Czechoslovak Mathematical Journal

Enrico Obrecht

Evolution operators for higher order abstract parabolic equations

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 2, 210–222

Persistent URL: http://dml.cz/dmlcz/102085

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

EVOLUTION OPERATORS FOR HIGHER ORDER ABSTRACT PARABOLIC EQUATIONS

ENRICO OBRECHT, Bologna[1])

(Received April 27, 1984)

1. INTRODUCTION

In this paper, we show the existence of an evolution operator for a higher order abstract parabolic equation with variable coefficients. More precisely, we exhibit an operator – valued function $U(t, s)$ which solves the Cauchy problem

$$\sum_{k=0}^{n} A_k(t)\, \partial_t^k U(t, s) = 0\,, \quad (\partial_t^k U(t, s))_{t=s} = 0\,, \quad k = 0, \ldots, n-2\,,$$
$$(\partial_t^{n-1} U(t, s))_{t=s} = I\,.$$

The usefulness of evolution operators in the study of the Cauchy problem for a first order equation is well known in the parabolic as well in the hyperbolic case.

Higher order abstract differential equations have been largely studied in the last decade but almost always either in the constant coefficient case or for equations of a particular type (for detailed references, see [1], sect. 2.5. (c)). The techniques we use are similar to Tanabe's method, bur further technical difficulties are involved and some new phenomena arise. The evolution operator will be seen to have the form

$$U(t, s) = V_0(t, s) + \int_s^t V_0(t, \tau)\, R(\tau, s)\, d\tau\,, \tag{1.1}$$

where $V_0(\cdot, s)\, x$ is the solution of the constant coefficient Cauchy problem

$$\sum_{k=0}^{n} A_k(s)\, u^{(k)}(t) = 0\,, \quad s < t \leqq T\,,$$
$$u^{(h)}(s) = 0\,, \quad h = 0, \ldots, n-2\,, \quad u^{(n-1)}(s) = x\,,$$

while R solves a suitable Volterra integral equation.

Let us now give a plan of the paper. In section 2 we formulate our hypotheses and we state the main result. In section 3 we define the operators V_0 and R and study their properties. Section 4 is devoted to the integral operator appearing in (1.1), while in section 5 we show that (1.1) really defines an evolution operator. Finally

[1]) Partially supported by G.N.A.F.A. of C.N.R. Italy.

in section 6 we give some applications to Cauchy problems for abstract parabolic equations. Here we limit ourselves to situations where evolution operators are more directly applicable, deferring to a subsequent paper the consideration of more general Cauchy problems as well the problem of uniqueness of solutions.

2. HYPOTHESES AND THE MAIN RESULT

Let X be a complex Banach space, let n be a fixed positive integer and let T be a fixed positive real number.

For every $t \in [0, T]$ and for every $k \in \{0, 1, \ldots, n-1\}$, we are given a linear operator $A_k(t)$, whose domain and range are contained in X.

In the sequel we shall always suppose the following hypotheses hold true.

I) The operators $A_k(t)$ are densely defined and closed, $\forall t \in [0, T]$ and $\forall k \in \{0, 1, \ldots, n-1\}$; furthermore, the domain of $A_k(t)$ is independent of t (we shall denote it by $\mathscr{D}(A_k)$) and there exist $K_1, K_2 \in \mathbb{R}^+$, such that

$$K_1 \|A_k(t)\, x\| \leqq \|A_k(s)\, x\| \leqq K_2 \|A_k(t)\, x\|, \quad \forall s, t \in [0, T], \quad \forall x \in \mathscr{D}(A_k).$$

Let us set $A_n(t) = I$, $\forall t \in [0, T]$, and

$$P(\lambda, t) = \sum_{k=0}^{n} \lambda^k A_k(t), \quad \lambda \in \mathbb{C}, \quad t \in [0, T].$$

$P(\lambda, t)$ will be an operator in X with domain $\mathscr{D}(P) \equiv \bigcap_{k=0}^{n-1} \mathscr{D}(A_k)$. We shall say that $\lambda \in \mathbb{C}$ belongs to the resolvent set of the operator pencil $P(\lambda, t)$ (and we write $\lambda \in \varrho(P(\cdot, t))$), if and only if the linear operator $P(\lambda, t)$ has an everywhere defined bounded inverse, which we shall denote by $P^{-1}(\lambda, t)$.

II) There exist $M_1 \in \mathbb{R}^+$ and $\theta \in]\pi/2, \pi[$, such that:

a) $S_\theta = \{z \in \mathbb{C};\ |\arg z| \leqq \theta\} \cup \{0\} \subset \varrho(P(\cdot, t))$, $\forall t \in [0, T]$;

b) $\|A_k(t)\, P^{-1}(\lambda, t)\| \leqq M_1(|\lambda| + 1)^{-k}$, $\forall \lambda \in S_\theta$, $\forall k \in \{0, \ldots, n\}$, $\forall t \in [0, T]$.

III) There exist $M_2 \in \mathbb{R}^+$ and $\alpha \in]0, 1]$, such that:

$$\|(A_k(t) - A_k(s))\, P^{-1}(\lambda, s)\| \leqq M_2 |t - s|^\alpha (|\lambda| + 1)^{-k},$$
$$\forall \lambda \in S_\theta, \quad \forall k \in \{0, \ldots, n-1\}, \quad \forall s, t \in [0, T].$$

If $n = 1$, Hypotheses I)–III) above coincide with the well known assumptions of Sobolevskiĭ [4] and Tanabe [5], assuring the existence of an evolution operator.

The main result of this paper is the following one.

Theorem. *Suppose Hypotheses* I)–III) *hold true. Then,* $\forall (t, s) \in \Delta = \{(\tau, \sigma) \in \mathbb{R}^2;\ 0 \leqq \sigma < t \leqq T\}$, *there exists a bounded linear operator* $U(t, s)$, *such that*:

i) $U \in \mathscr{C}(\Delta; \mathscr{L}(X))$;

ii) *U is partially differentiable with respect to t $n - 1$ times in the uniform operator topology and n times in the strong operator topology*;

iii) $\forall x \in X$, $\partial_t^k U(t, s)\, x \in \mathscr{D}(A_k)$ *and the functions* $(t, s) \to A_k(t)\, \partial_t^k U(t, s)\, x$ *are continuous in* Δ, *if* $k \in \{0, \dots, n\}$;

iv) $\forall x \in X$ *and* $\forall (t, s) \in \Delta$,

$$\sum_{k=0}^{n} A_k(t)\, \partial_t^k U(t, s)\, x = 0\,;$$

v) $\forall k \in \{0, \dots, n-2\}$, $\partial_t^k U(t, s) \to 0$, as $(t, s) \to (s_0, s_0)$, *in the uniform operator topology, while*

$$\partial_t^{n-1} U(t, s) \to I\,, \quad \textit{as} \quad (t, s) \to (s_0, s_0)\,,$$

strongly.

Such an operator will be called an *evolution operator for the equation*

$$\sum_{k=0}^{n} A_k(t)\, u^{(k)} = 0\,.$$

3. THE OPERATORS $V_0(t, s)$ AND $R(t, s)$

Let $\theta_0 \in]\pi/2, \theta[$ be fixed. We shall always denote by Γ the curve in the complex plane formed up by the two half-lines $\{\varrho \exp(\pm i\theta_0);\ \varrho \geqq 0\}$ and oriented in such a way that the imaginary part of a complex number increases as it moves along Γ.

Definition 3.1. Let us set:

$$V_p(t, s) = (2\pi i)^{-1} \int_\Gamma \lambda^p \exp[\lambda(t - s)]\, P^{-1}(\lambda, s)\, d\lambda\,,$$

where p is any nonnegative integer, if $(t, s) \in \Delta$, while $p \in \{0, \dots, n-2\}$, if $(t, s) \in \bar{\Delta}$; $V_{n-1}(t, t) = I$, $\forall t \in [0, T]$.

The following Proposition exhibits all required properties of the operators V_p.

Proposition 3.2.

a) $V_p(t, t) = 0$, $\forall t \in [0, T]$ *and* $\forall p \in \{0, \dots, n-2\}$;

b) $V_p(t, s) \in \mathscr{L}(X)$, $\forall (t, s) \in \Delta$ *and* $\forall p \in \mathbb{Z}_+ \cup \{0\}$;

c) $V_p(t, s) \in \mathscr{L}(X)$, $\forall (t, s) \in \bar{\Delta}$ *and* $\forall p \in \{0, \dots, n-1\}$;

d) $V_p \in \mathscr{C}(\Delta; \mathscr{L}(X))$, $\forall p \in \mathbb{Z}_+ \cup \{0\}$;

e) $\forall p \in \mathbb{Z}_+ \cup \{0\}$ *and* $\forall k \in \{0, \dots, n\}$, *there exist* $C_{p,k} \in \mathbb{R}^+$, *such that*

$$\|A_k(\tau)\, V_p(t, s)\| \leqq C_{p,k}(t - s)^{k-p-1}\,, \quad \forall (t, s) \in \Delta\,, \quad \forall \tau \in [0, T]\,,$$

$$\|(A_k(\tau) - A_k(\sigma))\, V_p(t, s)\| \leqq C_{p,k}(t - s)^{k-p-1}(\tau - \sigma)^\alpha\,, \quad \forall (t, s) \in \Delta\,,$$

$$\forall \tau, \sigma \in [0, T]\,;$$

f) $V_p \in \mathscr{C}(\bar{\Delta}; \mathscr{L}(X))$, $\forall p \in \{0, \dots, n-2\}$;

g) V_{n-1} *is strongly continuous in* $\bar{\Delta}$;

h) $\partial_t^p V_0(t, s) = V_p(t, s)$ *in the uniform operator topology*, $\forall (t, s) \in \Delta$ *and* $\forall p \in N$; *if* $p \leqq n - 2$, *the assertion holds true* $\forall (t, s) \in \bar{\Delta}$;

i) $(\partial_t^{n-1} V_0(t, s)\, x)_{t=s} = x$, $\forall x \in X$, $\forall s \in [0, T[$;

j) $V_p(t, s)\, x \in \mathscr{D}(P)$, $\forall x \in X$, $\forall (t, s) \in \Delta$ *and* $\forall p \in \mathbb{Z}_+ \cup \{0\}$; *if* $p \leqq n - 2$, *the assertion holds true* $\forall (t, s) \in \bar{\Delta}$;

k) *the function* $(\tau, t, s) \to A_k(\tau)\, V_p(t, s)$ *is continuous in* $[0, T] \times \Delta$ *in the uniform operator topology* $\forall k \in \{0, 1, \ldots, n - 1\}$ *and* $\forall p \in \mathbb{Z}_+ \cup \{0\}$; *the same assertion holds true in* $[0, T] \times \bar{\Delta}$ *if* $k \geqq p + 2$; *furthermore, the function* $(\tau, t, s) \to \to A_{p+1}(\tau)\, V_p(t, s)$ *is strongly continuous in* $[0, T] \times \bar{\Delta}$, *if* $p \in \{0, \ldots, n - 2\}$.

l) $\forall x \in X$, $V_{n-1}(t, t - \varepsilon)\, x \to x$, *as* $\varepsilon \to 0+$, *uniformly on compact subsets of* $]0, T]$.

Proof. a) follows straightforwardly by Cauchy's theorem, if we keep in mind the growth estimate of $P^{-1}(\lambda, t)$.

b)–c)–d) are obvious.

e) We have:

$$A_k(\tau)\, V_p(t, s) = (2\pi i)^{-1} \int_\Gamma \lambda^p \exp[\lambda(t - s)]\, A_k(\tau)\, P^{-1}(\lambda, \tau)\, P(\lambda, \tau)\, P^{-1}(\lambda, s)\, d\lambda .$$

Since $P(\lambda, \tau)\, P^{-1}(\lambda, s) = \sum_{k=0}^{n-1} \lambda^k (A_k(\tau) - A_k(s))\, P^{-1}(\lambda, s) + I$, we get that $P(\lambda, \tau)\, P^{-1}(\lambda, s)$ is uniformly bounded.

If we now repeat the argument in the proof of Lemma 6 in [3], we get the first assertion. The second one is proved quite analogously.

f) By d) we need to prove the assertion only at points (t, t), where $t \in [0, T]$. Let $(\tau, \sigma) \in \bar{\Delta}$; if $\tau = \sigma$, we have $V_p(\tau, \tau) = 0$, by a), while, if $\sigma < \tau$, by e) we get $\|V_p(\tau, \sigma)\| \leqq C_{p,n}(\tau - \sigma)^{n-p-1} \to 0$, as $(\tau, \sigma) \to (t, t)$.

g) By d), we need to prove the assertion only at points (t, t), where $t \in [0, T]$. Let $(\tau, \sigma) \in \bar{\Delta}$. If $\sigma = \tau$, then $V_{n-1}(\tau, \tau)\, x = x$, $\forall x \in X$. If $t = \sigma < \tau$, the proof is the same as that in Corollary 3 in [3].

Let, finally, be $\sigma < \tau$, $\sigma \neq t$; if we suppose $x \in \mathscr{D}(P)$, we get:

$$V_{n-1}(\tau, \sigma)\, x - x = (2\pi i)^{-1} \int_\Gamma \lambda^{-1} \exp(\lambda(\tau - \sigma))\, (\lambda^n P^{-1}(\lambda, \sigma)\, x - x)\, d\lambda =$$

$$= -(2\pi i)^{-1} \sum_{k=0}^{n-1} \int_\Gamma \lambda^{k-1} \exp(\lambda(\tau - \sigma))\, P^{-1}(\lambda, \sigma)\, A_k(\sigma)\, x\, d\lambda .$$

Hence, by e) and Hypothesis I),

$$\|V_{n-1}(\tau, \sigma)\, x - x\| \leqq C_1(\tau - \sigma) \sum_{k=0}^{n-1} \|A_k(\sigma)\, x\| \leqq$$

$$\leqq C_2 \sum_{k=0}^{n-1} \|A_k(0)\, x\|\, (\tau - \sigma) \to 0 , \quad \text{as} \quad (\tau, \sigma) \to (t, t) .$$

As $V_{n-1}(\tau, \sigma)$ is uniformly bounded, by e), we get $V_{n-1}(\tau, \sigma)\, x \to x$, as $(\tau, \sigma) \to \to (t, t)$, $\forall x \in X$.

h) Is an easy consequence of Hypothesis II) and of the theorem on differentiation under the integral sign.

i) Follows immediately from g) and h). The proof of j) is the same as that of Lemma 3 in [3].

k) The assertion in $[0, T] \times \Delta$ follows straightforwardly by the dominated convergence theorem. The continuity in the uniform operator topology in $[0, T] \times \bar{\Delta}$ follows immediately by a) and e). If $x \in \mathscr{D}(P)$ and $(t, s) \in \Delta$, we have:

$$\|A_{p+1}(\tau)\, V_p(t, s)\, x\| =$$
$$= \left\|(2\pi i)^{-1} \int_\Gamma \lambda^{p-n} \exp(\lambda(t-s))\, A_{p+1}(\tau)\, P^{-1}(\lambda, s) \sum_{h=0}^{n-1} \lambda^h A_h(s)\, x \, d\lambda\right\| \to 0\,,$$

as $(t, s) \to (s_0, s_0)$, by e). By the uniform boundedness of $A_{p+1}(\tau)\, V_p(t, s)$, the assertion holds true $\forall x \in X$.

l) Let us first suppose that $x \in \mathscr{D}(P)$. Then,

$$V_{n-1}(t, t-\varepsilon)\, x = -(2\pi i)^{-1} \int_\Gamma \lambda^{-1} \exp(\lambda\varepsilon)\, P^{-1}(\lambda, t-\varepsilon) \sum_{k=0}^{n-1} \lambda^k A_k(t-\varepsilon)\, x \, d\lambda\,.$$

Hence, by e) $\|V_{n-1}(t, t-\varepsilon)\, x - x\| \leqq C \sum_{k=0}^{n-1} \varepsilon^{n-k} \|A_k(t-\varepsilon)\, x\| \leqq C'\varepsilon \sum_{k=0}^{n-1} \|A_k(0)\, x\|$, where C, C' are independent of t and ε. Let now be $x \in X$ and fix $\eta \in \mathbb{R}^+$. Set $K = \max\{1, \sup_\Delta \|V_{n-1}(t, s)\|\}$ and choose $y \in \mathscr{D}(P) \setminus \{0\}$, such that $\|x - y\| < (3K)^{-1}\eta$. Then,

$$\|V_{n-1}(t, t-\varepsilon)\, x - x\| \leqq \|V_{n-1}(t, t-\varepsilon)(x - y)\| +$$
$$+ \|V_{n-1}(t, t-\varepsilon)\, y - y\| + \|y - x\| < (2/3)\,\eta + C'\varepsilon \sum_{k=0}^{n-1} \|A_k(0)\, y\|\,.$$

If $\varepsilon < (3C' \sum_{k=0}^{n-1} \|A_k(0)\, y\|)^{-1} \eta$, we get $\|V_{n-1}(t, t-\varepsilon)\, x - x\| < \eta$, so proving the assertion.

A formal calculation shows that, in order the operator $U(t, s)$ in (1.1) be an evolution operator, the integral kernel $R(t, s)$ must solve the Volterra integral equation

$$R(t, s) = R_1(t, s) + \int_s^t R_1(t, \tau)\, R(\tau, s)\, d\tau\,, \quad (t, s) \in \Delta\,, \tag{3.1}$$

where $R_1(t, s) = -\sum_{k=0}^{n} A_k(t)\, V_k(t, s) = -\sum_{k=0}^{n-1} (A_k(t) - A_k(s))\, V_k(t, s)$.

Lemma 3.3. *The operator R_1 is continuous from Δ to $\mathscr{L}(X)$; furthermore, there exists $K \in \mathbb{R}^+$, such that*

$$\|R_1(t, s)\| \leqq K(t - s)^{\alpha - 1}\,, \quad \forall (t, s) \in \Delta\,. \tag{3.2}$$

Proof. The first assertion follows by Proposition 3.2.k). Furthermore, Hypothesis III) and an argument quite analogous to that in Lemma 6 in [3] give (3.2).

Proposition 3.4. *The integral equation (3.1) has a unique solution R which is* *continuous from Δ to $\mathscr{L}(X)$; furthermore, there exists $K' \in \mathbb{R}^+$, such that*

$$\|R(t,s)\| \leqq K'(t-s)^{\alpha-1}, \quad \forall (t,s) \in \Delta. \tag{3.3}$$

The proof is a straightforward consequence of Lemma 3.3 and of the theory of Volterra integral equations with weakly singular kernels (see, e.g., [2] Ch. II, 4.2).

Lemma 3.5. *Let $\beta \in]0, \alpha[$; then, $\forall t, \tau, \sigma \in \mathbb{R}$ such that $0 \leqq s < \tau < t \leqq T$, we have*

$$\begin{aligned} \|R_1(t,s) - R_1(\tau,s)\| &\leqq C_1(t-\tau)^\beta (\tau-s)^{\alpha-\beta-1}, \\ \|R(t,s) - R(\tau,s)\| &\leqq C_2(t-\tau)^\beta (\tau-s)^{\alpha-\beta-1}. \end{aligned} \tag{3.4}$$

Proof. We have

$$\begin{aligned} R_1(t,s) - R_1(\tau,s) = -\sum_{k=0}^{n-1} (A_k(t) - A_k(\tau)) V_k(t,s) - \\ -\sum_{k=0}^{n-1} (A_k(\tau) - A_k(s)) (V_k(t,s) - V_k(\tau,s)) = I_1 + I_2. \end{aligned}$$

By Prop. 3.2 e), we get $\|I_1\| \leqq K_1(t-\tau)^\alpha (t-s)^{-1}$,

$$\|I_2\| \leqq K_2((\tau-s)^\alpha (t-s)^{-1} + (\tau-s)^{\alpha-1}) \leqq K_3(\tau-s)^{\alpha-1}. \tag{3.5}$$

On the other hand, by Prop. 3.2. h),

$$I_2 = -\sum_{k=0}^{n-1} \int_\tau^t (A_k(\tau) - A_k(s)) V_{k+1}(\sigma,s)\, d\sigma.$$

Hence, by Prop. 3.2. e),

$$\|I_2\| \leqq K_4(\tau-s)^\alpha \int_\tau^t (\sigma-s)^{-2}\, d\sigma \leqq K_4(\tau-s)^{\alpha-2}(t-\tau). \tag{3.6}$$

Now, by (3.5) and (3.6),

$$\|I_2\| \leqq K_5(t-\tau)^\alpha (\tau-s)^{-1}. \tag{3.7}$$

So,

$$\|R_1(t,s) - R_1(\tau,s)\| \leqq K_6(t-\tau)^\alpha (\tau-s)^{-1}. \tag{3.8}$$

On the other hand, by Lemma 3.3,

$$\|R_1(t,s) - R_1(\tau,s)\| \leqq \|R_1(t,s)\| + \|R_1(\tau,s)\| \leqq K_7(\tau-s)^{\alpha-1}. \tag{3.9}$$

If $\beta \in]0, \alpha[$, by (3.8) and (3.9) we get $\|R_1(t,s) - R_1(\tau,s)\| \leqq K_8(t-\tau)^\beta (\tau-s)^{\alpha-\beta-1}$.

So the first assertion is proved.

As

$$\begin{aligned} R(t,s) - R(\tau,s) = R_1(t,s) - R_1(\tau,s) + \int_\tau^t R_1(t,\sigma) R(\sigma,s)\, d\sigma + \\ + \int_s^\tau (R_1(t,\sigma) - R_1(\tau,\sigma)) R(\sigma,s)\, d\sigma, \end{aligned}$$

by what has already been proved and by Lemma 3.3 and Prop. 3.4, we get

$$\|R(t, s) - R(\tau, s)\| \leqq K_8(t - \tau)^\beta (\tau - s)^{\alpha-\beta-1} +$$

$$+ K_9 \int_\tau^t (t - \sigma)^{\alpha-1} (\sigma - s)^{\alpha-1} \, d\sigma + K_{10} \int_s^\tau (t - \tau)^\beta (\tau - \sigma)^{\alpha-\beta-1} (\sigma - s)^{\alpha-1} \, d\sigma =$$

$$= \sum_{k=1}^{3} J_k .$$

Now, $J_2 \leqq K_{11}(t - \tau)^\alpha (\tau - s)^{\alpha-1} \leqq K_{12}(t - \tau)^\beta (\tau - s)^{\alpha-\beta-1}$, while $J_3 \leqq \leqq K_{13}(t - \tau)^\beta (\tau - s)^{2\alpha-\beta-1} \leqq K_{14}(t - \tau)^\beta (\tau - s)^{\alpha-\beta-1}$.

So the assertion is completely proved.

4. THE OPERATOR $\int_s^t V_0(t, \tau) R(\tau, s) \, d\tau$

Definition 4.1. Let us set, $\forall p \in \{0, 1, \ldots, n - 1\}$,

$$W_p(t, s) = \int_s^t V_p(t, \tau) R(\tau, s) \, d\tau , \quad \text{if} \quad (t, s) \in \Delta ,$$

$$W_p(t, t) = 0 .$$

By Props. 3.2. e) and 3.4, we have if $(t, s) \in \Delta$:

$$\|A_k(t) V_p(t, \tau) R(\tau, s)\| \leqq C(t - \tau)^{k-p-1} (\tau - s)^{\alpha-1} ; \tag{4.1}$$

hence, the integral defining $W_p(t, s)$ is absolutely convergent and $W_p(t, s) \in \mathscr{L}(X)$.

Proposition 4.2. *We have:*

a) $\forall (t, s) \in \Delta$, $\|W_p(t, s)\| \leqq C(t - s)^{n-p+\alpha-1}$, $p = 0, \ldots, n - 1$;

b) W_p *is continuous in* $\bar{\Delta}$ *in the uniform operator topology,* $\forall p \in \{0, \ldots, n - 1\}$; *furthermore,* W_0 *is* $n - 1$ *times continuously partially differentiable with respect to* t *in the uniform operator topology in all of* $\bar{\Delta}$ *and we have*

$$\partial_t^p W_0(t, s) = W_p(t, s) , \quad \forall (t, s) \in \bar{\Delta} , \quad p = 1, \ldots, n - 1 ;$$

c) $\forall x \in X$, $\quad W_p(t, s) \, x \in \bigcap_{k=p}^{n-1} \mathscr{D}(A_k)$ *and* $\quad \|A_k(t) W_p(t, s)\| \leqq C(t - s)^{k-p+\alpha-1}$, $\forall (t, s) \in \Delta$, $\forall p \in \{0, \ldots, n - 1\}$ *and* $\forall k \in \{p + 1, \ldots, n\}$; *furthermore, the function* $(t, s) \to A_k(t) W_p(t, s)$ *is continuous in the uniform operator topology in* $\bar{\Delta}$, *if* $p \in \in \{0, \ldots, n - 2\}$ *and* $k \geqq p + 1$, *while* $(t, s) \to A_p(t) W_p(t, s)$ *is strongly continuous in* Δ;

d) W_0 *is* n *times strongly continuously partially differentiable with respect to* t *in* Δ *and we have*

$$\partial_t^n W_0(t, s) \, x = R(t, s) \, x + \int_s^t V_n(t, \tau) R(\tau, s) \, x \, d\tau , \quad \forall x \in X \quad \textit{and} \quad \forall (t, s) \in \Delta ,$$

where the integral must be considered as an improper Riemann integral.

Proof. a) follows immediately by (4.1).

b) The assertion at points in Δ follows by the estimate (4.1), which assures the uniform convergence on the compact subsets of Δ of the integral defining W_p. By a), the assertion holds true at points (t, t), too.

To prove c) let us first establish the following result.

Lemma 4.3. *Let* $x \in X$; *then the integrals*

$$\int_s^t A_p(t)\, V_p(t, \tau)\, R(\tau, s)\, x\, d\tau\,, \quad p = 0, \dots, n\,,$$

exist as improper Riemann integrals; *furthermore, the functions* $(t, s) \to \int_s^t A_p(t)\,.$ $.\, V_p(t, \tau)\, R(\tau, s)\, d\tau$ *are strongly continuous in* Δ.

Proof. If $\varrho \in]0, T[$, set $\Delta_\varrho = \{(t. s) \in \Delta;\ t - s \geqq \varrho\}$ and, if $\varepsilon \in]0, \varrho/2[$, $x \in X$, set, $\varphi_{k,\varepsilon}(t, s) = \int_s^{t-\varepsilon} V_k(t, \tau)\, R(\tau, s)\, x\, d\tau$, $k = 0, \dots, n$.

Since $\|A_k(t)\, V_k(t, \tau)\, R(\tau, s)\| \leqq C(t - \tau)^{-1} (\tau - s)^{\alpha - 1}$, then

$$\left\| \int_s^{t-\varepsilon} A_k(t)\, V_k(t, \tau)\, R(\tau, s)\, d\tau \right\| \leqq C' \varepsilon^{-1} (t - \varepsilon - s)^\alpha\,, \quad \forall (t, s) \in \Delta_\varrho\,;$$

so it turns out that, if $k \leqq n - 1$, $\varphi_{k,\varepsilon}(t, s) \in \mathscr{D}(A_k)$ and

$$A_k(t)\, \varphi_{k,\varepsilon}(t, s) = \int_s^{t-\varepsilon} A_k(t)\, V_k(t, \tau)\, R(\tau, s)\, x\, d\tau\,.$$

It is not difficult to prove that $A_k \varphi_{k,\varepsilon}$ is continuous in Δ_ϱ, so we only have to show that $A_k(t)\, \varphi_{k,\varepsilon}(t, s)$ converges uniformly in Δ_ϱ, as $\varepsilon \to 0+$. We have:

$$\begin{aligned}
&A_k(t)\, \varphi_{k,\varepsilon}(t, s) = \\
&= \int_s^{t-\varepsilon} \left((2\pi i)^{-1} \int_\Gamma \lambda^k \exp(\lambda(t - \tau))\, A_k(t)\, P^{-1}(\lambda, t)\, d\lambda \right) (R(\tau, s)\, x - R(t, s)\, x)\, d\tau\, + \\
&\quad + \int_s^{t-\varepsilon} \left((2\pi i)^{-1} \int_\Gamma \lambda^k \exp(\lambda(t - \tau))\, A_k(t)\, P^{-1}(\lambda, t)\, d\lambda \right) R(t, s)\, x\, d\tau\, + \\
&+ \int_s^{t-\varepsilon} \left((2\pi i)^{-1} \int_\Gamma \lambda^k \exp(\lambda(t - \tau))\, A_k(t)\, (P^{-1}(\lambda, \tau) - P^{-1}(\lambda, t))\, R(\tau, s)\, x\, d\lambda \right) d\tau = \\
&= \sum_{j=1}^{3} I_{k,j,\varepsilon}(t, s)\,.
\end{aligned}$$

By Prop. 3.2. e) and Lemma 3.5, we get, if $\beta \in]0, \alpha[$:

$$\left\| (2\pi i)^{-1} \int_\Gamma \lambda^k \exp(\lambda(t - \tau))\, A_k(t)\, P^{-1}(\lambda, t)\, d\lambda (R(\tau, s)\, x - R(t, s)\, x) \right\| \leqq$$
$$\leqq C(t - \tau)^{\beta - 1} (\tau - s)^{\alpha - \beta - 1}\,;$$

hence, the integral

$$\int_s^t \left((2\pi i)^{-1} \int_\Gamma \lambda^k \exp(\lambda(t - \tau))\, A_k(t)\, P^{-1}(\lambda, t)\, d\lambda \right) (R(\tau, s)\, x - R(t, s)\, x)\, d\tau\,,$$

which is absolutely convergent, is the uniform limit of $I_{k,1,\varepsilon}$ as $\varepsilon \to 0+$. Since $P^{-1}(\lambda, \tau) - P^{-1}(\lambda, t) = P^{-1}(\lambda, \tau)\sum_{h=0}^{n-1} \lambda^h(A_h(t) - A_h(\tau))\, P^{-1}(\lambda, t)$,

$$\left\|(2\pi i)^{-1} \int_\Gamma \lambda^k \exp(\lambda(t - \tau))\, A_k(t)\, (P^{-1}(\lambda, \tau) - P^{-1}(\lambda, t))\, R(\tau, s)\, x \, d\lambda)\, d\tau\right\| \leqq$$
$$\leqq C(t - \tau)^{\alpha - 1} (\tau - s)^{\alpha - 1} ;$$

we get that the integral

$$\int_s^t \left((2\pi i)^{-1} \int_\Gamma \lambda^k \exp(\lambda(t - \tau))\, A_k(t)\, (P^{-1}(\lambda, \tau) - P^{-1}(\lambda, t))\, R(\tau, s)\, x \, d\lambda\right) d\tau ,$$

which is absolutely convergent, is the uniform limit of $I_{k,3,\varepsilon}$ as $\varepsilon \to 0+$. Let us now handle $I_{k,2,\varepsilon}(t, s)$, when $k \geqq 1$. We have:

$$I_{k,2,\varepsilon}(t, s) = -\int_s^{t-\varepsilon} \partial_\tau \left((2\pi i)^{-1} \int_\Gamma \lambda^{k-1} \exp(\lambda(t - \tau))\, A_k(t)\, P^{-1}(\lambda, t)\, d\lambda\right) R(t, s)\, x \, d\tau =$$
$$= (2\pi i)^{-1} \int_\Gamma \lambda^{k-1} \exp(\lambda(t - s))\, A_k(t)\, P^{-1}(\lambda, t)\, R(t, s)\, x \, d\lambda -$$
$$- (2\pi i)^{-1} \int_\Gamma \lambda^{k-1} \exp(\lambda\varepsilon)\, A_k(t)\, P^{-1}(\lambda, t)\, R(t, s)\, x \, d\lambda .$$

Let $\eta \in \mathbb{R}^+$; then, by the uniform continuity of R in Δ_ϱ, there exists $\delta_\eta \in \mathbb{R}^+$, such that $\|R(t, s) - R(\tau, \sigma)\| < \eta$, $\forall (t, s), (\tau, \sigma) \in \Delta_\varrho$, such that $|(t - \tau, s - \sigma)| < \delta_\eta$.

Let $\{S((t_j, s_j), \delta_\eta) \cap \Delta_\varrho ;\ j = 1, \ldots, p\}$ be a finite cover of Δ_ϱ made up of relatively open balls of radius δ_η. By the density of $\mathscr{D}(P)$ in X, there exist $y_j \in \mathscr{D}(P)$, such that $\|R(t_j, s_j)\, x - y_j\| < \eta$, $j = 1, \ldots, p$. Then we have, if $k \leqq n - 1$:

$$-(2\pi i)^{-1} \int_\Gamma \lambda^{k-1} \exp(\lambda\varepsilon)\, A_k(t)\, P^{-1}(\lambda, t)\, R(t, s)\, x \, d\lambda =$$
$$= -(2\pi i)^{-1} \int_\Gamma \lambda^{k-1} \exp(\lambda\varepsilon)\, A_k(t)\, P^{-1}(\lambda, t)\, (R(t, s)\, x - R(t_j, s_j)\, x)\, d\lambda -$$
$$-(2\pi i)^{-1} \int_\Gamma \lambda^{k-1} \exp(\lambda\varepsilon)\, A_k(t)\, P^{-1}(\lambda, t)\, (R(t_j, s_j)\, x - y_j)\, d\lambda -$$
$$- (2\pi i)^{-1} \int_\Gamma \lambda^{k-1} \exp(\lambda\varepsilon)\, A_k(t)\, P^{-1}(\lambda, t)\, y_j \, d\lambda = \sum_{i=1}^{3} J_{i,k,\varepsilon,j}(t, s) .$$

By Prop. 3.2. e), if j is suitably chosen, we get

$$\|J_{1,k,\varepsilon,j}(t, s)\| < C_1\eta , \quad \|J_{2,k,\varepsilon,j}(t, s)\| < C_2\eta ,$$

where C_1, C_2 are independent of t, s, ε and j. Furthermore,

$$J_{3,k,\varepsilon,j}(t, s) = -(2\pi i)^{-1} \int_\Gamma \lambda^{k-n-1} \exp(\lambda\varepsilon)\, A_k(t)\, y_j \, d\lambda +$$

$$+ (2\pi i)^{-1} \sum_{h=0}^{n-1} \int_\Gamma \lambda^{k+h-n-1} \exp(\lambda\varepsilon) A_k(t) P^{-1}(\lambda, t) A_h(t) y_j \, d\lambda =$$

$$= -((n-k)!)^{-1} \varepsilon^{n-k} A_k(t) y_j +$$

$$+ (2\pi i)^{-1} \sum_{h=0}^{n-1} \int_\Gamma \lambda^{k+h-n-1} \exp(\lambda\varepsilon) A_k(t) P^{-1}(\lambda, t) A_h(t) y_j \, d\lambda .$$

Hence $J_{3,k,\varepsilon,j}(t, s) \to 0$, uniformly in Δ_ϱ, as $\varepsilon \to 0+$, by Prop. 3.2. e). A quite analogous argument shows that

$$-(2\pi i)^{-1} \int_\Gamma \lambda^{n-1} \exp(\lambda\varepsilon) P^{-1}(\lambda, t) R(t, s) x \, d\lambda + R(t, s) x \to 0 ,$$

uniformly in Δ_ϱ, as $\varepsilon \to 0+$.

Therefore, $I_{k,2,\varepsilon}(t, s)$ converges uniformly in Δ_ϱ, as $\varepsilon \to 0+$, if $k = 1, \ldots, n$.

On the other hand $I_{0,2,\varepsilon}(t, s) = -\sum_{h=1}^{n} I_{h,2,\varepsilon}(t, s)$; so, it converges uniformly in Δ_ϱ, as $\varepsilon \to 0+$, too.

The assertion is completely proved.

End of the proof of Prop. 4.2. By Lemma 4.3 we get immediately that $W_p(t, s) x \in \mathscr{D}(A_p)$, $\forall x \in X$, $\forall (t, s) \in \Delta$ and $\forall p \in \{0, \ldots, n-1\}$ and that $A_p W_p$ is strongly continuous in Δ. The estimate if $k > p$, follows by (4.1). The remaining assertions in c) follow by similar, but much simpler arguments.

To prove d), let us use the same notations as in the proof of Lemma 4.3. It is obvious that $\varphi_{n-1,\varepsilon}(t, s) \to W_{n-1}(t, s) x$, as $\varepsilon \to 0+$, and that

$$\partial_t \varphi_{n-1,\varepsilon}(t, s) = V_{n-1}(t, t-\varepsilon) R(t-\varepsilon, s) x + \int_s^{t-\varepsilon} V_n(t, \tau) R(\tau, s) x \, d\tau .$$

Since the last integral converges uniformly in Δ_ϱ to $\int_s^t V_n(t, \tau) R(\tau, s) x \, d\tau$ as $\varepsilon \to 0+$, we only have to prove that $V_{n-1}(t, t-\varepsilon) R(t-\varepsilon, s) x \to R(t, s) x$, uniformly in Δ_ϱ, as $\varepsilon \to 0+$. Now,

$$\|V_{n-1}(t, t-\varepsilon) R(t-\varepsilon, s) x - R(t, s) x\| \leqq$$

$$\leqq \|V_{n-1}(t, t-\varepsilon) (R(t-\varepsilon, s) x - R(t, s) x)\| +$$

$$+ \|V_{n-1}(t, t-\varepsilon) R(t, s) x - R(t, s) x\| = I_{1,\varepsilon}(t, s) + I_{2,\varepsilon}(t, s) .$$

By Prop. 3.2. e) and Lemma 3.5, if $\beta \in]0, \alpha[$, $I_{1,\varepsilon}(t, s) \leqq C_1 \varrho^{\alpha-\beta-1} \varepsilon^\beta$, which goes to zero uniformly in Δ_ϱ as $\varepsilon \to 0+$. To estimate $I_{2,\varepsilon}(t, s)$ let us fix $\eta \in \mathbb{R}^+$ and choose a finite collection of points $(t_j, s_j) \in \Delta_\varrho$ $(j = 1, \ldots, k)$ as in the proof of Lemma 4.2. Then,

$$I_{2,\varepsilon}(t, s) \leqq \|V_{n-1}(t, t-\varepsilon) (R(t, s) x - R(t_j, s_j) x)\| +$$

$$+ \|V_{n-1}(t, t-\varepsilon) R(t_j, s_j) x - R(t_j, s_j) x\| + \|R(t_j, s_j) x - R(t, s) x\| =$$

$$= \sum_{k=1}^{3} J_{k,j,\varepsilon}(t, s) .$$

If we choose j suitably, by Prop. 3.2. e), we get $J_{1,j,\varepsilon}(t, s) < C\eta$, $J_{3,j,\varepsilon}(t, s) < \eta$, where C is independent of t, s and ε.

On the other hand, by Prop. 3.2. l), there exists $\delta_\eta \in \mathbb{R}^+$, such that

$$J_{2,j,\varepsilon}(t, s) < \eta\,, \quad \text{if} \quad \varepsilon \in]0, \delta_\eta[\,, \quad \forall j \in \{1, \ldots, l\}\,.$$

Hence, if $\varepsilon \in]0, \delta_\eta[$, $I_{2,\varepsilon}(t, s) < (C + 2)\,\eta$, where C is independent of t, s and ε.

So the assertion is completely proved.

5. THE EVOLUTION OPERATOR

Let us set, $\forall (t, s) \in \Delta$,

$$U(t, s) = V_0(t, s) + W_0(t, s)\,.$$

We can now state and prove the main result of the paper.

Theorem 5.1. *Suppose Assumptions* I)–II)–III) *hold.*

Then:

a) $U \in \mathscr{C}(\Delta; \mathscr{L}(X))$;

b) *U is partially differentiable with respect to t $n - 1$ times in the uniform operator topology and we have* $\partial_t^k U(t, s) = V_k(t, s) + W_k(t, s), \forall (t, s) \in \Delta, k = 1, \ldots, n - 1$;

c) $\partial_t^k U(t, s)\, x \in \mathscr{D}(A_k)$, $\forall x \in X$, $\forall (t, s) \in \Delta$, $k = 0, \ldots, n - 1$;

d) *U is strongly partially differentiable with respect to t n times and we have*

$$\partial_t^n U(t, s)\, x = V_n(t, s)\, x + R(t, s)\, x + \int_s^t V_n(t, \tau)\, R(\tau, s)\, x \,\mathrm{d}\tau\,,$$

$\forall x \in X$, $\forall (t, s) \in \Delta$; *the integral must be considered as an improper Riemann integral*;

e) *the function* $(t, s) \to A_k(t)\, \partial_t^k U(t, s)\, x$ *is continuous in* Δ, $\forall x \in X$ *and* $\forall k \in \{0, \ldots, n - 1\}$;

f) $\forall x \in X$, $\forall (t, s) \in \Delta$,

$$\sum_{k=0}^{n} A_k(t)\, \partial_t^k U(t, s)\, x = 0\,;$$

g) $\forall (t, s) \in \Delta$, $\|A_k(t)\, \partial_t^p U(t, s)\| \leqq C(t - s)^{k-p-1}$,

$$\forall p \in \{0, \ldots, n - 1\} \quad \textit{and} \quad \forall k \in \{p + 1, \ldots, n\}\,;$$

hence, $\forall s_0 \in [0, T[$, $A_k(t)\, \partial_t^p U(t, s) \to 0$, *as* $(t, s) \to (s_0, s_0)$, *in the uniform operator topology if* $k \geqq p + 2$; *furthermore, the functions* $(t, s) \to A_{k+1}(t)\, \partial_t^k U(t, s)$ *have strong limit as* $(t, s) \to (s_0, s_0)$ *and* $\partial_t^{n-1} U(t, s) \to I$ *strongly as* $(t, s) \to (s_0, s_0)$.

Proof. a)–e) and g) follow by Props. 3.2 and 4.2.

To prove f), let us note that, by b) d) and Prop. 3.4, we have, $\forall x \in X$:

$$\sum_{k=0}^{n} A_k(t)\, \partial_t^k U(t, s) = \sum_{k=0}^{n} A_k(t)\, V_k(t, s)\, x + R(t, s)\, x +$$

$$+ \int_s^t \sum_{k=0}^{n} A_k(t)\, V_k(t, \tau)\, R(\tau, s)\, x \,\mathrm{d}\tau =$$

$$= -R_1(t, s)\, x + R(t, s)\, x - \int_s^t R_1(t, \tau)\, R(\tau, s)\, x \,\mathrm{d}\tau = 0\,.$$

By Theorem 5.1 we immediately get the following result.

Theorem 6.1. *Suppose Assumptions* I)–II)–III) *hold and let* $x \in X$. *Then the function* $v: t \to U(t, 0)\, x$ *is a solution of the Cauchy problem*

$$\sum_{k=0}^{n} A_k(t)\, u^{(k)}(t) = 0\,, \quad in \quad]0, T]\,,$$
$$u^{(h)}(0) = 0\,, \quad h = 0, \dots, n-2\,, \quad u^{(n-1)}(0) = x\,,$$

in the following sense:

i) $v \in \mathscr{C}^{(n)}(]0, T]; X) \cap \mathscr{C}^{(n-1)}([0, T]; X)$;

ii) $v^{(k)}(t) \in \mathscr{D}(A_k)$, $\forall t \in]0, T]$, $k = 0, \dots, n-1$;

iii) $A_k v^{(k)}$ *is continuous in* $]0, T]$, $k = 0, \dots, n-1$;

iv) $A_{k+1} v^{(k)}$ *is continuous in* $[0, T]$, $k = 0, \dots, n-2$;

v) $\sum_{k=0}^{n} A_k(t)\, v^{(k)}(t) = 0$, $\forall t \in]0, T]$;

vi) $v^{(h)}(0) = 0$, *if* $k = 0, \dots, n-2$; $v^{(n-1)}(0) = x$.

The following result allows to handle nonhomogeneous equations.

Theorem 6.2. *Suppose Assumptions* I)–II)–III) *hold and let* $f \in \mathscr{C}([0, T]; X)$, *such that there exist* $C \in \mathbb{R}^+$, $\beta \in]0, 1]$ *so that*

$$\|f(t) - f(\tau)\| \leqq C|t - \tau|^{\beta}\,, \quad \forall t, \ \tau \in [0, T]\,.$$

Then the function $w: t \to \int_0^t U(t, s) f(s)\, ds$ *is a solution of the Cauchy problem*

$$\sum_{k=0}^{n} A_k(t)\, u^{(k)}(t) = f(t)\,, \quad in \quad [0, T]\,, \quad u^{(h)}(0) = 0\,, \quad h = 0, \dots, n-1\,,$$

in the following sense:

i) $w \in \mathscr{C}^{(n)}([0, T]; X)$;

ii) $w^{(k)}(t) \in \mathscr{D}(A_k)$, $\forall t \in [0, T]$ *and* $\forall k \in \{0, \dots, n-1\}$;

iii) $A_k w^{(k)}$ *is continuous in* $[0, T]$, $\forall k \in \{0, \dots, n-1\}$;

iv) $\sum_{k=0}^{n} A_k(t)\, w^{(k)}(t) = f(t)$, $\forall t \in [0, T]$;

v) $w^{(k)}(0) = 0$, $k = 0, \dots, n-1$.

Proof. By Theorem 5.1, it is obvious that $w \in \mathscr{C}^{(n-1)}([0, T]; X)$, that $w^{(k)}(t) = \int_0^t \partial_t^k U(t, s) f(s)\, ds$, $\forall t \in]0, T]$ and $\forall k \in \{0, \dots, n-1\}$ and that w satisfies the initial conditions.

If $\varrho \in]0, T[$, set $I_\varrho = [\varrho, T]$ and, if $\varepsilon \in]0, \varrho/2[$,

$$\chi_{k,\varepsilon}(t) = \int_0^{t-\varepsilon} \partial_t^k U(t, s) f(s)\, ds\,, \quad k = 0, \dots, n\,.$$

Since, by Theorem 5.1. g),

$$\|A_k(t)\, \partial_t^k U(t, s) f(s)\| \leqq C(t - s)^{-1} \|f(s)\|\,,$$

then

$$\left\| \int_0^{t-\varepsilon} A_k(t)\, \partial_t^k U(t,s) f(s)\, ds \right\| \leqq C'\varepsilon^{-1} \max_{[0,T]} \|f\|\,, \quad \forall t \in I_\varrho\,;$$

so it turns out that $\chi_{k,\varepsilon}(t) \in \mathscr{D}(A_k)$ and that

$$A_k(t)\, \chi_{k,\varepsilon}(t) = \int_0^{t-\varepsilon} A_k(t)\, \partial_t^k U(t,s) f(s)\, ds\,, \quad \text{if} \quad k = 0, \ldots, n-1\,.$$

Furthermore, it is obvious that $A_k \chi_{k,\varepsilon}$ is continuous in I_ϱ even if $k = n$.

Let us prove that $A_k(t)\, \chi_{k,\varepsilon}(t)$ converges uniformly in I_ϱ, as $\varepsilon \to 0+$.

The proof is quite similar to that of Lemma 4.3, so we shall only sketch it. We have:

$$A_k(t)\, \chi_{k,\varepsilon}(t) = \int_0^{t-\varepsilon} A_k(t)\, V_k(t,s) f(s)\, ds\, +$$

$$+ \int_0^{t-\varepsilon} \left(\int_s^t A_k(t)\, V_k(t,\tau)\, R(\tau,s)\, d\tau \right) f(s)\, ds = I_{1,k,\varepsilon} + I_{2,k,\varepsilon}\,.$$

Exactly as in the proof of Lemma 4.3, we can show that $I_{1,k,\varepsilon}$ converges uniformly in I_ϱ as $\varepsilon \to 0+$, if we use the Hölder condition satisfied by f instead of Lemma 3.5. Furthermore, this limit is easily seen to go to zero as $t \to 0+$.

To handle $I_{2,k,\varepsilon}$, we decompose the integrand as in the proof of Lemma 4.3; it is then obvious that $I_{2,k,\varepsilon}$ converges uniformly in I_ϱ as $\varepsilon \to 0+$ and that this limit goes to zero as $t \to 0+$. So, iii) is proved.

Now, we have:

$$\chi'_{n-1,\varepsilon}(t) = \partial_t^{n-1} U(t, t-\varepsilon) f(t-\varepsilon) + \int_0^{t-\varepsilon} \partial_t^n U(t,s) f(s)\, ds\,.$$

The first term is shown to converge uniformly in I_ϱ to $f(t)$, as $\varepsilon \to 0+$, by an argument quite analogous to the one employed in the proof of Prop. 4.2. d). Hence, by what we have already proved, $\chi'_{n-1,\varepsilon}(t)$ converges uniformly in I_ϱ, as $\varepsilon \to 0+$. Furthermore, this limit goes to $f(0)$ as $t \to 0+$. This completes the proof of the assertion.

References

[1] *H. O. Fattorini:* The Cauchy Problem, Encyclop. of Math. and Appl., Vol. *18*, Addison-Wesley (1983).

[2] *S. G. Kreĭn:* Linear Differential Equations in Banach Space, Nauka (1963) (Russian); engl. transl.: Transl. Math. Monographs, Vol. *29*, Amer. Math. Soc. (1972).

[3] *E. Obrecht:* Sul problema di Cauchy per le equazioni paraboliche astratte di ordine *n*, Rend. Sem. Mat. Univ. Padova, *53* (1975), 231–256.

[4] *P. E. Sobolevskiĭ:* Equations of Parabolic Type in a Banach Space, Trudy Moskov. Mat. Obšč., *10* (1961), 297–350 (Russian); engl. transl.: Amer. Math. Soc. Transl., (2) *49* (1965), 1–62.

[5] *H. Tanabe:* On the Equations of Evolution in a Banach Space, Osaka Math. J., *12* (1960), 363–376.

Author's address: Dipartimento di Matematica dell' Università, Piazza di Porta S. Donato, 5, 40127 Bologna, Italy.