaction structure specified as in section 4.1 as a tuple $\langle [N], (S_i)_{i\in[N]}, (u_i)_{i\in[N]}, \mathcal{G}[N] \rangle$. We specialize this model to a family of games in which the utility function of the players consists of two parts. The first component is a *common payoff* term, which one may think of as the externalities the players exert on each other. The second component is an *idiosyncratic payoff* term which depends on the player's own choice, but varies from player to player in a random way. Suppose that all players have the same action set, which we denote (with an abuse of notation) by S. The set of action profiles is in this section denoted by S^N and a typical element of this set is a N-tuple $\vec{s} := (s_i)_{i\in[N]}$. The common payoff term from bilateral interaction is modeled by a *reward function* $\pi : S \times S \to \mathbb{R}$. In many applications it is conceivable that the common payoff terms displays a strong symmetry property of the form

$$\pi(s, s') = \pi(s', s) \quad \forall s, s' \in S.$$

The idiosyncratic payoff component is captured by a function $\tau_i : S \to \mathbb{R}$, where τ_i is an element of the set of functions $\Theta = \{\theta_1, \dots, \theta_m | \theta_l : S \to \mathbb{R}\}$. We think of the idiosyncratic component in the player's utility function as the *type* of the player. It is a fixed attribute and therefore will be thought of as being a parameter of the co-evolutionary process. Given an action profile $\vec{s} \in S^N$ and a profile of types $\tau \in \Theta^N$, the (ex-post) payoff of player i is assumed to be

$$u_i(s, G, \tau_i) = \sum_{j\neq i} A_{ij}(G)\pi(s_i, s_j) + \tau_i(s_i).$$

Interaction games with such a partnership structure capture situations where all agents have the same reward function, and the payoff function of every player is the sum of all per-interaction rewards. However, having the partnership structure does not imply that all agents earn the same payoff in the interaction game since the interaction model will in general prescribe different interactions to different players.

The co-evolutionary process is specified by the following data.

Action adjustment: Agents use the logit-choice function (Blume, 1993) to choose actions. This choice function is defined as

$$(\forall s \in S) : b_i^{\varepsilon}(s|x, \tau_i) = \frac{\exp\left[u_i((s, \sigma_{-i}(x)), G(x), \tau_i)/\varepsilon\right]}{\sum_{s'\in S} \exp\left[u_i((s', \sigma_{-i}(x)), G(x), \tau_i)/\varepsilon\right]}.$$

The *rate* of the transition $x = (\vec{s}, G) \to x' = ((s, \sigma_{-i}(x)), G(x))$ is

$$Q^{\varepsilon,\tau}(x, x') = b_i^{\varepsilon}(s|x, \tau_i).$$

Link creation: The attachment mechanism will be specified as function of the types of the agents. Hence, as in section 2.2.3, we specify the attachment mechanism by a collection of functions $\{\kappa^{\varepsilon}(s,s')\}_{(s,s')\in S\times S}$ as

$$\kappa^{\varepsilon}(s,s') = \frac{2}{N}\exp(\pi(s,s')/\varepsilon) \quad \forall s,s' \in S.$$

The *rate* of a transition $x = (\vec{s}, G) \to x' = (\vec{s}, G \oplus ij)$ is then given by

$$Q^{\varepsilon,\tau}(x,x') = \lambda(1 - A_{ij}(x))\kappa^{\varepsilon}(\sigma_i(x), \sigma_j(x)).$$

Link destruction: The volatility mechanism is specified as

$$v^{\varepsilon}_{ij}(x) = \delta^{\varepsilon}_{\tau_i,\tau_j}$$

where $\delta^{\varepsilon}_{k,l} \equiv \delta^{\varepsilon}_{l,k}$ is the *volatility rate* of a link between a player of type k and a player of type l.The rate of the transition $x = (\vec{s}, G) \to x' = (\vec{s}, G(x) \ominus ij)$ is then given by

$$Q^{\varepsilon,\tau}(x,x') = \xi A_{ij}(x)\delta^{\varepsilon}_{\tau_i,\tau_j}.$$

As shown in Staudigl (2012) this model can be completely analyzed using elementary arguments. The (strong) assumptions making the model tractable are the strong symmetry of the reward function π and the particular specification of the volatility and attachment mechanism. Working with these assumptions allows us to write down a simple and nice formula for the (unique) invariant distribution of the co-evolutionary process.

Theorem 4.6 (Staudigl (2011),Staudigl (2012))**.** *The unique invariant distribution of the co-evolutionary process $\{X^{\varepsilon,\tau}_N(t)\}_{t\geq 0}$ is the Gibbs measure*

$$\mu^{\varepsilon,\tau}_N(x) = \frac{\exp(\varepsilon^{-1}H^{\varepsilon}_N(x,\tau))}{\sum_{x'\in\mathcal{X}}\exp(\varepsilon^{-1}H^{\varepsilon}_N(x',\tau))} = \frac{\mu^{\varepsilon,\tau}_{0,N}(x)\exp\left(\varepsilon^{-1}\rho(x,\tau)\right)}{\sum_{x'\in\mathcal{X}}\mu^{\varepsilon,\tau}_{0,N}(x')\exp\left(\varepsilon^{-1}\rho(x',\tau)\right)}, \tag{9}$$

where, for all $x = (s,G) \in \mathcal{X}$,

$$\begin{aligned} H^{\varepsilon}_N(x,\tau) &:= \rho(x,\tau) + \varepsilon\log\mu^{\varepsilon,\tau}_{0,N}(x), \\ \mu^{\varepsilon,\tau}_{0,N}(x) &:= \prod_{i=1}^{N}\prod_{j>i}\left(\frac{2}{N\delta^{\varepsilon}_{\tau_i,\tau_j}}\right)^{A_{ij}(G)}, \\ \rho(x,\tau) &:= \frac{1}{2}\sum_{j\neq i}A_{ij}(G)\pi(\sigma_i(x),\sigma_j(x)) + \sum_i \tau_i(s_i). \end{aligned}$$

The Gibbs measure (9) is defined by a function $\mu^{\varepsilon,\tau}_{0,N}$ capturing the effect of the network formation process, and a function depending on the potential function ρ, which can be interpreted as a welfare function.[27]

The invariant measure $\mu_N^{\varepsilon,\tau}$ assigns to each state $x \in \mathcal{X}$ some probability. We can use this measure to derive a random graph measure over the $\vec{s}$-section $\mathcal{X}_{\vec{s}}$. We have a complete characterization for this measure, which can be described as follows. The random graph process $\{\gamma^{\varepsilon,\tau}(t)\}_{t\geq 0}$ defined on $\mathcal{G}[N]$ for a fixed profile of actions $\vec{s} \in S^N$ can be formally identified with a birth-death process with "birth rates" of the link ij given by $2\exp(\pi(s_i, s_j)/\varepsilon)$, and "death-rates" $\delta^{\varepsilon}_{\tau_i,\tau_j}$. Let us introduce the rate ratio

$$\varphi^{\varepsilon}_{k,l}(s,s') = \frac{2\varepsilon \exp(\pi(s,s')/\varepsilon)}{\delta^{\varepsilon}_{k,l}},$$

for $s, s' \in S$ and $1 \leq k, l \leq m$. Staudigl (2012) goes on in proving the following result, which is now a simple application of Proposition 4.5.

Proposition 4.7. *Consider the random graph process described above. This process is ergodic with unique invariant graph measure*

$$\mathsf{P}^{\varepsilon,\tau}_N(G|\vec{s}) = \prod_{i=1}^{N} \prod_{j>i} p^{\varepsilon}_{ij}(\vec{s},\tau)^{A_{ij}(G)} \left(1 - p^{\varepsilon}_{ij}(\vec{s},\tau)\right)^{(1-A_{ij}(G))},$$

where the edge-success probabilities are defined as

$$p^{\varepsilon}_{ij}(\vec{s},\tau) = \frac{\varphi^{\varepsilon}_{k,l}(s,s')}{\varphi^{\varepsilon}_{k,l}(s,s') + N\varepsilon} \quad \text{if } s_i = s, s_j = s',\ \tau_i = \theta_k, \tau_j = \theta_l$$

for all $i, j \in [N]$.

Staudigl (2012) goes on in characterizing the supports of the invariant measure $\mu_N^{\varepsilon,\tau}$ as $\varepsilon \to 0$ (the so-called stochastically stable states mentioned in section 3.6), and also derives the large deviation rate function of this measure in the limit of large player sets, i.e. where $N \to \infty$.

5. Summary and suggestions for future research

As seen in this survey, the literature on the evolution of networks has a long tradition in various disciplines, and there are many models available which are able to catch many

[27] Definition 3.4 introduced potential games formally. The function ρ can be shown to be even an exact potential function. See Staudigl (2012) for the details.

stylized facts observed in real social networks. Due to its interdisciplinary character the literature on social and economic networks is enormous. For this reason we have focused in this survey on two particular useful approaches to model network formation. Models based on random graph theory are very useful in order to describe the dynamic evolution of networks. In these models the network structure is encoded in the random graph measure, which describes the likelihood that a certain network structure will be observed in the long run of the dynamic process.

Models using tools from game theory are good in describing networks as outcomes of a strategic interaction. The equilibrium concepts are inherently static, but can be motivated using evolutionary approaches such as myopic best-response processes which give rise to improving paths. Moreover, economic reasoning is useful in the modeling stage of an evolutionary model by setting bounds to what a ″natural″ network formation process should look like. For this very reason we have devoted section 3 to describe the most popular game theoretic concepts to study network formation. From the view of "pure" network evolution there are several areas that we think are important for future research. We presented first steps to show the relation between assumptions on the utility functions and the structure of stable networks, but there is room for improvement. General characterizations of stable networks are so far missing (e.g. necessary conditions for existence and uniqueness or general conditions for emergence of particular networks). In particular, due to analytical tractability, game theoretic models (whether static or evolutionary) usually predict very stylized network structures. An attempt to overcome this is presented in Section 4 where game theoretic modeling and the statistical network approach are combined. We still think that also pure game theoretic models can be used to recover empirically observed facts of networks which is an interesting object for future research.

Section 4 exploited game theoretic reasoning in defining a class of dynamic network formation models which admit a firm economic foundation. As these models become quickly very difficult to analyze, we have restricted the discussion in this survey to models based on very strong assumptions permitting a simple analytic treatment of the model. Relaxing these assumptions, without destroying tractability, is an important topic for future research. It seems to be likely that once the assumptions imposed on the co-evolutionary process are relaxed we cannot obtain as detailed results as, for instance, those found in section 4.4.2. For this reason different mathematical techniques will be necessary to extract information from the random process. Stochastic approximation theory (Kushner and Yin, 1997; Benaïm, 1999) seems to be a useful tool to obtain accurate information on the statistical properties of co-evolutionary models, at least in the limit of large networks. Indeed, a common practice in statistical physics is to analyze the stochastic network formation dynamics by relying on so-called ”mean-field” approximation, whose accuracy is compared to numerical experiments. Only rarely a theoretical justification for this common practice is given. Hence, it is important to work out the exact conditions under which ”mean-field” models are ”correct” approximations.

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