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EVOLUTION OF SZEKERES'S COSMOLOGICAL MODELS

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SUMMARY

The models are solutions of Einstein's equations for dust with no Killing vectors. They depend on four arbitrary functions of one variable, and generalize both the Friedmann models and those of Kantowski & Sachs. The possibilities of evolution are diverse. One result is that a Friedmann open model can evolve from a variety of initial states depending on three arbitrary functions of one variable.

1. INTRODUCTION

This work was stimulated by the following question: Could the Universe as we now see it have arisen from a wide variety of initial states, as has been suggested by Misner (1), or were the initial conditions severely restricted, as has been claimed by Collins & Hawking (2)?

In a previous paper (3) it was shown that isotropic homogeneous models (which represent the Universe now) can evolve from a variety of initial states, some of them anisotropic and inhomogeneous. The models used were spherically symmetric, and the freedom of initial conditions was that conferred by one arbitrary function of the radial variable.

Recently Szekeres (4) has discovered some inhomogeneous irrotational dust models, with no spatial symmetry, which depend on four arbitrary functions of one variable. In this paper we study the evolution of Szekeres's models, partly out of interest to see what can happen in these rather general solutions, and partly for their relevance to the evolution of our Universe. Regarding the latter, the conclusion can be stated briefly: they bear out the results of the spherically symmetric models.

The models are all solutions of Einstein's equations for dust:

$$G_{ik} = -8\pi\rho u_i u_k \tag{1.1}$$

where G_{ik} is the Einstein tensor, u_i is the unit four-velocity, and ρ is the density, and the units are such that both the velocity of light and the constant of gravitation are equal to one. The coordinates used are comoving so $u_i = \delta_i{}^4$. The models are given and classified in Section 2, which is followed in Section 3 by a description of their general properties, and in Section 4 by a detailed study of their evolution. There is a concluding Section 5, in which the main results are summarized in Table I.

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2. THE MODELS

The models are those given in Section 5 of Szekeres's paper. We write them in a different notation which is more convenient for our purposes, and take the opportunity to correct some minor errors in Szekeres's solutions.

The metric in all cases is

$$ds^2 = -Q^2\,dx^2 - R^2(dy^2 + h^2\,dz^2) + dt^2 \tag{2.1}$$

where

$$Q = AR + T, \quad A = A(x, y, z), \quad R = R(t), \quad T = T(x, t), \quad h = h(y). \tag{2.2}$$

R is called a Friedmann function and satisfies

$$2R\ddot{R} + \dot{R}^2 = -k \tag{2.3}$$

where dot means $\partial/\partial t$ and $k = 0, \pm 1$; h satisfies

$$\frac{d^2h}{dy^2} = -kh; \tag{2.4}$$

and the values of A and T are given below in the various cases. Because of (2.4) the 2-spaces x = const., t = const., are of constant curvature k. If $k = -1$ the three solutions of (2.4) given by $h = \cosh y$, $h = \sinh y$, $h = e^y$ appear to lead to globally distinct solutions of (1.1). However, it is known that all spaces of the same dimension and signature with the same constant curvature are locally equivalent. Here we confine ourselves mainly to a local point of view, and therefore consider only one of these possibilities, namely $h = \cosh y$.

The models are classified according to the value of k, and will be denoted by a letter P (parabolic, $k = 0$), H (hyperbolic, $k = -1$) and E (elliptic, $k = +1$), corresponding to the behaviour of the homogeneous models with these values of k. Throughout, Greek letters, except ρ, π, τ, will denote arbitrary functions of x.

Parabolic models $k = 0$, $h = 1$

$$A = \beta(y^2 + z^2) + \sigma y + \nu z + \omega.$$

Model PI. $R = t^{2/3}$, $T = \frac{9}{5}\beta t^{4/3} + \mu t^{-1/3}$, $(t > 0)$,

$$8\pi\rho = \frac{4A}{3QR^2}.$$

Model PII. $R = 1$, $T = \beta t^2 + \mu t$, $(t > t_0(x, y, z)$: $Q(t_0, x, y, z) = 0)$,

$$8\pi\rho = -\frac{4\beta}{Q}.$$

Hyperbolic models $k = -1$, $h = \cosh y$

$$A = (\sigma \cosh z + \nu \sinh z) \cosh y + \omega \sinh y.$$

Model HI. $R = 2K \sinh^2 u/2$, $t = K(\sinh u - u)$, $(K > 0)$, $(t > 0)$,

$$T = \beta\left(\frac{u}{2}\coth\frac{u}{2} - 1\right) + \mu \coth\frac{u}{2},$$

$$8\pi\rho = \frac{6KA + \beta}{QR^2}.$$

Model HII. $R = 2K\cosh^2 v/2$, $t = K(v+\sinh v)$, $(K>0)$, $(-\infty<t<\infty)$,

$$T = \beta\left(\frac{v}{2}\tanh\frac{v}{2}-1\right)+\mu\tanh\frac{v}{2},$$

$$8\pi\rho = \frac{\beta-6KA}{QR^2}.$$

Model HIII. $R = t$, $(t>0)$,

$$T = \beta\log t+\mu,$$

$$8\pi\rho = \frac{2\beta}{QR^2}.$$

Elliptic model $k = 1$, $h = \sin y$

Model EI

$$A = [(\sigma\cos z+\nu\sin z)\sin y+\omega\cos y]$$

$$R = 2K\sin^2\frac{w}{2},\quad t = K(w-\sin w),\quad (K>0),\quad (0<t<2\pi K),$$

$$T = \beta\left(\frac{w}{2}\cot\frac{w}{2}-1\right)+\mu\cot\frac{w}{2}$$

$$8\pi\rho = \frac{6KA-\beta}{QR^2}.$$

The ranges of t have (except for PII) been dictated by the zeros of the Friedmann function R. Choice of ranges of spacelike variables depends on topological considerations, but the one adopted here will be:

$$-\infty<x, y, z<\infty$$

in all models except EI for which either

$$0\leqslant x\leqslant 2\pi,\quad 0<y<\pi,\quad 0<z<\pi, \tag{2.5}$$

or

$$-\infty<x<\infty,\quad 0<y<\pi,\quad 0\leqslant z\leqslant 2\pi, \tag{2.6}$$

as will be explained in Section 4.

By a transformation in x, one of the five arbitrary functions β, μ, σ, ν, ω, can be given some suitable assigned value, say ± 1. The models therefore depend on *four* distinct arbitrary functions of one variable, and in the case of HI, HII, EI on one arbitrary constant K.

Apart from notation the foregoing solutions agree with those of Szekeres except in the following respects:

(i) in the hyperbolic and elliptic models, Szekeres finds five distinct arbitrary functions instead of four;

(ii) several of the expressions for the density are different;

(iii) probably owing to a misprint, an exponent 2 is missing from Szekeres's formula for T (in his notation μ) in PII.

Dr Szekeres has kindly informed us that he accepts the solutions in this paper as correct.

3. GENERAL DESCRIPTION OF THE MODELS

As already remarked, the functions $R(t)$ are the ones occurring in the Friedmann models. That every model in Section 2 contains a Friedmann model as a special case follows from a theorem of Szekeres (4, page 56), which states that an irrotational dust flow in comoving coordinates is a Friedmann cosmology if the spatial sections of the metric are conformally static. All the models of Section 2 are conformally static if the arbitrary functions β and μ are put zero. The 3-spaces t = constant are then spaces of constant curvature k. The Friedmann specializations are:

PI: Einstein–de Sitter;
PII: flat;
HI: hyperbolic open;
HII: hyperbolic open without big bang, but with negative density;
HIII: flat (Milne model);
EI: elliptic closed.

The shear of the models (5) is

$$s = \sqrt{\frac{2}{3}}\left|\frac{\dot{R}T - R\dot{T}}{R(AR+T)}\right|$$

and one can easily see, using Szekeres's theorem, that if s vanishes everywhere the model is a Friedmann one.

By a different set of specializations one obtains various homogeneous, anisotropic models. For instance, if in HI one puts

$$\sigma = \nu = \omega = 0, \quad \beta = \mu,$$

one obtains the open model 2(b) of Kantowski & Sachs (K–S). Similar specializations of HII and EI lead to models 2(a) and 1 of K–S. Since we want to study models as inhomogeneous as possible, we shall henceforth suppose that σ, ν and ω are not all zero. Singularities in the models occur where

$$R = 0 \quad \text{(a)}, \qquad Q = AR + T = 0 \quad \text{(b)}. \tag{3.2}$$

The singularities $R = 0$ occur on the hypersurface $t = 0$ and are familiar from cosmology. Those from $Q = 0$ will not usually occur on a hypersurface t = constant, but will admit great variety because Q contains four independent arbitrary functions of x. The situation is similar to, but more complicated than the spherically symmetric case discussed in (3), (7). (3.2) are strong singularities in the sense that the density becomes infinite at them. Granted suitable behaviour of the arbitrary functions, the models (except EI—see next section) appear to have no other physical singularities: the density and the physical components of the Riemann tensor are well behaved except at (3.2).

For the sake of definiteness, and to obtain positive density wherever possible, we shall suppose the arbitrary functions chosen, if possible, so that $Q > 0$ in the ranges of t given in Section 2. This amounts to the assumption that at each spatial point the singularity (3.2b) occurs before (3.2a) and so does not enter our models. If this assumption is not made, the initial behaviour of the models may be different from that described in Section 4.

In models PI, HI, EI, the density can be made positive throughout by appropriate choices of arbitrary functions. In PII, HII and HIII a section at least of the space time must contain negative density.

If none of the five arbitrary functions is zero the space times contain no Killing vectors; this will be proved in a later paper. In the general case, therefore, they do not reduce to Friedmann, Kantowski–Sachs, or Bianchi cosmologies.

4. EVOLUTION

The evolution depends on the two functions of x, β and μ. In all cases the 2-surfaces x = constant, t = constant follow the history determined by the function R, but the expansion in the x-direction is different. In this section, when we refer to models at $t = 0+$ or as $t \to \infty$, we shall always mean that only the leading terms are being considered.

Model PI. The function $R = t^{2/3}$ is that of the Einstein–de Sitter model.

$\mu = 0$

To obtain positive density when t is small we need $A/\mu > 0$ and we shall assume $A > 0$, $\mu > 0$. Then we can make $\mu = +1$ by a transformation of x. The metric at $t = 0+$ is

$$ds^2 = -t^{-2/3}\,dx^2 - t^{4/3}(dy^2 + dz^2) + dt^2. \tag{4.1}$$

This is a Kasner metric (**8**) which is empty, non-flat and anisotropic. The density at $t = 0+$ is

$$\rho = \frac{A}{6\pi t} \tag{4.2}$$

which is not homogeneous. This shows that during the early stages the density and expansion are unconnected. This is not a cause for surprise since it corresponds with the freedom to prescribe independently the initial density and velocity in a classical fluid flow.

Examining the situation as $t \to \infty$ and assuming $\beta > 0$ we find it more convenient to choose x so that $9\beta/5 = +1$. Then

$$ds^2 \sim -t^{8/3}\,dx^2 - t^{4/3}(dy^2 + dz^2) + dt^2, \tag{4.3}$$

which is a metric for a non-empty anisotropic universe. The actual density as $t \to \infty$

$$\rho = \frac{A}{6\pi t^{8/3}} \tag{4.4}$$

which is again inhomogeneous. This model is entirely different from the Einstein–de Sitter model.

However, if $\beta = 0$ the spatial sections t = constant are flat, and the density tends as $t \to \infty$ to that of the Einstein–de Sitter model. This case is interesting as an example of a space time evolving to a Friedmann cosmology from a beginning of extreme inhomogeneity and anisotropy.

$\mu = 0$, $\beta \neq 0$

The metric is

$$ds^2 = -t^{4/3}\left[\left(A + \frac{9\beta}{5}\,t^{2/3}\right)^2 dx^2 + dy^2 + dz^2\right] + dt^2 \tag{4.5}$$

and it might be thought from an application of Szekeres's theorem (see Section 3) that in the limit as $t \to 0$ this is equivalent to the Einstein–de Sitter model. However, Szekeres's theorem applies to exact solutions of (1.1) and great care has to be taken in using it in approximate form. In fact the shear in model (4.5) tends to infinity as $t \to 0$, and the sections t = constant are not flat, so, although the density at $t = 0+$ is $(6\pi t^2)^{-1}$, the model is not the Einstein–de Sitter model in this limit. As $t \to \infty$ the model follows (4.3) and (4.4), becoming inhomogeneous and anisotropic.

The summary of the cases of evolution for PI is:

$\mu \neq 0$, $\beta \neq 0$. Inhomogeneous anisotropic throughout;

$\mu \neq 0$, $\beta = 0$. Inhomogeneous anisotropic $\to$ Einstein–de Sitter;

$\mu = 0$, $\beta \neq 0$. Homogeneous anisotropic $\to$ inhomogeneous anisotropic.

Model PII. The model is remarkable in that there is no expansion at all in the y and z directions. However, inspection shows that, whatever the choice of arbitrary functions, at every value of t there is negative density in some region of the model. Therefore, for physical reasons we do not examine it further.

Model HI. The function R is the same as that in the hyperbolic Friedmann model. Since for large t, $T/R \to 0$ the asymptotic form of the density is

$$\rho = \rho_0 \left(1 + \frac{\beta}{6KA}\right), \tag{4.6}$$

where $\rho_0 = 3K/4\pi R^3$ is the density in the corresponding Friedmann model. Thus the final density is the Friedmann density multiplied by a static factor, non-uniform if $\beta \neq 0$. As $t \to \infty$ the metric approaches

$$ds^2 = -R^2(A^2\,dx^2 + dy^2 + \cosh^2 y\,dz^2) + dt^2. \tag{4.7}$$

The 3-space in the bracket is of constant curvature -1 so this is the metric of the hyperbolic Friedmann model. This final condition will be called *almost-Friedmann*. No trace of the initial conditions is to be found in the metric, but the model remembers the arbitrary functions (except μ) in the density. Physically, since the particles of the model have more than the escape velocity as $t \to \infty$, the density is unaffected by gravitation, and contains the frozen static factor in (4.6). A somewhat similar situation occurs in the spherically symmetric hyperbolic models (3).

For $t = 0+$ we find, making suitable transformations of x:

$\mu \neq 0$. The metric can be put in the approximate form

$$ds^2 = -\left(\frac{4K}{3t}\right)^{2/3} dx^2 - \left(\frac{9Kt^2}{2}\right)^{2/3} (dy^2 + \cosh^2 y\,dz^2) + dt^2, \tag{4.8}$$

which resembles the Kasner metric (4.1) in being anisotropic but differs in that it is not a solution of the vacuum equations. In fact it is a K–S universe, Case 2b, for small t. The initial density is

$$\rho = \frac{6A + \beta K^{-1}}{3t}. \tag{4.9}$$

$\mu = 0$, $\beta \neq 0$. The metric for small t is

$$ds^2 = -R^2 \left[\left(A + \frac{\beta}{6K}\right)^2 dx^2 + dy^2 + \cosh^2 y\,dz^2\right] + dt^2, \tag{4.10}$$

and the situation is somewhat similar to that arising from (4.5). The shear tends to infinity as $t \to 0$ and the spatial sections are not of constant curvature, although the density is $(6\pi t^2)^{-1}$. The model at $t = 0+$ is homogeneous but anisotropic.

The possibilities of evolution of HI may therefore be summarized as follows:

(i) $\mu \neq 0$, $\beta \neq 0$. Inhomogeneous K–S → almost-Friedmann hyperbolic;

(ii) $\mu \neq 0$, $\beta = 0$. Inhomogeneous K–S → Friedmann hyperbolic;

(iii) $\mu = 0$, $\beta \neq 0$. Homogeneous anisotropic → almost Friedmann hyperbolic.

Model HII. The function R has no singularities, and by a choice of arbitrary functions it is possible to arrange that $RA + T \neq 0$ so the model has no big bangs. However, it is easy to see that for every value of t the density is negative in some region of space.

Model HIII. A space–time closely resembling this was discovered by Stephani (9). If $\beta = 0$ the model is flat.

The metric evolves from an anisotropic homogeneous form

$$ds^2 = -(\log t)^2\, dx^2 - t^2(dy^2 + dz^2) + dt^2 \tag{4.11}$$

at $t = 0+$ to flatness as $t \to \infty$. The density is initially uniform and negative

$$\rho = (4\pi t^2 \log t)^{-1} \tag{4.12}$$

but eventually can become universally positive (subject to appropriate choice of arbitrary functions) and ultimately is approximately $2\beta/At^3$.

Model EI. It is necessary to consider the singularity at $y = 0, \pi$, where the determinant of the metric vanishes. The specialization

$$\sigma = \omega = \beta = \mu = 0, \quad \nu = 1$$

reduces the metric to

$$ds^2 = -R^2[dy^2 + \sin^2 y\, dz^2 + \sin^2 y \sin^2 z\, dx^2] + dt^2, \tag{4.13}$$

which is the closed Friedmann model. With this specialization, x, y, z are angular variables, which was the reason for the ranges chosen in (2.5). Reverting to the full metric for EI

$$ds^2 = -R^2(dy^2 + \sin^2 y\, dz^2) - \{R[(\sigma \cos z + \nu \sin z) \sin y + \omega \cos y] + T\}^2 \times dx^2 + dt^2, \tag{4.14}$$

and adopting a similar interpretation of the coordinates, we must treat $z = 0, \pi$ as a rotation axis. Consider now the circles $y = y_0$, $t = t_0$, $z = z_0$, $0 \leq x \leq 2\pi$, where y_0, t_0, z_0 are constant: the circumference of these must tend to zero as $z_0 \to 0, \pi$, whatever the values of y_0 and t_0, otherwise we must suppose there is a singularity on the $z = 0, \pi$ axis, i.e.

$$\mathrm{Lt}_{z_0 \to 0 \text{ or } \pi} \int_0^{2\pi} (AR + T)\, dx = 0, \quad (y = y_0,\ z = z_0,\ t = t_0).$$

From (4.14) this enforces

$$\sigma = \omega = \beta = \mu = 0$$

and reduces EI to the Friedmann metric (4.13) after a transformation of x. It therefore appears, with this interpretation of coordinates, that the general model EI contains line singularities.

We therefore adopt the ranges (2.6), since in this case $y = 0, \pi$ are nothing more than the usual coordinate singularities at the poles of a sphere. EI should now be looked upon not as a generalized Friedmann model but as a generalization of K–S, Case 1, to which it reduces if $\sigma = \nu = \omega = 0$ and β, μ are constants.

We can avoid big bangs in the period $0 < t < 2\pi K$ if we choose $\beta < 0$, $\mu > 0$ and $\pi\beta + \mu \leq 0$. At $t = 0+$ the model is very similar to HI and a detailed discussion need not be given. However, since it is of finite duration its later evolution is quite different. We consider it as $t \to 2\pi K$, and put

$$\tau = 2\pi K - t,$$

letting $\tau \to 0$ from above. There are two distinct cases:
$\pi\beta + \mu < 0$. After a transformation of x to make $\pi\beta + \mu = -1$, we find that the metric approaches

$$ds^2 = -\left(\frac{4K}{3\tau}\right)^{2/3} dx^2 - \left(\frac{9K\tau^2}{2}\right)^{2/3} (dy^2 + \sin^2 y\, dz^2) + dt^2,$$

and the density $(6A - \beta K^{-1})(24\pi\tau)^{-1}$, which is an anisotropic inhomogeneous condition.

Table I

Model	Friedmann type	Kantowski–Sachs (K–S) type	Sub-cases	Evolution From	Evolution To
PI	Einstein–de Sitter	—	$\mu \neq 0, \beta \neq 0$	Inhomogeneous Kasner	Inhomogeneous, anisotropic
			$\mu \neq 0, \beta = 0$	Inhomogeneous Kasner	Einstein–de Sitter
			$\mu = 0, \beta \neq 0$	Homogeneous anisotropic	Inhomogeneous anisotropic
PII	Flat	—		Contains negative density at all times	
HI	Hyperbolic open	2(b)	$\mu \neq 0, \beta \neq 0$	Inhomogeneous K–S	Almost-Friedmann hyperbolic
			$\mu \neq 0, \beta = 0$	Inhomogeneous K–S	Friedmann hyperbolic
			$\mu = 0, \beta \neq 0$	Homogeneous anisotropic	Almost-Friedmann hyperbolic
HII	Hyperbolic open with $R > 0$	2(a)		Contains negative density at all times	
HIII	Flat (Milne model)	—		Homogeneous anisotropic	Non-uniform density, flat metric
EI	Elliptic closed	1(a)	$\pi\beta + \mu < 0$	Inhomogeneous K–S	Inhomogeneous anisotropic
			$\pi\beta + \mu = 0$	Inhomogeneous K–S	Homogeneous anisotropic

$\pi\beta + \mu = 0$. The asymptotic form of the density is $(6\pi\tau^2)^{-1}$, but the shear tends to infinity, and the spatial sections are not of constant curvature so the model does not tend to a Friedmann space time, but to a homogeneous, anisotropic condition.

To sum up, the possibilities for EI under the assumptions $\beta < 0$, $\mu > 0$, $\pi\beta + \mu \leq 0$, are:

(i) $\pi\beta + \mu < 0$. Inhomogeneous anisotropic throughout, and

(ii) $\pi\beta + \mu = 0$. Inhomogeneous anisotropic $\rightarrow$ homogeneous anisotropic.

5. CONCLUSION

The results are summarized in Table I. It will be seen that the possibilities of evolution are quite diverse. As in the spherically symmetric case (3), a Friedmann open model can develop from a variety of initial states, depending on a choice of three independent arbitrary functions (i.e. three of σ, ν, ω, μ; the function β must be zero in both cases of evolution to a Friedmann model). This work is an advance on (3) in that the models considered here have no symmetry.

The most general space times studied here have four independent arbitrary functions of one variable. Though of considerable generality they represent very special cases of the general initial value problem. In this one should be able to prescribe two functions of three variables x, y, z corresponding to the initial values of ρ and $\partial\rho/\partial t$ on a hypersurface t = constant.

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