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**EX-DIVIDEND STOCK PRICE BEHAVIOR AND  
ARBITRAGE OPPORTUNITIES**

by

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## ABSTRACT

Given a no arbitrage opportunity economy, this paper provides a theorem which gives necessary and sufficient conditions for the ex-dividend stock price drop to be equal to the amount of the dividend. Plausible conditions are seen to exist where the ex-dividend stock price drop is not equal to the dividend and yet no arbitrage opportunities exist. The empirical implications of this theorem for both option pricing and Kalay's "short term trader" hypothesis are examined.

## EX-DIVIDEND STOCK PRICE BEHAVIOR AND ARBITRAGE OPPORTUNITIES

### 1. Introduction

The proposition that "the stock price should drop on the ex-dividend date by approximately the amount of the dividend" has been part of market lore for over 30 years, see Campbell and Beranek [1955]. The academic literature investigating this phenomena also spans these same thirty years, see Campbell and Beranek [1955], Durand and May [1960], Elton and Gruber [1970], Kalay [1982], Hess [1982], Lakonishok and Vermaelen [1983], Eades, Hess, and Kim [1984], Elton, Gruber, and Rentzler [1984], and Kalay [1984]. Nowhere in this literature, surprisingly, is there a rigorous proof of this proposition in an economy with risk averse investors. The first explicit proof of this proposition can be found in Kalay [1982] in an economy populated with risk neutral investors. In a subsequent paper, Kalay [1984] (in a series of footnotes) argued that this proposition need not hold in a risk averse setting. In contrast, Copeland and Weston [1979; p. 352], Rubinstein and Cox [1983; p. 12], and Lakonishok and Vermaelen [1983; p. 1166] all give heuristic (non-rigorous) arguments which state that this proposition must be true in a frictionless market to avoid the existence of arbitrage opportunities. These economies are consistent with risk averse investors. Further still, Jarrow and Rudd [1983; p. 48] argue (heuristically) that the proposition is true in a frictionless no arbitrage opportunity economy only if prices follow a continuous sample path. Where does the truth lie? The purpose of this paper is to eliminate this confusion by providing a theorem which gives necessary and sufficient conditions for the validity of this proposition in a frictionless, no arbitrage opportunity economy.

This theorem has important implications for two contemporary research questions in finance. The first implication relates to the Kalay [1982, 1984] versus Elton, Gruber, Rentzler [1984] exchange concerning the "short term traders" hypothesis and estimates of the marginal tax bracket. We argue below that the short term trader hypothesis is probably not relevant, independent of transaction costs. The second implication relates to American call option valuation and its empirical testing. We show below that the generalized form of the ex-dividend stock process used by Roll [1977], Geske [1979], Whaley [1981] and tested by Whaley [1982] and Stark [1982] is inconsistent with a no arbitrage opportunity economy. This observation necessarily calls into question the conclusions reached in those empirical studies.

An outline of this paper is as follows. Section 2 presents the model, while section 3 characterizes a no arbitrage opportunity economy and provides necessary and sufficient conditions for the existence of no arbitrage opportunities. Section 4 applies this characterization to ex-dividend stock price behavior. Finally, section 5 concludes the paper with a discussion of the empirical implications of the theorems.

## 2. The Model

For brevity of presentation, we will utilize a simplified version of the securities market model as contained in Harrison and Kreps [1979]. The notation and terminology of their original paper will be adhered to in order that the critical reader can more easily verify our results.

Without loss of generality we focus our attention on a market consisting of only two assets, a riskless bond and a risky stock. These assets are traded in frictionless markets over the time interval  $[0, T]$ .

The uncertainty in the economy is characterized by a state space  $\Omega$  and an information structure  $(F_t)_{t \in [0, T]}$ .<sup>1</sup> Each element of  $(F_t)$  can be thought of as the information set available in the economy at time  $t$ . We assume that  $F_s \subseteq F_t$  for  $s \leq t$ , consequently, information increases over time. The economy is populated with a finite number of investors who have probability beliefs  $P_i$  over  $F_T$ . These probability beliefs are homogeneous to the following extent:<sup>2</sup> given  $A \in F_T$ ,

$$P_i(A) = 0 \text{ if and only if } P_j(A) = 0 \quad (1)$$

for all investors  $i, j$ .

Let us take  $P$  to be the average of all investor's probability beliefs, i.e.  $P = \sum_i P_i / I$  where  $I$  represents the number of investors in the economy. Given condition (1), an event occurs with  $P$  probability one if and only if it occurs with  $P_i$  probability one for all investors  $i$ .

Trades can take place in these two assets at any time over the interval  $[0, T]$ . The price process for the riskless asset will be denoted by  $Z_0(t, w) = 1$  for all  $t \in [0, T]$  and  $w \in \Omega$ . This is without loss of generality (this makes  $Z_0$  and  $X$  prices relative to the price of the riskless asset, see Harrison and Kreps [1979; p. 401]).  $Z_0(t, w)$  represents the price of 1 unit of the riskless asset at time  $t$  under state  $w$ . We denote by  $X(t, w)$  the price of 1 share of the risky stock at time  $t$  under state  $w$ . By assumption, we require that  $E_p[|X(t, w)|] < +\infty$  and that the price  $X(t, w)$  is included in the information set  $F_t$  at time  $t$ .<sup>3</sup>

This stock  $X(t, w)$  pays a random dividend of  $d$  dollars per share at the ex-dividend date, time  $\tau$ . The dividend is surely known at time  $\tau$ , and perhaps even earlier.<sup>4</sup> We have

$$\begin{aligned} X(t) \text{ for } t < \tau \text{ trades cum-dividend} & \quad (2) \\ \text{and } X(t) \text{ for } t \geq \tau \text{ trades ex-dividend.} & \end{aligned}$$

The question to be studied in this paper is:

$$\text{Does } \lim_{h \downarrow 0} X(\tau-h, w) = X(\tau, w) + d(w) \text{ with probability 1?} \quad (3)$$

That is, does the stock fall by the amount of the dividend at the ex-dividend date.

To facilitate the study of this question, let us define a new stochastic process  $Z_1(t, w)$  by

$$Z_1(t, w) = \begin{cases} X(t, w) & \text{if } t < \tau \\ X(t, w) + d(w) & \text{if } t \geq \tau \end{cases} \quad (4)$$

Prior to the ex-dividend date, the price  $Z_1(t, w)$  represents the price of the stock. After the ex-dividend date,  $Z_1(t, w)$  represents the price of a portfolio consisting of the stock plus investing the dividend in the riskless bond. The transformed stock,  $Z_1(t)$ , therefore represents the stock but without any cash flows over  $[0, T]$ .

The difference between the processes  $X(t)$  and  $Z_1(t)$  is diagrammed in Figure 1 for a fixed  $w$ .

Using the transformed stock process  $\{Z_1(t)\}$ , condition (3) is equivalent to:

$$\text{Does } \lim_{h \downarrow 0} Z_1(\tau-h, w) = Z_1(\tau, w)? \quad (5)$$

In Figure 1, this would correspond to  $Z_1(t, w)$  having a continuous sample path at  $\tau$ . The next section studies this question further.

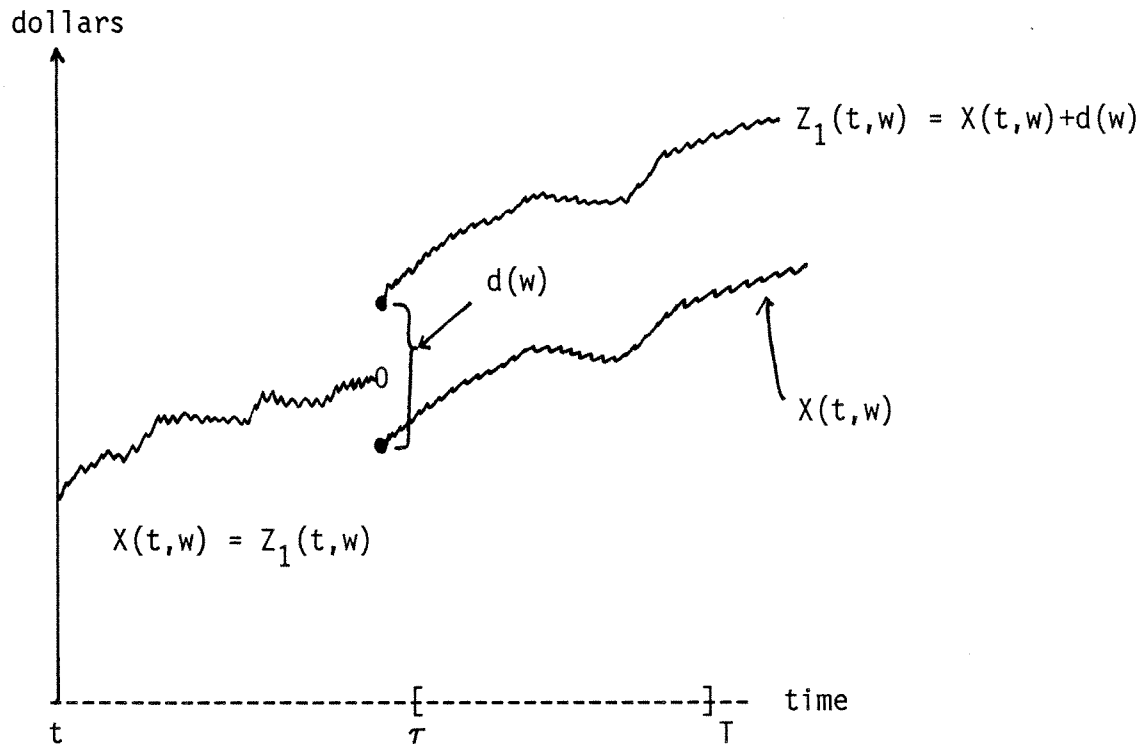


Figure 1: A Graphical Description of the Stock Process  $X(t)$  and  $Z_1(t)$  Around the Ex-dividend Date  $\tau$ .

### 3. Characterization of Arbitrage Pricing

To investigate whether the ex-dividend proposition ((3) or (5)) is true in an economy without arbitrage opportunities, one needs to characterize what it means for an economy to have no arbitrage opportunities. We use the characterization as given in Harrison and Kreps [1979; p. 392, Corollary (b)]. This characterization is the relevant concept since we are not interested in extending prices from a marketed set of assets to the set of contingent claims. In this latter case, uniqueness of the price extension would be crucial (see Harrison and Kreps [1979; p. 392, Corollary (c)]).

A security market is said to have no arbitrage opportunities if and only if there exists a probability belief (measure)  $Q$  on  $(\Omega, F_T)$  such that

$$(a) \quad \text{For } A \in F_T, \quad Q(A) = 0 \text{ if and only if}$$

$$P(A) = 0,$$

and

$$(b) \quad E_Q(Z_1(t) | F_s) = Z_1(s) \text{ where } s \leq t$$

$$\text{for all } s, t \in [0, T],$$

(6)

The probability measure  $Q$  is called an equivalent martingale measure. Condition (a) is the equivalent part. It states that events of  $P$  probability one are events of  $Q$  probability one (and vice-versa). Condition (b) is the martingale part. It states that under the probability  $Q$ ,  $Z_1(t)$  is a martingale.<sup>5</sup> Condition (6) gives a probabilistic characterization of no arbitrage opportunities.

An economic interpretation of this characterization is contained in Harrison and Kreps [1979]. For understanding, we paraphrase the economic intuition. Consider a simple dynamic trading strategy involving the stock and bond where we can adjust the portfolio only a finite number of times



over  $[0, T]$ . Let this strategy be self-financing, i.e. generating no cash inflows or outflows except at times 0 and  $T$ . Condition (6) then implies that there exists no such strategies which simultaneously satisfy:

- (i) positive or zero cash inflow at time 0,
- (ii) with  $P$  probability one, the cash inflow at time  $T$  is non-negative, and
- (iii) with positive  $P$  probability, the cash inflow at time  $T$  is strictly positive.

These strategies (if they existed) would be called an arbitrage opportunity or free lunch. The perhaps surprising aspect of (6) is that if markets are complete, these implications of (6) are also sufficient.<sup>6</sup> Given this characterization, the following theorem provides necessary and sufficient conditions for the existence of an equivalent martingale measure.

Theorem 1. (Existence of an Equivalent Martingale Measure)<sup>7</sup>

There is an equivalent martingale measure for  $\{Z_1(t), t \in [0, T]\}$  if and only if

- (i)  $Z_1(\tau-) = \lim_{t \uparrow \tau} Z_1(t)$  exists a.e.  $P$ .
- (ii) the "jump"  $V$  defined by  $V = Z_1(\tau) - Z_1(\tau-)$  satisfies  $P\{V > 0 | \mathcal{F}_{\tau-}\}(w) > 0$  if and only if  $P\{V < 0 | \mathcal{F}_{\tau-}\}(w) > 0$  for almost all  $w \in \Omega$ .
- (iii) the process  $Y(t)$  defined by  $Y(t) = Z_1(t) - V 1_{\{t \geq \tau\}}$  has an equivalent martingale measure.

Proof: In the appendix. ///

Given there are no arbitrage opportunities in the economy (i.e. there is an equivalent martingale measure for  $Z_1(t)$ ), theorem 1 demonstrates that the stock price need not fall by the amount of the dividend on the ex-dividend date. To see this interpretation, one must recognize that the stock price drops by the dividend if and only if  $V = (Z_1(\tau) - Z_1(\tau-)) \equiv 0$  for all  $\omega \in \Omega$  (by expression (5)). Yet,  $V \neq 0$  is consistent with no arbitrage opportunities by theorem 1 as long as two conditions are satisfied. First, the jump  $V$  must have a positive probability of being both positive and negative. This says that the stock price drop cannot be always below or always above the dividend. The economic intuition behind this condition is straightforward. To obtain a riskless position to take advantage of a price drop different from the dividend, an arbitrageur must take a position at  $\tau-$  and hold it until  $\tau$ . To know whether to buy or to sell, he must know whether the price will drop by less than or by more than the dividend. Otherwise, the position taken will not be riskless. The second condition says that the process without the jump included,  $Y(t) = Z_1(t) - V1_{\{t \geq \tau\}}$ , must have no arbitrage opportunities associated with it.

Another way to understand this result is to realize that if there exists an equivalent martingale measure for  $Z_1(t)$ , then from standard probability theorems, see Ikeda and Watanabe [1981; p. 33], we know that  $Z_1(t)$  has an equivalent version which is right continuous with left limits existing.<sup>8</sup> However, it need not be left-continuous at  $\tau$ . This is what expression (5) requires. Alternatively stated, the existence of an equivalent probability measure only implies that

$$E_Q(Z_1(\tau) | F_{\tau-h}) = Z_1(\tau-h) \text{ for all } h \geq 0 \quad (7)$$

We see that condition (5) holds in an expectational sense, not with probability one.

#### 4. Escrowed Dividend Stock Processes

This section of the paper provides a specific class of stock price processes to clarify the previous section's results. Let us suppose (without loss of generality) that the time period  $[0, T]$  occurs after the dividend declaration date. Having been declared, we assume that the dividend,  $d > 0$ , is no longer stochastic. The dividend component of the stock price is then separated out as follows:

$$X(t, w) = \begin{cases} S(t, w) + d & \text{if } t < \tau \\ S(t, w) - \alpha(w)d & \text{if } t \geq \tau \end{cases} \quad (8)$$

where  $\alpha: \Omega \rightarrow R$  is known at time  $\tau$ ,

$S: [0, T] \times \Omega \rightarrow R$ ,  $S(t, w)$  is known at time  $t$ ,<sup>9</sup>

$E_p(|S(t, w)|) < +\infty$  for all  $t \in [0, T]$ ,  $E_p(|\alpha(w)|) < +\infty$ ,

and  $\alpha(w)$  is not perfectly correlated to  $[S(\tau, w) - S(\tau-, w)]$ .

One can understand this decomposition of the stock price process as follows. After the declaration of the dividend, the dividend is "escrowed" and riskless. The risky component of the stock is  $S(t, w)$ . The dividend trades as part of the stock until the ex-dividend date  $\tau$ . At time  $\tau$ , the dividend is paid. The stock price drops by  $(1 + \alpha(w))$  of the dividend. Since the fraction  $\alpha(w)$  is random, this coefficient allows for the stock price to drop by less than  $d$  and in a random fashion. Furthermore, we assume that the random movement due to dividends is not perfectly correlated to the instantaneous change in the stock process over  $(\tau-, \tau]$ .

This process is common to option pricing, for example, if  $\alpha$  is constant,  $0 \leq \alpha \leq -1$  and  $\{S(t), t \in [0, T]\}$  is a geometric Brownian motion, we obtain the process underlying Roll's American call formula, see Roll [1977], Geske [1979], and Whaley [1981]. We will return to this process shortly.

The modified  $Z_1(t)$  process is now

$$Z_1(t, w) = \begin{cases} S(t, w) + d & \text{if } t < \tau \\ S(t, w) + [1 - \alpha(w)]d & \text{if } t \geq \tau \end{cases} \quad (9)$$

We now state the main theorem of this section.

Theorem 2. (Ex-dividend Stock Price Drop)

Under the assumptions above, in a no arbitrage opportunity economy, the stock price drops by precisely the amount of the dividend, i.e.

$P(Z_1(\tau^-) = Z_1(\tau)) = 1$  if and only if

- (i)  $S(t)$  is continuous at  $\tau$  a.e.  $P$  and
- (ii) given the information at time  $\tau^-$ , either  $\alpha \geq 0$  for sure or  $\alpha \leq 0$  for sure, i.e. for  $P$ -almost every  $w \in \Omega$ , either  $P(\alpha \in [0, \infty) | \mathcal{F}_{\tau^-})(w) = 1$  or  $P(\alpha \in (-\infty, 0] | \mathcal{F}_{\tau^-})(w) = 1$ .

Proof:

The hypothesis is that there exists an equivalent martingale measure for  $Z_1(t)$ . If  $S(\tau)$  is continuous a.e.  $P$ , (ii) implies by theorem 1 that  $\alpha(w) = 0$ . This implies  $Z_1(\tau^-) = Z_1(\tau)$  a.e.  $P$ . Conversely, if  $Z_1(\tau^-) = Z_1(\tau)$  a.e.  $P$  then  $S_1(\tau^-) - S_1(\tau) = -\alpha(w)d$  a.e.  $P$ . This contradicts the absence of a perfect negative correlation between  $S_1(\tau^-) - S_1(\tau)$  and  $\alpha(w)$  unless  $S_1(\tau^-) - S_1(\tau) = 0$  a.e.  $P$  and  $\alpha(w) = 0$  a.e.  $P$ . ///

This theorem states that in a no arbitrage opportunity economy on the ex-dividend date the stock price falls by the dividend if and only if:

- (i) the risky component of the stock process is continuous at  $\tau$ , and
- (ii) the random component of the drop in the stock price due to dividends  $\alpha(w)$  given the information at time  $\tau^-$  is known to be either always nonnegative or always nonpositive.

From condition (8), an  $\alpha > 0$  would correspond to a stock price drop of more than the dividend while an  $\alpha < 0$  would correspond to a stock price drop of less than the dividend. The economic intuition for these conditions is straightforward. To obtain a riskless position to take advantage of a price drop different from the dividend, the arbitrageur must take a position at  $\tau^-$  and hold it until  $\tau$ . To know whether to buy or sell, he must know whether the price will drop by less than the dividend or more than the dividend. If both are possible, the position to take is indeterminate. Furthermore, if a position is taken but there is a jump possible over the instant  $(\tau^-, \tau]$ , then this position is risky. The position will not present an arbitrage opportunity.

## 5. Conclusion

The preceding section gives necessary and sufficient conditions for the stock price to drop by the dividend on the ex-dividend date in a frictionless, no arbitrage opportunity economy (theorem 2). This section examines the implications of this theorem for two current research topics in finance. The two topics are option pricing and the estimation of a marginal tax bracket.

Recently, in the option pricing literature, Roll [1977], Geske [1979], and Whaley [1981] have developed a closed form solution for an American call option's value in a frictionless market where the underlying stock process satisfies condition (8) with  $\alpha(w)$  constant,  $-1 \leq \alpha \leq 0$ , and  $dS(t) = \delta S(t)dt + \sigma S(t)dB(t)$  where  $\delta, \sigma$  are constants and  $\{B(t), t \in [0, T]\}$  is a standard Brownian motion on  $(\Omega, (F_t), P)$ . This process satisfies the two hypotheses of theorem 2, hence,

$$Z_1(\tau) = Z_1(\tau-) \text{ a.e. } P$$

Equivalently, using (9) and the continuity of  $S(t)$ , this implies that  $\alpha = 0$ .

From a theoretical perspective, Roll's option pricing model is internally consistent if and only if  $\alpha = 0$ . Unfortunately, the existing empirical literature testing this model, see Whaley [1982] and Sterk [1982], utilize an  $\alpha \in (-1, 0)$ . The validity of the conclusions drawn in those studies is open to criticism since the model employed is internally inconsistent.

This theorem also has implications for the "short term traders" hypothesis of Kalay [1982], see also Kalay [1984] and Elton, Gruber, Rentzler [1984]. Let us briefly paraphrase this argument. Dividends and capital gains are not differentially taxed by short term traders. Excluding market frictions, if the stock price process satisfies the necessary and sufficient conditions in theorem 2, then in the absence of arbitrage opportunities the stock must fall by the dividend on the ex-dividend date. Otherwise, short term traders can generate arbitrage profits. In this case, the marginal tax rate is not reflected in the ex-dividend price drop. However, if the conditions of theorem 2 are not satisfied, then the short term trading hypothesis is not valid.

Unfortunately, the available evidence is inconsistent with the conditions of theorem 2. The stock price process does not appear to be continuous, see Oldfield, Rogalski, and Jarrow [1977] or Jarrow and Rosenfeld [1984]. Perhaps more damaging, however, is the observation that cross-sectionally  $\alpha(w)$  appears to be on either side of 0 with positive probability, see Campbell and Beranek [1955], Durand and May [1960], Elton and Gruber [1970], and Kalay [1982]. Hence, even in the absence of transaction costs, it appears that the short term trader hypothesis is probably not relevant. The ex-dividend stock price drop will most likely reflect both a risk premium and the marginal tax bracket.

FOOTNOTES

1. Formally,  $(F_t)_{t \in [0, T]}$  is taken to be an increasing family of sub  $\sigma$ -fields of  $F_T$  where  $F_0 = \{\emptyset, \Omega\}$  and  $(F_t)_{t \in [0, T]}$  is right continuous (see Ikeda and Watanabe [1981; p. 20] for the relevant definitions).
2. Formally,  $P_i: F_T \rightarrow [0, 1]$  is a probability measure over  $(\Omega, F_T)$ . Condition (1) requires  $P_i$  to be absolutely continuous with respect to  $P_j$  for all  $i$  and all  $j$ . Harrison and Kreps [1979] contain homogeneous beliefs, but condition (1) is an easy extension of their analysis, see Sethi [1984] or Jarrow [1984], by defining  $P = \sum_i P_i / I$  where  $I$  represents the number of investors in the economy.
3. Formally  $\{X(t): t \in [0, T]\}$  is adapted to  $(F_t)_{t \in [0, T]}$ , i.e.  $X(t)$  is  $F_t$  measurable for all  $t$ . We point out that Harrison and Kreps [1979]'s main analysis requires that  $X$  be square integrable, however, they state that this assumption could be relaxed (see also Sethi [1984]).
4. Formally,  $d: \Omega \rightarrow \mathbb{R}$  and  $d$  is  $F_T$  measurable.
5. Harrison and Kreps [1979] also require  $dQ/dP$ , the Radon-Nikodym derivative of  $Q$  with respect to  $P$ , to satisfy  $E_P[dQ/dP^2] < +\infty$ . Since our  $X$  is not square integrable, we do not impose this restriction.
6. The free lunch condition is itself necessary and sufficient for (6) if the market is complete in the sense that the trading strategies defined generate all elements of  $L^1(\Omega, F_T, P)$  at time  $T$ . This is



possible, for example, if we expand the number of assets trading at time 0, see Jarrow and Green [1984].

7. Two symbols need to be defined. First,  $F_{\tau^-} \equiv \bigcup_{t < \tau} F_t$ , i.e. this is the largest  $\sigma$ -field containing  $F_t$  for  $t < \tau$ . Second,  $P\{\cdot | F_{\tau^-}\}(w)$  is the conditional probability with respect to  $F_{\tau^-}$ .
8. This depends on  $(F_t)_{t \in [0, T]}$  being right continuous, see footnote 1.
9. Formally,  $\alpha$  is  $F_{\tau}$  measurable and  $\{S_t : t \in [0, T]\}$  is adapted to  $(F_t)_{t \in [0, T]}$ .

Appendix: Proof of Theorem 1.

Suppose  $Z_1$  has an equivalent martingale measure,  $Q$ . The following is known in this case.

- a)  $Z_1(\tau^-) = \lim_{t \uparrow \tau} Z_1(t)$  exists a.e.  $P$ .
- b)  $Z_1(\tau^-) = E_Q(Z_1(t) | \mathcal{F}_{\tau^-})$  for all  $t \geq \tau$ . Hence in particular,  $E_Q(V | \mathcal{F}_{\tau^-}) = 0$  a.e., which implies condition (ii) of the theorem.
- c) Set  $\tilde{V}(t) = V 1_{\{t > \tau\}}$ . Using the fact in b) above, it is easy to see that  $\tilde{V}$  is a martingale. But then  $Y = Z_1 - \tilde{V}$  must be a martingale.

This completes the proof in one direction.

Conversely, suppose conditions (i), (ii), and (iii) of the theorem are satisfied. Note that (i) implies  $\lim_{h \downarrow 0} Y(\tau-h)$  exists a.e. Let the equivalent martingale measure of  $Y$  be called  $\tilde{P}$ .

The basic idea is to modify  $\tilde{P}$  to get an equivalent probability measure  $Q$  so that  $Y$  remains a martingale and the process  $\tilde{V} = V 1_{\{t > \tau\}}$  is also a martingale. Under this condition,  $Z_1 = Y + \tilde{V}$  will be a martingale with respect to the equivalent probability measure  $Q$  and the proof will be complete.

To this end, set:

$$W(w) = \begin{cases} 1 & \text{if } \tilde{P}\{V=0 | \mathcal{F}_{\tau^-}\} = 1 \\ \frac{\tilde{E}(V^- | \mathcal{F}_{\tau^-}) 1_{\{V \geq 0\}} + \tilde{E}(V^+ | \mathcal{F}_{\tau^-}) 1_{\{V < 0\}}}{\tilde{E}(V^- | \mathcal{F}_{\tau^-}) \tilde{P}(V \geq 0 | \mathcal{F}_{\tau^-}) + \tilde{E}(V^+ | \mathcal{F}_{\tau^-}) \tilde{P}(V < 0 | \mathcal{F}_{\tau^-})} & \text{otherwise} \end{cases}$$

where

$$V^+ = \max(V, 0) \text{ and } V^- = -\max(-V, 0).$$

Now set  $Q = \tilde{W}P$ . This implies that  $Q$  is a probability measure equivalent to  $\tilde{P}$ . (Since  $\tilde{P}$  is equivalent to  $P$ , this gives  $Q$  equivalent to  $P$ .)

We claim that  $Y$  and  $\tilde{V}$  are  $Q$  martingales. For ease of exposition, we shall pretend that  $W$  is always given by the second line of its definition. Set  $W(s) = \tilde{E}(W|F_s)$ . Notice that  $W(s) = 1$  for  $s < \tau$  and  $W(s) = W$  for  $s \geq \tau$ . It is well known that showing  $Y$  or  $\tilde{V}$  is a  $Q$  martingale is equivalent to showing  $W(s)Y(s)$  or  $W(s)\tilde{V}(s)$  is a  $\tilde{P}$  martingale.

For  $\tilde{V}$ : We must show that if  $s < t$ ,  $\tilde{E}(\tilde{V}(t)W(t)|F_s) = \tilde{V}(s)W(s)$ . Consider three cases:

Case 1:  $\tau \leq s < t$ .

$$\tilde{E}(\tilde{V}(t)W(t)|F_s) = \tilde{E}(VW|F_s) = VW \text{ since } W \text{ and } V \text{ are } F_s\text{-measurable.}$$

Case 2:  $s < \tau \leq t$ .

$$\tilde{E}(\tilde{V}(t)W(t)|F_s) = \tilde{E}(VW|F_s) = \tilde{E}(\tilde{E}(VW|F_{\tau-})|F_s). \text{ Now compute:}$$

$$\tilde{E}(VW|F_{\tau-}) = \frac{\tilde{E}(V^-|F_{\tau-})\tilde{E}(V^+|F_{\tau-}) - \tilde{E}(V^+|F_{\tau-})\tilde{E}(V^-|F_{\tau-})}{\text{denominator}}$$

which is clearly zero, and hence equal to  $\tilde{V}(s)W(s)$ .

Case 3:  $s < t < \tau$ .

$$\tilde{V}(t) = 0 \text{ in this case, so } \tilde{E}(\tilde{V}(t)W(t)|F_s) = 0 = \tilde{V}(s)W(s).$$

Hence  $\tilde{V}$  is a  $Q$ -martingale.

For  $Y$ , the only interesting case is  $s < \tau \leq t$ . Notice that  $Y$  is continuous at time  $\tau$ , so that  $Y(\tau)$  is  $F_{\tau-}$  measurable.

Now,  $\tilde{E}(Y(t)W(t) | F_S) = \tilde{E}(Y(t)W | F_S) = \tilde{E}(\tilde{E}(Y(t)W | F_{\tau}) | F_S)$ . But  $\tilde{E}(WY(t) | F_{\tau}) = W\tilde{E}(Y(t) | F_{\tau}) = WY(\tau)$  since  $Y$  is a  $\tilde{P}$  martingale. Hence  $\tilde{E}(Y(t)W(t) | F_S) = \tilde{E}(WY(\tau) | F_S) = \tilde{E}(\tilde{E}(WY(\tau) | F_{\tau-}) | F_S)$ . But we know  $Y(\tau)$  is  $F_{\tau-}$  measurable, so  $\tilde{E}(WY(\tau) | F_{\tau-}) = Y(\tau-)\tilde{E}(W | F_{\tau-}) = Y(\tau-)$ . Summarizing,  $\tilde{E}(Y(t)W(t) | F_S) = \tilde{E}(Y(\tau-) | F_S) = Y(s) = Y(s)W(s)$ . ///

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