

Exact connections between effective magnetostriction and effective elastic moduli of fibrous composites and polycrystals

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The effective magnetostriction of a two-phase fibrous composite and a two-dimensional polycrystal assembled from cylindrical magnetostrictive cubic crystals is studied. For the considered systems, we show that there exist exact microstructure-independent relations between the effective magnetostriction and the effective elastic moduli. The fibers could be aligned identically or randomly oriented in the transverse plane. There is no restriction on the cross-sectional shapes of the fibers, nor on the arrangement of transverse geometry of the composite aggregate. These connections imply that knowledge of the effective elastic moduli will readily provide the effective magnetostriction of the composite medium. © 2003 American Institute of Physics.

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I. INTRODUCTION

When an object is subjected to a magnetic field, its dimensions change. This deformation is referred to as a magnetostriction strain, denoted by λ .¹ The magnetostriction λ varies nonlinearly with the applied magnetic field \mathbf{H} and attains a saturation value for a sufficiently large \mathbf{H} .² In this work, we are concerned with the magnetostrictive response associated with the saturation state of composite aggregates. Magnetostrictive composites are mixtures of a magnetostrictive phase together with a nonmagnetostrictive (or magnetostrictive) matrix to gain magnetostrictive effect and, at the same time, to possess good mechanical toughness and strength. These composites are of potential applications in a few technological devices, such as magnetoactive sensors and actuators. To gain insights into the understanding of their physical behavior, it is important to characterize the overall magnetostriction of composites in terms of their constituent properties and microstructure information in a rigorous manner. In the analysis, typically one may regard the magnetostriction as a stress free strain due to a magnetic field, much like the spirit of the thermal strain induced from the uniform temperature change. Nevertheless, the determination of the effective magnetostriction and the effective thermal expansion of composites is not exactly alike. The magnetostriction is intrinsically orientation dependent, while the thermal strain is not. Much progress has been made in the last few years, including some experimental works.³⁻⁵ Among the analytical studies, one branch is directed toward the deriva-

tion of the upper and lower bounds for the effective magnetostriction of polycrystals⁵ and composites. The other branch is to provide an estimate for the effective λ based on various micromechanical schemes.⁶⁻¹⁰ This article, on the other hand, is to pursue exact microstructure independent relations on the effective magnetostriction. Specifically, we find that there are situations in which the effective magnetostriction can be exactly linked with the effective elastic moduli of a composite medium. Explicit connections are derived for a two-phase composite made of an elastic matrix with fibrous magnetostrictive phase, or a polycrystal assembled from cylindrical magnetostrictive cubic crystals. The fibers could be aligned identically or randomly oriented in the transverse plane. There is no restriction on the cross-sectional shapes of the fibers, nor on the transverse arrangement of the composite. The derivation is based on a construction of uniform fields,^{11,12} originally devised for finding effective thermal expansions of two-phase composites and later generalized to polycrystals.^{13,14} We derive exact connections for three different preferential orientations, with one of the crystallographic axes $\langle 100 \rangle$, $\langle 111 \rangle$ and $\langle 112 \rangle$ of the cubic magnetostrictive crystal being placed along the x_3 direction. These results are theoretically rigorous and microstructurally independent. They are of particular value in that knowledge of the effective elastic moduli, either by theoretical analysis or through experimental measurement, will readily provide an estimate for the effective magnetostriction.

II. DERIVATION OF THE EXACT CONNECTIONS

We consider a magnetostrictive crystal with cubic symmetry, which corresponds to the most commonly reported

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class with giant magnetostriction. The constitutive equation can be written as, based on the crystallographic coordinate \mathbf{x}'

$$\boldsymbol{\sigma}' = \mathbf{L}'(\boldsymbol{\varepsilon}' - \boldsymbol{\lambda}'), \quad (1)$$

where $\boldsymbol{\sigma}'$ is the stress, $\boldsymbol{\varepsilon}'$ is the strain and \mathbf{L}' is the elastic moduli. The term $\boldsymbol{\lambda}'$ is the magnetostriction which depends on the magnetic field intensity \mathbf{H} . As the crystal possesses cubic symmetry, the elastic moduli can be characterized by three independent moduli,¹⁵ which are conveniently defined as $L'_{11} \equiv a$, $L'_{12} \equiv b$, $L'_{44} \equiv c$. Let us now consider a material coordinate x_i which is related to the crystallographic axes x'_i by the transformation $x_i = \alpha_{ij}x'_j$. For an external magnetic field H_3 applied along the x_3 axis of the material sample, the magnetostrictive strain along the crystallographic axes of a microcrystallite is given as¹⁶

$$\lambda'_{ij} = \begin{cases} \frac{3}{2}\lambda_{100}\left(\alpha_{3i}^2 - \frac{1}{3}\right), & \text{for } i=j, \\ \frac{3}{2}\lambda_{111}\alpha_{3i}\alpha_{3j}, & \text{for } i \neq j, \end{cases} \quad (2)$$

where λ_{100} and λ_{111} are the magnetostriction constants of the cubic microcrystallite. The constitutive Eq. (1) can be recast as

$$\boldsymbol{\sigma} = \mathbf{L}(\boldsymbol{\varepsilon} - \boldsymbol{\lambda}), \quad (3)$$

based on the material coordinate \mathbf{x} , in which all field variables are represented by unprimed quantities and their components follow the tensor transformation rules, $L_{ijkl} = \alpha_{ip}\alpha_{jq}\alpha_{km}\alpha_{ln}L'_{pqmn}$ and $\lambda_{ij} = \alpha_{ip}\alpha_{jq}\lambda'_{pq}$. If the phase is isotropic, then $\mathbf{L} = \mathbf{L}'$ and there exists a further reduction on the elastic constants, $a = b + 2c$. In the absence of body force, the stresses must fulfill the divergence free equation. At interfaces, perfect bonding conditions require that the displacement and traction be continuous across the interfaces. If the composite is statistically homogeneous, the aggregate can be regarded as a macroscopically homogeneous medium, whose constitutive equations, based on \mathbf{x} , can be characterized by similar relations

$$\bar{\boldsymbol{\sigma}} = \mathbf{L}^*(\bar{\boldsymbol{\varepsilon}} - \boldsymbol{\lambda}^*), \quad (4)$$

where the over bar denotes the volume average over the representative volume element and the asterisked quantities are the effective physical constants.

We first consider a two-phase particle reinforced composite consisting of inclusions and matrix. Both phases could be magnetostrictive. The constitutive relation of the inclusions is given by Eq. (3) and the matrix follows the same form, but with its field quantities distinguished by a subscript "0." Crystallites are *all identically aligned*. We now separate the phases of the fiber and the matrix apart and apply a sufficiently large magnetic intensity field in the x_3 axis, to reach the saturation state in both phases. This will cause a stress-free strain $\boldsymbol{\lambda}$ in the fiber and $\boldsymbol{\lambda}_0$ in the matrix. Since the two quantities are in general different, the phases cannot be assembled together without inducing stresses. Let us now apply a constant strain $\hat{\boldsymbol{\varepsilon}}$ through a homogeneous displacement condition on the boundaries of the fiber and of the matrix, and at the same time require that $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 \equiv \hat{\boldsymbol{\sigma}}$. This

will automatically guarantee the satisfaction of stress equilibrium and displacement compatibility throughout the medium. By simple algebra it is found that

$$\hat{\boldsymbol{\varepsilon}} = (\mathbf{L} - \mathbf{L}_0)^{-1}(\mathbf{L}\boldsymbol{\lambda} - \mathbf{L}_0\boldsymbol{\lambda}_0). \quad (5)$$

Now with the applied mechanical strain $\hat{\boldsymbol{\varepsilon}}$ together with the phase magnetostriction, the whole composite medium has a uniform strain $\hat{\boldsymbol{\varepsilon}}$ and stress $\hat{\boldsymbol{\sigma}}$ throughout. This constitutes a particular set of uniform field solution, valid without any regard to the transverse microstructure of the composite. Use of $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\sigma}}$ in Eq. (4) will give an exact link between effective magnetostriction and effective moduli

$$\mathbf{L}^*\boldsymbol{\lambda}^* = \mathbf{L}\boldsymbol{\lambda} + (\mathbf{L}^* - \mathbf{L})(\mathbf{L} - \mathbf{L}_0)^{-1}(\mathbf{L}\boldsymbol{\lambda} - \mathbf{L}_0\boldsymbol{\lambda}_0). \quad (6)$$

The exact correspondence (6) is valid for two-phase composites, without any restriction on the geometric shapes of the particles nor on the elastic symmetry of constituent materials. It is noted that Eq. (6) takes the same form as the well-known exact correspondence between the effective thermal and mechanical effects found by Levin,¹⁷ Benveniste and Dvorak,¹⁸ and Milton,¹⁹ among others. This is due to our assumption that the reinforced particles are all aligned identically, and thus each inclusion has a constant magnetostriction with respect to \mathbf{x} , similar to the thermal expansion tensor.

For technological applications the assumption that all magnetostrictive crystallites are all identically aligned is quite restrictive, and indeed not very realistic. To explore its potential for wider applications we now consider two composite systems: (i) two-dimensional polycrystals made of cylindrical magnetostrictive cubic crystallites and (ii) two-phase fibrous (or rod-like) composites consisting of magnetostrictive fibers and isotropic matrix. The fibers, with one of the preferential crystallographic axes being aligned in the x_3 axis, are *randomly oriented* in the transverse x_1-x_2 plane. The transverse cross sections of the fibers could be arbitrary in shape. Thus on a macroscopic scale the heterogeneous medium is effectively transversely isotropic. Again, we start from the constitutive Eq. (3) based on the material axes \mathbf{x} , in which the components of \mathbf{L} and $\boldsymbol{\lambda}$ can be written in terms of $a, b, c, \lambda_{100}, \lambda_{111}$. Let us now separate each fiber and the matrix apart and apply a sufficiently large magnetic intensity field in the x_3 direction to attain the saturation state. Now since each fiber has its own orientation with respect to the x_3 axis, its magnetostriction is generally a function of one rotation angle versus the x_3 axis, say θ . This will cause a stress-free magnetostriction strain $\boldsymbol{\lambda}(\theta)$ in each fiber and $\boldsymbol{\lambda}'$ in the matrix. Apparently, they are not compatible and cannot be assembled together. Since the fibers are arbitrarily oriented, to ensure the conditions of stress equilibrium and compatibility conditions at the interfaces, the fields need to be *hydrostatic* in the transverse plane. Thus, we now apply a homogeneous strain field $\hat{\boldsymbol{\varepsilon}} = [\hat{\varepsilon}_1, \hat{\varepsilon}_1, \hat{\varepsilon}_3, 0, 0, 0]$, in which

$$\begin{aligned} \varepsilon_1 = \varepsilon_2 = \varepsilon_1^0 = \varepsilon_2^0 = \hat{\varepsilon}_1, \quad \varepsilon_3 = \varepsilon_3^0 = \hat{\varepsilon}_3, \\ \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_4^0 = \varepsilon_5^0 = \varepsilon_6^0 = 0. \end{aligned} \quad (7)$$

In the same time, we request that

$$\begin{aligned} \sigma_1 = \sigma_2 = \sigma_1^0 = \sigma_2^0 = \hat{\sigma}_1, \\ \sigma_4 = \sigma_5 = \sigma_6 = \sigma_4^0 = \sigma_5^0 = \sigma_6^0 = 0 \end{aligned} \tag{8}$$

throughout the composite. Satisfaction of Eqs. (7) and (8) will constitute an admissible uniform field solution for the considered system. Evidently, these uniform fields are exactly the volume average field quantities in Eq. (4). The component $\bar{\sigma}_3$ simple follows $\nu\sigma_3 + \nu_0\sigma_3^0$, where ν and ν_0 are the volume fraction of the fiber and the matrix, respectively. For two-dimensional polycrystals, the quantities associated with the matrix phase in Eqs. (7) and (8) are taken out and $\bar{\sigma}_3$ simple follows σ_3 . Upon substituting the uniform field solutions in Eq. (4), one will find the exact connections between λ^* and L^* and, in some cases, we also construct exact connections between the components of L^* . One of the famous known results of the latter kind is the universal connection found by Hill,²⁰ in which he showed that among the five effective constants of an effective transversely isotropic fibrous composite only three of them are independent. See also Ref. 21 for a unified generalization to a broader context.

We now give exact connections for three different cases, with one of the crystallographic directions $\langle 100 \rangle$, $\langle 111 \rangle$ and $\langle 112 \rangle$ of the cubic crystals being placed along the x_3 direction. For simplicity, we define the following short notations: $\Delta v \equiv v - v_0$ and $\langle v \rangle \equiv \nu v + \nu_0 v_0$.

(A) $\langle 100 \rangle$ along the x_3 axis. We find that the components of L^* are not all independent, which are necessarily connected by

$$\begin{aligned} \alpha_1(L_{11}^* + L_{12}^*) - L_{13}^* &= \alpha_1(a + b) - b, \\ 2\alpha_1 L_{13}^* - L_{33}^* &= 2\alpha_1 \langle b \rangle - \langle a \rangle. \end{aligned} \tag{9}$$

Further λ^* can be directly linked with L^* through

$$\begin{bmatrix} (L_{11}^* + L_{12}^*) & L_{13}^* \\ 2L_{13}^* & L_{33}^* \end{bmatrix} \begin{bmatrix} \lambda_1^* + \alpha_2 \\ \lambda_3^* \end{bmatrix} = \begin{bmatrix} \alpha_2(a + b) - \lambda_{100}(a - b)/2 \\ 2\alpha_2 \langle b \rangle + \nu \lambda_{100}(a - b) \end{bmatrix}, \tag{10}$$

where

$$\alpha_1 = \frac{\Delta b}{\Delta(a + b)}, \quad \alpha_2 = \frac{\lambda_{100}(a - b)}{2\Delta(a + b)}. \tag{11}$$

If the matrix is absent and the aggregate becomes a two-dimensional polycrystal, the following exact results can be identified: $L_{11}^* + L_{12}^* = a + b$, $L_{13}^* = b$, $L_{33}^* = a$, $\lambda_1^* = \lambda_2^* = -\lambda_{100}/2$, $\lambda_3^* = \lambda_{100}$.

(B) $\langle 111 \rangle$ along the x_3 axis. We find that

$$\begin{aligned} \alpha_3(L_{11}^* + L_{12}^*) - L_{13}^* &= \frac{2}{3}\alpha_3(a + 2b + c) - \frac{1}{3}(a + 2b - 2c), \\ 2\alpha_3 L_{13}^* - L_{33}^* &= \frac{2}{3}\alpha_3 \langle a + 2b - 2c \rangle - \frac{1}{3} \langle a + 2b + 4c \rangle, \end{aligned} \tag{12}$$

$$\begin{bmatrix} (L_{11}^* + L_{12}^*) & L_{13}^* \\ 2L_{13}^* & L_{33}^* \end{bmatrix} \begin{bmatrix} \lambda_1^* + 3\alpha_4 \\ \lambda_3^* \end{bmatrix} = \begin{bmatrix} 2\alpha_4(a + 2b + c) - c\lambda_{111} \\ 2\alpha_4 \langle a + 2b - 2c \rangle + 2\nu c\lambda_{111} \end{bmatrix}, \tag{13}$$

where

$$\alpha_3 = \frac{\Delta(a + 2b - 2c)}{\Delta(2a + 4b + 2c)}, \quad \alpha_4 = \frac{c\lambda_{111}}{\Delta(2a + 4b + 2c)}. \tag{14}$$

For two-dimensional polycrystals, namely $\nu = 1$, exact moduli are determined as $\lambda_1^* = -\lambda_{111}/2$, $\lambda_3^* = \lambda_{111}$, $L_{11}^* + L_{12}^* = \frac{2}{3}(a + 2b + c)$, $L_{13}^* = \frac{1}{3}(a + 2b - 2c)$, $L_{33}^* = \frac{1}{3}(a + 2b + 4c)$.

(C) $\langle 112 \rangle$ along the x_3 axis. Only the magneto-mechanical correspondence can be found

$$\begin{bmatrix} (L_{11}^* + L_{12}^*) & L_{13}^* \\ 2L_{13}^* & L_{33}^* \end{bmatrix} \begin{bmatrix} (\lambda_1^* + \alpha_5) \\ (\lambda_3^* - \alpha_6) \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{3}a + \frac{4}{3}b + \frac{2}{3}c\right)\alpha_5 - \left(\frac{1}{3}a + \frac{2}{3}b - \frac{2}{3}c\right)\alpha_6 - c\lambda_{111}, \\ \left\langle \frac{a}{2} + \frac{3b}{2} - c \right\rangle \alpha_5 - \left\langle \frac{a}{2} + \frac{b}{2} + c \right\rangle \alpha_6 + \beta_0 \nu \end{bmatrix}, \tag{15}$$

where

$$\begin{aligned} \alpha_5 &= \frac{p}{2} \frac{\Delta(a + 2b - 2c)}{\Delta(a + 2b)} \lambda_{100} + q \frac{\Delta b}{\Delta(a + 2b)} \lambda_{111}, \\ \alpha_6 &= p \frac{\Delta(a + 2b + c)}{\Delta(a + 2b)} \lambda_{100} + q \frac{\Delta(a + b)}{\Delta(a + 2b)} \lambda_{111} \end{aligned} \tag{16}$$

and $\beta_0 = \lambda_{100}(a - b)/4 + 3c\lambda_{111}/2$, $p = (a - b)/(a - b - 2c)$, $q = -3c/(a - b - 2c)$.

For two-dimensional polycrystals, we have $L_{11}^* + L_{12}^* + L_{13}^* = 2L_{13}^* + L_{33}^* = a + 2b$, and

$$\begin{bmatrix} L_{11}^* + L_{12}^* & L_{13}^* \\ 2L_{13}^* & L_{33}^* \end{bmatrix} \begin{bmatrix} \lambda_1^* + \xi \\ \lambda_3^* \end{bmatrix} = \begin{bmatrix} p(a + 2b + c)\lambda_{100} + q(a + b)\lambda_{111} \\ p(a + 2b - 2c)\lambda_{100} + 2bq\lambda_{111} \end{bmatrix}, \tag{17}$$

where $\xi = \frac{3}{2}p\lambda_{100} + q\lambda_{111}$.

In all three cases the effective magnetostriction of the composites can be exactly calculated once the effective moduli of $L_{11}^* + L_{12}^*$, L_{13}^* , L_{33}^* are known. This linkage is theoretically exact and is independent of the micromechanical models used to evaluate the effective moduli. For example, if one uses Voigt's estimate to calculate L^* , our Eqs. (10), (13) and (15) will give exactly Voigt's effective λ^* which was previously formulated as²²

$$\lambda_{\text{Voigt}}^* = \nu(\nu \langle L \rangle_{\text{orien}} + \nu_0 L_0)^{-1} \langle L \lambda \rangle_{\text{orien}}. \tag{18}$$

Here $\langle L \rangle_{\text{orien}}$ means the orientational average of all crystallites about the axial axis. Likewise, if one uses Mori-Tanaka or self-consistent method to calculate L^* , our (10), (13) and (15) will give, respectively, MT and self-consistent λ^* . In fact, an experimental measurement of these moduli $L_{11}^* + L_{12}^*$, L_{13}^* , L_{33}^* will readily provide a quick estimate for λ^* . Note, however, that in the construction of uniform fields we have enforced that the axial strain in the crystallites be the same with that of the matrix to ensure the required axial compatibility for fibrous aggregates. Thus, the Reuss model, which assumes constant stress, apparently violates this requirement and should not be adopted in our connections.

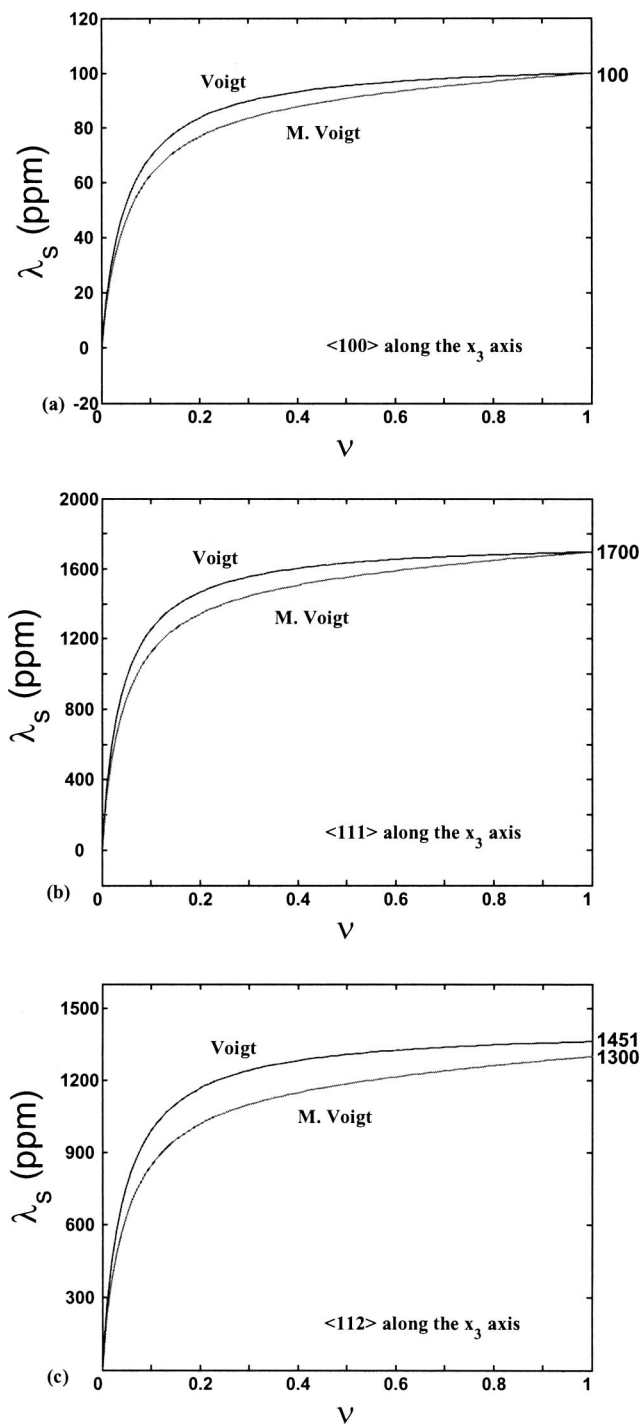


FIG. 1. (a)–(c): Effective engineering magnetostriction $\lambda_s = 2(\lambda_3^* - \lambda_1^*)/3$ vs the volume fraction of the magnetostrictive crystallites. The effective elastic moduli of Terfenol-D/epoxy transversely isotropic composites are estimated based on Voigt and a modified Voigt model. (a) illustrates the effective magnetostriction of the considered system in which the crystallographic direction $\langle 100 \rangle$ of the magnetostrictive crystallites is aligned along the x_3 axis, (b) for $\langle 111 \rangle$ direction and (c) for $\langle 112 \rangle$ direction. The effective magnetostrictions of two-dimensional polycrystals are indicated at $\nu=1$, in which they coincide with exact value for cases A and B.

Any other micromechanical models, which employ Green's function and/or Eshelby's tensors, as well as Voigt's model, will automatically fulfill the equal axial strain requirement and thus will provide reasonable estimates for λ^* through Eqs. (10), (13) and (15).

III. RESULTS AND DISCUSSION

To illustrate our results, we calculate the effective magnetostriction for fibrous composite systems made of Terfenol-D magnetostrictive crystallites and epoxy matrix. When the volume fraction of the magnetostrictive crystallites ν equals one, the calculated value gives the effective magnetostriction for two-dimensional polycrystals. The properties⁸ of the constituent phases used for calculations are $a=101$ GPa, $b=40$ GPa, $c=38$ GPa, $\lambda_{111}=1700$ ppm, $\lambda_{100}=100$ ppm for Terfenol-D, and $a_0=6.5$ GPa, $b_0=3.5$ GPa, $c_0=1.5$ GPa for epoxy. The effective elastic moduli of composites and polycrystals are evaluated based on two simple models: Voigt's constant strain model and a modified Voigt's assumption. Here the modified Voigt's model assumes that a constant axial strain prevails in the composite, while in the remaining directions the stress components are uniform throughout. Figures 1(a)–1(c) show the effective engineering magnetostriction $\lambda_s = \frac{2}{3}(\lambda_3^* - \lambda_1^*)$ for three different cases (A), (B) and (C). We have correctly verified that our calculations based on Eqs. (10), (13) and (15) using Voigt's effective moduli are exactly the same as those directly calculated from Eq. (18). We have also verified analytically that, when $\nu \rightarrow 1$, the effective magnetostrictions from Eqs. (10) and (13) recover the exact results for $\langle 100 \rangle$ and $\langle 111 \rangle$ polycrystals. The latter effect can be observed in the Figs. 1(a) and 1(b) that the Voigt and modified Voigt curves coincide with each other at $\nu=1$. Also we remark the result of Eq. (15) yields the correct value of Eq. (17) when $\nu=1$. Note that in Fig. 1(c) at $\nu=1$, λ_s takes slightly different values for the two estimates. This is due to the fact that the two models give different effective elastic constants for case (C).

In conclusion, we have derived exact connections between the effective magnetostriction and the effective elastic moduli for fibrous composite systems and two-dimensional polycrystals. The magnetostrictive crystallites are cubic symmetry with one of the directions $\langle 100 \rangle$, $\langle 111 \rangle$ and $\langle 112 \rangle$ being aligned in the x_3 axis. These results are of theoretical and technological value in that knowledge of the effective elastic moduli readily provides an estimate for the effective magnetostriction of the composite medium.

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¹ All bold-faced letters denote vectors or tensors.

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