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# Exact and Approximation Algorithms for Minimum-Width Cylindrical Shells*

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**Abstract.** Let $S$ be a set of $n$ points in $\mathbb{R}^3$. Let $\omega^*$ be the width (i.e., thickness) of a minimum-width infinite cylindrical shell (the region between two co-axial cylinders) containing $S$. We first present an $O(n^5)$-time algorithm for computing $\omega^*$, which as far as we know is the first nontrivial algorithm for this problem. We then present an $O(n^{2+\delta})$-time algorithm, for any $\delta > 0$, that computes a cylindrical shell of width at most $56\omega^*$ containing $S$.

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## 1. Introduction

Given a line $\ell$ in $\mathbb{R}^3$ and two real numbers $0 \le r \le R$, the *cylindrical shell* $\Sigma(\ell, r, R)$ is the closed region lying between the two co-axial cylinders of radii $r$ and $R$ with $\ell$ as their axis, i.e.,

$$\Sigma(\ell, r, R) = \{p \in \mathbb{R}^3 \mid r \le d(p, \ell) \le R\},$$

where $d(p, \ell)$ is the Euclidean distance between point $p$ and line $\ell$. The *width* of $\Sigma(\ell, r, R)$ is $R - r$. Let $S$ be a set of $n$ points in $\mathbb{R}^3$. One can measure how well $S$ fits a cylindrical surface by computing a cylindrical surface $\mathcal{C} = \mathcal{C}(S)$ so that the maximum distance between any point of $S$ and $\mathcal{C}$ is minimized. If $\ell$ and $\rho$ are the axis and the radius of $\mathcal{C}$ and $\theta$ is the maximum distance between $\mathcal{C}$ and $S$, then $S \subset \Sigma(\ell, \rho - \theta, \rho + \theta)$. Hence, the problem of approximating $S$ by a cylindrical surface is equivalent to computing a cylindrical shell $\Sigma^*(S)$ of minimum width that contains $S$.

The main motivation for computing a minimum-width cylindrical shell comes from computational metrology. In order to measure the quality of a manufactured cylinder $\Gamma$, we sample a set $S$ of points on the surface of $\Gamma$ using coordinate measuring machines and then fit a cylindrical surface through $S$ so that the maximum distance between the points of $S$ and the cylinder is minimized. For example, this is one of the criteria suggested in the recent ASME Y14.5M standard to determine how closely $\Gamma$ resembles a cylinder [17], [18].

In the last few years much work has been done on measuring the circularity of a planar point set, which is defined as the width of the thinnest annulus that contains the point set [2], [5], [11]–[14]. The best known exact algorithm runs in $O(n^{3/2+\delta})$ time, for any $\delta > 0$ [5], and near-linear approximation algorithms are proposed in [2], [9], and [11]. In three dimensions, Chan [9] has shown that the minimum-width spherical shell (a region enclosed between two concentric spheres) containing an $n$-element point set $S$ can be computed in time $O(n^2)$. The same paper also presents linear-time algorithms that compute an approximation to the minimum-width enclosing spherical shell in any dimension; see also [2]. There has also been some work on computing the smallest cylinder enclosing a point set in $\mathbb{R}^3$ [1], [15]. Agarwal et al. [1] developed an $O(n^{3+\delta})$-time algorithm, for any $\delta > 0$, for computing the smallest enclosing cylinder. They also proposed a $(1 + \varepsilon)$-approximation algorithm (i.e., an algorithm that produces an enclosing cylinder whose radius is at most $(1 + \varepsilon)$ times the minimum radius) that runs in $O(n/\varepsilon^2)$ time. This has been improved by Chan [9] to $O(n/\varepsilon)$ or to $O(n + 1/\varepsilon^3)$.

Finding the minimum-width cylindrical shell $\Sigma^*(S)$ that contains a given point set is harder than computing a minimum-width enclosing spherical shell, computing a smallest enclosing cylinder, or computing a thinnest annulus containing a planar point set. Actually, the second and third problems are special cases of computing a thinnest cylindrical shell—finding a smallest enclosing cylinder is the same as finding a minimum-width cylindrical shell whose inner radius is 0; and finding a thinnest cylindrical shell with the axis parallel to a given direction $\mathbf{n}$ is the same as finding a thinnest annulus containing the projection of $S$ in direction $\mathbf{n}$ onto a plane orthogonal to $\mathbf{n}$. Since a cylindrical shell is specified by six parameters—four parameters define the axis of the shell and the remaining two define the inner and outer radii of the shell—$\Sigma^*(S)$ is in general "defined" by a subset $A \subset S$ of six points, in the sense that $\Sigma^*(S)$ is one of the $O(1)$ cylindrical shells that contain $A$ on their inner and outer boundaries. This suggests the following naïve procedure for computing $\Sigma^*(S)$: For each subset $A \subseteq S$ of size six, compute the $O(1)$ cylindrical shells containing $A$ on their inner and outer boundary. For each such shell $\Sigma$, check in $O(n)$ time whether $S \subset \Sigma$. Return the thinnest among those shells that contain $S$. This naïve approach leads to an $O(n^7)$-algorithm for computing $\Sigma^*(S)$ under an appropriate model of computation in which the roots of a fixed-degree polynomial can be computed in $O(1)$ time. As the first result of this paper, we describe, in Section 2, an improved $O(n^5)$-time algorithm for computing $\Sigma^*(S)$. We are not aware of any faster algorithm for the exact problem. Recently, Devillers and Preparata proposed a linear-time constant-factor approximation algorithm for the minimum-width cylindrical shell problem under the assumption that the points are "almost" cylindrical [10].

Since computing $\Sigma^*(S)$ is so expensive, we develop a more efficient approximation algorithm for computing a cylindrical shell that contains $S$ and has width at most $c\omega^*$, where $\omega^*$ is the width of $\Sigma^*(S)$ and $c$ is a constant. We first prove in Section 3 a *Helly-type* theorem for $\Sigma^*(S)$, which we believe to be of independent interest, and which asserts roughly the following: Let $A \subseteq S$ be a subset of four points so that the volume of the tetrahedron spanned by $A$ is close to the largest volume of a tetrahedron spanned by any four points of $S$. For a direction $\mathbf{n} \in \mathbb{S}^2$ and a point set $X$, let $\omega^*(X, \mathbf{n})$ denote the minimum width of a cylindrical shell containing $X$ and with axis direction $\mathbf{n}$. Then for any direction $\mathbf{n}$, $\omega^*(S, \mathbf{n}) \le c \cdot \max_{p \in S} \omega^*(A \cup \{p\}, \mathbf{n})$, for an absolute constant $c > 1$. The constant that our analysis yields is about 56, but we believe that the theorem also holds with a much smaller constant. Using this observation, we develop in Section 4 an $O(n^{2+\delta})$-time algorithm, for any $\delta > 0$, to compute a cylindrical shell of width at most about $56\omega^*$ that contains $S$.

## 2. Computing $\Sigma^*(S)$ Exactly

In this section we describe an $O(n^5)$-time algorithm for computing $\Sigma^*(S)$. Without loss of generality assume that the axis of $\Sigma^*(S)$ is not parallel to the $xy$-plane; the case of a horizontal axis can be handled by a simpler algorithm, whose details are omitted. A cylinder $C$ with a nonhorizontal axis $a$ can be parametrized by a 5-tuple $(a_1, a_2, a_3, a_4, r)$, where $r$ is the radius of $C$ and where the axis of $C$ is the line $a = \{p + tq \mid t \in \mathbb{R}\}$, $p = (a_1, a_2, 0)$ is the intersection point of $a$ with the $xy$-plane, and $q = (a_3, a_4, 1)$ is the direction vector of $a$. Let $x$ be a point in $\mathbb{R}^3$. The orthogonal projection of $x$ to the line $a$ is $p + ((x - p) \cdot q/\|q\|)q/\|q\| = p + ((x - p) \cdot q/\|q\|^2)q$. Hence, the distance between $x$ and $a$ is

$$d(x, a) = \left\| (p - x) - \frac{(p - x) \cdot q}{\|q\|^2} q \right\|.$$

Since $x$ lies in the cylinder $C$ if and only if $d(x, a) \le r$, after some algebraic manipulation we obtain that $x = (x_1, x_2, x_3)$ lies inside $C$ if and only if

$$f(x_1, x_2, x_3, a_1, a_2, a_3, a_4) \le (a_3^2 + a_4^2 + 1)r^2,$$

where

$$\begin{aligned} f(x_1, x_2, x_3, a_1, a_2, a_3, a_4) \\ &= [(a_4^2+1)a_1^2 + (a_3^2+1)a_2^2 - 2a_1a_2a_3a_4] + 2[a_2a_3a_4 - a_1(a_4^2+1)]x_1 \\ &\quad + 2[a_1a_3a_4 - a_2(a_3^2+1)]x_2 + 2[a_1a_3 + a_2a_4]x_3 - 2[a_3a_4]x_1x_2 \\ &\quad - 2[a_3]x_1x_3 - 2[a_4]x_2x_3 + [1](x_1^2+x_2^2) + [a_3^2](x_2^2+x_3^2) \\ &\quad + [a_4^2](x_1^2+x_3^2). \end{aligned} \tag{2.1}$$

Hence, a point $x$ lies in a cylindrical shell $\sigma = (a_1, a_2, a_3, a_4, r, R)$ with axis $a = (a_1, a_2, a_3, a_4)$, parametrized as above, inner radius $r$, and outer radius $R$ if and only if

$$r^2(a_3^2 + a_4^2 + 1) \le f(x_1, x_2, x_3, a_1, a_2, a_3, a_4) \le R^2(a_3^2 + a_4^2 + 1). \tag{2.2}$$

We set

$$\begin{aligned} \varphi_1(\sigma) &= a_2a_3a_4 - a_1(a_4^2+1), \\ \varphi_2(\sigma) &= a_1a_3a_4 - a_2(a_3^2+1), \\ \varphi_3(\sigma) &= a_1a_3 + a_2a_4, \\ \varphi_4(\sigma) &= a_3a_4, \\ \varphi_5(\sigma) &= a_3, \\ \varphi_6(\sigma) &= a_4, \\ \varphi_7(\sigma) &= a_3^2, \\ \varphi_8(\sigma) &= a_4^2, \\ \varphi_9(\sigma) &= r^2(a_3^2+a_4^2+1) - (a_4^2+1)a_1^2 - (a_3^2+1)a_2^2 + 2a_1a_2a_3a_4, \\ \varphi_{10}(\sigma) &= R^2(a_3^2+a_4^2+1) - (a_4^2+1)a_1^2 - (a_3^2+1)a_2^2 + 2a_1a_2a_3a_4, \end{aligned}$$

$$\begin{aligned} \psi_0(x) &= x_1^2 + x_2^2, & \psi_1(x) &= 2x_1, \\ \psi_2(x) &= 2x_2, & \psi_3(x) &= 2x_3, \\ \psi_4(x) &= -2x_1x_2, & \psi_5(x) &= -2x_1x_3, \\ \psi_6(x) &= -2x_2x_3, & \psi_7(x) &= x_2^2 + x_3^2, \\ \psi_8(x) &= x_1^2 + x_3^2. \end{aligned}$$

Then the constraint (2.2) can be rewritten as a linear constraint:

$$H_x(\sigma)\colon\ \varphi_9(\sigma) \le \psi_0(x) + \sum_{i=1}^{8} \varphi_i(\sigma)\psi_i(x) \le \varphi_{10}(\sigma).$$

For any point $p \in \mathbb{R}^3$, define the wedge $H_p \subset \mathbb{R}^{10}$, formed by the intersection of two halfspaces, as

$$H_p = \left\{ (y_1, \ldots, y_{10}) \mid y_9 \le \psi_0(p) + \sum_{i=1}^{8} y_i\psi_i(p) \le y_{10} \right\}.$$

Set $\varphi(\sigma) = \langle \varphi_1(\sigma), \ldots, \varphi_{10}(\sigma) \rangle \in \mathbb{R}^{10}$. Let $P = \bigcap_{p \in S} H_p$ be the convex polyhedron defined by the intersection of the $2n$ corresponding halfspaces. $P$ has $O(n^5)$ faces and can be computed in $O(n^5)$ time [8]. A cylindrical shell (with nonhorizontal axis) $\sigma$ contains $S$ if and only if $\varphi(\sigma) \in P$.

Let $\Psi \subseteq \mathbb{R}^4 \times (\mathbb{R}^+)^2$ denote the six-dimensional set of all cylindrical shells (with nonhorizontal axis) that contain $S$. Then $\varphi(\Psi)$ is the intersection of $P$ with the six-dimensional surface $\Phi = \{\varphi(\sigma) \mid \sigma \in \mathbb{R}^4 \times (\mathbb{R}^+)^2\}$. After having computed $P$, $\Psi$ can be computed in $O(n^5)$ time, e.g., by triangulating $P$ into $O(n^5)$ simplices and then, for every simplex $\tau$ in the triangulation, computing $\tau \cap \Phi$. Finally, for each simplex $\tau$, we compute in $O(1)$ time (under an appropriate model of computation in which the roots of a constant-degree polynomial can be computed in $O(1)$ time) the minimum-width cylindrical shell $\sigma$ such that $\varphi(\sigma) \in \tau \cap \varphi(\Psi)$. Hence, we have established the following result.

**Theorem 2.1.** *Given a set $S$ of $n$ points in $\mathbb{R}^3$, a minimum-width cylindrical shell containing $S$ can be computed in $O(n^5)$ time.*

## 3. A Helly-Like Property of Cylindrical Shells

Let $S$ be a set of $n$ points in $\mathbb{R}^3$, and let $t > 1$ be a constant. For any finite point set $X \subset \mathbb{R}^3$ of at least four points, let $\mu(X)$ denote the maximum volume of a simplex spanned by four points of $X$. Let $T$ be a tetrahedron spanned by points of $S$ whose volume is $\mu(S)/t$. Let $A = \{a_1, \ldots, a_4\} \subseteq S$ denote the set of vertices of $T$. The simplex $T$ has the following useful property.

**Lemma 3.1.** *Let $f$ be any $k$-flat, for $k = 0, 1, 2$. Then for any $p \in S$ we have*

$$d(p, f) \leq (4t - 1) \cdot \max_{1 \leq i \leq 4} d(a_i, f). \tag{3.1}$$

*Proof.* Let $\Delta \subset \mathbb{R}^3$ be the locus of all points $q$ so that each of the simplices $a_1a_2a_3q$, $a_1a_2a_4q$, $a_1a_3a_4q$, and $a_2a_3a_4q$ has volume at most $t \cdot \mathrm{Vol}(T)$; see Fig. 1. By assumption,

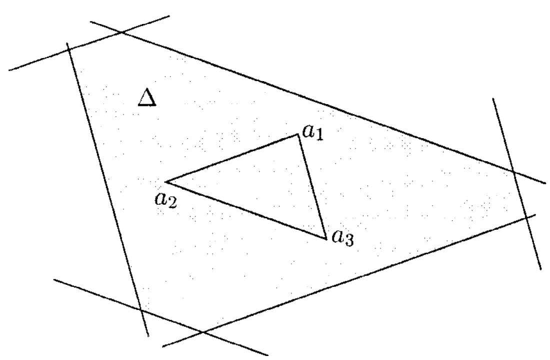


**Fig. 1.** A two-dimensional version of the region $\Delta$, for $t$ slightly larger than 1.

we have $S \subset \Delta$. Let $h_i$ be the plane containing $A \setminus \{a_i\}$, and let $\Lambda_i$ be the slab bounded by two planes parallel to $h_i$ and at distance $t \cdot d(a_i, h_i)$ from it. Then $\Delta = \bigcap_{i=1}^{4} \Lambda_i$; see Fig. 1. Using barycentric coordinates, we can represent any point $q \in \Delta$ as $q = \sum_{i=1}^{4} \lambda_i a_i$, where $\sum_{i=1}^{4} \lambda_i = 1$ and $|\lambda_i| \le t$, for $i = 1, \ldots, 4$. For $i = 1, \ldots, 4$, let $b_i$ be the point in $f$ nearest to $a_i$, and put $q^* = \sum_{i=1}^{4} \lambda_i b_i \in f$. We then have

$$\begin{aligned} d(q, f) &\le d(q, q^*) \\ &= d\left(\sum_{i=1}^{4} \lambda_i a_i, \sum_{i=1}^{4} \lambda_i b_i\right) \\ &= \left\| \sum_{i=1}^{4} \lambda_i (a_i - b_i) \right\| \\ &\le \sum_{i=1}^{4} |\lambda_i| d(a_i, f) \\ &\le (4t - 1) \cdot \max_{1 \le i \le 4} d(a_i, f), \end{aligned}$$

for each $q \in \Delta$, where the last inequality follows by observing that $\max \sum_{i=1}^{4} |\lambda_i|$, subject to $\sum_{i=1}^{4} \lambda_i = 1$ and $|\lambda_i| \le t$ for $i = 1, \ldots, 4$, is $4t - 1$. This implies the assertion of the lemma. □

Fix a direction $\mathbf{n} \in \mathbb{S}^2$, the unit sphere of directions, and let $\pi = \pi^{(\mathbf{n})}$ be the plane normal to $\mathbf{n}$ and passing through the origin. For a point $x \in \mathbb{R}^3$, let $x^*$ denote its orthogonal projection to $\pi$. Set $S^* = \{p^* \mid p \in S\}$. Similarly, define $A^*$ to be the projection of $A$ to $\pi$.

**Corollary 3.2.**

(i) *Let $o$ and $\rho$ be the center and radius of the smallest disk enclosing $A^*$. Then $S^*$ is contained in the disk of radius $(4t - 1)\rho$ centered at $o$.*

(ii) *For any line $\ell$ lying in $\pi$,*

$$\max_{p \in S} d(p^*, \ell) \le (4t - 1) \max_{a \in A} d(a^*, \ell).$$

*Proof.* Part (i) follows by applying Lemma 3.1 to the line in direction $\mathbf{n}$ and passing through $o$. The second part is proved by applying Lemma 3.1 to the plane orthogonal to $\pi$ and passing through $\ell$. □

The following geometric lemma lies at the heart of the main result of this section. Let $D(x, \delta)$ denote the disk of diameter $\delta$ centered at a point $x$.

**Lemma 3.3.** *Let $\triangle abc$ be a triangle in the plane, and let $\tau \ge 1$ and $0 < \omega <$ $\mathrm{Width}(\triangle abc)/3.4$ be two parameters. Define $\Delta = \Delta(\tau)$ to be the locus of all points $x$ such that the area of each of the triangles $\triangle abx$, $\triangle acx$, $\triangle bcx$ is at most $\tau$ times the*

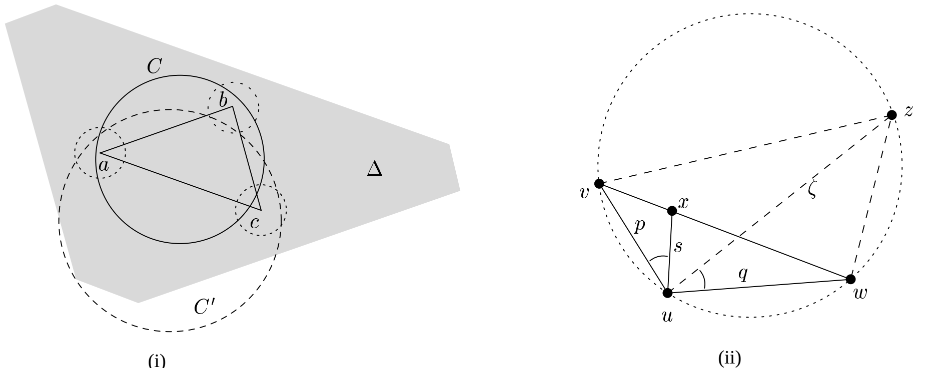


**Fig. 2.** (i) Setup of the lemma; (ii) geometric interpretation of the inversion.

*area of* $\triangle abc$. *Let* $C$ *and* $C'$ *be two circles, each of which meets all three disks* $D(a, \omega)$, $D(b, \omega)$, $D(c, \omega)$. *Then for any* $z \in C \cap \Delta$ *we have*

$$d(z, C') \le (6.95\tau + 3.5)\omega$$

(*see Fig.* 2(i)).

**Remark 3.4.** Informally, the lemma asserts that if two circles are close to each other near the three points $a, b, c$, then they remain close to each other within $\Delta$. Without confinement to $\Delta$, the assertion may fail, as is easily checked.

*Proof.* We parametrize points on $C$ using *inversion*, as follows. Pick points $u \in C \cap D(a, \omega)$, $v \in C \cap D(b, \omega)$, $w \in C \cap D(c, \omega)$. (Note that the condition on $\omega$ implies that the disks $D(a, \omega)$, $D(b, \omega)$, $D(c, \omega)$ are pairwise disjoint.) Without loss of generality, we may assume that the order of $u$, $v$, $w$, and $z$ along $C$ in the clockwise direction is $u, v, z, w$. Write $v = u + p$, $w = u + q$, and $z = u + \zeta$. Apply an inversion to the plane that takes $u$ to infinity. For example, using complex numbers, we may use the transformation $\xi \mapsto 1/(\xi - u)$. This transformation maps $C$ to a straight line containing the images $1/p$, $1/q$, and $1/\zeta$ of $v$, $w$, and $z$, respectively, so that $1/\zeta$ lies *between* $1/p$ and $1/q$. Hence there is a real parameter $\lambda \in [0, 1]$, such that

$$\frac{1}{\zeta} = \frac{\lambda}{p} + \frac{1-\lambda}{q} \tag{3.2}$$

or

$$\zeta = \frac{pq}{\lambda q + (1-\lambda)p}.$$

The following geometric interpretation will be useful in the subsequent analysis. Put $s = \lambda q + (1-\lambda)p$ and $x = u + s$. The point $x$ lies on the edge $vw$ of the triangle $uvw$ and splits it in the ratio $\lambda{:}(1-\lambda)$; that is $|x - v| = \lambda|w - v|$ and $|x - w| = (1-\lambda)|w - v|$. Since $pq = \zeta s$ (or $p/s = \zeta/q$), the triangles $\triangle vux$ and $\triangle zuw$ are similar. Analogously, $\triangle wux$ and $\triangle zuv$ are similar. See Fig. 2(ii).

This implies that

$$\frac{\lambda|w-v|}{|s|}=\frac{|w-z|}{|q|} \quad \text{and} \quad \frac{(1-\lambda)|w-v|}{|s|}=\frac{|v-z|}{|p|}. \tag{3.3}$$

Since $u, v, z, w$ are cocircular, $\measuredangle vuw=\pi-\measuredangle vzw$, therefore $\sin(\measuredangle vuw)=\sin(\measuredangle vzw)$. Multiplying the two equalities in (3.3), we obtain

$$\begin{aligned}\lambda(1-\lambda)|w-v|^2 &= |s|^2\cdot\frac{|v-z||w-z|}{|p||q|}\\ &= |s|^2\cdot\frac{|v-z|\cdot|w-z|\sin(\measuredangle vzw)}{|p|\cdot|q|\sin(\measuredangle vuw)}\\ &= |s|^2\cdot\frac{\text{Area}(\triangle vwz)}{\text{Area}(\triangle uvw)}.\end{aligned}$$

We prove below in Corollary 3.6 that

$$\text{Area}(\triangle vwz)\le 4.05\tau\cdot\text{Area}(\triangle uvw). \tag{3.4}$$

Intuitively, this is to be expected because the area of $\triangle uvw$ (resp. $\triangle vwz$) is a good approximation of the area of $\triangle abc$ (resp. $\triangle bcz$); a rigorous proof is given in Lemma 3.5 below.

We thus have

$$\lambda(1-\lambda)|w-v|^2\le 4.05\tau|s|^2. \tag{3.5}$$

Let $\theta=\angle uvw$. Using the law of cosines, we have

$$|s|^2=|p|^2+\lambda^2|w-v|^2-2\lambda|p||w-v|\cos\theta$$

and

$$|q|^2=|p|^2+|w-v|^2-2|p||w-v|\cos\theta.$$

Eliminating $\cos\theta$ from the last two equations, we obtain

$$|s|^2=\lambda|q|^2+(1-\lambda)|p|^2-\lambda(1-\lambda)|w-v|^2. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\lambda|q|^2+(1-\lambda)|p|^2\le(4.05\tau+1)\,|s|^2. \tag{3.7}$$

Apply a symmetric transformation to parametrize $C'$: Pick points $u'\in C'\cap D(a,\omega)$, $v'\in C'\cap D(b,\omega)$, $w'\in C'\cap D(c,\omega)$. Write $v'=u'+p'$, $w'=u'+q'$, and put

$$z'=u'+\frac{p'q'}{\lambda q'+(1-\lambda)p'}\in C'.$$

Set

$$\delta=\frac{pq}{\lambda q+(1-\lambda)p}-\frac{p'q'}{\lambda q'+(1-\lambda)p'}.$$

Put $\xi = p' - p$ and $\eta = q' - q$. Observe that $|\xi|, |\eta| \le \omega$. We have

$$\begin{aligned} |\delta| &= \left| \frac{pq}{\lambda q + (1-\lambda)p} - \frac{(p+\xi)(q+\eta)}{\lambda(q+\eta) + (1-\lambda)(p+\xi)} \right| \\ &\le \frac{|\lambda q + (1-\lambda)p| \cdot |\xi| \cdot |\eta| + \lambda |q|^2 |\xi| + (1-\lambda)|p|^2 |\eta|}{|\lambda q + (1-\lambda)p| \cdot |\lambda(q+\eta) + (1-\lambda)(p+\xi)|}. \end{aligned}$$

The denominator in the last expression is at least $|s|(|s| - \omega)$. Moreover, $|s|$ is larger than the height to $vw$ in $\triangle uvw$. As we show below in Lemma 3.5, this height is at least $\mathrm{Width}(\triangle abc) - \omega \ge 2.4\omega$ (again, this holds because $\triangle uvw$ is a good approximation of $\triangle abc$). Therefore

$$|\delta| \le \frac{|s|\omega^2 + \omega(\lambda|q|^2 + (1-\lambda)|p|^2)}{|s|(|s| - \omega)}.$$

Using (3.7) and the fact that $|s| \ge 2.4\omega$, we obtain

$$\begin{aligned} |\delta| &\le \left( \frac{1}{|s|/\omega - 1} + \frac{4.05\tau + 1}{1 - \omega/|s|} \right) \omega \\ &\le \left( \frac{5}{7} + \frac{12\,(4.05\tau + 1)}{7} \right) \omega \\ &< (6.95\tau + 2.5)\,\omega. \end{aligned}$$

Therefore,

$$\begin{aligned} d(z, C') &\le d(z, z') \le d(u, u') + |\delta| \\ &\le (6.95\tau + 3.5)\,\omega. \end{aligned}$$

This completes the proof of the lemma. □

We still need to establish the following lemma.

**Lemma 3.5.**

(a) $\mathrm{Area}(\triangle uvw) \ge \frac{94}{289}\,\mathrm{Area}(\triangle abc)$.
(b) $\mathrm{Area}(\triangle vwz) \le \left(\frac{22}{17}\tau + \frac{25}{1156}\right) \mathrm{Area}(\triangle abc)$.
(c) $|\mathrm{Width}(\triangle uvw) - \mathrm{Width}(\triangle abc)| \le \omega$.

*Proof.* We have

$$2\,\mathrm{Area}(\triangle abc) = |\vec{ab} \times \vec{ac}|$$

and

$$2\,\mathrm{Area}(\triangle uvw) = |\vec{uv} \times \vec{uw}| = |(\vec{ab} + \vec{ua} + \vec{bv}) \times (\vec{ac} + \vec{ua} + \vec{cw})|.$$

Put $\vec{p} = \vec{ua} + \vec{bv}$ and $\vec{q} = \vec{ua} + \vec{cw}$, and note that $|\vec{p}|, |\vec{q}| \le \omega$. We thus have

$$\begin{aligned} 2|\mathrm{Area}(\triangle uvw) - \mathrm{Area}(\triangle abc)| &\le |\vec{p} \times \vec{ac}| + |\vec{ab} \times \vec{q}| + |\vec{p} \times \vec{q}| \\ &\le \omega(|\vec{ab}| + |\vec{ac}| + \omega). \end{aligned}$$

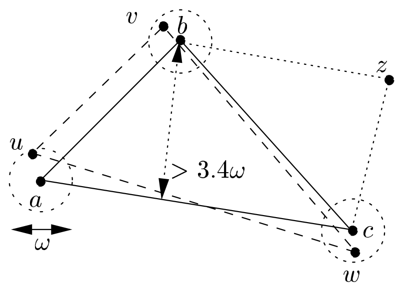


**Fig. 3.** Illustration to Lemma 3.5.

Let $h_{ab}$, $h_{ac}$ denote the heights of $\triangle abc$ to the sides $ab$, $ac$, respectively. By assumption, we have

$$h_{ab},\ h_{ac},\ |ab|,\ |ac| \geq 3.4\omega, \tag{3.8}$$

which implies that

$$\begin{aligned} 2|\mathrm{Area}(\triangle uvw) - \mathrm{Area}(\triangle abc)| &\leq \tfrac{5}{17}(|ab| \cdot h_{ab} + |ac| \cdot h_{ac} + \tfrac{5}{17}|ab| \cdot h_{ab}) \\ &= \tfrac{390}{289}\,\mathrm{Area}(\triangle abc), \end{aligned}$$

or

$$\mathrm{Area}(\triangle uvw) \geq (1 - \tfrac{195}{289})\,\mathrm{Area}(\triangle abc) = \tfrac{94}{289}\,\mathrm{Area}(\triangle abc).$$

This establishes (a).

To prove (b), we note that $\mathrm{Area}(\triangle vwz)$ is maximized when $z$ is a vertex of the region $\Delta(\tau)$. Using the fact that the slope of $uv$ is almost the same as that of $bc$, it can be shown that the point $z$ maximizing $\mathrm{Area}(\triangle vwz)$ must coincide with an endpoint of the edge of $\Delta(\tau)$ parallel to $bc$ and lying on the opposite side of $a$; see Fig. 2(i). In this case $d(z, \ell_{bc}) = \tau d(a, \ell_{bc})$ and $\mathrm{Area}(\triangle bcz) = \tau\,\mathrm{Area}(\triangle abc)$.

Arguing as in (a), we have

$$2\,\mathrm{Area}(\triangle bcz) = |\vec{zb} \times \vec{zc}|$$

and

$$2\,\mathrm{Area}(\triangle vwz) = |\vec{zv} \times \vec{zw}| = |(\vec{zb} + \vec{bv}) \times (\vec{zc} + \vec{cw})|.$$

Note that $|\vec{bv}|, |\vec{cw}| \leq \omega/2$. We thus have

$$\begin{aligned} 2|\mathrm{Area}(\triangle vwz) - \mathrm{Area}(\triangle bcz)| &\leq |\vec{bv} \times \vec{zc}| + |\vec{zb} \times \vec{cw}| + |\vec{bv} \times \vec{cw}| \\ &\leq \frac{\omega}{2}\left(|\vec{zb}| + |\vec{zc}| + \frac{\omega}{2}\right). \end{aligned}$$

The two vertices $z_1$, $z_2$ of $\Delta(\tau)$ where $z$ can lie satisfy $\vec{bz_1} = \tau\vec{ac}$ and $\vec{cz_2} = \tau\vec{ab}$. Consider the vertex $z_1$ (the treatment of $z_2$ is fully symmetric). We have

$$|\vec{bz_1}| = \tau|\vec{ac}|$$

and

$$|\vec{cz_1}| = |\vec{cb} + \vec{bz_1}| = |\vec{cb} + \tau\vec{ac}| = |(\tau - 1)\vec{ac} + \vec{ab}| \le (\tau - 1)|\vec{ac}| + |\vec{ab}|.$$

Hence

$$\frac{\omega}{2}\left(|\vec{z_1 b}| + |\vec{z_1 c}| + \frac{\omega}{2}\right) \le \frac{\omega}{2}\left((2\tau - 1)|\vec{ac}| + |\vec{ab}| + \frac{\omega}{2}\right).$$

Using the inequalities (3.8), we obtain, as in (a),

$$2|\text{Area}(\triangle vwz) - \text{Area}(\triangle bcz)| \le \left(\frac{5(2\tau - 1)}{17} + \frac{5}{17} + \frac{25}{578}\right)\text{Area}(\triangle abc),$$

or

$$\text{Area}(\triangle vwz) \le \text{Area}(\triangle abc)\left(\tau + \frac{5\tau}{17} + \frac{25}{1156}\right),$$

as asserted, thus establishing (b).

Finally, to prove (c), suppose that the width of $\triangle abc$ is the height $h_{bc}$ to the edge $bc$. Then $\triangle abc$ is contained in the strip $\sigma$ of width $h_{bc}$ whose boundary lines pass through the edge $bc$ and the vertex $a$. The strip of width $h_{bc} + \omega$, obtained by translating each line of $\sigma$ by $\omega/2$ away from $\sigma$, contains $u$, $v$, and $w$. Therefore,

$$\text{Width}(\triangle uvw) \le \text{Width}(\triangle abc) + \omega.$$

The reverse inequality is proved in exactly the same manner. □

The first two parts of the above lemma along with the fact that $\tau \ge 1$ imply the following.

**Corollary 3.6.** $\text{Area}(\triangle vwz) < 4.05\tau \cdot \text{Area}(\triangle uvw)$.

We are now in position to prove the main result of this section.

**Theorem 3.7.** *Suppose there exists $\omega > 0$ such that for each $p \in S^*$ there exists an annulus of width $\omega$ that encloses $A^* \cup \{p\}$. Then there exists an annulus of width at most $55.6t\omega$ that encloses $S^*$.*

*Proof.* If $\text{Width}(A^*) \le 6.95\omega$, then Corollary 3.2(ii) implies that the width of $S^*$ is at most $6.95(4t - 1)\omega$. Since a slab can be regarded as a degenerate annulus, $S^*$ can be enclosed by an annulus of width at most $55.6t\omega$. So assume that $\text{Width}(A^*) \ge 6.95\omega$.

Suppose, without loss of generality, that $\triangle a_1^* a_2^* a_3^*$ is the largest-area triangle spanned by three of the points of $A^*$. We have

$$\text{Width}(\triangle a_1^* a_2^* a_3^*) \ge \text{Width}(A^*)/2 > 3.4\omega.$$

Fix a point $q \in S^*$. By Corollary 3.2(ii), the area of each of the triangles $\triangle a_1^* a_2^* q$, $\triangle a_1^* a_3^* q$, $\triangle a_2^* a_3^* q$ is at most $(4t - 1) \cdot \text{Area}(\triangle a_1^* a_2^* a_3^*)$. Let $\mathcal{A}$ be an annulus of width $\omega$ that contains $A^* \cup \{q\}$, and let $C$ be the mid-circle of $\mathcal{A}$. Let $\mathcal{A}^*$ be the annulus of width

$55.6t\omega$ that has $C$ as its mid-circle. We claim that $\mathcal{A}^*$ contains $S^*$. Indeed, let $q'$ be any point of $S^*$, and let $\mathcal{A}'$ be an annulus of width $\omega$ that contains $A^* \cup \{q'\}$. Let $C'$ be the mid-circle of $\mathcal{A}'$. Clearly, $C$, $C'$, and $\triangle a_1^* a_2^* a_3^*$ satisfy the conditions in Lemma 3.3 (with $\tau = 4t - 1$), which implies

$$d(q, C) \leq (6.95(4t - 1) + 3.5)\omega \leq 27.8t\omega,$$

implying that $q \in \mathcal{A}^*$, as claimed. □

## 4. Approximating $\boldsymbol{\Sigma}^*(\boldsymbol{S})$

In this section we apply the results of the preceding section to obtain an algorithm for computing a cylindrical shell of width at most $O(\omega^*(S))$ that encloses an $n$-element point set $S \subset \mathbb{R}^3$. We first describe an algorithm for computing a subset $A \subseteq S$ of four points so that $\mu(A) \geq (1 - \varepsilon)\mu(S)$, for some constant $\varepsilon > 0$; recall that $\mu(X)$ is the maximum volume of a simplex spanned by the points of $X$.

**Lemma 4.1.** *Given a set of $n$ points in $\mathbb{R}^3$ and a parameter $\varepsilon > 0$, we can compute in $O(n \log(1/\varepsilon) + (1/\varepsilon)^{4.5} \log(1/\varepsilon))$ time a subset $A$ of four points so that $\mu(A) \geq (1 - \varepsilon)\mu(S)$.*

*Proof* (*Sketch*). We first compute a box $B$ enclosing $S$ whose volume is at most $1 + \varepsilon$ times the minimum volume of any box containing $S$. This can be done in $O(n + 1/\varepsilon^{4.5})$ time using the algorithm of Barequet and Har-Peled [7]. Suppose, with no loss of generality, that $B$ is axis-aligned and the coordinates of the endpoints of its main diagonal are $(0, 0, 0)$ and $(l_x, l_y, l_z)$. Choose a sufficiently large constant $c > 1$ and set $\alpha = \varepsilon/c$. Draw a three-dimensional grid

$$\{[i\alpha l_x, (i + 1)\alpha l_x] \times [j\alpha l_y, (j + 1)\alpha l_y] \times [k\alpha l_z, (k + 1)\alpha l_z] \mid 0 \leq i, j, k \leq \lceil 1/\alpha \rceil\}$$

of size $O(1/\alpha^3)$. Let $Q$ be the set of grid vertices adjacent to the grid cells that contain at least one point of $S$. $Q$ can be computed in $O(n \log(1/\varepsilon) + 1/\varepsilon^3)$ time. For each pair $1 \leq i, j \leq \lceil 1/\alpha \rceil$, if there are more than two points in $Q$ whose $x$- and $y$-coordinates are $i$ and $j$, respectively, we keep only two of them—the ones with the maximum and minimum values of $k$. $Q$ now has at most $O(1/\alpha^2)$ points. We then compute, in $O((1/\alpha^2) \log(1/\alpha))$ time, the set $V \subseteq Q$ of vertices of the convex hull of $Q$. By a result of Andrews [6], $|V| = O(1/\alpha^{3/2})$. Next, we compute in $O(|V|^3 \log|V|)$ time the largest volume tetrahedron $q_1q_2q_3q_4$ spanned by $V$ (we omit details of the rather straightforward algorithm for doing so). Let $a_i \in S$ be a nearest neighbor of $q_i$, for $i = 1, \ldots, 4$. We return $A = \{a_1, a_2, a_3, a_4\}$. Using a somewhat tedious analysis, similar to the one in [7], it can be shown that $\mu(A) \geq (1 - \varepsilon)\mu(S)$. □

Set $\varepsilon = \frac{1}{140}$ and compute in $O(n)$ time a set $A \subseteq S$ of four points such that $\mu(A) \geq (1 - \varepsilon)\mu(S)$, using the above lemma. Let $\mathbb{S}^2$ denote the unit sphere of directions in $\mathbb{R}^3$. For each $q \in S$ we define a real-valued function $F_q$ on $\mathbb{S}^2$, so that, for $\mathbf{n} \in \mathbb{S}^2$, $F_q(\mathbf{n})$ is the width of a thinnest annulus within the plane $\pi^{(\mathbf{n})}$ that contains the orthogonal projections of $A \cup \{q\}$ on the plane $\pi^{(\mathbf{n})}$. Clearly, $F_q$ is a piecewise-algebraic function of "constant description complexity" (in the terminology of [16]). Let $E$ denote the pointwise maximum of $\{F_q\}_{q \in S}$, let $\mathbf{n} \in \mathbb{S}^2$ be a direction that minimizes $E$, and let $\omega = E(\mathbf{n})$.

**Lemma 4.2.** $\omega \le \omega^*(S) \le 56\omega$.

*Proof.* The fact that $\omega = \min_{\mathbf{v} \in \mathbb{S}^2} \max_{q \in S} F_q(\mathbf{v})$ implies that, for each $\mathbf{v} \in \mathbb{S}^2$, there exists $q \in S$ such that any cylindrical shell that contains $A \cup \{q\}$ and has axis-direction $\mathbf{v}$ must have width at least $\omega$. Hence the minimum width of a cylindrical shell that encloses $S$ is at least $\omega$.

On the other hand, since $\mu(A) \ge (1 - \varepsilon)\mu(S)$, which corresponds to setting $t = 1/(1 - \varepsilon) = 7/6.95$ in Lemma 3.3, Theorem 3.7 implies that there exists a cylindrical shell with axis-direction $\mathbf{n}$ and width at most $55.6 \cdot t\omega = 56\omega$ that contains $S$. $\square$

The algorithm is now straightforward. We compute $E$ in $O(n^{2+\delta})$ time, for any $\delta > 0$, using, e.g., the algorithm of [4], and then examine each vertex, edge, and face of (the graph of) $E$ to find the global minimum of $E$. Suppose the minimum is attained at some direction $\mathbf{n}$ by a point $q \in S$. We project $S$ orthogonally onto $\pi^{(\mathbf{n})}$, and compute the minimum-width annulus $\mathcal{A}$ within $\pi^{(\mathbf{n})}$ that contains the projected set $S^*$. This can be done in additional time $O(n^2)$ [12]. (Alternatively, we can compute in $O(1)$ time the radius $\rho$ and the mid-circle $C^*$ of the minimum width annulus containing $A^{(\mathbf{n})} \cup \{q^{(\mathbf{n})}\}$ and set $\mathcal{A}$ to be the annulus of width $56\rho$ and with mid-circle $C^*$.) We then "lift" $\mathcal{A}$ in the direction $\mathbf{n}$ to obtain a cylindrical shell, of the same width, that encloses $S$. By the preceding analysis, we obtain the following.

**Theorem 4.3.** *Given a set $S$ of $n$ points in $\mathbb{R}^3$, one can compute, in $O(n^{2+\delta})$ time, for any $\delta > 0$, a cylindrical shell that contains $S$, whose width is at most $56\omega^*(S)$.*

**Remark 4.4.** We believe that our approach can be strengthened to give a near-linear-time algorithm. Intuitively, we need to show that one does not have to search over all directions $\mathbf{n} \in \mathbb{S}^2$. Instead, we conjecture that it suffices to search over the one-dimensional locus of axis directions of cylinders that pass through four points of $S$ that span a "large-volume" simplex. However, at present we do not know whether this holds.

## 5. Conclusions

In this paper we presented a constant-factor approximation algorithm for the minimum-width cylindrical shell problem that runs in near-quadratic time. We also presented an algorithm for computing the thinnest cylindrical shell containing a point set. We conclude by mentioning two open problems:

1. Is there a faster algorithm for computing the minimum-width cylindrical shell containing a point set in $\mathbb{R}^3$?
2. Develop a $(1 + \varepsilon)$-approximation algorithm for the minimum-width cylindrical shell problem that runs in near-linear time (or even in near-quadratic time).

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*Note added in proof.* Recently Har-Peled and Varadarajan have developed a $(1+\varepsilon)$-approximation algorithm for the minimum-width cylindrical shell problem whose running time is $n/\varepsilon^{O(1)}$.