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By

E.G. Zweibel and A.H. Boozer

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PLASMA PHYSICS LABORATORY

PRINCETON UNIVERSITY
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EVOLUTION OF TWISTED MAGNETIC FIELDS

Ellen G. Zweibel[†]

and

Allen H. Boozer

Plasma Physics Laboratory, Princeton Univesity

P.O. Box 451

Princeton, New Jersey 08544

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ABSTRACT

The magnetic field of the solar corona evolves quasistatically in response to slowly changing photospheric boundary conditions. The magnetic topology is preserved by the low resistivity of the solar atmosphere. We show that a magnetic flux coordinate system simplifies the problem of calculating field evolution with invariant topology. As an example, we calculate the equilibrium of a thin magnetic flux tube with small twist per unit length.

[†]Permanent Address: Department of Astrophysical, Planetary, and Atmospheric Sciences, University of Colorado, Boulder, CO 80309

I. INTRODUCTION

It is generally accepted that the magnetic field of the solar corona consists of flux that has emerged from the solar interior, and remains connected to the subsurface field. Thus, the evolution of the coronal field and the photospheric field are strongly coupled together. Three reasonably well established properties of the solar atmosphere govern the nature of this coupling.

First, the inertia of the photospheric gas-field system much exceeds that of the coronal gas-field system. Photospheric dynamics are generally dominated by the fluid, which is capable of advecting the field. In the corona, the energy density in the field is greater than that in the ambient fluid, but is much lower than the photospheric energy density.

Second, typical velocities in the photosphere ($\lesssim 1$ km s^{-1}) are much smaller than the coronal Alfvén speed (several hundred km s^{-1}). Likewise, the Alfvén transit time (10 – 100 s) in a typical coronal structure is considerably shorter than the time scale for photospheric motions (*e.g.*, 10 minutes for granule turnover).

Third, the solar atmosphere is essentially perfectly conducting over the basic length scales and time scales of motion. The field can be treated as frozen into the fluid, except in the neighborhood of any current sheets that may be present.

These three properties, taken together, imply that the coronal magnetic field evolves quasistatically in response to photospheric motions, so that the field remains nearly in equilibrium as it adjusts to slowly changing boundary conditions. The surface conditions are determined by the dynamics of the interior, with negligible feedback from the corona. The successive equilibrium states that the field assumes are connected to each other by flux preserving motions, and all have the same magnetic topology. Finally, to the extent that the corona is magnetically dominated, the equilibrium states are force free.

The magnetic evolution problem is of major importance in understanding both the state of the quiet corona and eruptive phenomena such as flares and coronal transients. Parker (1983) has proposed that the random motion of magnetic field line endpoints leads to the formation of current sheets, in which enough energy is dissipated to heat the corona. It has been suggested by a number of authors (e.g., Low, 1977; Birn et al., 1978) that eruptive events are to be identified as points in which one parameter families of equilibria terminate or become singular.

The problem of calculating coronal magnetic field evolution with the field line endpoints specified, i.e., with invariant magnetic topology, was first posed by Sturrock and Woodbury (1967), who suggested that the field would evolve to an unstable, flaring state. The topological constraint leads to a difficult nonlinear problem, which has been reviewed recently by Low (1982). Numerical calculations of evolving field structure have been published by Sakurai (1979, 1981). Parker (1972, 1979) has suggested that the conditions for equilibrium are so exceptional that they are rarely satisfied, and that singular layers (or current sheets) develop in an evolving field.

In this paper, we formulate the magnetic field evolution problem in a magnetic flux coordinate system. The flux coordinates embody the invariant magnetic field topology, making the field line connectivity constraint easier to apply than in the usual Cartesian coordinate formulation. The problem is still nonlinear, but we derive and solve a linear equation that is valid for long, thin magnetic flux tubes with a small twist per unit length. A stability analysis suggests that the field becomes magnetohydrodynamically unstable when twisted by an amount that is small compared to the maximum twist that can be studied in linear theory. Therefore, linear theory appears adequate for the study of these models.

We introduce flux coordinates, pose the equilibrium problem, and derive the linear approximation in Sec. II. In Sec. III, we consider one example of a twisted field. We solve for the magnetic field, calculate the stored energy, and briefly discuss the MHD stability of the model. In Sec. IV, we give conclusions and a discussion.

II. MAGNETIC FLUX COORDINATES AND THE EQUILIBRIUM PROBLEM

A) Flux Coordinate System

Magnetic flux coordinates are widely used in fusion plasma physics because they have been found to simplify both equilibrium and stability calculations. Solov'ev and Shafranov (1970) and Bateman (1978) discuss flux coordinates. Boozer (1984) discusses magnetic field evolution and flux coordinates. We give a somewhat specialized but physically motivated development here, including all the properties of flux coordinates that are necessary for the calculations in this paper.

Consider an axisymmetric magnetic field

$$\vec{B} = \hat{r}B_r(r,z) + \hat{\theta}_c B_\theta(r,z) + \hat{z}B_z(r,z) \tag{1}$$

with B_z always positive. Here, θ_c denotes the usual cylindrical angle, to be distinguished from the magnetic angle θ introduced below. The vector potential $\vec{A}$ for $\vec{B}$ is not uniquely defined, because of freedom of gauge. However, because $\vec{B}$ is axisymmetric and $\vec{A}$ must be single valued, the component A_θ, which determines B_r and B_z, is unique:

$$B_r = -\frac{\partial A_\theta}{\partial z} \quad ; \quad B_z = \frac{1}{r}\frac{\partial}{\partial r} rA_\theta \quad .$$

Let $\psi \equiv rA_\theta$. The function $\psi(r,z)$ has two important properties. First, ψ is constant on magnetic field lines; $B \cdot \nabla\psi = 0$. Second, the magnetic flux through a circle of radius r at constant z is simply $2\pi\psi(r,z)$:

$$\int_0^r rdr \int_0^{2\pi} d\theta_c \; B_z = 2\pi\psi(r,z) \quad .$$

The equations for the magnetic field lines traced by the field given in Eq. (1) are

$$\frac{dr}{dz} = \frac{B_r}{B_z} \quad , \quad r\frac{d\theta_c}{dz} = \frac{B_\theta}{B_z} \quad ,$$

so the field can be written

$$\vec{B} = B_z \frac{\partial \vec{x}}{\partial z} \tag{2}$$

$$\vec{x} \equiv r\hat{r}(\theta_c) + z\hat{z} \quad ,$$

where the derivative is understood to be taken along the magnetic field line.

Now consider a set of coordinates ψ, θ, z. The function $\psi(r,z)$ is the flux function introduced above. The variable θ is a proper angle, meaning that the points θ and $\theta + 2\pi$ are identified, and can be written

$$\theta = \theta_c - \omega(r,z) \quad .$$

The variable z is the same as z in cylindrical coordinates.

We can express the magnetic field in flux coordinates in the following way. The gradient of any scalar function f can be written

$$\vec{\nabla} f = \left.\frac{\partial f}{\partial \psi}\right|_{\theta,z} \vec{\nabla}\psi + \left.\frac{\partial f}{\partial \theta}\right|_{\psi,z} \vec{\nabla}\theta + \left.\frac{\partial f}{\partial z}\right|_{\psi,\theta} \vec{\nabla} z \quad .$$

Taking the scalar product of $\vec{\nabla} f$ with $\vec{\nabla}\psi \times \vec{\nabla}\theta$ enables us to solve for $\partial f/\partial z|_{\psi,\theta}$:

$$\left.\frac{\partial f}{\partial z}\right|_{\psi,\theta} = \frac{\vec{\nabla} f \cdot \left(\vec{\nabla}\psi \times \vec{\nabla}\theta\right)}{\vec{\nabla} z \cdot \left(\vec{\nabla}\psi \times \vec{\nabla}\theta\right)} \quad . \tag{3}$$

The quantity $\vec{\nabla} z \cdot (\vec{\nabla}\psi \times \vec{\nabla}\theta)$ is simply the Jacobian determinant which transforms the cylindrical coordinate volume element into the flux coordinate volume element:

$$d\psi d\theta dz = r dr\, d\theta_c dz \;\; \vec{\nabla} z \cdot \left(\vec{\nabla}\psi \times \vec{\nabla}\theta\right) \quad .$$

In the present case

$$\vec{\nabla} z \cdot \left(\vec{\nabla}\psi \times \vec{\nabla}\theta\right) = \frac{1}{r}\frac{\partial \psi}{\partial r} = B_z \quad .$$

If we let f in Eq. (3) be any component of the position vector $\vec{X}$, we see that

$$\left.\frac{\partial \vec{X}}{\partial z}\right|_{\psi,\theta} = \frac{1}{B_z} \; \vec{\nabla}\psi \times \vec{\nabla}\theta \quad .$$

Therefore, according to Eq. (2)

$$\vec{B} = \frac{1}{J}\frac{\partial\vec{x}}{\partial z} = \vec{\nabla}\psi \times \vec{\nabla}\theta \tag{4}$$

$$J \equiv \frac{\partial\vec{x}}{\partial\psi} \cdot \left(\frac{\partial\vec{x}}{\partial\theta} \times \frac{\partial\vec{x}}{\partial z}\right) = r\,\frac{\partial r}{\partial\psi} \quad .$$

Equation (4) shows the important fact that θ as well as ψ is constant on magnetic field lines. We will refer to the ψ, θ, z coordinates as a magnetic coordinate system.

The advantage of a magnetic coordinate system becomes apparent when we consider the problem of specifying the endpoints of the magnetic field line. Imagine an initially uniform field $\hat{z}B$ in a perfectly conducting medium between two perfectly conducting surfaces at $z = \pm L/2$. Suppose that the field line endpoints at $z = L/2$ are rotated about some axis (taken to be $r = 0$) by an angle $\Delta\theta(r_o)$ and displaced radially by an amount $\Delta r(r_o)$, where r_o is the initial radius. The connectivity conditions expressed in cylindrical coordinates are given by the integrals along field lines

$$\Delta r(r_o) = \int_{-L/2}^{L/2} dz\, \frac{B_r(r,z)}{B_z(r,z)}$$

$$\Delta\theta(r_o) = \int_{-L/2}^{L/2} dz\, \frac{1}{r}\frac{B_\theta(r,z)}{B_z(r,z)} \quad ,$$

which the magnetic field must satisfy. In magnetic coordinates, we can write

$$r\left(\psi, \frac{L}{2}\right) = r\left(\psi, -\frac{L}{2}\right) + \Delta r(\psi)$$

$$\omega\left(\psi, \frac{L}{2}\right) = \omega\left(\psi, -\frac{L}{2}\right) + \Delta\theta(\psi) \tag{5}$$

because ψ and θ are constant along the field line. To discover the explicit transformation between the cylindrical and the magnetic coordinates, however, we must consider the equilibrium conditions.

B) Magnetostatic Equilibrium

We continue to consider the problem of calculating the magnetic field between two plates. Suppose the pressure of the ambient medium is negligible. Then, the equilibrium must be force free. It is well known (Kruskal and Kulsrud, 1958) that the equations of magnetostatic equilibrium *can be derived from a variational principle by extremizing the energy of the* system. We derive force-free equilibria when the only energy is magnetic

$$W = \int d^3x \, \frac{B^2}{8\pi} \quad .$$

The equilibria derived by varying W depend on the imposed constraints. If $\nabla \cdot B = 0$ is the only constraint, the minimum energy state is a potential field. If constancy of the magnetic helicity

$$K \equiv \int d^3x \, \vec{A} \cdot \vec{B}$$

is imposed as a constraint, Woltjer (1958) showed that the resulting minimum energy state has

$$\vec{\nabla} \times \vec{B} = \alpha \vec{B}$$

where α is constant. In our problem, we wish to constrain the magnetic topology by imposing the field line connectivity conditions. This can be done by finding the position vector $\vec{x}(\phi,\theta,z)$ which minimizes the magnetic energy and satisfies the boundary conditions at $z = \pm L/2$. We therefore use Eq. (4) to write the magnetic energy as

$$W = \int \frac{d\phi d\theta dz}{J} \left[1 + \left(\frac{\partial r}{\partial z}\right)^2 + r^2\left(\frac{\partial \omega}{\partial z}\right)^2\right] \quad . \tag{6}$$

We then vary W to solve for r and ω as functions of ϕ and z, imposing the boundary conditions (5). Once r and ω are known, we solve for $\vec{B}$, using Eq. (4). This magnetic field is the minimum energy state subject to the field line connectivity condition.

The Euler equations for r and ω obtained by varying W are

$$-\frac{\partial}{\partial \phi}\left[\frac{1+(\partial r/\partial z)^2+r^2(\partial \omega/\partial z)^2}{2r(\partial r/\partial \phi)^2}\right] + \frac{\partial}{\partial z}\frac{(\partial r/\partial z)}{r(\partial r/\partial \phi)} + \left[\frac{1+(\partial r/\partial z)^2-r^2(\partial \omega/\partial z)^2}{2r^2(\partial r/\partial \phi)}\right] = 0 \tag{7a}$$

$$\frac{\partial}{\partial z}\left(\frac{r\partial \omega/\partial z}{\partial r/\partial \phi}\right) = 0 \quad . \tag{7b}$$

Equation (7b) implies immediately that

$$\frac{r\ \partial \omega/\partial z}{\partial r/\partial \phi} = F(\phi) \tag{8}$$

In terms of $F(\phi)$, the Euler equation (7a) for r becomes

$$-\frac{\partial}{\partial \phi}\left[\frac{1+(\partial r/\partial z)^2}{2r(\partial r/\partial \phi)^2}\right] + \frac{\partial}{\partial z}\left[\frac{\partial r/\partial z}{r(\partial r/\partial \phi)}\right] + \frac{1+(\partial r/\partial z)^2}{2r^2(\partial r/\partial \phi)} = \frac{1}{2r}\frac{\partial F^2}{\partial \phi} \quad . \tag{9}$$

Equation (9) is a nonlinear partial differential equation for r, and, as such, is difficult to solve. We will therefore derive and solve a linearized version of Eq. (9) which allows us to study the equilibrium of magnetic flux tubes which are thin and which have a small twist per unit length.

C) Small Displacements of the Field lines

We assume that $r(\psi,z)$ can be written as

$$r = R + r_1(\psi,z) \; ; \quad r_1 << R$$
$$R = \left(\frac{2\psi}{B_o}\right)^{1/2} \qquad (10)$$

where B_o is a constant fiducial field strength that we will suppress to save writing. We first consider the function $F(\psi)$ defined in Eq. (8):

$$\int_{-L/2}^{L/2} dz \, \frac{F(\psi)}{r} \frac{\partial r}{\partial \psi} = \omega\left(\psi, \frac{L}{2}\right) - \omega\left(\psi \, , \, -\frac{L}{2}\right) = \Delta\theta(\psi) \quad , \qquad (11)$$

where Eq. (5) has been used, and the integral is evaluated at constant ψ. To lowest order in r_1,

$$F_o(\psi) = \frac{R^2}{L} \Delta\theta(R) \; . \qquad (12)$$

F_o is sufficient for the purpose of calculating r_1. However, we will need F to higher order when we calculate the energy of the distorted configuration. We write $F(\psi) = F_o(\psi) + F_1(\psi)$ and calculate F_1 by requiring that Eq. (11) be satisfied. This leads to

$$F_1(R) = -F_o(R)\, R\, \frac{\partial}{\partial R} \frac{1}{R} \int_{-L/2}^{L/2} dz\; r_1(R,z) \quad . \tag{13}$$

We now linearize Eq. (9). To first order in r_1 and its derivatives, we find that the linearized Euler equation for r_1 is

$$\frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial r_1}{\partial R} - \frac{r_1}{R^2} + \frac{\partial^2 r_1}{\partial z^2} = \frac{1}{2R^2}\frac{\partial F_o^2}{\partial R} \quad . \tag{14}$$

The solution of Eq. (14), subject to the boundary conditions

$$r_1\left(R, -\frac{L}{2}\right) = r_1^{(-)}(R)$$

$$r_1\left(R, \frac{L}{2}\right) = r_1^{(+)}(R)$$

is

$$r_1(R,z) = f(R) + \int_0^\infty kdk \int_0^\infty udu\; J_1(kR)J_1(ku)\Big[\big[r_1^{(-)}(u) + r_1^{(+)}(u)$$

$$- 2f(u)\big] \frac{\cosh kz}{2\cosh(kL/2)} - \big[r_1^{(-)}(u) - r_1^{(+)}(u)\big] \frac{\sinh kz}{2\sinh(kL/2)}\Big] \tag{15}$$

where

$$f(R) = \alpha R + \beta R^{-1} + \int^R ds\, \frac{1}{4s} \frac{\partial F_o^2}{\partial s} \left(\frac{R}{s} - \frac{s}{R}\right) \tag{16}$$

is a sum of the homogeneous and particular solutions to the equation

$$\frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial f}{\partial R} - \frac{f}{R^2} = \frac{1}{2R^2}\frac{\partial F^2}{\partial R} \quad .$$

The constants α and β are chosen to keep r_1 bounded as R approaches zero and infinity.

The magnetic field components, to first order in r_1, are

$$B_r = \frac{\partial r_1}{\partial z}$$

$$B_\theta = \frac{F_o}{R} \tag{17}$$

$$B_z = 1 - \frac{1}{R}\frac{\partial}{\partial R} R r_1 \quad .$$

Evidently, B_θ is of order $R\Delta\theta(R)/L$ relative to B_z, while B_r is of order $[\Delta\theta(r)]^2\ (R/L)^3$. Notice that the angle $\Delta\theta$ may itself be large as long as R/L, the ratio of the radius of the twisted region to its length, is small. Notice also that the twist per unit length is approximately constant.

It is interesting to see the extent to which the approximate solution (17) for $\vec{B}$ satisfies the force-free condition. We find that

$$\nabla \times B = -\hat{r}\,\frac{F_o}{R}\,\frac{\partial^2 r_1}{\partial R \partial z} + \hat{\theta}_c\,\frac{B_\theta}{R}\,\frac{\partial F_o}{\partial R} + \hat{z}\,\frac{B_z}{R}\,\frac{\partial F_o}{\partial R} \quad .$$

In order to make the radial component of $\vec{\nabla} \times \vec{B}$ proportional to B_r, we would have to work to higher order in r_1. Our primary interest here is in the shapes of the magnetic surfaces, and in the energy of the twisted configuration.

A similar expansion has been carried out by Sturrock and Uchida (1981) for a thin, twisted, force-free magnetic tube. Their solution satisfies $\vec{B} \times (\vec{\nabla} \times \vec{B}) = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$ up to terms of order R^2/L^2. In our formulation, $\nabla \cdot B = 0$ is satisfied automatically.

Two other comments on the validity of the linearization are in order.

The first is that the perturbed magnetic surfaces must not cross each other; i.e.,

$$R + r_1(R,z) = R + \delta R + r_1(R + \delta R,\ z)$$

is forbidden. To first order in δR, the condition that flux surfaces never cross is

$$\frac{\partial r_1}{\partial R} \neq -1 \quad .$$

The second validity condition is that $\partial r_1/\partial z < 1$ must hold. We will see in Sec. III that both these conditions limit the value of $\Delta\theta$ that can be studied within the linear theory.

III. A GAUSSIAN TWIST PROFILE

A) Magnetic Geometry

As an example of the foregoing techniques, we consider the effect of a pure twist with a Gaussian dependence on radius:

$$\Delta\theta(R) = \Delta\theta \exp\left(-R^2/2R_o^2\right)$$

$$r_1^{(+)}(R) = r_1^{(-)}(R) = 0$$

and we assume that $R_o/L \ll 1$. The Gaussian twist turns out to be analytically tractable and gives results which are qualitatively similar to those obtained with other twist profiles which are localized to a region of size R_o. The function F_o defined in Eq. (12) is

$$F_o(R) = RB_\theta = \Delta\theta \frac{R^2}{L} \exp\left(-R^2/2R_o^2\right)$$

and

$$r_1(R,z) = \frac{-(\Delta\theta)^2 R^2}{4L^2} \left[R \exp\left(\frac{-R^2}{R_o^2}\right) - \frac{R_o^4}{4} \int_o^\infty dk\, k^2 \exp\left(\frac{-k^2 R_o^2}{4}\right) \frac{\cosh kz}{\cosh(kL/2)}\right] ; \quad (18)$$

the integral over u in Eq. (15) can be done analytically.

Figure 1 is a plot of the flux surface contour r(z) for several values of R. The units are chosen such at $R_o/L = 0.05$ and $\theta_o = 12.7\ \pi$. This rather large value of $\Delta\theta$ was chosen to make the field line curvature visible.

Several interesting qualitative features appear in Fig. 1. First, the inner part (small R) of the field is pinched in by the twist. The inward displacement of the flux surfaces is nearly constant over most of the length of the field lines, with most of the curvature appearing at the ends. As expected, r_1 approaches zero as R approaches zero, and the maximum radial displacement of the field lines occurs for $R = R_o/\sqrt{2}$, $z = 0$, and has the value $0.107(\Delta\theta\ R_o/L)^2\ R_o$. The maximum value of B_θ occurs at $R = R_o$; $B_{\theta max} = 0.61(\Delta\theta\ R_o/L)B_o$.

For R greater than $R_o/\sqrt{2}$, the amplitude of r_1 decreases. B_r changes sign from positive to negative near the ends of the surface at $R \sim 1.7\ R_o$. For large R, r_1 is oscillatory, with an amplitude that decreases as $R^{-3/2}$.

The maximum value of $\partial r_1/\partial z$ is $3.97\ 10^4(\Delta\theta)^2(R_o/L)^6$ in this model, and occurs at $|z| = L/2$. Thus, for the value of $R_o/L = 0.05$ chosen here, $|\partial r_1/\partial z| < 1$ everywhere for $\Delta\theta < 12.8\ \pi$.

The maximum value of $\partial r_1/\partial R$ occurs at $R = 0$, $z = 0$, and is equal to $-(\Delta\theta R/2L)^2$. For $R/L = 0.05$, the flux surfaces do not cross if $\Delta\theta < 12.7\ \pi$.

Thus, the condition that flux surfaces do not cross is about as stringent as the condition that $|\partial r_1/\partial z| < 1$ in ensuring the validity of the linearization of Eq. (9). It is unclear at present whether the breakdown of the linear theory has any fundamental physical significance in terms of the properties of the real, nonlinear solutions.

B) Stored Magnetic Energy

We now consider the energy stored in the twisted field. Equation (6) must be expanded to second order in r_1, and F_1, the correction to F_o given in Eq. (13) must be included. The result is

$$\frac{W}{B_o^2} = \frac{1}{4}\int_o^\infty RdR \int_{-L/2}^{L/2} dz \left[1 + \frac{F_o^2}{R^2} + \frac{1}{R^2}\left(\frac{\partial}{\partial R} Rr_1\right)^2 + \left(\frac{\partial r_1}{\partial z}\right)^2 - \frac{F_o^2}{R}\frac{\partial}{\partial R}\frac{r_1}{R}\right] .$$

The first term of the integrand represents the unperturbed energy; W_o. The second term represents the effect of the twist; ΔW_T. The last three terms represent the effect of radial distortion; ΔW_R. For the Gaussian model, we find

$$\Delta W_T = B_o^2 \frac{\Delta\theta^2 R_o^4}{8L}$$

$$\Delta W_R = -B_o^2 \frac{\Delta\theta^4 R_o^6}{128L^3} = -\frac{\Delta W_T}{16}\left(\frac{\Delta\theta\ R_o}{L}\right)^2 \quad .$$

Thus, the radial distortion slightly lowers the stored magnetic energy below what it would be for a twisted tube with $B_r = 0$. Since $(\Delta\theta R_o/L)$ must be less than one for the linear theory to hold, the correction to the energy due to radial distortion is small compared to the twist energy in this Gaussian model.

c) Stability

We close this section with a few remarks on the stability of the twisted fields, deferring a detailed analysis to a future paper. There are two types of instability to consider; ideal MHD modes, which preserve the frozen flux condition, and resistive modes, which break the field lines. The ideal modes generally grow much more rapidly than the resistive modes, but the criteria for resistive instability are often milder than ideal instability criteria.

The theory of ideal MHD stability for infinitely long, cylindrical pinches was developed by Newcomb (1960) in a classic paper (see also Freidberg, 1982). Cylindrical pinches of finite length have been treated by Einaudi and van Hoven (1981). Neither analysis is strictly applicable to our problem, in which B_r is nonzero and the fields depend on z. Nevertheless, Newcomb's stability condition is of some interest, because B_r is small compared to B_z and is strongest near $|z| = L/2$, and we can take approximate account of the finite length of the structure by considering only axial wave numbers k with $kL > 2\pi$. We have applied Newcomb's procedure and find that the field becomes unstable when $\Delta\theta$ slightly exceeds 2π. Thus, the instability limit will lie well inside the range of $\Delta\theta$ for which the linearized equilibrium theory is valid. The most unstable modes are $m = 1$ kink modes, where $\exp(im\theta)$ is the angular dependence of the perturbation. On longer time scales, resistive instabilities may also be important.

The results of the stability analysis suggest that if the field is rotated beyond a critical angle of 2π near $r = 0$, the minimum energy state is a nonaxisymmetric one, in which the magnetic axis is wound into a helix instead of being straight (Taylor, 1975, Reiman, 1980).

IV. DISCUSSION AND CONCLUSIONS

We have considered the force-free equilibrium of the solar magnetic field as it evolves due to slowly changing photospheric boundary conditions. This problem was first posed in the solar context by Sturrock and Woodbury (1967), who pointed out the topological constraint on field line connectivity that holds in a perfectly conducting medium. The nonlinearity of the problem has prevented much progress toward its solution.

We formulated the problem in a flux coordinate system, taking as the basic geometry an axisymmetric cylinder which is twisted, dilated, or compressed at its ends. We showed that the flux coordinate system makes it possible to express the topological constraint in a simpler, more natural way than using cylindrical coordinates. The problem of solving for the force-free magnetic field is equivalent to solving for the transformation between flux coordinates and cylindrical coordinates.

The basic magnetostatic equations are nonlinear partial differential equations. We showed that the equilibrium of long thin magnetic flux tubes with a small twist per unit length can be studied by means of a linearized equation, which can be solved analytically up to a numerical integral. The magnetic field in the linear theory is divergence free, but force free only up to third order in the expansion parameter.

We gave an example of the equilibrium which results when the endpoints of the field lines are twisted according to a Gaussian profile. The twist per unit length is nearly uniform, and the field near the axis is pinched in. The radial component of the field is concentrated near the endpoints. The energy stored in the twisted field is enhanced by an amount of order $B_o^2(\Delta\theta)^2R_o^4/L$, where $\Delta\theta$ is the maximum twist angle, R_o the scale length for the twist, and L the separation between the boundaries. Our result is of the same order as that found by Sturrock and Uchida (1981).

A preliminary analysis suggests that the Gaussian twist field is unstable if the twist angle exceeds 2π. We conjecture that the minimum energy state of a field twisted beyond this critical value is nonaxisymmetric, and helical in form. The stability of the tubes to resistive modes is unresolved.

We suggest that the magnetic flux coordinate method will prove useful in calculating models of coronal magnetic field evolution due to photospheric motion. Such models provide insight into coronal structure and activity. We intend to pursue the stability of the solutions and the properties of nonlinear solutions in future papers.

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FIGURE CAPTION

FIG. 1. The flux surfaces $r(z)$ for a field with a Gaussian twist profile. The Gaussian scale length r_o is 0.05 L, the length of the tube, in this example, and $\Delta\Theta R_o/2L = 1$. Each curve is labeled by its position in the untwisted configuration.

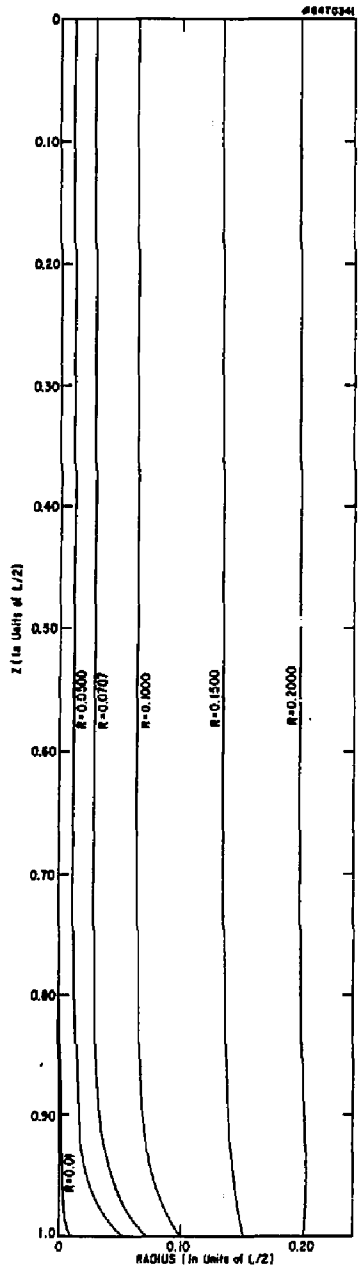
Z (In Units of L/2)
RADIUS (In Units of L/2)
R=0.01
R=0.0500
R=0.0707
R=0.1000
R=0.1500
R=0.2000
0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.0
0
0.10
0.20

Fig. 1

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