EXACT BOUNDARY CONTROLLABILITY OF THE LINEAR BIHARMONIC SCHRÖDINGER EQUATION WITH VARIABLE COEFFICIENTS

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Abstract. In this paper, we study the exact boundary controllability of the linear fourth-order Schrödinger equation, with variable physical parameters and clamped boundary conditions on a bounded interval. The control acts on the first spatial derivative at the left endpoint. We prove that this control system is exactly controllable at any time $T > 0$. The proofs are based on a detailed spectral analysis and on the use of nonharmonic Fourier series.

Contents

2010 *Mathematics Subject Classification.* 93B05, 93B07, 93B12, 93B60.
Key words and phrases. Biharmonic Schrödinger, boundary control, Fourier series, Riesz basis.

1. Introduction

The fourth-order cubic nonlinear Schrödinger equation or biharmonic cubic nonlinear Schrödinger equation reads is given by

$$i\partial_t y + \partial_x^4 y - \partial_x^2 y - \mu|y|^2 y = 0, \tag{1}$$

where y is a complex-valued function and μ is a real constant. This equation has been modeled by Karpman [22] and Karpman and Shagalov [23] in order to describe the propagation of intense laser beams in a bulk medium with Kerr nonlinearity when small fourth-order dispersion are taken into account. The fourth-order cubic nonlinear Schrödinger equation (1) could be found in various areas of physics, such that nonlinear optics, plasma physics, superconductivity and quantum mechanics, we refer to the book of Fibich [18], see also [10, 14, 30].

The well-posedness and the dynamic properties of the biharmonic Schrödinger equation (1) have been extensively studied from the mathematical perspective, see the paper by Pausader [31], see the papers by Capistrano-Filho et al. [15, 16], also [13, 32] and references therein.

In this work we are interested in studying the controllability properties of the linear biharmonic Schrödinger equation (1) for $\mu = 0$, with variable physical parameters on the bounded interval $(0,\ell)$, $\ell > 0$. More precisely, we consider the following control system

$$\begin{cases} i\rho(x)\partial_t y = -\partial_x^2\left(\sigma(x)\partial_x^2 y\right) + \partial_x(q(x)\partial_x y)_x,\ (t,x) \in (0,T)\times(0,\ell), \\ y(t,0) = \partial_x y(t,0) = y(t,\ell) = 0, \quad \partial_x y(t,\ell) = f(t), \in t \in (0,T), \\ y(0,x) = y^0(x),\ x \in (0,\ell), \end{cases} \tag{2}$$

where f is a control that acts at the left end $x = \ell$, and the functions y^0 is the initial condition. Throughout the paper, we assume the following assumptions on the coefficients:

$$\rho, \sigma \in H^2(0,\ell),\ q \in H^1(0,\ell), \tag{3}$$

and there exist constants $\rho_0,\ \sigma_0 > 0$, such that

$$\rho(x) \geq \rho_0, \quad \sigma(x) \geq \sigma_0, \quad q(x) \geq 0, \quad x \in [0,\ell]\,. \tag{4}$$

For system (2), the appropriate control notion to study is the exact controllability, which is defined as follows: system (2) is said to be exact controllable in time $T > 0$ if, given any initial state y^0, there exists control f such that the corresponding solution $y = y(t,x)$ satisfies $y(T,.) = 0$.

Let us now describe the existing results on stabilization and control of the Biharmonic Schrödinger system (2). When $\sigma \equiv 0$, we recover the classical second order Schrödinger equation with variable coefficients occupying the interval $(0,\ell)$. In this context, the stabilization of the second order Schrödinger equation been thoroughly studied, see for instance [4, 2, 8, 3]. We also refer to [5, 6, 7, 1] for related results on exact controllability of the second order Schrödinger equation, see also [17, 20], and references therein. The first result on exact controllability of the linear biharmonic Schrödinger equation (1) for $\mu = 0$ on a bounded domain $\Omega \subset \mathbb{R}^n,\ n \geq 1$, has been established by Zheng and Zhongcheng [35]. In that paper, the authors proved that the linearized system

$$
\begin{cases} i\partial_t y + \Delta^2 y = 0, & (t,x) \in (0,T)\times\Omega, \\ y = 0,\ \dfrac{\partial y}{\partial \nu} = f\chi_{\Gamma_0}, & (t,x)\in(0,T)\times\partial\Omega \\ y(0,x) = y^0, & x\in\Omega, \end{cases} \tag{5}
$$

is exactly controllable for any positive time T, where the control $f \in L^2\left((0,T)\times\Gamma_0\right)$ and $\Gamma_0 \subset \partial\Omega$. Their proof uses the Hilbert Uniqueness Method "Lions'HUM" (cf. Lions [26, 27]) and the multiplier techniques [24]. Later, Wen et al.[32] proved the well-posedness and the exact controllability for the linear fourth order Schrödinger system (5) with the boundary observation

$$
z(t,x) = -i\Delta\left(\left(\Delta^2\right)^{-1} y(t,x)\right),\ (t,x)\in(0,T)\times\Gamma_0.
$$

As consequence, they established the exponential stability of the closed-loop system under the output feedback $f = -kz$ for any $k > 0$. The same authors in [33], extended these results to the case of a linear fourth-order multi-dimensional Schrödinger equation with hinged boundary by either moment or Dirichlet boundary control and collocated observation, respectively.

The inverse problem of retrieving a stationary potential from boundary measurements for the one-dimensional linear system (2) with $\rho \equiv \sigma \equiv 1$ and $f \equiv 0$, was studied by Zheng [34]. To this end, the author proved a global Carleman estimate for the corresponding fourth order operator. Exact controllability result has been established recently by Gao [21] when the linear system (2) with $\rho \equiv \sigma \equiv 1$ and $q \equiv 0$, has a particular structure. In that reference, the author consider a forward and backward stochastic fourth order Schrödinger equation and, again, uses Carleman inequalities for the adjoint problem for proving the exact controllability result. More recently, the global stabilization and exact controllability properties have been studied by Capistrano-Filho et al. [14] for the biharmonic cubic non-linear Schrödinger equation (1) on a periodic domain $\mathbb{T}$ with internal control supported on an arbitrary sub-domain of $\mathbb{T}$. More precisely, by means of some properties of propagation of compactness and regularity in Bourgain spaces, first they showed that the system is globally exponentially stabilizable. Then they used this with a local controllability result to get the global controllability for the associated control system. In particular, for the proof of the local controllability result, they combined a perturbation argument and the fixed point theorem of Picard.

To our knowledge, the exact controllability of the fourth order Schrödinger equation with variable coefficients is still unknown. In this paper we prove that the linear control system (2) is exactly controllable in any time $T > 0$, where the control $f \in L^2(0,T)$ and the initial condition $y^0 \in H^{-2}(0,\ell)$. Our approach is essentially based on the qualitative theory of fourth-order linear differential equations, and on a precise asymptotic analysis of the eigenvalue and eigenfunction. Firstly, we prove that all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ associated to the control system (2) with $f(t) \equiv 0$ are allegorically simple. Moreover, we show that the second derivative of each eigenfunction $\phi_n,\ n \in \mathbb{N}^*$, associated with the uncontrolled system does not vanish at the end $x = \ell$. Secondly, by a precise computation of the asymptotics of the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$, we establish that the spectral gap

$$|\lambda_{n+1}-\lambda_n| \asymp n^3\left(\frac{\pi}{\gamma}\right)^4, \quad \text{as } n\to\infty, \gamma := \int_0^\ell \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt.$$

As a result of the theory of non-harmonic Fourier series and a variant of Ingham's inequality due to Beurling (e.g., [17]), we derive the following observability inequality

$$\int_0^T |\partial_x^2\tilde{y}(t,\ell)|^2 dt \asymp \|\tilde{y}^0\|^2_{H_0^2(0,\ell)}, \tag{6}$$

for any $T>0$, where $\tilde{y}$ is the solution of system (2) without control. Finally, we apply the Lions'HUM to deduce the exact controllability result for the system (2).

The rest of the paper is divided as follow: In Section 2, we establish the well-posedness of system (2) without control. In Section 3, we prove the simplicity of all the eigenvalues $(\lambda_n)_{n\geq 1}$ and we determinate the asymptotics of the associated spectral gap. In Section 4, we prove the observability inequality (6). Finally in Section 5, we prove the exact controllability result for the linear control problem (2).

2. Well-posedness of the uncontrolled system

In this section, we will see how solutions of system (2) without control can be developed in terms of Fourier series. As a consequence, we establish the existence and the uniqueness of solutions of the uncontrolled system (2) with $f(t)\equiv 0$. To this end, we consider the following system

$$\begin{cases} i\rho(x)\partial_t y = -\partial_x^2\left(\sigma(x)\partial_x^2 y\right) + \partial_x(q(x)\partial_x y)_x, & (t,x)\in(0,T)\times(0,\ell),\\ y(t,0)=\partial_x y(t,0)=y(t,\ell)=\partial_x y(t,\ell)=0, & t\in(0,T),\\ y(0,x)=y^0, & x\in(0,\ell), \end{cases} \tag{7}$$

First of all, let us define by $L^2_\rho(0,\ell)$ the space of functions y such that

$$\int_0^\ell |y(x)|^2\rho(x)dx < \infty.$$

Throughout this paper, we denote by $H^k(0,\ell)$ the $L^2_\rho(0,\ell)-$based Sobolev spaces for $k>0$. We consider the following Sobolev space

$$H_0^2(0,\ell) := \left\{y\in H^2(0,\ell) \ : \ y(0)=y'(0)=y(\ell)=y'(\ell)=0\right\}$$

endowed with the norm

$$\|u\|_{H_0^2(0,\ell)} = \|u''\|_{L^2_\rho(0,\ell)}.$$

It is easy to show by Rellich's theorem (e.g., [24]) that the space $H_0^2(0,\ell)$ is densely and compactly embedded in the space $L^2_\rho(0,\ell)$. In the sequel, we introduce the operator $\mathcal{A}$ defined in $L^2_\rho(0,\ell)$ by setting:

$$\mathcal{A}y = \rho^{-1}\left((\sigma y'')'' - (qy')'\right),$$

on the domain

$$\mathcal{D}(\mathcal{A}) = H^4(0,\ell)\cap H_0^2(0,\ell),$$

which is dense in $L^2_\rho(0,\ell)$.

Lemma 2.1. *The linear operator $\mathcal{A}$ is positive and self-adjoint such that $\mathcal{A}^{-1}$ is compact. Moreover, the spectrum of $\mathcal{A}$ is discrete and consists of a sequence of positive eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ tending to $+\infty$:*

$$0 < \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n \leq ..... \underset{n\to+\infty}{\longrightarrow} +\infty.$$

The corresponding eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ can be chosen to form an orthonormal basis in $L^2_\rho(0,\ell)$.

Proof. Let $y \in \mathcal{D}(\mathcal{A})$, then by integration by parts, we have

$$\begin{aligned} \langle \mathcal{A}y, y\rangle_{L^2_\rho(0,\ell)} &= \int_0^\ell \Big((\sigma(x)y''(x))'' - (q(x)y'(x))' \Big)\overline{y}(x)dx \\ &= \int_0^\ell \sigma(x)|y''(x)|^2dx + q(x)|y'(x)|^2dx. \end{aligned}$$

Since $\sigma > 0$ and $q \geq 0$, then

$$\langle \mathcal{A}y, y\rangle_{L^2_\rho(0,\ell)} > 0 \text{ for } y \not\equiv 0,$$

and hence the quadratic form has a positive real values, which implies that the linear operator $\mathcal{A}$ is symmetric. Furthermore, it is easy to show that $Ran(\mathcal{A} - iId) = L^2_\rho(0,\ell)$, and this means that $\mathcal{A}$ is selfadjoint. Since the space $H^2_0(0,\ell)$ is continuously and compactly embedded in the space $L^2_\rho(0,\ell)$, then $\mathcal{A}^{-1}$ is compact in $L^2_\rho(0,\ell)$. The lemma is proved. □

Now, we give a characterization of some fractional powers of the linear operator $\mathcal{A}$ which will be useful to give a description of the solutions of problem (7) in terms of Fourier series. According to Lemma 2.1, the operator $\mathcal{A}$ is positive and self-adjoint, and hence it generates a scale of interpolation spaces $\mathcal{H}_\theta$, $\theta \in \mathbb{R}$. For $\theta \geq 0$, the space $\mathcal{H}_\theta$ coincides with $\mathcal{D}(\mathcal{A}^\theta)$ and is equipped with the norm $\|u\|^2_\theta = \langle \mathcal{A}^\theta u, \mathcal{A}^\theta u\rangle_{L^2_\rho(0,\ell)}$, and for $\theta < 0$ it is defined as the completion of $L^2_\rho(0,\ell)$ with respect to this norm. Furthermore, we have the following spectral representation of space $\mathcal{H}_\theta$,

$$\mathcal{H}_\theta = \left\{ u(x) = \sum_{n\in\mathbb{N}^*} c_n\Phi_n(x) \ : \ \|u\|^2_\theta = \sum_{n\in\mathbb{N}^*} \lambda_n^{2\theta}|c_n|^2 < \infty \right\}, \tag{8}$$

where $\theta \in \mathbb{R}$, and the eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ are defined in Lemma 2.1. In particular,

$$\mathcal{H}_0 = L^2_\rho(0,\ell) \text{ and } \mathcal{H}_{1/2} = H^2_0(0,\ell).$$

Obviously, the solutions of problem (7) can be written as

$$y(t,x) = \sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n(x)$$

where the Fourier coefficients are given by

$$c_n := \int_0^\ell y^0(x)\overline{\Phi_n}(x)\rho(x)dx, \ \ n \in \mathbb{N}^*,$$

and $(c_n) \in \ell^2(\mathbb{N}^*)$. Let us denote by $\mathcal{E}_\theta$ the energy associated to the space $\mathcal{H}_\theta$, then

$$\begin{aligned} \mathcal{E}_\theta(t) = \|y\|^2_\theta &= \sum_{n\in\mathbb{N}^*} \lambda_n^{2\theta}|c_n e^{i\lambda_n t}|^2 \\ &= \sum_{n\in\mathbb{N}^*} \lambda_n^{2\theta}|c_n|^2 = \mathcal{E}_\theta(0), \end{aligned}$$

which establishes the conservation of energy along time. As consequence, we have the following existence and uniqueness result for problem (7).

Proposition 2.2. *Let $\theta \in \mathbb{R}$ and $y^0 \in \mathcal{H}_\theta$. Then problem (7) has a unique solution $y \in C([0,T], \mathcal{H}_\theta)$ and is given by the following Fourier series*

$$y(t,x) = \sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n(x), \tag{9}$$

where $y^0 = \sum_{n\in\mathbb{N}^*} c_n \Phi_n$. Moreover, the energy of the system (7) is conserved along the time.

3. Spectral analysis

In this section, we investigate the main properties of all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the operator $\mathcal{A}$. On one hand, we prove that all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ are algebraically simple, and then, the second derivatives of the corresponding eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ do not vanish at $x=\ell$. On another hand, we establish that the spectral gap "$\left|\lambda_{n+1}-\lambda_n\right|$" is uniformly positive. To this end, we consider the following spectral problem which arises by applying separation of variables to system (7),

$$\begin{cases} (\sigma(x)\phi'')'' - (q(x)\phi')' = \lambda\rho(x)\phi, \quad x\in(0,\ell), \\ \phi(0) = \phi'(0) = \phi(\ell) = \phi'(\ell) = 0. \end{cases} \tag{10}$$

It is clear that, problem (10) is equivalent to the following spectral problem

$$\mathcal{A}\phi = \lambda\phi, \quad \phi \in \mathcal{D}(\mathcal{A}),$$

i.e., the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the operator $\mathcal{A}$ and problem (10) coincide together with their multiplicities. One has:

Theorem 3.1. *All the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the spectral problem (10) are simple such that :*

$$0 < \lambda_1 < \lambda_2 < .... < \lambda_n < ..... \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Moreover, the corresponding eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ satisfy

$$\Phi_n''(\ell) \neq 0 \quad \forall n \in \mathbb{N}^*. \tag{11}$$

Our main tool in proving this is the following result [12, Lemma 3.2].

Lemma 3.2. *Let u be a nontrivial solution the linear fourth order differential equation defined on the interval $[a,b]$, $a > b$:*

$$(\sigma(x)u'')'' - (q(x)u')' - \rho(x)u = 0,$$

where the functions $\rho(x) > 0$, $\sigma(x) > 0$ and $q(x) \geq 0$. If u, u', u'' and $\mathcal{T}u = (\sigma(x)u'')' - q(x)u'$ are nonnegative at $x = a$ (but not all zero), then they are positive for all $x > a$. If $u, -u', u''$ and $(-\mathcal{T}u)$ are nonnegative at $x = b$ (but not all zero), then they are positive for all $x < b$.

Proof of Theorem 3.1. First, we prove that the set $\mathcal{E}_\lambda$, of solutions of the following boundary value problem

$$\begin{cases} (\sigma(x)\phi'')'' - (q(x)\phi')' = \lambda\rho(x)\phi, \quad x\in(0,\ell), \\ \phi(0) = \phi'(0) = \phi'(\ell) = 0, \end{cases} \tag{12}$$

is one-dimensional subspace for $\lambda > 0$, i.e., $\dim \mathcal{E}_\lambda = 1$. Suppose that there exist two linearly independent solutions ϕ_1 and ϕ_2 of problem (12). Both $\phi_1''(0)$ and $\phi_2''(0)$ must be different from zero since otherwise it would follow from the first statement of Lemma 3.2 that $\phi_i'(\ell) > 0$ $(i = 1, 2)$ which contradicts the last boundary condition in (12). In view of the assumptions about ϕ_1 and ϕ_2, the solution

$$\phi(x) = \phi_1''(0)\phi_2(x) - \phi_2''(0)\phi_1(x)$$

satisfies

$$\phi(0) = \phi'(0) = \phi''(0) = 0 \text{ and } \phi'(\ell) = 0.$$

This again contradicts the first statement of Lemma 3.2 unless $\phi \equiv 0$. Therefore,

$$\dim \mathcal{E}_\lambda = 1,$$

and then, all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of problem (10) are geometrically simple. On the other hand, by Lemma 2.1, the operator $\mathcal{A}$ is self-adjoint in $L^2_\rho(0, \ell)$, and this implies that all the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ are algebraically simple. Now, we prove (11). Let $\{\lambda_n, \Phi_n\}$ $(n \geq 1)$ be an eigenpair of problem (10), and assume that $\Phi_n''(\ell) = 0$, for some $n \in \mathbb{N}^*$. Then the eigenfunctions Φ_n satisfy the boundary conditions

$$\Phi_n(\ell) = \Phi_n'(\ell) = \Phi_n''(\ell) = 0, \text{ for some } n \in \mathbb{N}^*,$$

and then, by standard theory of differential equations

$$\mathcal{T}\Phi_n(\ell) = (\sigma(\ell)\,\Phi_n(\ell)'')' - q(\ell)\,\Phi_n(\ell)' \neq 0, \text{ for some } n \in \mathbb{N}^*.$$

Without loss of generality, let $\mathcal{T}\Phi_n(\ell) < 0$ for some $n \in \mathbb{N}^*$. Since $\lambda_n > 0$, it follows from the second statement of Lemma 3.2, that

$$\varphi_n(x) > 0,\ \varphi_n'(x) < 0,\ \varphi_n''(x) > 0 \text{ and } \mathcal{T}\varphi_n(x) < 0,\ \forall\, x \in [0, \ell],$$

but this contradicts the boundary conditions $\Phi_n(0) = \Phi_n'(0) = 0$. Thus,

$$\Phi_n''(\ell) \neq 0 \quad \forall n \in \mathbb{N}^*,$$

and this finalizes the proof of the theorem. □

Next we establishes the asymptotic behavior of the spectral gap $\lambda_{n+1} - \lambda_n$ for large n. Namely, we have the following theorem:

Theorem 3.3. *The eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of the associated spectral problem (10) satisfy the following asymptotic:*

$$\sqrt[4]{\lambda_n} := \mu_n = \frac{\pi}{\gamma}\left(n - \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{\exp(n)}\right),\ \gamma = \int_0^\ell \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt. \tag{13}$$

Moreover,

$$|\lambda_{n+1} - \lambda_n| \asymp n^3 \left(\frac{\pi}{\gamma}\right)^4, \quad \text{as } n \to \infty. \tag{14}$$

Proof. It is known (e.g., [19, Chapter 5, p.235-239] and [29, Chapter 2]) that for $\lambda \in \mathbb{C}$, the fourth-order linear differential equation

$$(\sigma(x)\phi'')'' - (q(x)\phi')' = \lambda\rho(x)\phi, \quad x \in (0, \ell), \tag{15}$$

has four fundamental solutions $\{\phi_i(x,\lambda)\}_{i=1}^{i=4}$ satisfying the asymptotic forms

$$\left\{\begin{array}{l} \phi_i(x,\lambda) = \left([\rho(x)]^{\frac{3}{4}}\,[\sigma(x)]^{\frac{1}{4}}\right)^{-\frac{1}{2}} \exp\left\{\mu w_i \int_0^x \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt\right\}[1], \\ \qquad\qquad (16) \\ \phi_i^{(k)}(x,\lambda) = (\mu w_i)^k \left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{k}{4}} \left([\rho(x)]^{\frac{3}{4}}\,[\sigma(x)]^{\frac{1}{4}}\right)^{-\frac{1}{2}} \exp\left\{\mu w_i \int_0^x \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt\right\}[1], \end{array}\right.$$

where $\mu^4 = \lambda$, ${w_i}^4 = 1$, $\phi^{(k)} := \frac{\partial^k \phi}{\partial x^k}$ for $k \in \{1,2,3\}$, and $[1] = 1 + \mathcal{O}(\mu^{-1})$ uniformly as $\mu \to \infty$ in a sector $\mathcal{S}_\tau = \{\mu \in \mathbb{C}$ such that $0 \le \arg(\mu+\tau) \le \frac{\pi}{4}\}$ where τ is any fixed complex number. It is convenient to rewrite these asymptotes in the form

$$\phi_1(x,\lambda) = \left([\rho(x)]^{\frac{3}{4}}\,[\sigma(x)]^{\frac{1}{4}}\right)^{-\frac{1}{2}} \cos\left(\mu \int_0^x \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt\right)[1],$$

$$\phi_2(x,\lambda) = \left([\rho(x)]^{\frac{3}{4}}\,[\sigma(x)]^{\frac{1}{4}}\right)^{-\frac{1}{2}} \cosh\left(\mu \int_0^x \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt\right)[1],$$

$$\phi_3(x,\lambda) = \left([\rho(x)]^{\frac{3}{4}}\,[\sigma(x)]^{\frac{1}{4}}\right)^{-\frac{1}{2}} \sin\left(\mu \int_0^x \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt\right)[1],$$

$$\phi_4(x,\lambda) = \left([\rho(x)]^{\frac{3}{4}}\,[\sigma(x)]^{\frac{1}{4}}\right)^{-\frac{1}{2}} \sinh\left(\mu \int_0^x \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt\right)[1].$$

Hence every solution $\phi(x,\lambda)$ of equation (15) can be written in the following asymptotic form

$$\phi(x,\lambda) = \zeta(x)\Big(C_1\cos(\mu X) + C_2\cosh(\mu X) + C_3\sin(\mu X) + C_4\sinh(\mu X)\Big)[1] \tag{17}$$

and from (16), we have also

$$\begin{aligned} \phi^{(k)}(x,\lambda) \;=\; & \mu^k \zeta(x)\left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{k}{4}} \Big(C_1\cos^{(k)}(\mu X) + C_2\cosh^{(k)}(\mu X) + C_3\sin^{(k)}(\mu X) \\ & + C_4\sinh^{(k)}(\mu X)\Big)[1], \quad \text{as} \quad \mu \to \infty, \quad k \in \{1,2,3\}, \end{aligned} \tag{18}$$

where $C_i, i = 1,2,3,4$ are constants and

$$\zeta(x) = \left([\rho(x)]^{\frac{3}{4}}\,[\sigma(x)]^{\frac{1}{4}}\right)^{-\frac{1}{2}} \quad \text{and} \quad X = \int_0^x \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt. \tag{19}$$

If $\phi(x,\lambda)$ satisfies the boundary conditions $\phi(0,\lambda) = \phi'(0,\lambda) = 0$, then by the asymptotics (17) and (18), we obtain for large positive μ the asymptotic estimate

$$\left\{\begin{array}{l} \zeta(0)\,(C_1 + C_2)\,[1] = 0, \\ \mu\zeta(0)\left(\frac{\rho(0)}{\sigma(0)}\right)^{\frac{1}{4}}(C_3 + C_4)\,[1] = 0. \end{array}\right.$$

and then,

$$\phi(x,\lambda) = C_1\zeta(x)\left(\cos(\mu X) - \cosh(\mu X)\right)[1] + C_3\left(\sin(\mu X) - \sinh(\mu X)\right)[1] \tag{20}$$

and

$$\phi'(x,\lambda) = \mu\zeta(x)\left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{1}{4}}\Big(C_1\left(\sinh(\mu X) - \sin(\mu X)\right) + C_3\left(\cos(\mu X) - \cosh(\mu X)\right)\Big)[1], \tag{21}$$

From the boundary conditions $\phi(\ell,\lambda) = \phi'(\ell,\lambda) = 0$, and the above asymptotics one has:

$$\left\{\begin{array}{l} C_1\left(\cos(\mu\gamma) - \cosh(\mu\gamma)\right)[1] + C_3\left(\sin(\mu\gamma) - \sinh(\mu\gamma)\right)[1] = 0, \\ C_1\left(-\sin(\mu\gamma) - \sinh(\mu\gamma)\right)[1] + C_3\left(\cos(\mu\gamma) - \cosh(\mu\gamma)\right)[1] = 0, \end{array}\right. \tag{22}$$

where the constant γ is defined by

$$\gamma = \int_0^l \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt. \tag{23}$$

This homogeneous system of equations in the unknowns C_1 and C_2 admits a non-trivial solution if and only if the corresponding determinant is zero, i.e.,

$$\left((\cos(\mu\gamma) - \cosh(\mu\gamma))^2 + \sin^2(\mu\gamma) - \sinh^2(\mu\gamma)\right)[1] = 0$$

Equivalently

$$\mu\zeta(\ell)\left(\frac{\rho(\ell)}{\sigma(\ell)}\right)^{\frac{1}{4}}\left(\cos(\mu\gamma)\cosh(\mu\gamma) - 1\right)[1] = 0.$$

Then by (21), one gets that the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ are solution of following asymptotic characteristic equation

$$\mu\zeta(\ell)\left(\frac{\rho(\ell)}{\sigma(\ell)}\right)^{\frac{1}{4}}\exp(\mu\gamma)\left(\cos(\mu\gamma) - \frac{1}{\exp(\mu\gamma)}\right)[1] = 0,$$

which can also be rewritten as

$$\cos(\mu\gamma) + \mathcal{O}\left(\frac{1}{\exp(\mu\gamma)}\right) = 0. \tag{24}$$

Since the solutions of the equation $\cos(\mu\gamma) = 0$ are given by

$$\widetilde{\mu_n} = \frac{\pi}{\gamma}\left(n - \frac{1}{2}\right),\ n = 0, 1, 2, ...,$$

it follows from Rouché's theorem that the solutions of (24) satisfy the following asymptotic

$$\begin{aligned} \mu_n &= \widetilde{\mu_n} + \delta_n \\ &= \frac{\pi}{\gamma}\left(n - \frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{\exp(n)}\right), \end{aligned} \tag{25}$$

which proves (13). Furthermore,

$$\begin{aligned} \sqrt{\lambda_n} &= \frac{\pi^2}{\gamma^2}\left(n - \frac{1}{2}\right)^2 + \mathcal{O}\left(\frac{n}{\exp(n)}\right), \\ &= \frac{\pi^2}{\gamma^2}\left(n^2 - n\right) + \mathcal{O}(1). \end{aligned}$$

and hence

$$\begin{aligned} \lambda_{n+1} - \lambda_n &= \left(\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n}\right)\left(\sqrt{\lambda_{n+1}} + \sqrt{\lambda_n}\right) \\ &= \frac{\pi^4}{\gamma^4}\left((n+1)^2 - n^2 + \mathcal{O}(1)\right)\left((n+1)^2 + n^2 - 2n + \mathcal{O}(1)\right) \\ &= \frac{\pi^4}{\gamma^4}n^3 + \mathcal{O}\left(n^2\right). \end{aligned}$$

The theorem is proved. □

We conclude this section with the following result about the asymptotics of the eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ of the spectral problem (10).

Proposition 3.4. *Let us normalize the eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ of the spectral problem (10) in the sense that $\lim_{n\to\infty} \|\Phi_n\|_{L^2_\rho(0,\ell)} = 1$. One has, the following asymptotic estimates:*

$$\begin{aligned} \Phi_n(x) = {} & \frac{2\zeta(x)}{\gamma \exp(\mu_n\gamma)} \left(\cos(\mu_n\gamma) - \cosh(\mu_n\gamma)\right)\left(\cos(\mu_n X) - \cosh(\mu_n X)\right)[1] \\ & + \frac{2\zeta(x)}{\gamma \exp(\mu_n\gamma)} \left(\sin(\mu_n\gamma) + \sinh(\mu_n\gamma)\right)\left(\sin(\mu_n X) - \sinh(\mu_n X)\right)[1], \end{aligned} \tag{26}$$

where the quantities ζ, X and γ are given by (19) and (23), respectively. Furthermore,

$$\lim_{n\to\infty} \frac{|\Phi_n''(\ell)|}{\sqrt{\lambda_n}} = \left(\frac{2\zeta(\ell)}{\gamma} \left(\frac{\rho(\ell)}{\sigma(\ell)} \right)^{\frac{1}{2}} \right). \tag{27}$$

Proof. If μ_n satisfies (24), then, by solving the homogeneous system of two equations (22), one gets

$$\begin{cases} C_1 = C\left(\cos(\mu_n\gamma) - \cosh(\mu_n\gamma)\right)[1] \\ C_3 = C\left(\sin(\mu_n\gamma) + \sinh(\mu_n\gamma)\right)[1]. \end{cases} \tag{28}$$

for some constant $C \neq 0$. From this, (13) and (21), we obtain the following asymptotic estimate for the eigenfunctions $\phi(x,\lambda_n)$ of the problem (10):

$$\begin{aligned} \phi(x,\lambda_n) = {} & C\zeta(x)\{\left(\cos(\mu_n\gamma) - \cosh(\mu_n\gamma)\right)\left(\cos(\mu_n X) - \cosh(\mu_n X)\right)\}[1] \\ & + C\zeta(x)\{\left(\sin(\mu_n\gamma) + \sinh(\mu_n\gamma)\right)\left(\sin(\mu_n X) - \sinh(\mu_n X)\right)\}[1]. \end{aligned} \tag{29}$$

By (25) and (28)

$$C_1 \asymp \frac{\exp(\mu_n\gamma)}{2} \asymp C_3,$$

and then,

$$\begin{aligned} \Phi_n(x) \quad & \sim \quad C\zeta(x)\frac{\exp(\mu_n\gamma)}{2}\left(\sin(\mu_n X) - \cos(\mu_n X) + \cosh(\mu_n X) - \sinh(\mu_n X)\right) \\ & \sim \quad C\zeta(x)\frac{\exp(\mu_n\gamma)}{2}\left(\sin(\mu_n X) - \cos(\mu_n X)\right), \quad \text{as } n\to\infty, \end{aligned}$$

where γ is defined by (23). By the change of variables $t = X$, one has

$$\begin{aligned} \int_0^\ell \xi^2(x)\sin^2(\mu_n X)\,\rho(x)dx &= \int_0^\ell \sin^2\left(\mu_n \int_0^x \sqrt[4]{\frac{\rho(t)}{\sigma(t)}}dt\right)\sqrt[4]{\frac{\rho(x)}{\sigma(x)}}dx, \\ &= \int_0^\gamma \sin^2(\mu_n t)\,dt = \frac{\gamma}{2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\int_0^\ell \xi^2(x)\cos^2(\mu_n X)\,\rho(x)dx = \frac{\gamma}{2}, \\ &\int_0^\ell \xi^2(x)\sin(\mu_n X)\cos(\mu_n X)\,\rho(x)dx = \frac{\sin^2(\mu_n\gamma)}{2\mu_n}[1]. \end{aligned}$$

Consequently, one gets

$$\lim_{n\to\infty} \|\phi(x,\lambda_n)\|_{L^2_\rho(0,l)} = |C|\frac{\gamma\exp(\mu_n\gamma)}{2}. \tag{30}$$

We set

$$\Phi_n(x) := \frac{\phi(x,\lambda_n)}{\lim\limits_{n\to\infty}\|\phi(x,\lambda_n)\|_{L^2_\rho(0,l)}}. \tag{31}$$

Then, $(\Phi_n(x))_{n\in\mathbb{N}^*}$ are the normalized eigenfunctions of problem (10) so that, $\lim\limits_{n\to\infty}\|\Phi_n\|_{L^2_\rho(0,l)} = 1$. Therefore, by (29) and (30)-(31), we get (26).

In a similar way, from the asymptotics (18), (25) and (30), a straightforward computation yields

$$\begin{aligned}\Phi_n''(x) =& \frac{-2\mu^2\zeta(x)}{\gamma\exp(\mu_n\gamma)}\left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{1}{2}}(\cos(\mu_n\gamma)-\cosh(\mu_n\gamma))(\cos(\mu_nX)+\cosh(\mu_nX))[1]\\ &-\frac{2\mu^2\zeta(x)}{\gamma\exp(\mu_n\gamma)}\left(\frac{\rho(x)}{\sigma(x)}\right)^{\frac{1}{2}}(\sin(\mu_n\gamma)+\sinh(\mu_n\gamma))(\sin(\mu_nX)+\sinh(\mu_nX))[1].\end{aligned}$$

As consequence, one has

$$|\Phi_n''(\ell)| = \frac{4\mu_n^2\zeta(\ell)}{\gamma\exp(\mu_n\gamma)}\left(\frac{\rho(\ell)}{\sigma(\ell)}\right)^{\frac{1}{2}}|\sin(\mu_n\gamma)\sinh(\mu_n\gamma)|[1].$$

Therefore, from this and the asymptote (13), we get (27). The proof is complete. $\square$

4. Observability

In this section, we prove some observability results which are consequences of the asymptotic properties of the previous section. The reason to study these properties is that, by means of the Lions'HUM [27], controllability properties can be reduced to suitable observability inequalities for the adjoint system. As (2) is a self-adjoint system, we are reduced to the same system, without control. Therefore, consider system (2) without control, i.e.,

$$\begin{cases} i\rho(x)\partial_t\tilde{y} = -\partial_x^2\left(\sigma(x)\partial_x^2\tilde{y}\right) + \partial_x(q(x)\partial_x\tilde{y})_x, & (t,x)\in(0,T)\times(0,\ell),\\ \tilde{y}(t,0) = \partial_x\tilde{y}(t,0) = \tilde{y}(t,\ell) = \partial_x\tilde{y}(t,\ell) = 0, & t\in(0,T),\\ \tilde{y}(0,x) = \tilde{y}^0, & x\in(0,\ell).\end{cases} \tag{32}$$

One has:

Proposition 4.1. *Let $T>0$ and $\tilde{y}^0 \in H_0^2(0,\ell)$. Then*

$$\int_0^T |\partial_x^2\tilde{y}(t,\ell)|^2 dt \asymp \|\tilde{y}^0\|^2_{H_0^2(0,\ell)}, \tag{33}$$

where $\tilde{y}$ is the solution of problem (32).

In order to prove Proposition 4.1, we need the following variant of Ingham's inequality due to Beurling (e.g., [17]).

Lemma 4.2. [17] *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence satisfying for some $\delta>0$ the condition*

$$|\lambda_{n+1}-\lambda_n| > \delta,\ \forall\ n\in\mathbb{Z}.$$

Then, for any $T > 2\pi D^{+}(\lambda_n)$, the family $\left(e^{i\lambda_n t}\right)_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(0,T)$, that is

$$\int_0^T \left|\sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t}\right|^2 dt \asymp \sum_{n\in\mathbb{Z}} |c_n|^2,$$

where $D^{+}(\lambda_n) := \lim_{r\to\infty} \dfrac{n^{+}(r,\lambda_n)}{r}$ is the Beurling upper density of the sequence $(\lambda_n)_{n\in\mathbb{N}^*}$, with $n^{+}(r,\lambda_n)$ denotes the maximum number of terms of the sequence $(\lambda_n)_{n\in\mathbb{N}^*}$ contained in an interval of length r.

Proof of Proposition 4.1. It follows from the spectral representation (8) of the space $\mathcal{H}_\theta$, that

$$\begin{aligned}\mathcal{H}_{1/2} &= \left\{ u(x) = \sum_{n\in\mathbb{N}^*} c_n \Phi_n(x) \ : \ \|u\|_\theta^2 = \sum_{n\in\mathbb{N}^*} \lambda_n |c_n|^2 < \infty \right\} \\ &= \mathcal{D}\left(\mathcal{A}^{1/2}\right) = H_0^2(0,\ell),\end{aligned}$$

where the eigenfunctions $(\Phi_n)_{n\in\mathbb{N}^*}$ are given in Proposition 3.4. By Proposition 2.2, the solution $\tilde{y}$ of problem (32) has the form

$$\tilde{y}(t,x) = \sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n(x),$$

where $\tilde{y}^0 = \sum_{n\in\mathbb{N}^*} c_n \phi_n$. Consequently,

$$\int_0^T |\partial_x^2 \tilde{y}(t,\ell)|^2 dt = \int_0^T \Big|\sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n''(\ell)\Big|^2 dt. \tag{34}$$

Thus by the first statement of Theorem 3.1 and the gap condition (14), Beurling's Lemma 4.2 states that for any $T > D^{+}(\lambda_n)$, the family $\left(e^{i\lambda nt}\right)_{n\in\mathbb{N}^*}$ forms a Riesz basis in $L^2(0,T)$, where $D^{+}(\lambda_n)$ is the Beurling upper density of the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$. Furthermore, for every $T > D^{+}(\lambda_n)$, one has

$$\int_0^T \Big|\sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n''(\ell)\Big|^2 dt \asymp \sum_{n\in\mathbb{N}^*} \left|c_n \Phi_n''(\ell)\right|^2. \tag{35}$$

From the asymptote (13) and the characteristic equation (24), we find that the Beurling upper density of the eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$,

$$D^{+}(\lambda_n) = \lim_{n\to\infty} \frac{\gamma^4}{\pi^4\left(n-\frac{1}{2}\right)^3} = 0.$$

By the second statement of Theorem 3.1, we have

$$\Phi_n'(\ell) \neq 0 \text{ for all } n\in\mathbb{N}^*,$$

and then by (27), we deduce that there exists $C_1,\ C_2 > 0$ such that

$$C_1\lambda_n \leq \left|\Phi_n''(\ell)\right|^2 \leq C_2\lambda_n, \quad \text{as } n\to\infty.$$

Therefore from the above and (35), for any $T > 0$

$$\int_0^T \Big|\sum_{n\in\mathbb{N}^*} c_n e^{i\lambda_n t}\Phi_n''(\ell)\Big|^2 dt \asymp \sum_{n\in\mathbb{N}^*} \lambda_n \left|c_n\right|^2.$$

Thus from this and (34), we get (33). This completes the proof. $\square$

5. Exact boundary controllability

In this section, we prove the exact boundary controllability of the control problem (2).

5.1. Well-posedness. Since we are dealing with boundary control, we need to introduce the weaker notion of "solution defined by transposition" in the spirit of [24, 28].

Let $\tilde{y}$ be the solution to problem (32) satisfying (9). Now let $f \in C^\infty(0,T)$ (or $f \in L^2(0,T)$ since $C^\infty(0,T)$ is dense in $L^2(0,T)$) and let $y \in C^4\left([0,T];(0,\ell)\right)$ be a function satisfying (2). Then we multiply (7) by y and integrate on $(0,T)\times(0,\ell)$ to obtain

$$i\int_0^\ell \int_0^T \partial_t \tilde{y} y(t,x) dt \rho(x) dx + \int_0^\ell \int_0^T \left(\partial_x^2\left(\sigma(x)\partial_x^2\tilde{y}\right) - \partial_x(q(x)\partial_x\tilde{y})_x\right) y(t,x) dt dx = 0.$$

Then integrate by parts and using the boundary conditions in (2) and (7), we get

$$\begin{aligned} i\int_0^\ell \Big[\tilde{y}y(t,x)\Big]_0^T \rho(x)dx &= \sigma(\ell)\int_0^T \partial_x^2\tilde{y}\left(t,\ell\right)f(t)dt + i\int_0^\ell\int_0^T \partial_t y\tilde{y}(t,x)dt\rho(x)dx \\ &\quad - \int_0^\ell \int_0^T \left(\partial_x^2\left(\sigma(x)\partial_x^2 y\right) - \partial_x(q(x)\partial_x y)_x\right)\tilde{y}(t,x)dtdx \end{aligned}$$

and then

$$i\int_0^\ell \tilde{y}y(T,x)\rho(x)dx = \sigma(\ell)\int_0^T \partial_x^2\tilde{y}\left(t,\ell\right)f(t)dt + i\int_0^\ell \tilde{y}^0 y^0\rho(x)dx. \tag{36}$$

Let us define the spaces

$$\mathcal{S} := H_0^2(0,\ell) \text{ and } \mathcal{S}' := H^{-2}(0,\ell),$$

and the linear functional $\mathcal{L}_T$ on S by

$$\mathcal{L}_T(\tilde{y}^0) = i\langle y^0, \tilde{y}^0\rangle_{\mathcal{S}',\mathcal{S}} + \sigma(\ell)\int_0^T \partial_x^2\tilde{y}\left(t,\ell\right)f(t)dt. \tag{37}$$

Moreover, we have

$$\|\mathcal{L}_T\| \leq C\left(\|y^0\|_{H^{-2}(0,\ell)} + \|f\|_{L^2(0,T)}\right). \tag{38}$$

Using (36), we may rewrite the identity (37) in the following form

$$\mathcal{L}_T(\tilde{y}^0) = i\langle y(T,x), \tilde{y}(T,x)\rangle_{\mathcal{S}',\mathcal{S}}. \tag{39}$$

This motivates the following definition.

Definition 5.1. *We say that y is a weak solution to problem (2) in the sense of transposition if $y \in C\left([0,T];H^{-2}(0,\ell)\right)$ satisfies (39) for all $T>0$ and for every $\tilde{y}^0 \in \mathcal{S}$.*

Then we have the following:

Proposition 5.2. *Let $T>0$, and $f \in L^2(0,T)$. Then for any $y^0 \in H^{-2}(0,\ell)$, there exists a unique weak solution of system (2) in the sense of transposition, satisfying*

$$y \in C\left([0,T];H^{-2}(0,\ell)\right). \tag{40}$$

Moreover, there exists a constant $C>0$ such that

$$\|y\|_{L^\infty([0,T];H^{-2}(0,\ell))} \leq C\left(\|y^0\|_{H^{-2}(0,\ell)} + \|f\|_{L^2(0,T)}\right). \tag{41}$$

Proof. It follows from Proposition 2.2, that for any $T > 0$ the linear map
$$\tilde{y}(T, .) \longmapsto \tilde{y}^0$$
is an isomorphism from $H_0^2(0, \ell)$ into itself. Hence, by Proposition 4.1 we deduce that the linear map
$$\tilde{y}(T, .) \longmapsto \mathcal{L}_T(\tilde{y}^0)$$
is continuous on $H_0^2(0, \ell)$. Therefore, by duality, Equation (39) defines $y(T, x)$, as a unique element in $H^{-2}(0, \ell)$. Moreover from (38) it follows that (41) holds. The continuity with respect to time in (40) is proved by density argument. The proof is complete. □

5.2. **Exact controllability.** We are now ready to state our main controllability result. Thanks to the reversibility in time of (2), this system is exactly controllable if and only if the system is null controllable. One has:

Theorem 5.3. *Assume that the coefficients ρ, σ and q satisfy (3) and (4). Given $T > 0$ and $y^0 \in H^{-2}(0, \ell)$, there exists a control $f \in L^2(0, T)$ such that the solution y of the control problem (2) satisfies*
$$y(T, x) = 0, \quad x \in [0, \ell] .$$

Proof. By the Lions'HUM [27], solving the exact controllability problem is equivalent to proving an observability inequality for the backward problem. The backward problem is
$$\left\{ \begin{array}{ll} i\rho(x)\partial_t y = -\partial_x^2 \left(\sigma(x)\partial_x^2 y\right) + \partial_x(q(x)\partial_x y)_x, & (t, x) \in (0, T) \times (0, \ell), \\ y(t, 0) = \partial_x y(t, 0) = y(t, \ell) = 0, \ \partial_x y(t, \ell) = \partial_x^2 \tilde{y}(t, \ell), & t \in (0, T), \\ y(T, x) = 0, & x \in (0, \ell), \end{array} \right. \tag{42}$$
where $\tilde{y}$ is the solution of the uncontrolled system (32). By Proposition 5.2, problem (42) has a unique weak solution y, satisfying $y^0 := y(0, x) \in H^{-2}(0, \ell)$. Hence the linear map
$$\Lambda \ : \ H_0^2(0, \ell) \longrightarrow H^{-2}(0, \ell), \quad \tilde{y}^0 \longmapsto -iy^0$$
is continuous from $H_0^2(0, \ell)$ into $H^{-2}(0, \ell)$. Furthermore, if Λ is shown to be surjective then there exists a control of the form $f(t) = \partial_x^2 \tilde{y}(t, \ell)$ which drives the system (2) to rest in time T. Since $y(T, x) = 0$, then for the choice of $f(t) = \partial_x^2 \tilde{y}(t, \ell)$ by (37), one has
$$-i\langle y^0, \tilde{y}^0 \rangle_{\mathcal{S}', \mathcal{S}} = \sigma(\ell) \int_0^T \partial_x^2 \tilde{y}\,(t, \ell)\, dt.$$
Equivalently
$$\langle \Lambda(\tilde{y}^0), \tilde{y}^0 \rangle_{\mathcal{S}', \mathcal{S}} = \sigma(\ell) \int_0^T \partial_x^2 \tilde{y}\,(t, \ell)\, dt.$$
By Proposition 4.1, for every $T > 0$ and $\tilde{y}^0 \in H_0^2(0, \ell)$, we have
$$\int_0^T |\partial_x^2 \tilde{y}(t, \ell)|^2 dt \asymp \|\tilde{y}^0\|^2_{H_0^2(0, \ell)}.$$
Consequently from the above, for every $T > 0$,
$$\langle \Lambda(\tilde{y}^0), \tilde{y}^0 \rangle_{\mathcal{S}', \mathcal{S}} \asymp \|\tilde{y}^0\|^2_{H_0^2(0, \ell)}.$$
Therefore by the Lax–Milgram Theorem, Λ is surjective. This implies that there exists a control of the form $f(t) = \partial_x^2 \tilde{y}(t, \ell)$ which drives the system (2) to rest in time $T > 0$, and this completes the proof of Theorem 5.3. □

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