EX-HOMOTOPY THEORY I

BY
I. M. JAMES

This is the first of a series of studies of a generalization of ordinary homotopy theory. The basic notion is simple enough and seems to have occurred more or less simultaneously to others as well as myself. I understand that Heller and Hodgkin, independently, have given applications to the Eilenberg-Moore spectral sequence. Also McLendon [4] has announced a generalization of the Adams spectral sequence on these lines. My own work is directed towards applications of a different type. Some of these are contained in [3], to which the present note is closely related. Others will be given subsequently.

1. Basic notions

Let B be a space. By an *ex-space* (over B) we mean a triple (X, σ, ρ), where X is a space and

$$B \xrightarrow{\sigma} X \xrightarrow{\rho} B$$

are maps such that $\rho\sigma = 1$. Normally it will be sufficient to denote the ex-space by X. We refer to σ as the *section*, to ρ as the *projection*. Together they constitute an *ex-structure* on the *total space* X over the *base space* B. Notice that B can always be regarded as an ex-space over itself, with $\rho = 1 = \sigma$. We refer to this as the *trivial ex-space* over the given base.

We describe an ex-space X as *proper* if σB is a closed subspace of X. When this condition is satisfied we can embed B in X, by means of σ, so that ρ constitutes a retraction. Instead of regarding B as a retract of X we regard X as an "extract" of B. This change of view opens up the prospect of the following development.

We shall outline a theory which reduces to ordinary homotopy theory when B is a point. The generalization proceeds on formal lines, for the most part; whenever we meet a basepoint, in the ordinary theory, we replace it by B, using the section and projection in an appropriate way.

Starting from the given base space we have begun to construct a new category out of the category of topological spaces. The objects in the new category are ex-spaces. We now define the morphisms. Let X_i $(i = 0, 1)$ be an ex-space over B with section σ_i and projection ρ_i. By an *ex-map* $f : X_0 \to X_1$ we mean an ordinary map such that

$$(1.1) \qquad f\sigma_0 = \sigma_1, \qquad \rho_1 f = \rho_0,$$

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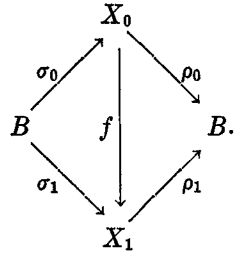
as shown in the following diagram:



By the *trivial ex-map* of X_0 to X_1 we mean $\sigma_1\rho_0$, which satisfies (1.1) since $\rho_i\sigma_i = 1$. Note that the composition of any ex-map with a trivial ex-map, on either side, is again a trivial ex-map. Thus the category of ex-spaces and ex-maps is a pointed category. The equivalences in the category will be called *ex-homeomorphisms*.

Let X be an ex-space with section σ and projection ρ. We describe a subspace X' of X as *admissible* if it contains the image of σ. Under this condition we can regard X' as an ex-space, with section σ' obtained by restricting the codomain of σ, and projection ρ' obtained by restricting the domain of ρ. We refer to this as the *relative* ex-structure and to X', with this ex-structure, as a *subspace* of the ex-space X. Note that the inclusion map $X' \to X$ is an ex-map.

Let X'' be the space obtained from X by identifying points of X' which have the same image under the projection. When X' is admissible we can give X'' ex-structure so that the natural projection $X \to X''$ is an ex-map. We denote the ex-space thus obtained by X/X' and refer to X'' as the ex-space obtained from X by collapsing X'. When $X' = \sigma B$, in particular, we have $X'' = X$.

In our category products are defined as follows. Let X_i $(i = 0, 1)$ be an ex-space over B. The *direct product* $X_0 \times X_1$ is the subspace of the ordinary topological product consisting of pairs (x_0, x_1) such that $\rho_0 x_0 = \rho_1 x_1$, with the section σ and projection ρ given by

$$\sigma b = (\sigma_0 b, \sigma_1 b) \qquad (b \in B)$$

$$\rho(x_0, x_1) = \rho_0 x_0 = \rho_1 x_1 \qquad (x_i \in X_i).$$

The *inverse product*, or *wedge sum*, $X_0 \vee X_1$ can be defined as the subspace of the direct product consisting of pairs (x_0, x_1) such that $x_0 = \sigma_0 \rho_1 x_1$ or $x_1 = \sigma_1 \rho_0 x_0$. Structural ex-maps

$$X_0 \leftarrow X_0 \times X_1 \rightarrow X_1, \qquad X_0 \rightarrow X_0 \vee X_1 \leftarrow X_1$$

are defined in the obvious way. The *smash product* $X_0 \wedge X_1$ is defined to be the ex-space obtained from $X_0 \times X_1$ by collapsing $X_0 \vee X_1$. Direct, inverse and smash products of ex-maps are similarly defined.

If X is an ex-space over B then the structural ex-maps constitute ex-homeomorphisms $X \times B \to X \to X \vee B$. Moreover $X \wedge B$ is ex-homeomorphic to B.

Pull-backs and push-outs are also defined in our category. Thus let X, X_i $(i = 0, 1)$ be ex-spaces, over B, and let $f_i : X_i \to X$ be an ex-map. Then the pull-back X' of (f_0, f_1) is defined to be the subspace of the direct product ex-space $X_0 \times X_1$ consisting of pairs (x_0, x_1) such that $f_0 x_0 = f_1 x_1$, where $x_i \in X_i$. The structural ex-maps of the direct product determine, by restriction, ex-maps $g_i : X' \to X_i$, with the appropriate formal properties to complete the pull-back structure. Push-outs are similarly defined, using the inverse rather than the direct product.

The basic procedure we have followed can be applied to any category $\mathcal{C}$. Having chosen an object B of $\mathcal{C}$ we define the corresponding ex-category $\mathcal{C}_B$ of ex-objects and ex-morphisms. The ex-category is pointed, in any case. If $\mathcal{C}$ admits pull-backs and direct products, or push-outs and inverse products, then $\mathcal{C}_B$ does the same. If $\mathcal{C}$ is a pointed category, moreover, then functors from $\mathcal{C}$ into $\mathcal{C}_B$ are defined by taking the direct or inverse product with B. Although these functors are not unimportant in the topological case, more interesting functors into the category of ex-spaces can be defined as follows.

Let G be a compact Lie group with identity e, and let A be a completely regular space. Suppose that G acts on A as a topological transformation group. Then we describe A as a *G-space*. The image of $x \in A$ under $g \in G$ will be denoted by $g \cdot x$. We describe A as a *pointed* G-space if A is a pointed space, in the ordinary sense, such that

$$g \cdot x_0 = x_0 \qquad (g \in G) \tag{1.2}$$

where $x_0 \in A$ is the basepoint. For example the sphere S^n is a pointed $O(n)$-space, provided the basepoint is taken to be one of the poles.

We adopt the basic terminology of G-spaces, as given in [5], with appropriate modifications where the basepoint is concerned. Thus if A_i $(i = 0, 1)$ is a pointed G-space then a pointed G-map $f : A_0 \to A_1$ means an ordinary pointed map such that

$$f(g \cdot x) = g \cdot f(x) \qquad (g \in G,\ x \in A_0). \tag{1.3}$$

This condition is satisfied when f is constant. Hence the category of pointed G-spaces and pointed G-maps is a pointed category. The direct or inverse product of pointed G-spaces is defined in the obvious way.

Now let P be a principal G-bundle with base space B. Given a pointed G-space A, consider the associated bundle E over B with A as fibre. Since (1.2) is satisfied we have a canonical cross-section of E, defined as in (9.3) of [7], and so E determines an ex-space $P_*(A)$ over B. Moreover let $f : A_0 \to A_1$ be a pointed G-map, where A_i $(i = 0, 1)$ is a pointed G-space. Let E_i denote the associated bundle with fibre A_i. The bundle map $E_0 \to E_1$ associated with f respects the canonical cross-sections and hence determines an ex-map

$$P_*(f) : P_*(A_0) \to P_*(A_1).$$

Thus P_* constitutes a functor from the category of pointed G-spaces to the category of ex-spaces over B. Moreover P_* respects the direct and inverse products in the sense that the transform of the product is naturally ex-homeomorphic to the product of the transforms.

2. Ex-homotopy

Let X, Y be ex-spaces, over B. By an *ex-homotopy*

$$f_t : X \to Y \qquad (t \in I)$$

we mean an ordinary homotopy such that f_t is an ex-map for all values of t. We write $f_0 \simeq f_1$, when such an ex-homotopy exists. This defines an equivalence relation, on the set of ex-maps of X into Y, and we denote by $\pi(X, Y)$ the set of ex-homotopy classes thus obtained. The class of the trivial ex-map is denoted by 0.

Let Z be an ex-space, over B. Composition on the left with an ex-map $g : Y \to Z$ determines a function

$$g_* : \pi(X, Y) \to \pi(X, Z),$$

while composition on the right with an ex-map $f : X \to Y$ determines a function

$$f^* : \pi(Y, Z) \to \pi(X, Z).$$

Both functions send 0 into 0.

Let E be an ex-space. We say that a sequence of ex-maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact for $\pi(E, \ )$ if we have image f_* = kernel g_* in the induced sequence

$$\pi(E, X) \xrightarrow{f_*} \pi(E, Y) \xrightarrow{g_*} \pi(E, Z).$$

We say that the sequence of ex-maps is exact for $\pi(\ , E)$ if we have image g^* = kernel f^* in the induced sequence

$$\pi(X, E) \xleftarrow{f^*} \pi(Y, E) \xleftarrow{g^*} \pi(Z, E).$$

Similar definitions are made in the case of longer sequences of ex-maps.

Let π_i $(i = 0, 1)$ denote the structural ex-maps of the direct product of ex-spaces X_i; thus

$$X_0 \xleftarrow{\pi_0} X_0 \times X_1 \xrightarrow{\pi_1} X_1.$$

An ex-map $h : X \to X_0 \times X_1$ determines, and is determined by, its component ex-maps $(\pi_0 h, \pi_1 h)$, where $\pi_i h : X \to X_i$, and ex-homotopies behave in the same way. Thus we obtain a natural equivalence

$$\pi(X, X_0 \times X_1) \leftrightarrow \pi(X, X_0) \times \pi(X, X_1).$$

Similarly in the case of the inverse product we obtain a natural equivalence

$$\pi(X_0 \vee X_1, X) \leftrightarrow \pi(X_0, X) \times \pi(X_1, X).$$

We say that an ex-map $f : X \to Y$ is an *ex-homotopy equivalence* if there exists an ex-map $g : Y \to X$ such that $fg \simeq 1$, $gf \simeq 1$. When such a pair of ex-maps exists we say that X and Y have the same *ex-homotopy type*. If X has the same ex-homotopy type as the base space B, we say that X is *ex-contractible*. For example, take X to be the total space of a vector bundle over B, with ρ the fibration and σ the zero cross-section. Then $f_t : X \to X$ constitutes an ex-homotopy where

$$f_t(x) = tx \qquad (x \in X,\ t \in I).$$

Since f_0 is the trivial ex-map and f_1 is the identity this shows that X is ex-contractible.

By a formal modification of the proof of Lemma 3 of [6] we obtain

LEMMA (2.1). *Let X be an ex-space with an admissible sub-space X'. Suppose that the identity ex-map on X is ex-homotopic to an ex-map whose restriction to the ex-space X' is trivial. Then the natural ex-map $X \to X/X'$ is an ex-homotopy equivalence.*

In some cases the set $\pi(X, Y)$ can be expressed in terms of ordinary homotopy theory. For example, let C be a space with basepoint c_0 and let TC denote the ordinary topological product $B \times C$ with ex-structure given by

$$\rho(b, c) = b, \qquad \sigma b = (b, c_0),$$

where $b \in B$, $c \in C$. Let X be any proper ex-space, over B, and let B be embedded in X through the section. Then there is a (1-1)-correspondence between ex-maps $X \to TC$ and ordinary maps

$$(X, B) \to (C, c_0).$$

Furthermore there is a (1–1)-correspondence between such maps and maps

$$(Z, z_0) \to (C, c_0),$$

where Z denotes the space obtained from the space X by collapsing the subspace B to the point z_0. Similar remarks apply in the case of homotopies and so $\pi(X, TC)$ is equivalent to the set of homotopy classes of maps of Z into C in the ordinary basepoint-preserving sense. Note that Z can be interpreted as a Thom space when X is the ex-space associated with a sphere-bundle over B.

As an example, with arbitrary C, take $B = S^n$ and $X = TS^q$. Then Z has the homotopy type of $S^{n+q} \vee S^q$ and so $\pi(TS^q, TC)$ is equivalent to the set

$$\pi_{n+q}(C) \oplus \pi_q(C).$$

A similar analysis can be made of the set $\pi(B \vee C, Y)$, for any ex-space Y, where $B \vee C$ denotes the ordinary inverse product with the obvious ex-structure.

Let G be a compact Lie group. The notion of pointed G-homotopy between pointed G-maps is defined in the obvious way. If P is a principal G-bundle over B, as in §1, then the functor P_* transforms pointed G-homotopies into ex-homotopies. Hence P_* transforms pointed G-homotopy equivalences into ex-homotopy equivalences.

3. The suspension functor

Let X be a proper ex-space, over B. We embed B in X, by means of the section σ, so that the projection ρ constitutes a retraction of X on B. By the *suspension* of X, in the ex-category, we mean the ex-space (SX, σ', ρ') defined as follows. Consider the ordinary cylinder $X \times I$, and write

$$\pi(x, t) = \rho x \qquad (x \in X, t \in I).$$

Then SX is obtained from $X \times I$ by identifying points of $B \times I \cup X \times \dot{I}$ which have the same image under π. The section σ' is given by $\sigma' b = (b, t)$, for any t, and the projection ρ' is induced by π. Note that SX is a proper ex-space.

It is easy to check that SX is ex-homeomorphic to the smash product $X \wedge TS^1$, in the ex-category, where T is the functor defined at the end of §2. Also the suspension of the wedge-sum is ex-homeomorphic to the wedge-sum of the suspensions.

Suspension of ex-maps and ex-homotopies is similarly defined. We denote by

$$S_* : \pi(X_0, X_1) \to \pi(SX_0, SX_1)$$

the function thus obtained.[1]

Now form the wedge sum of SC with itself and consider the structural ex-maps

$$SX \xrightarrow{u_0} SX \vee SX \xleftarrow{u_1} SX.$$

There is an ex-map $m : SX \to SX \vee SX$, defined by

$$\begin{aligned} m(x, t) &= u_0(x, 2t) \qquad (0 \leq t \leq \tfrac{1}{2}), \\ &= u_1(x, 2t - 1) \quad (\tfrac{1}{2} \leq t \leq 1). \end{aligned}$$

Using the natural identification

$$\pi(SX \vee SX, Y) = \pi(SX, Y) \times \pi(SX, Y),$$

where Y is any ex-space, we can regard the induced function m^* as a binary operation on the set $\pi(SX, Y)$. We call this operation *track addition* and

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[1] In §7 of [3] a suspension theorem, of the Freudenthal type, is proved for S_* under certain conditions.

normally write

$$m^*(\alpha_0, \alpha_1) = \alpha_0 + \alpha_1 \qquad (\alpha_i \in \pi(SX, Y))$$

without meaning to suggest that the operation is commutative. The formal properties of m are the same as in the ordinary theory. Thus the ex-homotopy-associativity property

$$(1 \vee m)m \simeq (m \vee 1)m$$

is established, by the same argument as in the ordinary case, and it follows that track addition is associative. Similarly for the other properties, so that we finally obtain

THEOREM (3.1). *Under track addition the set $\pi(SX, Y)$ forms a group. If X has the same ex-homotopy type as SX', for some ex-space X', then the group is abelian.*

Note that if $f : Y \to Z$ is an ex-map then the induced function

$$f_* : \pi(SX, Y) \to \pi(SX, Z)$$

is a homomorphism of track groups.

If A is an ex-space over a point, i.e. a pointed space, then SA is the usual (reduced) suspension. Suppose that A is a pointed G-space, where G is a compact Lie group. Then SA is also a pointed G-space, with the action of $g \in G$ on SA defined to be the suspension of the action of g on A. Let P be a principal G-bundle over B, where B is regular and locally compact. I assert that $SP_*(A)$ is naturally ex-homeomorphic to $P_*(SA)$. In other words, the suspension functor commutes with P_*, to within a natural equivalence in the category of ex-spaces. Some other propositions of this type occur in what follows and can be proved in a similar way.[2]

To prove the assertion consider the associated bundle E over B with fibre A. As shown in (3.2) of [7], we can construct E as an identification space of a disjoint union W of spaces

$$W_j = (j) \times V_j \times A \qquad (j \in J)$$

where J is an indexing set and $\{V_j\}$ is a covering of the base space B. The identifications are given in terms of the maps

$$g_{ij} : V_i \cap V_j \to G \qquad (i, j \in J).$$

If $x \in V_i \cap V_j$ and $y, z \in A$ we identify (i, x, y) with (j, x, z) when $g_{ij}(x) \cdot y = z$. Let E' denote the bundle constructed by making corresponding identifications in the disjoint union W' of the spaces

$$W'_j = (j) \times V_j \times SA \qquad (j \in J).$$

[2] I have not been able to establish a "continuous functor" lemma, on the lines of (1.2) of [1].

The product of the identity on $(j) \times V_j$ and the identification map

$$A \times I \rightarrow SA$$

constitutes a map $W_j \times I \rightarrow W'_j$. We form the union f of these product maps, as shown in the following diagram, where p, p' are the natural maps of the bundle construction and g is induced by f:

$$\begin{array}{ccc} W \times I & \xrightarrow{p \times 1} & E \times I \\ \downarrow f & & \downarrow g \\ W' & \xrightarrow[p']{} & E'. \end{array}$$

Without change of notation, consider the ex-spaces of B determined by E, E', with their canonical cross-sections. The suspension is defined, as above, through making identifications on $E \times I$. It is easy to check that g is consistent with these identifications and induces a bijection $h : SE \rightarrow E'$. Moreover, h is an ex-map, and has the relevant naturality properties. Up to this stage the restriction on B is not required.

In the usual version of fibre bundle theory, as in [7], the co-ordinate neighbourhoods are open sets of the base space. But it is also possible to develop a theory where the co-ordinate neighbourhoods are compact subspaces whose interiors form a covering. Every bundle in this theory is a bundle in the usual sense, and the converse is true when the base space is regular and locally compact. By hypothesis, B satisfies this condition and so we can take each of the neighbourhoods V_j to be compact. Then f, in our diagram, is an identification map, and it follows at once that h is a homeomorphism. Furthermore, h is an ex-homeomorphism since h is an ex-map. This proves our assertion, since

$$E = P_*(A), \qquad E' = P_*(SA).$$

4. Well-based ex-spaces

Given an ex-space (X, σ, ρ), over B, we construct an ex-space $(\hat{X}, \hat{\sigma}, \hat{\rho})$, over B, as follows. We form $\hat{X}$ from the union of X and the cylinder $B \times I$ by identifying $\sigma b \in X$ with $(b, 0)$ for all $b \in B$. The section $\hat{\sigma}$ is given by $\hat{\sigma} b = (b, 1)$. Let $\pi : \hat{X} \rightarrow X$ be given on X by the identity and on $B \times I$ by $\pi(b, t) = b$. Then $\pi\hat{\sigma} = \sigma$ and we take $\hat{\rho} = \rho\pi$, so as to make π an ex-map. Note that X is not an admissible subspace of $\hat{X}$.

We describe the ex-space X as *well-based* if the ex-map $\pi : \hat{X} \rightarrow X$ is an ex-homotopy equivalence. It is a simple exercise to prove

THEOREM (4.1). *For any ex-space X, the ex-space $\hat{X}$ is well-based.*

Let $f : X \rightarrow Y$ be an ex-map, where X, Y are ex-spaces over B. Then $\pi\hat{f} = f\pi$, where $\hat{f} : \hat{X} \rightarrow \hat{Y}$ is the ex-map given by f on X, by the identity on

$B \times I$. Similarly with ex-homotopies, and so we obtain

THEOREM (4.2). *If X is well-based and Y has the same ex-homotopy type as X then Y is well-based.*

Note that the well-based ex-spaces over a point are the well-pointed spaces of ordinary homotopy theory.

One of the uses of the present construction is to facilitate comparisons between ex-spaces with the same total space and projection but with different sections. We prove

THEOREM (4.3). *Let X be a space, let $\rho : X \to B$ be a map, and let $\sigma_t : B \to X$ be a homotopy such that $\rho\sigma_t = 1$. Then $\hat{X}_0$ and $\hat{X}_1$ have the same ex-homotopy type, where $X_t = (X, \sigma_t, \rho)$.*

COROLLARY (4.4). *If X_0 and X_1 are well-based, in* (4.3), *then X_0 and X_1 have the same ex-homotopy type.*

Let $h : \hat{X}_0 \to \hat{X}_1$ be given by the identity on X and on $B \times I$ by

$$h(b, t) = \sigma_{2t} b \qquad (0 \leq t \leq \tfrac{1}{2}),$$
$$= (b, 2t - 1) \quad (\tfrac{1}{2} \leq t \leq 1).$$

Let $k : \hat{X}_1 \to \hat{X}_0$ be similarly defined, using σ_{1-t} instead of σ_t. Then h and k are ex-maps and it is easy to check that kh and hk are ex-homotopic to the identity ex-maps. This proves the theorem.

Let G be a compact Lie group and let A be a pointed G-space with basepoint $x_0 \in A$. Form $\hat{A}$ from the union of A and $x_0 \times I$, as above, and extend the action of G on A to an action of G on $\hat{A}$ so that points of $x_0 \times I$ are left fixed. The natural projection $\pi : \hat{A} \to A$ is a pointed G-map. We say that A is a well-pointed G-space if π is a pointed G-homotopy equivalence.[3] Consider the functor P_* determined by a principal G-bundle P over B, where B is regular and locally compact. Proceeding in much the same way as in the case of the suspension functor we find that $P_*(\hat{A})$ is ex-homeomorphic to $\hat{E}$, where $E = P_*(A)$, and that the ex-homeomorphism carries the transform of $\pi : \hat{A} \to A$ into $\pi : \hat{E} \to E$. When A is well-pointed, as a G-space, the transform of π is an ex-homotopy equivalence and so E is well-based, as an ex-space.

5. Ex-cofibrations

Let $f : X \to Y$ be an ex-map, where X, Y are ex-spaces over B. We say that f is an *ex-cofibration* if f has the following *ex-homotopy extension property*. Let Z be an ex-space, let $h : Y \to Z$ be an ex-map, and let $g_t : X \to Z$ be an ex-homotopy such that $g_0 = hf$. Then there exists an ex-homotopy $h_t : Y \to Z$ such that $h_0 = h$ and $g_t = h_t f$. When f has this property we define the *ex-cofibre* of the ex-cofibration to be the push-out of $f : X \to Y$ and $\rho : X \to B$.

[3] This is the case, for example, if A is a differentiable G-space (i.e. a paracompact differentiable manifold with G acting differentiably).

An important special case is when Y contains an admissible subspace Y' such that the inclusion $Y' \subset Y$ is an ex-cofibration. In that case we prefer to say that the pair (Y, Y') has the ex-homotopy extension property. A necessary and sufficient condition for this property is for $Y \times 0 \cup Y' \times I$ to be an ex-retract of $Y \times I$. Note that Y/Y' is the ex-cofibre in this case. Just as in Theorem 2 of [6] we obtain

THEOREM (5.1). *Suppose that Y' is ex-contractible, and that (Y, Y') has the ex-homotopy extension property. Then the natural projection $Y \to Y/Y'$ is an ex-homotopy equivalence.*

In ordinary homotopy theory complexes play a special role, but it seems doubtful whether there is a satisfactory generalisation to ex-homotopy theory. We need a reasonably comprehensive class of ex-spaces with various desirable properties. After some experiment I propose the following, which is adequate for the type of application I have in mind.

Let A be a G-space, where G is a compact Lie group. We recall that a subspace A' of A is described as *invariant* if $gA' \subset A'$ for all $g \in G$. When this condition is satisfied we regard A' as a G-space with the induced action. As in (1.6.1) of [5] we say that A is a *G-ANR* (absolute neighbourhood retract) if, given a normal G-space Y and a G-map f of a closed invariant subspace F of Y into A, there exists an extension of F to a G-map of an invariant neighbourhood of F. It has been shown (see p. 27 of [5] for references) that every compact differentiable G-space is a *G-ANR*. Note that the (finite) product of *G-ANR*'s is again a *G-ANR*.

Let A be a normal pointed G-space which is a *G-ANR*. Let A' be a closed invariant pointed subspace of A. Make $A \times I$ into a G-space, with action given by

$$g_\bullet (x, t) = (g_\bullet x, t) \qquad (g \in G, x \in A, t \in I).$$

I assert that $A \times 0 \cup A' \times I$ is a G-retract of $A \times I$. For since $A \times I$ is a *G-ANR* there exists an invariant neighbourhood U of $A \times 0 \cup A' \times I$ in $A \times I$, such that $A \times 0 \cup A' \times I$ is a G-retract of U. Let V be an invariant neighbourhood of A' in A such that $V \times I \subset U$. By (1.1.7) of [5] there exists an invariant map $p : A \to I$ such that $p(x) = 0$ if $x \notin V$ and $p(x) = 1$ if $x \in A'$. If f is a G-retraction of U on $A \times 0 \cup A' \times I$ then f' is a G-retraction of $A \times I$, where

$$f'(x, t) = f(x, tp(x)) \qquad (x \in A, t \in I).$$

This proves the assertion.

Now consider the ex-spaces $E = P_*(A)$, $E' = P_*(A')$, where P is a principal G-bundle over B. Since $A \times 0 \cup A' \times I$ is a G-retract of $A \times I$ it follows that $E \times 0 \cup E' \times I$ is an ex-retract of $E \times I$, and so the pair (E, E') has the ex-homotopy extension property. Suppose, furthermore, that A' is G-contractible. Then E' is ex-contractible and hence the natural projection of

E onto E/E' is an ex-homotopy equivalence, by (5.1). We shall use this in the next section.

6. Spaces over B

By a space over B we mean an ordinary space X with a map $\rho : X \to B$. Usually it is sufficient to denote the pair (X, ρ) by X alone. By a section of (X, ρ) we mean a map $\sigma : B \to X$ such that $\rho\sigma = 1$. An ex-space over B can be regarded as a space over B with a section. Maps and homotopies over B are defined in the obvious way.

The suspension $(\tilde{S}X, \rho')$ of a proper space (X, ρ) over B is defined as follows. The space $\tilde{S}X$ is formed from the union of $X \times I$ and $B \times \dot{I}$ by identifying (x, t) with $(\rho x, t)$ for $x \in X$, $t \in \dot{I}$. The map ρ' is given by

$$\rho'(x, t) = \rho x \qquad (x \in X,\ t \in I),$$

$$\rho'(b, t) = b \qquad (b \in B,\ t \in \dot{I}).$$

We have $\rho'\sigma_t = 1$, where $\sigma_t : B \to \tilde{S}X$ is defined by $\sigma_t b = (b, t)$ for $t = 0, 1$. The ex-space $(\tilde{S}X, \sigma_t, \rho')$ will be denoted by $\tilde{S}_t X$. Note that $\tilde{S}_0 X$ is ex-homeomorphic to $\tilde{S}_1 X$.

Now suppose that X admits a section σ, with $\rho\sigma = 1$, and so constitutes an ex-space. The definition of σ_t can be extended to all $t \in I$, by $\sigma_t b = (\sigma b, t)$. Thus a family $\tilde{S}_t X$ of ex-spaces is derived from X. When $0 < t < 1$ it is easy to see, by reparametrization, that $\tilde{S}_t X$ is independent of the choice of t, to within an ex-homeomorphism.

For any $t \in I$, the identity function on $X \times I$ determines an ex-map

$$\varphi : \tilde{S}_t X \to SX.$$

Now $B \times I$ constitutes an admissible subspace of $\tilde{S}_t X$, and if the pair

$$(\tilde{S}_t X, B \times I)$$

has the ex-homotopy extension property it follows at once from (5.1) that φ is an ex-homotopy equivalence. When this is the case we can replace $\tilde{S}_t X$ by SX, for many purposes.

When the base space is a point $\tilde{S}A$ is the unreduced suspension of the space A. Suppose that A is a G-space, and regard $\tilde{S}A$ as a G-space in the obvious way. Suppose that A is a compact differentiable pointed G-space. Then it follows[4] from (2.7.9) of [5] that $\tilde{S}A$ is a G-ANR, and hence that the natural projection $\varphi : \tilde{S}_t A \to SA$ is a G-homotopy equivalence.

Let P be a principal G-bundle over B, where B is regular and locally compact. If A satisfies the above conditions, so that φ is a G-homotopy equivalence, then the transform

$$P_*(\varphi) : P_* \tilde{S}_t(A) \to P_* S(A)$$

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[4] It seems reasonable to conjecture that $\tilde{S}A$ is a G-ANR if A is a G-ANR.

lent to $SP_*(A)$, under an ex-homomorphism. A similar argument shows that is an ex-homotopy equivalence. We have shown in §3 that $P_* S(A)$ is equiva- $P_* \tilde{S}_t(A)$ is equivalent to $\tilde{S}_t P_*(A)$ and that $P_*(\varphi)$ is equivalent to

$$\varphi : \tilde{S}_t P_*(A) \to SP_*(A).$$

Hence φ is an ex-homotopy equivalence. Of course this can also be deduced from (5.1).

It is convenient to take $t = \frac{1}{2}$ as standard, and henceforth $\tilde{S}X$ will mean the ex-space with this section. When φ is an ex-homotopy equivalence we identify $\pi(SX, Y)$ with $\pi(\tilde{S}X, Y)$ under φ^*, so that the track addition defined in §3 is transferred to the latter set. Thus $\pi(\tilde{S}X, Y)$ obtains a natural group structure, which is abelian if $X = \tilde{S}X'$ for some ex-space X'.

In the applications we usually begin with a euclidean bundle V over B, and take P to be the associated principal bundle. Consider the Whitney sum $V \oplus r$ of V and the trivial r-plane bundle, where $r = 1, 2, \cdots$. If we denote the associated sphere-bundle by square brackets then $[V \oplus r]$ can be identified with the iterated suspension $\tilde{S}_r[V]$. Thus

$$\pi([V \oplus r], Y)$$

constitutes a group for $r \geq 1$, an abelian group for $r \geq 2$. In the sequel to this note we shall study the structure of these ex-homotopy groups in some detail.

The unreduced suspension is a special case of the join[5] operation, which can be discussed on similar lines. Thus let X_i $(i = 0, 1)$ be a space over B. In this category the direct product $X_0 \times X_1$ is the subspace of the ordinary topological product consisting of pairs (x_0, x_1) $(x_i \in X_i)$ such that $\rho_0 x_0 = \rho_1 x_1$, with projection ρ given by $\rho(x_0, x_1) = \rho_i x_i$. Let $X_0 * X_1$ denote the space formed from the union of the cylinder $(X_0 \times X_1) \times I$ and X_0, X_1 by identifying (x_0, x_1, i) with x_i for $i = 0, 1$. The join of X_0 and X_1 in the category of spaces over B is defined to be this space $X_0 * X_1$ with projection ρ' given by

$$\rho'(x_0, x_1, t) = \rho_i x_i \qquad (x_i \in X_i,\ t \in I).$$

If $X_1 = I \times B$, and ρ_1 is right projection, then $X_0 * X_1$ is equivalent to $\tilde{S}X_0$.

Next suppose that X_1 admits a section σ_1, and so constitutes an ex-space. Then we can define a section σ' of the join $X_0 * X_1$ by $\sigma' b = (x_0, \sigma_1 b, 1)$, where $b \in B$ and where $x_0 \in X_0$ is arbitrary. Thus $X_0 * X_1$ also constitutes an ex-space.

Finally, suppose that X_i admits a section σ_i, for $i = 0, 1$, and so constitutes an ex-space. Then we can define a family σ_t' of sections of $X_0 * X_1$, for $t \in I$, by

$$\sigma_t' b = (\sigma_0 b, \sigma_1 b, t) \qquad (b \in B).$$

Choose a value of t and regard $X_0 * X_1$ as an ex-space with σ_t' as section. In the ex-category, the smash product $X_0 \wedge X_1$ is defined and the identity function

[5] In fibre bundle theory this is known as the fibre-join. Given a pair of euclidean bundles, over B, the fibre-join of their associated sphere-bundles is equivalent to the sphere-bundle associated with their Whitney sum.

on the cylinder $(X_0 \times X_1) \times I$ determines an ex-map

$$\varphi : X_0 * X_1 \to S(X_0 \wedge X_1).$$

This natural projection is an ex-homotopy equivalence, under certain conditions. For example, let G_i be a compact Lie group and let P_i be a principal G_i-bundle over B. Suppose that $X_i = P_{i*}(A_i)$, where A_i is a differentiable G_i-space. Then an argument similar to the one used in the case of $\tilde{S}$ shows that φ is an ex-homotopy equivalence, provided B is regular and locally compact. For applications in which the join operation plays a major role see the latter part of [3].

7. The Puppe sequence

Let X be an ex-space over B. By the *cone* on X, in the ex-category, we mean the ex-space (CX, σ', ρ') defined as follows. Consider the ordinary cylinder $X \times I$ and write

$$\pi(x, t) = \rho x \qquad (x \in X,\ t \in I).$$

Then CX is obtained from $X \times I$ by identifying points of $B \times I \cup X \times 0$ which have the same image under π. The section σ' is given by $\sigma' b = (b, t)$, for any t, and the projection ρ' is induced by π. It is easy to check that CX is ex-contractible. An ex-map $u : X \to CX$ is given by $ux = (x, 1)$.

Given an ex-map $f : X \to Y$ we define the *ex-map cone* C_f to be the push-out of (u, f). It is simple exercise to check that the structural ex-map $v : Y \to C_f$ is an embedding, in the topological sense, so that Y can be regarded as a subspace of C_f. Furthermore, following almost literally the argument in §1 of [6], we obtain

LEMMA (7.1). *If $f : X \to Y$ is an ex-map then the structural ex-map $v : Y \to C_f$ is an ex-cofibration, with ex-cofibre $C_f/Y = SX$.*

Let E be any ex-space. It is a simple exercise to show that the sequence

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ v\ } C_v$$

is exact for $\pi(\ \ , E)$. Substitute u for f and consider the corresponding exact sequence

$$Y \to C_f \to C_\varphi .$$

By (7.1), u is an ex-cofibration and so it follows as in Theorem 2 of [6] that C_v has the same ex-homotopy type as the ex-cofibre SX of v. Pursuing this argument in exactly the same way as Puppe does in Theorem 5 of [6], we arrive at

THEOREM (7.2). *Let $f : X \to Y$ be an ex-map, and let E be any ex-space. Then the sequence*

$$X \to Y \to C_f \to SX \to SY \to \cdots$$

with appropriate ex-maps, is exact for $\pi(\ \ , E)$.

Thus we have a generalised form of Puppe's mapping sequence, and following Puppe we can deduce various corollaries. As an example we state

COROLLARY (7.3). *Let* X_i $(i = 0, 1)$ *be a well-based ex-space. Then* $S(X_0 \times X_1)$ *has the same ex-homotopy type as* $SX_0 \vee SX_1 \vee S(X_0 \wedge X_1)$.

The dual notion of ex-fibration will be discussed in the second paper in this series.

REFERENCES

1. M. F. ATIYAH, *K-theory*, Lecture Notes, Harvard, 1964.
2. B. ECKMANN AND P. J. HILTON, *Group-like structures in general categories I*, Math. Ann. vol. 145 (1962), pp. 227–255.
3. I. M. JAMES, *Bundles with special structure I*, Ann. of Math., vol. 89 (1969), pp. 359–390.
4. J. F. MCLENDON, *A spectral sequence for classifying liftings in fiber spaces*, Bull. Amer. Math. Soc., vol. 74 (1968), pp. 982–984.
5. R. S. PALAIS, *The classification of G-spaces*. Mem. Amer. Math. Soc., no. 36, Providence, R. I., 1960.
6. D. PUPPE, *Homotopiemengen und ihre induzierten abbildungen I*, Math. Zeitschr., vol. 69 (1958), pp. 299–344.
7. N. E. STEENROD, *The topology of fibre bundles*, Princeton Univ. Press, Princeton 1951.

OXFORD UNIVERSITY MATHEMATICAL INSTITUTE.
OXFORD, ENGLAND