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EXACT CONTROLLABILITY AND STABILIZATION OF A VIBRATING STRING WITH AN INTERIOR POINT MASS

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ABSTRACT

In this article we examine the problems of boundary control and stabilization for a one-dimensional wave equation with interior point masses. We show that singularities in waves are "smoothed one order" as they cross a point mass. Thus in the case of one interior point mass, with e.g., L^2-Dirichlet control at the left end, the most general reachable space (from 0) which one can expect is $L^2 \times H^{-1}$ to the left of the mass and $H^1 \times L^2$ to the right of the mass. We show that this is in fact the optimal result (modulo certain compatibility conditions). Several related results for both control and stabilization of such systems are also given.

Key words. boundary control, hyperbolic system, hybrid system, vibrating network

AMS(MOS) subject classifications. 35P10, 35P20, 35L20, 73K03, 93C20

Short title: CONTROLLABILITY OF A STRING-MASS SYSTEM

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1. INTRODUCTION AND MAIN RESULTS

In recent years there has been much interest in the topic of control and stabilization of so called "hybrid systems" in which the dynamics of elastic systems and possibly rigid structures are related through some form of coupling. To mention a few examples; see [1] for serially connected beams, [8],[9],[10] for beams with end masses and [4],[12],[14] for networks of strings and beams.

In this article we examine a simple model for an elastic string involving an interior point mass. We obtain a precise description of the space of exact controllability when control is active at one or both ends of the string and also describe the best possible stabilization results via velocity feedback at one or both ends. We refer to [14] for a discussion on the modelling and well-posedness of networks of strings containing in particular, point masses. Approximate controllability results for these networks were announced in [13].

It will be convenient to regard the string-mass system as two separate strings in which one end of each string is attached to a common point mass. Thus assume the first string occupies $\Omega_1 = (-\ell_1, 0) \subset \mathbf{R}$ and the second one $\Omega_2 = (0, \ell_2) \subset \mathbf{R}$, where ℓ_1 and ℓ_2 are positive.

For simplicity of the exposition we suppose both strings to be homogeneous. The deformations of the first and second string will be described respectively by the functions

$$u = u(x, t), \quad x \in \Omega_1, \quad t > 0$$
$$v = v(x, t), \quad x \in \Omega_2, \quad t > 0.$$

The position of the mass (which is attached to the strings at the point $x = 0$) is described by the function $z = z(t)$ for $t > 0$.

To fix ideas we suppose that the strings satisfy Dirichlet boundary conditions at the end points $(x = -\ell_1, \ell_2)$. Then the equations modelling the dynamics of this system in the absence of controls are as follows:

(1.1)
$$\begin{cases} \rho_1 u_{tt} = \sigma_1 u_{xx}, & x \in \Omega_1, \ t > 0 \\ \rho_2 v_{tt} = \sigma_2 v_{xx}, & x \in \Omega_2, \ t > 0 \\ M z_{tt}(t) + \sigma_1 u_x(0, t) - \sigma_2 v_x(0, t) = 0, & t > 0 \\ u(-\ell_1, t) = v(\ell_2, t) = 0, & t > 0 \\ u(0, t) = v(0, t) = z(t), & t > 0. \end{cases}$$

The constants $\rho_1 > 0$ and $\rho_2 > 0$ represent the density of each string and $M > 0$ represents the mass of the point mass. The tensions in each string are assumed positive and denoted by σ_1 and σ_2. If the only forces acting on the point mass are those of the strings then $\sigma_1 = \sigma_2$, however, if an external force is present (for example, gravity acting along the x-axis) then the two tensions will be different.

Note that when $M = 0$, system (1.1) describes the motion of a string with a piecewise constant wave speed.

Of course, in order to determine the solution of (1.1) in a unique way we have to add some initial conditions at time $t = 0$ that will be represented by

(1.2)
$$\begin{cases} u(x, 0) = u^0(x), & u_t(x, 0) = u^1(x), & x \in \Omega_1 \\ v(x, 0) = v^0(x), & v_t(x, 0) = v^1(x), & x \in \Omega_2 \\ z(0) = z^0, & z_t(0) = z^1. \end{cases}$$

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As usual, depending on the regularity properties and the compatibility conditions these initial data satisfy, we may expect a different degree of regularity of solutions.

Let us introduce the energy

(1.3)
$$E_M(t) = \frac{1}{2} \int_{-\ell_1}^{0} [\, \rho_1 |u_t(x,t)|^2 + \sigma_1 |u_x(x,t)|^2 \,] \, dx + \frac{M}{2} |z_t(t)|^2$$
$$+ \frac{1}{2} \int_{0}^{\ell_2} [\, \rho_2 |v_t(x,t)|^2 + \sigma_2 |v_x(x,t)|^2] dx.$$

In the absence of controls, i.e. for solutions of (1.1), this energy is constant in time.

We are interested in the controllability properties of this system when control is active at one or both of the end points of the string-mass system. We will discuss these control problems from two different point of views. One of which consists of finding suitable observability estimates and then applying HUM (cf. J.L. Lions [6], [7]) and the other one is based on the use of nonharmonic Fourier series and moment problems (cf. D.L. Russell [11]).

We will see that the presence of the point mass introduces some important changes on the behavior of the system with respect to the observability properties.

Let us recall what one has concerning the observability of (1.1) in the absence of the mass (i.e. when $M = 0$):

(a) If $T > \ell_1 \sqrt{\frac{\rho_1}{\sigma_1}} + \ell_2 \sqrt{\frac{\rho_2}{\sigma_2}}$ there exists $C(T) > 0$ such that

(1.4)
$$E_0(0) \le C \int_{0}^{T} [\, |u_x(-\ell_1,t)|^2 + |v_x(\ell_2,t)|^2] dt.$$

(b) If $T > 2 \left(\ell_1 \sqrt{\frac{\rho_1}{\sigma_1}} + \ell_2 \sqrt{\frac{\rho_2}{\sigma_2}} \right)$ there exists $C(T) > 0$ such that

(1.5)
$$E_0(0) \le C \int_{0}^{T} |v_x(\ell_2,t)|^2 dt$$

We will see that an estimate of the form (1.4) holds for any $M > 0$ but only if $T > 2 \max \left(\ell_1 \sqrt{\frac{\rho_1}{\sigma_1}}, \ell_2 \sqrt{\frac{\rho_2}{\sigma_2}} \right)$. Thus when controlling at both extremes $x = -\ell_1, \ell_2$ we will get the analogue of the results one can prove for two serially connected strings without the point mass but only for $T > 2 \max \left(\ell_1 \sqrt{\frac{\rho_1}{\sigma_1}}, \ell_2 \sqrt{\frac{\rho_2}{\sigma_2}} \right)$.

However, concerning the observability inequality (1.5) the situation is completely different when $M > 0$. By an explicit computation (see Proposition 2.5) one sees that when a wave starting from initial data

$$\begin{cases} u^0 = \varphi^0 \in H_0^1(\Omega_1), & v^0 = 0 \\ u^1 = \varphi^1 \in L^2(\Omega_1), & v^1 = 0 \\ z^0 = z^1 = 0 \end{cases}$$

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crosses the point mass, part of the wave is reflected off the point mass and part is transmitted. The part which is reflected keeps the same regularity as the initial data but the part that crosses the mass is regularized by one degree (i.e. $v(\cdot, t) \in H^2(\Omega_2)$ for all $t > 0$). Of course this is a phenomena due to the presence of the mass that does not occur when $M = 0$.

This phenomena explains why, if we want to observe the initial energy of the solution (in this case the H^1-norm of φ) we need an estimate on $v_{xt}(\ell_2, t)$ in $L^2(0, T)$ and not only on $v_x(\ell_2, t)$.

When $M > 0$ and $T > 2\left(\ell_1\sqrt{\frac{\rho_1}{\sigma_1}} + \ell_2\sqrt{\frac{\rho_2}{\sigma_2}}\right)$ we are able to prove that

$$(1.6) \qquad E_M(0) + ||v_{xx}(\cdot, 0)||^2_{L^2(\Omega_1)} + ||v_{xt}(\cdot, 0)||^2_{L^2(\Omega_2)} \le C \int_0^T |v_{xt}(\ell_2, t)|^2 dt$$

and this inequality is sharp in the sense that the reverse one holds for all $T > 0$.

As a consequence of (1.6) we deduce that when controlling from only the end $x = \ell_2$ with $L^2(0, T)$ controls, the controllability is achieved in a space which is smaller than the usual one (the one we have for $M = 0$) since the components corresponding to the first string are of one more degree of regularity. The well-posedness of the system (1.1) in this asymmetric space is due to the presence of the point mass and does not hold when $M = 0$.

It is also interesting to understand this phenomena from the point of view of nonharmonic Fourier series. Let us consider, for simplicity, the case $\rho_1 = \rho_2 = \ell_1 = \ell_2 = \sigma_1 = \sigma_2 = 1$. By a careful study of the spectrum of the elliptic operator involved in system (1.1) we can see that $v_x(\ell_2, t)$ is given by the nonharmonic Fourier series

$$(1.7) \qquad v_x(1, t) = \sum_{k=-\infty}^{\infty} (a_k e^{ik\pi t} + b_k e^{i\omega_k t})$$

where a_k and b_k are complex numbers related to the Fourier coefficients of the initial data and $i\omega_k$ is a sequence of eigenvalues such that

$$(1.8) \qquad \begin{cases} |\omega_k - k\pi| \to 0 \text{ as } k \to \infty \\ \omega_k \ne k\pi, \forall k \in \mathbf{Z}. \end{cases}$$

When $M = 0$ this second sequence of eigenvalues is $\omega_k = k\pi + \frac{\pi}{2}$ and $v_x(1, t)$ is given by a (harmonic) Fourier series of the form $v_x(1, t) = \sum a_k e^{i\pi kt} + b_k e^{i(\pi k + \pi/2)t}$.

In terms of (1.7) the inequality (1.5) (that we already know does not hold when $M > 0$) would be equivalent to

$$(1.9) \qquad \sum_{k=-\infty}^{\infty} [\,|a_k|^2 + |b_k|^2\,] \le C \int_0^T |v_x(1, t)|^2 dt.$$

However all results concerning inequality (1.9) for nonharmonic Fourier series existing in the literature require an asymptotic spectral gap (cf. [2],[3] and [16]) that, in view of (1.8),

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does not hold in our case. Instead, of (1.9), employing a result of D. Ullrich [15] we get the following weaker version of (1.9)

$$(1.10) \qquad \sum_{k=-\infty}^{\infty} [\,|a_k + b_k|^2 + (w_k - k\pi)^2 |a_k - b_k|^2] \le C \int_0^T |v_x(1,t)|^2 dt$$

which holds for $T \ge 4$, and is just the Fourier version of the observability inequality (1.6).

When controlling at both extremes $x = -\ell_1, \ell_2$ but with

$$(1.11) \qquad \ell_1 \sqrt{\frac{\rho_1}{\sigma_1}} + \ell_2 \sqrt{\frac{\rho_2}{\sigma_2}} < T < 2\max\left(\ell_1 \sqrt{\frac{\rho_1}{\sigma_1}}, \ell_2 \sqrt{\frac{\rho_2}{\sigma_2}}\right)$$

(which is only possible if $\ell_1 \sqrt{\rho_1/\sigma_1} \ne \ell_2 \sqrt{\rho_2/\sigma_2}$) we again obtain, as above, an asymmetric controllability space since the solution components corresponding the string with the longest propagation time has one more degree of regularity over a portion of that string. Thus we also obtain controls with different regularities, namely, the control on the side of the mass with the smoother solution belongs to $H_0^1(0,T)$ and the control at the other end belongs to $L^2(0,T)$.

Therefore, even if we control at both ends this new phenomena appears in which the controllability space is asymmetric when T satisfies (1.11).

We will also discuss briefly the stabilization problem concernings two different situations. First we prove that by introducing boundary damping at both extremes $x = -\ell_1, \ell_2$ the energy of solutions decays exponentially uniformly. Moreover trajectories converge exponentially to a constant equilibrium that can be determined in terms of the initial data. We then consider the case where the boundary damping acts only at one extreme point. In this case the energy of every solution converges to zero but there is not a uniform exponential decay since a sequence of eigenvalues of the system approaches the imaginary axis. This phenomena is similar to that founded by B. Lee and Y.C. You [5] and W. Littman and L. Markus [10] when studying the stability properties of strings and beams damped at one extreme through a point mass.

This paper will be devoted to the particular case of two homogeneous strings with a point mass but the techniques and ideas involved in it are rather general and may be used to discuss non-homogeneous strings, other boundary conditions at the extremes, other dynamics at the point mass and also other situations where more than two strings and one mass are present. A more detailed discussion of some of these extensions will be given in the last section.

The rest of the paper is organized as follows. In section 2 we give some preliminary results concerning the existence, regularity and uniqueness of solutions of both the uncontrolled and the controlled system. In section 3 we examine the controllability problem for the case where the control acts at both extremes. In section 4, by using energy methods, we prove estimate (1.7) and its consequence concerning controllability. In section 5 we carefully analyze the problem of controlling from only one extreme by means of moment problems and nonharmonic Fourier series. In particular we discuss and prove inequality (1.10). In section 6 we discuss the boundary-stabilization problem. Finally, in section 7, we give some extension of our main results.

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Throughout this paper C will denote a positive constant that may vary from line to line. We will only make explicit the dependence of these constants with respect to the various parameters of the problem when this becomes necessary.

2. Existence, uniqueness and regularity of solutions

In this section we give some preliminary results concerning the existence, uniqueness and regularity of solutions. First we consider the system (1.1) without controls and then the case of non-homogeneous boundary conditions.

2.1. Homogeneous boundary conditions. Let us introduce the vector spaces

$$\vartheta_1 = \{\varphi \in H^1(\Omega_1) : \varphi(-\ell_1) = 0\}$$
$$\vartheta_2 = \{\psi \in H^1(\Omega_2) : \psi(\ell_2) = 0\}$$
$$\vartheta = \{(\varphi, \psi) \in \vartheta_1 \times \vartheta_2 : \varphi(0) = \psi(0)\}$$

endowed with the norms

$$\|\varphi\|_{\vartheta_i}^2 = \int\limits_{\Omega_i} |\varphi_x(x)|^2 \, dx, \qquad i = 1, 2$$
$$\|(\varphi, \psi)\|_{\vartheta}^2 = \|\varphi\|_{\vartheta_1}^2 + \|\psi\|_{\vartheta_2}^2.$$

Note that the space ϑ is algebraically and topologically equivalent to $H_0^1(-\ell_1, \ell_2)$. However since we are considering a system made of two different strings it is convenient to think of ϑ as a subspace of $\vartheta_1 \times \vartheta_2$.

Let us also consider the following closed subspace of $\vartheta \times \mathbf{R}$:

$$W_1 = \{(\varphi, \psi, z) \in \vartheta \times \mathbf{R} : \varphi(0) = \psi(0) = z\},$$

which is continuously and compactly embedded in the space

$$W_0 = L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbf{R}.$$

Define the Hilbert space $\mathcal{H}$ by
$$\mathcal{H} = W_1 \times W_0$$

with the product topology. In terms of the vector-valued function

$$y = (u, v, z, \dot{u}, \dot{v}, \dot{z})^t$$

(the superscript ^{t} denotes transposition), we may define an unbounded operator $\mathcal{A}$ on $\mathcal{H}$ by

$$\mathcal{A}y = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} y; \quad A = \begin{pmatrix} \frac{\sigma_1}{\rho_1} d^2 & 0 & 0 \\ 0 & \frac{\sigma_2}{\rho_2} d^2 & 0 \\ \frac{\sigma_1}{M} d\delta_0 & \frac{-\sigma_2}{M} d\delta_0 & 0 \end{pmatrix}$$

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where d denotes the (distributional) derivative operator and δ_0 denotes the Dirac delta function with mass at $x = 0$. The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ y \in \mathcal{H} : u \in H^2(\Omega_1), v \in H^2(\Omega_2), (\dot{u}, \dot{v}, \dot{z}) \in W_1 \right\}.$$

When $\mathcal{D}(\mathcal{A})$ is endowed with the graph-norm topology

$$\|y\|_{\mathcal{D}(\mathcal{A})} = \{\|y\|_{\mathcal{H}}^2 + \|\mathcal{A}y\|_{\mathcal{H}}^2\}^{1/2}$$

it becomes a Hilbert space with continuous and compact embedding in $\mathcal{H}$.

System (1.1)–(1.2) can be written as

$$(2.1) \qquad \frac{dy}{dt} = \mathcal{A}y, \qquad y(0) = y^0 = (u^0, v^0, z^0, u^1, v^1, z^1)^t.$$

It is easy to see that $\mathcal{A}$ is skew-adjoint and m-dissipative on $\mathcal{H}$ and therefore generates a strongly continuous group of isometries on $\mathcal{H}$. Therefore we have the following existence and uniqueness result for (1.1)–(1.2):

PROPOSITION 2.1. (i) *For every* $y^0 = (u^0, v^0, z^0, u^1, v^1, z^1)^t \in \mathcal{H}$ *there exists a unique solution of (1.1)–(1.2) in the class*

$$(2.2) \qquad (u, v, z) \in C([0, T]; W_1) \cap C^1([0, T]; W_0).$$

Furthermore, the energy E_M *remains constant along this solution trajectory.*

(ii) *If* $y^0 \in \mathcal{D}(\mathcal{A})$ *then the corresponding solution has the following additional regularity*

$$(2.3) \qquad \begin{cases} u \in C([0, T]; H^2(\Omega_1)) \cap C^1([0, T]; H^1(\Omega_1)) \\ v \in C([0, T]; H^2(\Omega_2)) \cap C^1([0, T]; H^1(\Omega_2)). \end{cases}$$

Let us denote those solutions satisfying (2.2) by finite-energy solutions.

We can also prove the following regularity result for finite-energy solutions:

PROPOSITION 2.2. *For every* $T > 0$ *there exists some constant* $C(T) > 0$ *such that the following inequality holds for every finite-energy solution:*

$$(2.4) \qquad \int_0^T [\,|u_x(-\ell_1, t)|^2 + |v_x(\ell_2, t)|^2]dt \leq C E_M(0).$$

PROOF: It is well known by now (cf. [7]) that this estimate is of local nature. Therefore does not depend at all on whether or not there is a point mass on the string. However we have to use the conservation of the energy $E_M(t)$ in (1.3) to obtain the upper bound in terms of the initial energy. ∎

It is also convenient to consider system (1.1) in the presence of some external distributed force:

$$(2.5) \qquad \begin{cases} \rho_1 u_{tt} = \sigma_1 u_{xx} + f(x, t), & x \in \Omega_1,\ 0 < t < T \\ \rho_2 v_{tt} = \sigma_2 v_{xx} + g(x, t), & x \in \Omega_2,\ 0 < t < T \\ M z_{tt}(t) + \sigma_1 u_x(0, t) - \sigma_2 v_x(0, t) = h(t), & 0 < t < T \\ u(0, t) = v(0, t) = z(t), & 0 < t < T \\ u(-\ell_1, t) = v(\ell_2, t) = 0, & 0 < t < T. \end{cases}$$

By standard semigroup methods we get:

PROPOSITION 2.3. *For every $y^0 \in \mathcal{H}$ and $(f, g, h) \in L^1(0, T; W_0)$ there exists a unique finite-energy solution of (2.5), (1.2) in the class (2.2). Moreover, there exists a constant $C > 0$ such that*

$$
(2.6) \quad \int_0^T \left[|u_x(-\ell_1, t)|^2 + |v_x(\ell_2, t)|^2 \right] dt \leq C \left[\|y^0\|_{\mathcal{H}}^2 + \|f\|_{L^1(0, T; L^2(\Omega_1))}^2 \right.
$$

$$
\left. + \|g\|_{L^1(0, T; L^2(\Omega_2))}^2 + \|h\|_{L^1(0, T)}^2 \right]
$$

We will also need the following result:

PROPOSITION 2.4. *Suppose that $y^0 = 0$ and*

$$
f = \frac{\partial F}{\partial t}, \quad g = \frac{\partial G}{\partial t}, \quad h = \frac{dH}{dt}
$$

where $(F, G, H) \in L^1(0, T; W_1)$. Then the solution (u, v, z) of (2.5), (1.2) is such that

$$
(u, v, z) = (U_t, V_t, Z_t)
$$

with

$$
(U, V, Z, U_t, V_t, Z_t)^t \in C([0, T]; D(\mathcal{A}))
$$

and therefore, in particular,

$$
(2.7) \quad \begin{cases} (u, v, z) \in C([0, T]; W_1) \\ (u_t, v_t, z_t) \in C([0, T]; W_0) + L^1(0, T; W_1) \\ (U, V) \in C([0, T]; H^2(\Omega_1) \times H^2(\Omega_2)) \end{cases}
$$

Moreover, there exists a constant $C > 0$ such that

$$
(2.8) \quad \int_0^T [|u_x(-\ell_1, t)|^2 + |v_x(\ell_2, t)|^2] dt \leq C \|(F, G, H)\|_{L^1(0, T; W_1)}^2
$$

PROOF: We have $(u, v, z) = (U_t, V_t, Z_t)$ where (U, V, Z) is the solution of (2.5), (1.2) with zero initial data and $(f, g, h) = (F, G, H)$. We have

$$
(U, V, Z, U_t, V_t, Z_t)^t \in C([0, T]; D(\mathcal{A})).
$$

and therefore (2.7) holds.

The regularity property (2.8) is more subtle and can be proved proceeding as in [7], Chapter I, Th. 4.2, p. 46. ∎

Let us now study the regularity of solutions where the initial data belong to a space where the regularity is not the same in each of the strings. More precisely, let us consider initial data

$$
(2.9) \quad y^0 \in \mathcal{H}
$$

such that

$$
(2.10) \quad u^0 \in H^2(\Omega_1), \quad u^1 \in \vartheta_1, \quad u^1(0) = z^1.
$$

Of course, (2.9)–(2.10) do not imply that $y^0 \in D(\mathcal{A})$ and therefore we cannot apply the regularity Proposition 2.1 (ii) provides. However, we can prove the following result.

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PROPOSITION 2.5. *Suppose that the initial data y^0 satisfies (2.9)–(2.10). Then the solution of (1.1)–(1.2) is such that, in addition to (2.2), we have*

(2.11) $$u \in C([0,T]; H^2(\Omega_1) \cap C^1([0,T]; \vartheta_1).$$

Moreover, there exists $C > 0$ such that

(2.12) $$\|u\|^2_{L^\infty(0,T;H^2(\Omega_1))} + \|u_t\|^2_{L^\infty(0,T;\vartheta_1)} \le C[E_M(0) + \|u^0\|^2_{H^2(\Omega_1)} + \|u^1\|^2_{\vartheta_1}]$$

for every solution with initial data satisfying (2.9)–(2.10).

PROOF: It is sufficient to prove the existence of some $\tau > 0$ and $C > 0$ such that (2.11) and (2.12) hold in the time interval $[0,\tau]$. By scaling the spatial variable in Ω_1 and changing the time scale we may assume $\ell_1 = 1$ and in the wave equation u satisfies, that $\rho_1 = \sigma_1 = 1$. Likewise, by changing the length of the second string we may assume $\rho_2 = \sigma_2 = 1$ in the wave equation v satisfies. However the conditions at the point mass change and we are lead to consider the following system:

$$\begin{cases} u_{tt} = u_{xx}, & -1 < x < 0, \quad 0 < t < T \\ v_{tt} = v_{xx}, & 0 < x < \ell, \quad 0 < t < T \\ mz_{tt}(t) + u_x(0,t) - \gamma v_x(0,t) = 0, & 0 < t < T \\ u(-1,t) = v(\ell,t) = 0, & 0 < t < T \\ u(0,t) = v(0,t) = z(t), & 0 < t < T \end{cases}$$

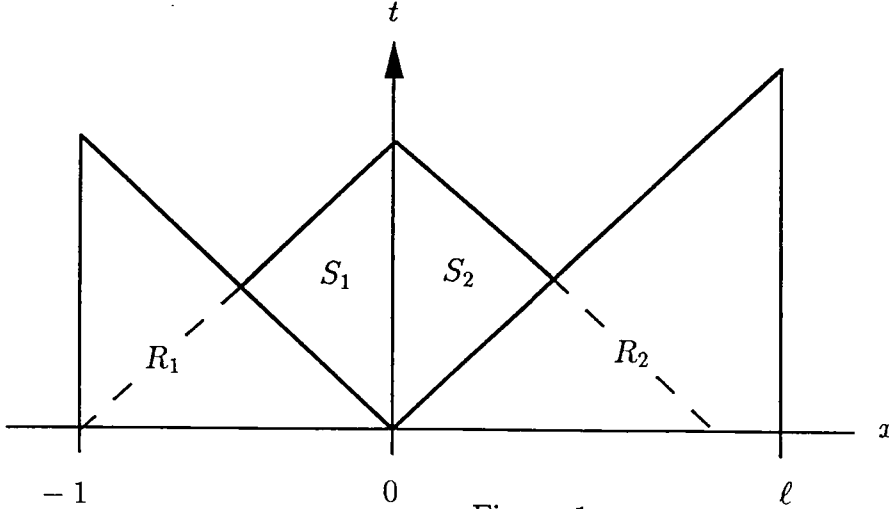with $\gamma > 0$ and $m > 0$, ℓ being the length of the second string.

Figure 1

The value of u and v in the region

$$R_1 = \{(x,t) \in (-1,0) \times (0,1) : t < -x\}$$

and

$$R_2 = \{(x,t) \in (0,\ell) \times (0,\ell) : t < x\}$$

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respectively do not depend on the point mass because of the finite speed of propagation (see Figure 1).

Therefore, in R_1 and R_2, u and v remain as smooth as in the absence of mass. In particular $u \in C([0,1]; H^2(R_1(t)) \cap C^1([0,1]; H^1(R_1(t)))$ where $R_1(t) = (-1, -t)$. Moreover

$$\|u\|_{L^\infty(0,1;H^2(R_1(t)))} + \|u_t\|_{L^\infty(0,1;H^1(R_1(t)))} \leq C \left[\|u^0\|_{H^2(-1,0)} + \|u^1\|_{H^1(-1,0)} \right].$$

Let us compute now u and v in S_1 and S_2 respectively where

$$S_1 = \left\{ (x,t) \in \left(-\frac{\mu}{2}, 0 \right) \times (0, \mu) : |2t - \mu| < \mu + 2x \right\}$$

$$S_2 = \left\{ (x,t) \in \left(0, \frac{\mu}{2} \right) \times (0, \mu) : |2t - \mu| < \mu - 2x \right\}$$

with $\mu = \min(\ell, 1)$ ($\mu = 1$ in figure 1).

We have from D'Alembert's formula

$$(2.13) \qquad u(x,t) = \frac{1}{2}[z(t-x) + z(t+x)] + \frac{1}{2} \int_{t-x}^{t+x} u_x(0,s)ds \quad \text{in } S_1$$

$$(2.14) \qquad v(x,t) = \frac{1}{2}[z(t-x) + z(t+x)] + \frac{1}{2\gamma} \int_{t-x}^{t+x} [u_x(0,s) + mz''(s)]dx \quad \text{in } S_2.$$

Likewise for (x,t) in S_1 and S_2, u and v, respectively, are given in terms of the initial data by D'Alembert's formula. By Proposition 2.1 we know that the solution is continuous, in particular, along the rays $t = |x|$, $(-\mu/2 < x < \mu/2)$. Imposing continuity in the expressions for u and v along these rays leads to

$$(2.15) \qquad z_t(t) + \frac{(1+\gamma)}{m} z(t) = z^1 + \frac{1}{m}[L(t) + \gamma R(t)], \quad z(0) = z^0$$

and

$$(2.16) \qquad u_x(0,t) = z_t(t) + u_x^0(-t) - u^1(-t)$$

where

$$(2.17) \qquad L(t) = u^0(t) - \int_0^{-t} u^1(s)ds; \quad R(t) = v^0(t) + \int_0^{t} v^1(s)ds.$$

From the conditions the initial data satisfy we deduce from (2.17) that L and R belong to $H^1(0, \mu)$. It thus follows from (2.15) that $z = z(t)$ belongs to $H^2(0, \mu)$. Then, from (2.10) and (2.16) we deduce that $u_x(0,t) \in H^1(0, \mu)$.

Now, from (2.13) we easily deduce that

$$(2.18) \qquad u \in C\left(\left[0, \frac{\mu}{2} \right]; H^2(-t, 0) \right) \cap C^1\left(\left[0, \frac{\mu}{2} \right]; H^1(-t, 0) \right).$$

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Finally, it is easy to check that the expressions we have for u in R_1 (as solution of the wave equation) and in S_1 (by (2.13)) are such that u_x and u_t are continuous across $x = -t$. This concludes the proof of the proposition. ∎

REMARK 2.5: It is obvious from the proof of this Proposition that we cannot replace in the hypothesis (2.10) and in the conclusions (2.11)–(2.12) the space $H^2(-1,0) \times H^1(-1,0)$ by any $H^s(-1,0) \times H^{s-1}(-1,0)$ with $s > 2$. In other words, the most extra regularity degree we may keep in one of the strings is one.

2.2. Non-homogeneous boundary data.

In this section we prove the existence and uniqueness of weak solutions when we introduce $L^2(0,T)$-Dirichlet controls at the extreme points $x = -\ell_1, \ell_2$.

Let us consider the system

$$(2.19) \quad \begin{cases} \rho_1 u_{tt} = \sigma_1 u_{xx}, & x \in \Omega_1,\ 0 < t < T \\ \rho_2 v_{tt} = \sigma_2 v_{xx}, & x \in \Omega_2,\ 0 < t < T \\ M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, & 0 < t < T \\ u(0,t) = v(0,t) = z(t), & 0 < t < T \\ u(-\ell_1,t) = p(t), & 0 < t < T \\ v(\ell_2,t) = q(t), & 0 < t < T \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in \Omega_1 \\ v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x), & x \in \Omega_2 \\ z(0) = z^0, \quad z_t(0) = z^1. \end{cases}$$

with $p, q \in L^2(0,T)$, $u^0 \in L^2(\Omega_1)$, $v^0 \in L^2(\Omega_2)$, the initial velocity u^1 and v^1 belonging respectively to the dual spaces ϑ_1' and ϑ_2' and $z^0, z^1 \in \mathbf{R}$.

The solution (u,v,z) of (2.19) has to be understood in the sense of transposition. Let us give its precise definition. For that, consider the following system with homogeneous boundary conditions:

$$(2.20) \quad \begin{cases} \rho_1 \varphi_{tt} = \sigma_1 \varphi_{xx} + f & x \in \Omega_1,\ 0 < t < T \\ \rho_2 \psi_{tt} = \sigma_2 \psi_{xx} + g & x \in \Omega_2,\ 0 < t < T \\ M \zeta_{tt}(t) + \sigma_1 \varphi_x(0,t) - \sigma_2 \psi_x(0,t) = h(t) & 0 < t < T \\ \varphi(-\ell_1,t) = \psi(\ell_2,t) = 0 & 0 < t < T \\ \varphi(0,t) = \psi(0,t) = \zeta(t) & 0 < t < T \\ \varphi(x,T) = \varphi_t(x,T) = 0 & x \in \Omega_1 \\ \psi(x,T) = \psi_t(x,T) = 0 & x \in \Omega_2 \\ \zeta(T) = \zeta_t(T) = 0. \end{cases}$$

For every $(f,g,h) \in L^1(0,T;W_0)$ in view of the time-reversibility of the system and as a consequence of Proposition 2.3, system (2.20) has a unique solution

$$(2.21) \quad \begin{cases} (\varphi, \psi, \zeta) \in C([0,T]; W_1) \\ (\varphi_t, \psi_t, \zeta_t) \in C([0,T]; W_0) \end{cases}$$

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satisfying

$$(2.22) \qquad \int_0^T [\,|\varphi_x(\ell_1,t)|^2 + |\psi_x(\ell_2,t)|^2]dt \leq C\|(f,g,h)\|_{L^1(0,T;W_0)}^2.$$

Multiplying by φ and ψ in the equations satisfied by u and v in (2.19) and integrating formally by parts with respect to x and t we obtain the following identity:

$$
\begin{aligned}
(2.23) \qquad & \int_0^T \int_{-\ell_1}^0 uf\,dx\,dt + \int_0^T \int_0^{\ell_2} vg\,dx\,dt + \int_0^T zh\,dt = -\rho_1 \int_{-\ell_1}^0 u^0 \varphi_t(x,0)dx \\
& - \rho_2 \int_0^{\ell_2} v^0 \psi_t(x,0)dx + \rho_1\langle u^1, \varphi(\cdot,0)\rangle_{\Omega_1} + \rho_2\langle v^1, \psi(\cdot,0)\rangle_{\Omega_2} \\
& + \sigma_1 \int_0^T p(t)\varphi_x(-\ell_1,t)dt - \sigma_2 \int_0^T q(t)\psi_x(\ell_2,t)dt + Mz^1\zeta(0) - Mz^0\zeta_t(0).
\end{aligned}
$$

We adopt this identity as the definition for weak solutions of (2.19) in the sense of transposition, i.e. (u,v,z) is said to be weak solution of (2.19) (in the sense of transposition) if (2.23) holds for every $(f,g,h) \in L^1(0,T;W_0)$.

In (2.23) observe that the initial velocities (u^1,v^1,z^1) are applied (in the sense of the duality in W_1) to the elements $(\varphi(\cdot,0),\psi(\cdot,0),\zeta(0))$ of W_1. Therefore two initial data that coincide in $W_0 \times W_1'$ (note that W_1' is a quotient space of $(\vartheta_1 \times \vartheta_2 \times \mathbf{R})'$) give rise to the same solution.

We have the following result.

PROPOSITION 2.6. *For every* $p,q \in L^2(0,T)$, $(u^0,v^0) \in L^2(\Omega_1) \times L^2(\Omega_2)$, $(u^1,v^1) \in (\vartheta_1' \times \vartheta_2')$ *and* $z^0,z^1 \in \mathbf{R}$ *there exists a solution (in the sense of transposition) of (2.19) in the class*

$$(2.24) \qquad\qquad (u,v,z) \in C([0,T];W_0)$$

$$(2.25) \qquad\qquad (u_t,v_t,z_t) \in C^1([0,T];\vartheta_1' \times \vartheta_2' \times \mathbf{R}).$$

Moreover, there is a one-to-one correspondence between the initial data as elements of the quotient space $W_0 \times W_1'$ *and the solutions of (2.19) in the class (2.24)–(2.25).*

PROOF: In view of Proposition 2.3, the right hand side of (2.23) defines a linear and continuous form on $(f,g,h) \in L^1(0,T;W_0)$. Therefore, there exists a unique

$$(2.26) \qquad\qquad (u,v,z) \in L^\infty(0,T;W_0)$$

satisfying (2.23). Furthermore, there exists $C > 0$ such that

$$
\begin{aligned}
(2.27) \qquad \|(u,v,z)\|_{L^\infty(0,T;W_0)} \leq C\,\big\{ & \|p\|_{L^2(0,T)} + \|q\|_{L^2(0,T)} + \|u^0\|_{L^2(\Omega_1)} + \\
& + \|v^0\|_{L^2(\Omega_2)} + \|u^1\|_{\vartheta_1'} + \|v^1\|_{\vartheta_2'} + |z^0| + |z^1| \big\}.
\end{aligned}
$$

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When the data are smooth, the solution of (2.19) satisfies (2.24). By a density argument using (2.27), we deduce that (2.24) holds for our weak solution.

Suppose now that

$$(f, g, h) = \left(\frac{\partial F}{\partial t}, \frac{\partial G}{\partial t}, \frac{dH}{dt} \right)$$

with $(F, G, H) \in \mathcal{D}((0, T); W_1)$ (C^∞ and compactly supported with respect to time). In this case the solution of (2.20) can be written as

$$(\varphi, \psi, \zeta) = \left(\frac{\partial \Phi}{\partial t}, \frac{\partial \Psi}{\partial t}, \frac{d\Sigma}{dt} \right)$$

where (Φ, Ψ, Σ) is the solution of (2.20) with data (F, G, H) instead of (f, g, h).

In view of Proposition 2.4 we have

$$(2.28) \qquad \begin{aligned} \|\varphi_x(-\ell_1, t)\|_{L^2(0,T)} + \|\psi_x(\ell_2, t)\|_{L^2(0,T)} + \|\varphi(\cdot, 0)\|_{\vartheta_1} + \|\psi(\cdot, 0)\|_{\vartheta_2} + |\zeta(0)| \\ \leq C \|(F, G, H)\|_{L^1(0, T, W_1)}. \end{aligned}$$

On the other hand

$$\rho_1 \varphi_t(x, 0) = \rho_1 \Phi_{tt}(x, 0) = \sigma_1 \Phi_{xx}(x, 0) + F(x, 0) = \sigma_1 \Phi_{xx}(x, 0) \in L^2(\Omega_1)$$

$$\rho_2 \psi_t(x, 0) = \rho_2 \Psi_{tt}(x, 0) = \sigma_2 \Psi_{xx}(x, 0) + G(x, 0) = \sigma_2 \Psi_{xx}(x, 0) \in L^2(\Omega_2)$$

$$M \zeta_t(0) = M \Sigma_{tt}(0) = -\sigma_2 \Phi_x(0, 0) + \sigma_2 \Psi_x(0, 0) \in \mathbf{R}$$

with the bound

$$(2.29) \qquad \|(\varphi_t(\cdot, 0), \psi_t(\cdot, 0), \zeta_t(0))\|_{W_0} \leq C \|(F, G, H)\|_{L^1(0, T; W_1)}.$$

As a consequence of (2.28)–(2.29) we deduce that

$$(u, v, z) \in W^{1, \infty}(0, T; (\vartheta_1)' \times (\vartheta_2)' \times \mathbf{R}).$$

The continuity in time of (u_t, v_t, z_t) with values in $(\vartheta_1)' \times (\vartheta_2)' \times \mathbf{R}$ can be proved again by density. ∎

Let us finally consider these weak solutions when the initial and boundary data corresponding to the first string have one more degree of regularity, i.e.

$$(2.30) \qquad p \in H^1(0, T), \ u^0 \in H^1(-\ell_1, 0), \ u^1 \in L^2(\Omega_1), \ u^0(0) = z^0, \ p(0) = u^0(-\ell_1).$$

We have the following result.

PROPOSITION 2.7. *Suppose that the initial and boundary data in Proposition 2.6 satisfy the further regularity and compatibility conditions (2.30). Then, in addition to (2.24)–(2.25) we have*

$$(2.31) \qquad u \in C([0, T]; H^1(\Omega_1)) \cap C^1([0, T]; L^2(\Omega_1)).$$

Furthermore, $u_x(-\ell_1, t) \in L^2(0, T)$ and there exists some $C > 0$ such that

$$(2.32) \qquad \begin{aligned} \int_0^T |u_x(-\ell_1, t)|^2 dt \leq C \Big[&\|q\|_{L^2(0,T)}^2 + \|p\|_{H^1(0,T)}^2 + \|u^0\|_{H^1(\Omega_1)}^2 + \|u^1\|_{L^2(\Omega_1)}^2 + \\ &+ \|v^0\|_{L^2(\Omega_2)}^2 + \|v^1\|_{(\vartheta_2)'}^2 + |z^0|^2 + |z^1|^2 \Big]. \end{aligned}$$

PROOF: The proof of (2.31) is very similar to that of Proposition 2.5. As we pointed out in Proposition 2.2, estimate (2.32) is of local nature and holds from (2.31). ∎

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3. Control at both extremes

In this section we consider the problem of controlling our system from only both ends $x = -\ell_1, \ell_2$. The system reads now:

$$(3.1) \quad \begin{cases} \rho_1 u_{tt} = \sigma_1 u_{xx}, & x \in \Omega_1, \ 0 < t < T \\ \rho_2 v_{tt} = \sigma_2 v_{xx}, & x \in \Omega_2, \ 0 < t < T \\ M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, & 0 < t < T \\ u(-\ell_1, t) = p(t), & 0 < t < T \\ v(\ell_2, t) = q(t), & 0 < t < T \\ u(0,t) = v(0,t) = z(t), & 0 < t < T \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in \Omega_1 \\ v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x), & x \in \Omega_2 \\ z(0) = z^0, \quad z_t(0) = z^1. \end{cases}$$

We have the following results.

Theorem 3.1. *Suppose that* $T > 2 \max \left(\ell_1 \sqrt{\frac{\rho_1}{\sigma_1}}, \ell_2 \sqrt{\frac{\rho_2}{\sigma_2}} \right)$. *Then, for every*

$$(3.2) \quad \begin{cases} (u^0, v^0, z^0) \in W_0 \\ (u^1, v^1, z^1) \in (\vartheta_1)' \times (\vartheta_2)' \times \mathbf{R} \end{cases}$$

there exist controls $p, q \in L^2(0, T)$ *such that the solution of (3.1) satisfies*

$$(3.3) \quad \begin{cases} u(x,T) = u_t(x,T) = 0, & x \in \Omega_1 \\ v(x,T) = v_t(x,T) = 0, & x \in \Omega_2 \\ z(T) = z_t(T) = 0. \end{cases}$$

Remark 3.1: Concerning Thoerem 3.1 we have the following:

(1) As a consequence of Proposition 2.6 the solution of (3.1) (which is defined by transposition) satisfies (2.24)–(2.25).

(2) Due to the linearity and time-reversibility of system (3.1), we deduce that for any initial data as in (3.2) and final data $(\tilde{u}^0, \tilde{v}^0, \tilde{z}^0, \tilde{u}^1, \tilde{v}^1, \tilde{z}^1)$ satisfying the same properties there exist controls $p, q \in L^2(0, T)$ such that the solution of (3.1) satisfies

$$(3.4) \quad \begin{cases} u(x,T) = \tilde{u}^0(x), \quad u_t(x,T) = \tilde{u}^1(x), & x \in \Omega_1 \\ v(x,T) = \tilde{v}^0(x), \quad v_t(x,T) = \tilde{v}^1(x), & x \in \Omega_2 \\ z(T) = \tilde{z}^0, \quad z_t(T) = \tilde{z}^1 \end{cases}$$

(3) The controllability space (3.2) is that same as what one obtains when $M = 0$, however the control time is strictly larger when $\ell_1 \sqrt{\rho_1/\sigma_1} \neq \ell_2 \sqrt{\rho_2/\sigma_2}$.

THEOREM 3.2. *Suppose that* $T > \ell_1\sqrt{\frac{\rho_1}{\sigma_1}} + \ell_2\sqrt{\frac{\rho_2}{\sigma_2}}$ *and for instance,* $\ell_1\sqrt{\frac{\rho_1}{\sigma_1}} > \ell_2\sqrt{\frac{\rho_2}{\sigma_2}}$. *Then, for every initial data as in (3.2) satisfying the additional regularity and compatibility properties*

$$(3.5) \qquad u^0 \in \vartheta_1, \ u^1 \in L^2(\Omega_1), \ u^0(0) = z^0$$

there exist controls $p \in H_0^1(0,T)$ *and* $q \in L^2(0,T)$ *such that the solution of (3.1) satisfies (3.3).*

REMARK 3.2: Concerning Theorem 3.2 we have the following:

(1) In addition to satisfying (2.24)-(2.25), by Proposition 2.7, the solution of (3.1) also satisfies (2.31).

(2) Due to the linearity, time-reversibility and well-posedness of the system in the (asymmetric) space (3.2),(3.5), as a consequence of Theorem 3.2, we deduce that we can drive system (3.1) from any initial state in the class (3.2),(3.5) to any terminal state in the same class.

(3) The control time we obtain is the same as in the absence of the mass ($M = 0$). However we only get controllability in the usual space (3.2) when $T > 2\max\left(\ell_1\sqrt{\frac{\rho_1}{\sigma_1}}, \ell_2\sqrt{\frac{\rho_2}{\sigma_2}}\right)$.

The proof of Theorem 3.2 will be given at the end of section 4.

Applying Lions' HUM (cf. [6],[7]) Theorem 3.1 is a direct consequence of the following observability result for solution of the uncontrolled problem:

$$(3.6) \qquad \begin{cases} \rho_1\varphi_{tt} = \sigma_1\varphi_{xx} & x \in \Omega_1, \quad 0 < t < T \\ \rho_2\psi_{tt} = \sigma_2\psi_{xx} & x \in \Omega_2, \quad 0 < t < T \\ M\zeta_{tt}(t) + \sigma_1\varphi_x(0,t) - \sigma_2\psi_x(0,t) = 0 & 0 < t < T \\ \varphi(-\ell_1, t) = \psi(\ell_2, t) = 0 & 0 < t < T \\ \varphi(0,t) = \psi(0,t) = \zeta(t) & 0 < t < T \end{cases}$$

PROPOSITION 3.3. *Let* $T_0 = 2\max\left(\ell_1\sqrt{\frac{\rho_1}{\sigma_1}}, \ell_2\sqrt{\frac{\rho_2}{\sigma_2}}\right)$ *and suppose that* $T > T_0$. *Then*

$$(3.7) \quad (T - T_0)E_M(0) \le \left(\frac{\max(\ell_1, \ell_2)}{2} + \frac{M}{2(\rho_1 + \rho_2)}\right) \int_0^T [\sigma_1|\varphi_x(-1,t)|^2 + \sigma_2|\psi_x(1,t)|^2]dt$$

for every finite-energy solution of (3.6).

PROOF OF PROPOSITION 3.3: We proceed in several steps.
Step 1. Consider the x-dependent energy:

$$(3.8) \qquad e_1(x) = \frac{1}{2}\int_{(x+\ell_1)\tau_1}^{T-(x+\ell_1)\tau_1} \left[\rho_1|\varphi_t(x,t)|^2 + \sigma_1|\varphi_x(x,t)|^2\right] dt, \quad -\ell_1 \le x \le 0$$

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where $\tau_1 = \sqrt{\rho_1/\sigma_1}$. It is easy to check that $e_1(\cdot)$ is non-increasing. Thus

$$(3.9) \qquad e_1(x) \le e_1(-\ell_1) = \frac{\sigma_1}{2} \int_0^T |\varphi_x(-\ell_1, t)|^2 dt, \quad -\ell_1 \le x \le 0.$$

Step 2. Consider

$$(3.10) \qquad e_2(x) = \frac{1}{2} \int_{(\ell_2-x)\tau_2}^{T-(\ell_2-x)\tau_2} \left[\rho_2 |\psi_t(x,t)|^2 + \sigma_2 |\psi_x(x,t)|^2 \right] dt, \quad 0 \le x \le \ell_2$$

where $\tau_2 = \sqrt{(\rho_2/\sigma_2)}$. This energy is non-decreasing and therefore

$$(3.11) \qquad e_2(x) \le e_2(\ell_2) = \frac{\sigma_2}{2} \int_0^T |\psi_x(\ell_2, t)|^2 dt, \quad 0 \le x \le \ell_2.$$

Step 3. From (3.9), (3.11) we deduce in particular that

$$\int_\mu^{T-\mu} \int_{-\ell_1}^0 \left[\rho_1 |\varphi_t(x,t)|^2 + \sigma_1 |\varphi_x(x,t)|^2 \right] dx\,dt + \int_\mu^{T-\mu} \int_0^{\ell_2} \left[\rho_2 |\psi_t(x,t)|^2 + \sigma_2 |\psi_x(x,t)|^2 \right] dx\,dt$$

$$\le \max(\ell_1, \ell_2) \int_0^T \left[\sigma_1 |\varphi_x(-\ell_1, t)|^2 + \sigma_2 |\psi_x(\ell_2, t)|^2 \right] dt$$

where $\mu = \max\left(\tau_1 \ell_1, \tau_2 \ell_2\right)$.

Moreover, since $\varphi(0,t) = \psi(0,t) = \zeta(t)$, we have

$$M \int_\mu^{T-\mu} |\zeta_t(t)|^2 dt \le M \Big(r \frac{e_1(0)}{\rho_1} + (2-r) \frac{e_2(0)}{\rho_2} \Big) \Big|_{r=\frac{2\rho_1}{\rho_1+\rho_2}}$$

$$\le \frac{M}{\rho_1 + \rho_2} \int_0^T [\sigma_1 |\varphi_x(-1, t)|^2 + \sigma_2 |\psi_x(1, t)|^2] dt$$

Thus

$$(T - 2\mu) E_M(0) = \int_\mu^{T-\mu} E_M(t) dt$$

$$\le \left(\frac{\max(\ell_1, \ell_2)}{2} + \frac{M}{2(\rho_1 + \rho_2)} \right) \int_0^T [\sigma^2 |\varphi_x(-\ell_1, t)|^2 + \sigma_2 |\psi_x(\ell_2, t)|^2] dt. \quad \blacksquare$$

REMARK 3.3: (1) Inequality (3.7) provides explicit constants. In particular, we have explicitly the dependence of the observability constant with respect to the mass. As $M \to 0$ we obtain the usual constant one has for a wave equation with piecewise constant coefficients.

(2) The reverse inequality of (3.7) has been proved in Proposition 2.2.

4. CONTROL AT ONE EXTREME

In this section we consider the problem of controlling our system from only one extreme-point, for instance, $x = \ell_2$. The system reads now:

$$(4.1) \quad \begin{cases} \rho_1 u_{tt} = \sigma_1 u_{xx}, & x \in \Omega_1,\ 0 < t < T \\ \rho_2 v_{tt} = \sigma_2 v_{xx}, & x \in \Omega_2,\ 0 < t < T \\ M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, & 0 < t < T \\ u(0,t) = v(0,t) = z(t), & 0 < t < T \\ u(-\ell_1, t) = 0, & 0 < t < T \\ v(\ell_2, t) = q(t), & 0 < t < T \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in \Omega_1 \\ v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x), & x \in \Omega_2 \\ z(0) = z^0, \quad z_t(0) = z^1. \end{cases}$$

We have the following result.

THEOREM 4.1. *Suppose that $T > 2\left(\ell_1\sqrt{\frac{\rho_1}{\sigma_1}} + \ell_2\sqrt{\frac{\rho_2}{\sigma_2}}\right)$. Then, for every*

$$(4.2) \quad \begin{cases} (u^0, u^1) \in \vartheta_1 \times L^2(\Omega_1) \\ (v^0, v^1) \in L^2(\Omega_2) \times (\vartheta_2)' \\ (z^0, z^1) \in \mathbf{R}^2, \end{cases}$$

such that

$$(4.3) \quad u^0(0) = z^0$$

there exists a control function $q \in L^2(0,T)$ such that the solution of (4.1) satisfies (3.3).

REMARK 4.1: (1) As a consequence of Proposition 2.6 the solution of (4.1) satisfies (2.24)–(2.25). Furthermore, by Proposition 2.7, it also satisfies (2.31).

(2) Due to the linearity, time-reversibility and well-posedness of the system in the (asymmetric) space (4.2)–(4.3), as a consequence of Theorem 4.1, we deduce that we can drive system (4.1) from any initial state to any final state in the class (4.2)–(4.3).

(3) The control time we obtain is the same as for the case $M = 0$. However, notice that, when $M = 0$, the controllable space is larger and is given by (3.2). The controllable space we obtain for $M > 0$ is the optimal one.

By HUM, Theorem 4.1 is equivalent to the following observability result for the solutions of the uncontrolled problem (3.6).

PROPOSITION 4.2. *Suppose that $T > 2\left(\ell_1\sqrt{\frac{\rho_1}{\sigma_1}} + \ell_2\sqrt{\frac{\rho_2}{\sigma_2}}\right)$. Then, there exists $C = C(T) > 0$ such that the following holds for every finite-energy solution of (3.6):*

$$(4.4) \quad \begin{aligned} &\|\varphi(\cdot,0)\|_{L^2(\Omega_1)}^2 + \|\varphi_t(\cdot,0)\|_{(\vartheta_1)'}^2 + \|\psi(\cdot,0)\|_{\vartheta_2}^2 + \|\psi_t(\cdot,0)\|_{L^2(\Omega_2)}^2 \\ &\qquad + |\zeta(0)|^2 + |\zeta_t(0)|^2 \le C \int_0^T |\psi_x(\ell_2, t)|^2 dt. \end{aligned}$$

REMARK 4.2: This inequality is sharp in the sense that, as pointed out in Proposition 2.7, the reverse one holds for all $T > 0$. As a consequence of this double inequality we deduce that the controllable space (4.2)–(4.3) is the optimal one.

PROOF: However, first note that by rescaling the spatial variables and renaming the densities ρ_1 and ρ_2 we can obtain the system (4.1) with $\ell_1 = \ell_2 = 1$. Thus it will suffice to prove Proposition 4.2 for the case $\Omega_1 = (-1, 0)$ and $\Omega_2 = (0, 1)$.

We proceed in several steps.

Step 1. Consider the space-dependent energy $e_2(\cdot)$ defined in (3.10). Since it is non-decreasing we have

$$e_2(x) \leq e_2(1) = \frac{\sigma_2}{2} \int_0^T |\psi_x(1, t)|^2 dt, \quad 0 \leq x \leq 1.$$

This implies, in particular, that

$$(4.5) \qquad \int_{\tau_2}^{T-\tau_2} \int_0^1 \left[\rho_2 |\psi_t(x, t)|^2 + \sigma_2 |\psi_x(x, t)|^2 \right] dx dt \leq \sigma_2 \int_0^T |\psi_x(1, t)|^2 dt,$$

where $\tau_2 = \sqrt{\rho_2 / \sigma_2}$. Taking into account that $\psi(1, t) = 0$ for $\tau_2 \leq t \leq T - \tau_2$ from Poincaré's inequality and (4.5) we deduce that

$$\int_{\tau_2}^{T-\tau_2} |\psi(0, t)|^2 dt \leq C \int_0^T |\psi_x(1, t)|^2 dt$$

and therefore, since $\varphi(0, t) = \psi(0, t) = \zeta(t)$,

$$(4.6) \qquad \int_{\tau_2}^{T-\tau_2} \left[|\varphi(0, t)|^2 + |\zeta(t)|^2 \right] dt \leq C \int_0^T |\psi_x(1, t)|^2 dt.$$

On the other hand, since $e_2(0) \leq e_2(1)$ we have

$$(4.7) \qquad \int_{\tau_2}^{T-\tau_2} \left[\rho_2 |\psi_t(0, t)|^2 + \sigma_2 |\psi_x(0, t)|^2 \right] dt \leq \sigma_2 \int_0^T |\psi_x(1, t)|^2 dt$$

and therefore, since $\zeta(t) = \psi(0, t)$,

$$(4.8) \qquad \int_{\tau_2}^{T-\tau_2} \rho_2 |\zeta_t(t)|^2 dt \leq \sigma_2 \int_0^T |\psi_x(1, t)|^2 dt.$$

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Furthermore, since $\sigma_1 \varphi_x(0,t) = -M\zeta_{tt}(t) + \sigma_2 \psi_x(0,t)$, in view of (4.7)–(4.8) we have

$$(4.9) \qquad \|\varphi_x(0,t)\|^2_{H^{-1}(\tau_2, T-\tau_2)} \leq C \int_0^T |\psi_x(1,t)|^2 dt.$$

Step 2. As a consequence of (4.6) and (4.9) we have

$$(4.10) \qquad \|\varphi(0,t)\|^2_{L^2(\tau_2, T-\tau_2)} + \|\varphi_x(0,t)\|^2_{H^{-1}(\tau_2, T-\tau_2)} \leq C \int_0^T |\psi_x(1,t)|^2 dt.$$

The well-posedness of the wave equation satisfied by φ in the x-variable allows us to prove that (see [17],[18] for details)

$$(4.11) \qquad \int_{-1}^{0} \int_{\tau_2 - \tau_1 x}^{T-\tau_2 + \tau_1 x} |\varphi(x,t)|^2 dt dx \leq C \left[\|\varphi(0,t)\|^2_{L^2(\tau_2, T-\tau_2)} + \|\varphi_x(0,t)\|^2_{H^{-1}(\tau_2, T-\tau_2)} \right],$$

where $\tau_1 = \sqrt{\rho_1/\sigma_1}$. Combining (4.10) and (4.11) we deduce that

$$(4.12) \qquad \int_{\tau_2 + \tau_1}^{T-(\tau_2+\tau_1)} \int_{1}^{0} |\varphi(x,t)|^2 dx dt \leq C \int_0^T |\varphi_x(1,t)|^2 dt.$$

Step 3. Set $\mu = \tau_1 + \tau_2$ and let $\varepsilon > 0$ be so that

$$(4.13) \qquad\qquad\qquad T - 2\varepsilon > 2\mu$$

and $\eta = \eta(t) \in C^1(\mu, T-\mu)$ such that

$$(4.14) \qquad \begin{cases} 0 \leq \eta(t) \leq 1, & \mu \leq t \leq T - \mu \\ \eta(\mu) = \eta(t-\mu) = 0 \\ \eta(t) = 1, & \mu + \varepsilon \leq t \leq T - \mu - \varepsilon \\ \frac{|\eta_t|^2}{\eta} \in L^\infty(\mu, T-\mu) \end{cases}$$

Let us introduce the function $\phi = \phi(x,t)$ such that

$$(4.15) \qquad \begin{cases} -\phi_{xx} = \varphi, & -1 < x < 0, \quad \mu \leq t \leq T - \mu \\ \phi(-1,t) = \phi_x(0,t) = 0, & \mu \leq t \leq T - \mu \end{cases}$$

We have

$$(4.16) \qquad -\int_{-1}^{0} \varphi_{xx}(x,t)\phi(x,t)dx = \int_{-1}^{0} |\varphi(x,t)|^2 dx - \varphi_x(0,t)\phi(0,t), \quad \mu \leq t \leq T - \mu$$

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and

$$(4.17) \quad \int_{-1}^{0} \varphi_t(x,t)\phi_t(x,t)dx = \int_{-1}^{0} |\phi_{tx}(x,t)|^2 dx = \|\varphi_t(\cdot,t)\|^2_{(\vartheta_1)'}, \quad \mu \le t \le T - \mu.$$

We multiply by $\phi\eta$ the equation satisfied by φ and integrate in the region $(x,t) \in \Omega_1 \times (\mu, T - \mu)$ to get, in view of (4.16)–(4.17),

$$(4.18) \quad \rho_1 \int_{\mu}^{T-\mu} \|\varphi_t(\cdot,t)\|^2_{\vartheta'_1}\eta(t)dt = -\rho_1 \int_{\mu}^{T-\mu}\int_{-1}^{0} \varphi_t\phi\eta_t dx dt + \sigma_1 \int_{\mu}^{T-\mu}\int_{-1}^{0} |\varphi|^2\eta dx dt$$
$$- \sigma_1 \int_{\mu}^{T-\mu} \varphi_x(0,t)\phi(0,t)\eta(t)dt.$$

On the other hand

$$(4.19) \quad \left| \int_{\mu}^{T-\mu}\int_{-1}^{0} \varphi_t\phi\eta_t dx dt \right| \le \frac{1}{2} \int_{\mu}^{T-\mu} \|\varphi_t(\cdot,t)\|^2_{\vartheta'_1}\eta(t)dt + \frac{1}{2} \int_{\mu}^{T-\mu} \|\phi(\cdot,t)\|^2_{\vartheta_1}\frac{|\eta_t|^2}{\eta}dt.$$

Note that

$$(4.20) \quad \|\phi(\cdot,t)\|_{\vartheta_1} = \|\varphi(\cdot,t)\|_{\vartheta'_1} \le C\|\varphi(\cdot,t)\|_{L^2(\Omega_1)}, \quad \mu \le t \le T - \mu.$$

Combining (4.18)–(4.20) we deduce that

$$(4.21) \quad \int_{\mu}^{T-\mu} \|\varphi_t(\cdot,t)\|^2_{\vartheta'_1}\eta(t)dt \le C_1 \left\{ \int_{\mu}^{T-\mu}\int_{-1}^{0} \varphi^2 dx dt + \sigma_1 \left| \int_{\mu}^{T-\mu} \varphi_x(0,t)\phi(0,t)\eta(t)dt \right| \right\}.$$

Since $\sigma_1\varphi_x(0,t) = \sigma_2\psi_x(0,t) - M\zeta_{tt}(t)$, we have

$$(4.22) \quad \sigma_1 \int_{\mu}^{T-\mu} \varphi_x(0,t)\phi(0,t)\eta(t)dt = \sigma_2 \int_{\mu}^{T-\mu} \psi_x(0,t)\phi(0,t)\eta(t)dt$$
$$- M \int_{\mu}^{T-\mu} \zeta_{tt}(t)\phi(0,t)\eta(t)dt.$$

On the other hand

$$(4.23) \quad \int_{\mu}^{T-\mu} \zeta_{tt}(t)\phi(0,t)\eta(t)dt = - \int_{\mu}^{T-\mu} \zeta_t(t)\phi_t(0,t)\eta(t)dt - \int_{\mu}^{T-\mu} \zeta_t(t)\phi(0,t)\eta_t(t)dt.$$

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Now, observe that

$$|\phi_t(0,t)| \le C\|\phi_t(\cdot,t)\|_{\vartheta_1} \le C\|\varphi_t(\cdot,t)\|_{\vartheta_1'} \quad \mu \le t \le T - \mu.$$

Therefore, there exists $C > 0$ such that

$$(4.24) \qquad \left| \int_\mu^{T-\mu} \zeta_t(t)\phi_t(0,t)\eta(t)dt \right| \le \frac{1}{2} \int_\mu^{T-\mu} \|\varphi_t(\cdot,t)\|_{\vartheta_1'}^2 \eta(t)dt + C \int_\mu^{T-\mu} |\zeta_t(t)|^2 dt.$$

Moreover

$$|\phi(0,t)| \le C\|\phi(\cdot,t)\|_{\vartheta_1} \le C\|\varphi(\cdot,t)\|_{L^2(-1,0)}$$

and thus

$$(4.25) \qquad \left| \int_\mu^{T-\mu} \zeta_t(t)\phi(0,t)\eta_t dt \right| \le C \left\{ \int_\mu^{T-\mu}\int_{-1}^0 \varphi^2 dx dt + \int_\mu^{T-\mu} |\zeta_t(t)|^2 dt \right\}$$

and

$$(4.26) \qquad \left| \int_\mu^{T-\mu} \psi_x(0,t)\phi(0,t)\eta dt \right| \le C \left\{ \int_\mu^{T-\mu}\int_{-1}^0 \varphi^2 dx dt + \int_\mu^{T-\mu} |\psi_x(0,t)|^2 dt \right\}.$$

Combining (4.21)–(4.26) with $\varepsilon > 0$ small enough we deduce that

$$(4.27) \qquad \begin{aligned} \int_{\mu+\varepsilon}^{T-\mu-\varepsilon} \|\varphi_t(\cdot,t)\|_{\vartheta_1'}^2 dt &\le \int_\mu^{T-\mu} \|\varphi_t(\cdot,t)\|_{\vartheta_1'}^2 \eta(t)dt \\ &\le C \left\{ \int_\mu^{T-\mu}\int_{-1}^0 \varphi^2 dx dt + \int_\mu^{T-\mu} [\,|\zeta_t(t)|^2 + |\psi_x(0,t)|^2]dt \right\}. \end{aligned}$$

Step 4. As a consequence of (4.5), (4.6), (4.8), (4.12) and (4.27) we have

$$(4.28) \qquad \begin{aligned} \int_{\mu+\varepsilon}^{T-\mu-\varepsilon} & \left[\int_{-1}^0 \varphi^2 dx + \|\varphi_t(\cdot,t)\|_{\vartheta_1'}^2 + \int_0^1 \left[\,|\psi_x|^2 + |\psi_t|^2\right] dx + |\zeta(t)|^2 + |\zeta_t(t)|^2 \right] dt \\ & \le C \int_0^T |\psi_x(1,t)|^2 dt. \end{aligned}$$

Recall that, in virtue of Proposition 2.7, system (3.6) is well-posed in the space

$$\begin{cases} \varphi \in L^2(-1,0), & \varphi_t \in \vartheta_1' \\ \psi \in \vartheta_2, & \psi_t \in L^2(0,1) \\ \zeta,\zeta_t \in \mathbb{R} \end{cases}$$

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with compatibility condition
$$\psi(0) = \zeta.$$
Therefore, (4.4) is an immediate consequence of (4.28) and the time-reversibility of system (3.6). ∎

We are now in a position to prove Theorem 3.2.

PROOF OF THEOREM 3.2: Theorem 3.2 is a direct consequence (by HUM) of the following observability result for the uncontrolled problem (3.6).

PROPOSITION 4.3. *Suppose that* $T > \ell_1\sqrt{\frac{\rho_1}{\sigma_1}} + \ell_2\sqrt{\frac{\rho_2}{\sigma_2}}$ *and* $\ell_1\sqrt{\frac{\rho_1}{\sigma_1}} > \ell_2\sqrt{\frac{\rho_2}{\sigma_2}}$. *Then, there exists* $C(T) > 0$ *such that the following holds for every finite energy solution of (3.6):*
(4.29)
$$\|\varphi(\cdot,0)\|^2_{L^2(\Omega_1)} + \|\varphi_t(\cdot,0)\|^2_{(\vartheta_1)'} + \|\psi(\cdot,0)\|^2_{\vartheta_2} + \|\psi_t(\cdot,0)\|^2_{L^2(\Omega_2)} +$$
$$+ |\xi(0)|^2 + |\xi_t(0)|^2 \le C\left[\|\varphi_x(-\ell_1,t)\|^2_{H^{-1}(0,T)} + \|\psi_x(\ell_2,t)\|^2_{L^2(0,T)}\right].$$

PROOF OF PROPOSITION 4.3: As in the proof of Proposition 4.2 we assume that $\ell_1 = \ell_2 = 1$ without loss of generality.

Let τ_1 and τ_2 be the same as in the proof of Proposition 4.2 and let $\mu = \frac{\tau_1+\tau_2}{2}$. For any sufficiently small $\varepsilon > 0$ we have

(4.30)
$$\int_{\mu+\varepsilon}^{T-\mu-\varepsilon} \left[\|\varphi(\cdot,t)\|^2_{L^2\left(-1,\frac{\tau_2-\tau_1}{2\tau_1}\right)} + \|\varphi_t(\cdot,t)\|^2_{H^{-1}\left(-1,\frac{\tau_2-\tau_1}{2\tau_1}\right)} \right] dt$$
$$\le C\|\varphi_x(-1,t)\|^2_{H^{-1}(0,T)}$$

and

(4.31)
$$\int_{\mu+\varepsilon}^{T-\mu-\varepsilon} \left[\|\varphi(\cdot,t)\|^2_{L^2\left(\frac{\tau_2-\tau_1}{2\tau_1},0\right)} + \|\varphi_t(\cdot,t)\|^2_{\left(H^1(\frac{\tau_2-\tau_1}{2\tau_1},0)\right)'} \right] dt$$
$$+ \int_{\mu+\varepsilon}^{T-\mu-\varepsilon}\int_0^1 [|\psi_t(x,t)|^2 + |\psi_x(x,t)|^2]dx\,dt + \int_{\mu+\varepsilon}^{T-\mu-\varepsilon} [|\xi(t)|^2 + |\xi_t(t)|^2]dt$$
$$\le C\int_0^T |\psi_x(1,t)|^2\,dt.$$

The inequality (4.30) is a standard estimate that holds for the wave equation (which we may apply here due to the finite speed of propogation) while (4.31) is proved in the same way that (4.28) was.

Since
$$\int_{\mu+\varepsilon}^{T-\mu-\varepsilon} \|\varphi_t(\cdot,t)\|^2_{(\vartheta_1)'} \le C \int_{\mu+\varepsilon}^{T-\mu-\varepsilon} \left[\|\varphi_t(\cdot,t)\|^2_{H^{-1}\left(-1,\frac{\tau_2-\tau_1}{2\tau_1}\right)} + \|\varphi_t(\cdot,t)\|^2_{\left(H^1(\frac{\tau_2-\tau_1}{2\tau_1},0)\right)'} \right] dt,$$
the inequalities (4.30) and (4.31) can be combined and we deduce easily (4.29) as in step 4 of the proof of Proposition 4.2. ∎

5. Representation of controllability spaces by nonharmonic Fourier Series

In this section we give a characterization of the controllable and observable spaces for (4.1) in terms of nonharmonic Fourier series. For simplicity, we limit our analysis to the case where

$$\text{(5.1)} \qquad \ell_1 = \ell_2 = \sigma_1 = \sigma_2 = \rho_1 = \rho_2 = 1,$$

although a similar analysis is valid, for example, when the above parameters are rational.

5.1. Spectral analysis. We begin with a spectral analysis of the operator $\mathcal{A}$ in (2.1). Since $\mathcal{A}$ is skew-adjoint with compact inverse, $\sigma(\mathcal{A})$ consists of a discrete sequence of imaginary eigenvalues. We seek nontrivial solutions y for

$$\text{(5.2)} \qquad \mathcal{A}y = i\omega y \quad \omega \in \mathbf{R}.$$

We obtain nontrivial solutions only when $\omega \in \sigma(\mathcal{A}) = S_1 \cup S_2$, where

$$\text{(5.3)} \qquad \begin{cases} S_1 = (ik\pi)_{k\in\mathbf{Z}\setminus\{0\}}, \\ S_2 = (i\omega_k)_{k\in\mathbf{Z}\setminus\{0\}}, \end{cases}$$

where for $k \in \mathbf{N}$, $\omega_k = -\omega_{-k}$ and ω_k is the kth positive root of

$$\text{(5.4)} \qquad M\omega = -2\cot\omega.$$

It is easy to see from (5.4) that $|\omega_k - k\pi| \to 0$ as $k \to \infty$. In fact, if we let

$$\text{(5.5)} \qquad \delta_k = k\pi - \omega_k \quad k \in \mathbf{N},$$

a simple calculation using Taylor's formula applied to (5.4) gives

$$\text{(5.6)} \qquad \delta_k = \frac{2}{Mk\pi} + \mathcal{O}(k^{-2}) \quad \text{as } k \to \infty.$$

The eigenfunctions of $\mathcal{A}$ corresponding to the eigenvalue ω are given by

$$\text{(5.7)} \qquad \varphi_\omega(x) = \begin{pmatrix} U_\omega \\ V_\omega \\ Z_\omega \\ \dot{U}_\omega \\ \dot{V}_\omega \\ \dot{Z}_\omega \end{pmatrix} = \begin{pmatrix} \frac{a}{\omega}\sin\omega(x+1) \\ \frac{1}{\omega}\sin\omega(x-1) \\ -\frac{(1-a)\cos\omega}{M\omega^2} \\ ia\sin\omega(x+1) \\ i\sin\omega(x-1) \\ -\frac{i(1-a)}{M\omega}\cos\omega \end{pmatrix},$$

where $a = 1$ when $\omega \in S_1$ and $a = -1$ when $\omega \in S_2$.

Under the above normalization there exist positive constants C_1, C_2 for which

$$\text{(5.8)} \qquad C_1 \le \|\varphi_\omega\|_{\mathcal{H}} \le C_2 \quad \forall \omega \in \sigma(\mathcal{A}).$$

REMARK 5.1: The form of the eigenfunctions in (5.7) shows that each spectral class, S_1 and S_2, has its own physical significance; namely, if $\omega \in S_1$ then φ_ω describes a sinusoidal motion of the string which does not move the mass, while if $\omega \in S_2$ then φ_ω describes an even, piecewise sinusoidal motion of the string with a jump in the spatial derivative at the point mass. It is insightful to note that, although $(\varphi_\omega)_{\omega\in\sigma(\mathcal{A})}$ forms an orthogonal set, $\|V_{k\pi} - V_{\omega_k}\|_{H^1(\Omega_2)} + \|\dot{V}_{k\pi} - \dot{V}_{\omega_k}\|_{L^2(\Omega_2)} \to 0$ as $k \to \infty$. It thus becomes increasingly difficult to distinguish such consecutive modes (as the frequency gets large) by only observing the v (or only the w) portion of the state. This explains why, in terms of the eigenfunctions, we obtain an asymmetric observability (and hence also controllability) space when we only observe (or control) at an endpoint.

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5.2. Reduction to moment problem. Let us consider a general class of control problems which will include the problem (4.1).

Let $b \in \mathcal{D}(\mathcal{A})'$ and consider the system

$$(5.9) \qquad \dot{y} = \mathcal{A}y + bg(t) \quad y(0) = 0,$$

where $g \in L^2(0, T)$. A unique mild solution $y \in C([0, T]; D(\mathcal{A})')$ is given by

$$(5.10) \qquad y(t) = \Phi_t g,$$

where for $0 \le t \le T$, $\Phi_t : L^2(0, T) \to \mathcal{D}(\mathcal{A})'$ is given by

$$(5.11) \qquad \Phi_t h = \int_0^t e^{\mathcal{A}(t-s)} bh(s)ds.$$

This notion of mild solution coincides with that of the weak solutions in the sense of transposition introduced in section 2. (Note that $\mathcal{D}(\mathcal{A})' = W_0 \times W_1'$.) We adopt here this notation and terminology of the theory of semigroups for the sake of clarity and brevity.

The central problem for controllability of (5.9) is to determine the range of Φ_t. By integrating (5.10) against φ_ω we obtain the modal solutions:

$$(5.12) \qquad y_\omega(t) = \int_0^t e^{i\omega(t-s)} b_\omega g(s)ds, \quad \omega \in \sigma(\mathcal{A}),$$

where

$$y_\omega(t) = (y(t), \varphi_\omega); \quad b_\omega = (b, \varphi_w),$$

and $(\cdot, \cdot)$ denotes the duality pairing of $\mathcal{D}(\mathcal{A})'$ and $\mathcal{D}(\mathcal{A})$ relative to $< \cdot, \cdot >_{\mathcal{H}}$. Thus $x = \sum_{\omega \in \sigma(\mathcal{A})} x_\omega \varphi_\omega \in \mathcal{R}(\Phi_T)$ if and only if there exist $g \in L^2(0, T)$ for which

$$(5.13) \qquad x_\omega = \int_0^T e^{i\omega s} b_w g(s)ds, \quad \forall \omega \in \sigma(\mathcal{A}).$$

Hence the problem of determining $\mathcal{R}(\Phi_T)$ has been replaced by one of determining the moment space (i.e., the sequences (x_ω) associated with some $g \in L^2(0, T)$ in (5.13)) of the moment problem (5.13).

5.3. Some general results on moment problems. Before determining the moment space of (5.13) we first give some background and some results of independent interest.

A Riesz basis for a Hilbert space X is the image of an orthonormal basis through a bounded, invertible operator $B : X \to X$.

The following is due to Ullrich [15].

THEOREM 5.2. *Let $(\sigma_n)_{n \in \mathbf{Z}}$ be a distinct sequence of complex numbers with $(\sigma_n) \cap \mathbf{Z} = \emptyset$ and $\lim_{|n| \to \infty} |\sigma_n - n| = 0$. Then*

$$(5.14) \qquad (e^{int})_{n \in \mathbf{Z}} \cup \left(\frac{e^{int} - e^{i\sigma_n t}}{n - \sigma_n} \right)_{n \in \mathbf{Z}}$$

forms a Riesz basis for $L^2(0, 4\pi)$.

COROLLARY 5.3. *For each $T \geq 4\pi$ the moment problem*

$$(5.15) \qquad \int_0^T e^{int} f(t)\, dt = a_n, \quad \int_0^T e^{i\sigma_n t} f(t)\, dt = b_n; \quad n \in \mathbf{Z},$$

has a solution $f \in L^2(0, T)$ if and only if

$$(5.16) \qquad \left(\frac{a_n - b_n}{n - w_n} \right) \in \ell^2, \quad (a_n) \in \ell^2, \quad (b_n) \in \ell^2.$$

PROOF: It is obvious that if the result is true when $T = 4\pi$, then it is true for $T > 4\pi$. Hence we may assume $T = 4\pi$. By Theorem 5.2 we know that

$$(5.17) \qquad \int_0^T e^{int} f(t)\, dt = c_k; \quad \int_0^T \frac{e^{int} - e^{i\sigma_n t}}{n - \sigma_n} f(t)\, dt = d_k$$

has a solution if and only if $(c_k) \in \ell^2$ and $(d_k) \in \ell^2$. Putting $a_n = c_n - d_n(n - \sigma_n)$ and $b_n = c_n$, $n \in \mathbf{Z}$ we recover (5.15).

Since

$$(c_n) \in \ell^2 \text{ and } (d_n) \in \ell^2 \Leftrightarrow (5.16) \text{ holds}$$

the result holds. ∎

When applied to the moment problem (5.13), Corollary 5.3 completely describes the moment space. However it will be more convenient to utilize a dual version of Corollary 5.3, which we give below after we develop some notation.

Let

$$\mathcal{M} = \{ (a_n) \cup (b_n) : (a_n), (b_n) \text{ satisfy } (5.16) \}.$$

Then $\mathcal{M}$ becomes a Hilbert space when endowed with the inner product

$$(5.18) \qquad \langle (a_n) \cup (b_n), (\hat{a}_n) \cup (\hat{b}_n) \rangle_{\mathcal{M}} = \langle (a_n), (\hat{a}_n) \rangle_{\ell^2} + \langle \left(\frac{a_n - b_n}{\sigma_n - n} \right), \left(\frac{\hat{a}_n - \hat{b}_n}{\sigma_n - n} \right) \rangle_{\ell^2}.$$

One easily computes $\mathcal{M}'$, the dual space of $\mathcal{M}$ relative to the ℓ^2 inner product, to be the Hilbert space of sequences $(c_k)_{k \in \mathbf{Z}} \cup (d_k)_{k \in \mathbf{Z}}$ with $(c_k + d_k) \in \ell^2$ and $([\sigma_k - k][c_k - d_k]) \in \ell^2$ with corresponding scalar product given by

$$(5.19) \qquad \begin{aligned} \langle (c_k) \cup (d_k), (\hat{c}_k) \cup (\hat{d}_k) \rangle_{\mathcal{M}'} &= \langle ([\sigma_k - k][c_k - d_k]), ([\sigma_k - k][\hat{c}_k - \hat{d}_k]) \rangle_{\ell^2} \\ &\quad + \langle (c_k + d_k), (\hat{c}_k + \hat{d}_k) \rangle_{\ell^2}. \end{aligned}$$

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COROLLARY 5.4. *For any $T > 0$ there exists $M > 0$ such that for any $N \in \mathbb{N}$*

$$(5.20) \qquad \int_0^T |\sum_{k=-N}^{N} a_k e^{ikt} + b_k e^{i\sigma_k t}|^2 dt \leq M \|(a_k) \cup (b_k)\|_{\mathcal{M}'}^2.$$

Furthermore, there exists $m > 0$ such that for any $N \in \mathbb{N}$

$$(5.21) \qquad \int_0^{4\pi} \left| \sum_{k=-N}^{N} a_k e^{ikt} + b_k e^{i\sigma_k t} \right|^2 dt \geq m \|(a_k) \cup (b_k)\|_{\mathcal{M}'}^2.$$

PROOF: Let $\mathcal{C}_T : L^2(0,T) \to \mathcal{M}$ by

$$\mathcal{C}_T f = (a_k) \cup (b_k) : \quad (5.15) \text{ holds .}$$

By Theorem 5.2 and the proof of Corollary 5.3, $\mathcal{C}_T$ is continuous for all $T > 0$ and becomes onto when $T \geq 4\pi$. Equation (5.20) is simply the statement that $\mathcal{C}_T^*$ is continuous for $T > 0$ while (5.21) states that C_T^* is bounded away from zero for $T \geq 4\pi$. ∎

5.4. Fourier description of observable space. We close this section with a description of the observable space in terms of nonharmonic Fourier series. In particular, we obtain a slight improvement of Theorem 4.1 for the case where (5.1) holds and also examine the dependence of the observability in terms of the frequency of the initial data.

PROPOSITION 5.5. *Assume (5.1) and let $y = (u, v, z, \dot{u}, \dot{v}, \dot{z})^t$ be a finite-energy solution of (2.1) with*

$$y^0 = \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k \varphi_{k\pi} + b_k \varphi_{\omega_k}.$$

Then for any $T > 0$ there exists $C > 0$ for which

$$(5.22) \qquad \int_0^T |v_x(1,t)|^2 dt \leq C \|(a_k) \cup (b_k)\|_{\mathcal{M}'}^2.$$

Furthermore, there exists $c > 0$ for which

$$(5.23) \qquad \int_0^4 |v_x(1,t)|^2 dt \geq c \|(a_k) \cup (b_k)\|_{\mathcal{M}'}^2.$$

In particular, Theorem 4.1 and Proposition 4.2 remain true when $T = 4$ and (5.1) holds.

PROOF: For $h = (u, v, z, \dot{u}, \dot{v}, \dot{z})^t \in \mathcal{D}(\mathcal{A})$ define $b^* : \mathcal{D}(\mathcal{A}) \to \mathbb{R}$ by

$$b^* h = v_x(1).$$

Then b can be viewed as an element of $\mathcal{D}(\mathcal{A})'$ and we are in the setting of (5.9). Under the normalization of the eigenfunctions taken in (5.7) we have

$$b^* \varphi_w = 1 \quad \forall w \in \sigma(\mathcal{A}).$$

It follows that for $T > 0$

(5.24)
$$\int_0^T |b^* y|^2 dt = \int_0^T \left| \sum_{k \in \mathbf{Z} \setminus \{0\}} a_k e^{ik\pi t} + b_k e^{i\omega_k t} \right|^2 dt.$$

Thus (5.22) and (5.23) follow from (5.24) and Corollary 5.4. Define E as the completion of $\mathcal{H}$ with respect to the norm

(5.25)
$$\left\| \sum_{k \in \mathbf{Z} \setminus \{0\}} a_k \varphi_{k\pi} + b_k \varphi_{\omega_k} \right\|_E = \|(a_k) \cup (b_k)\|_{\mathcal{M}'}.$$

Corollary 5.4 and (5.22) show that the map $E \to \mathbf{R}$ defined by

$$y^0 \to \left(\int_0^T |b^* y|^2 dt \right)^{1/2}$$

defines a norm on E for any $T \geq 4$. By Proposition 4.2, an equivalent norm on E is given by the left side of (4.4). It thus follows that the inequality (4.4) remains valid when ((5.1) holds and) $T \geq 4$. Consequently Theorem 4.1 remains true when ((5.1) holds and) $T \geq 4$. ∎

Let us now give a characterization of the observable space E defined in (5.25) in terms of the eigenfunctions.

Let

$$p_k = \frac{\varphi_{k\pi} + \varphi_{w_k}}{2}, \quad q_k = \frac{\varphi_{k\pi} - \varphi_{w_k}}{2\delta_k}.$$

LEMMA 5.6. $(p_k)_{k \in \mathbf{Z} \setminus \{0\}} \cup (q_k)_{k \in \mathbf{Z} \setminus \{0\}}$ forms a Riesz basis for the space E defined by (5.25). In particular,

$$E = \left\{ \sum_{k \in \mathbf{Z} \setminus \{0\}} c_k p_k + d_k q_k : (c_k) \in \ell^2, (d_k) \in \ell^2 \right\}.$$

PROOF: From (5.6) and (5.19), an equivalent representation for $\mathcal{M}'$ as it applies to our problem is

$$\mathcal{M}' = \{(a_k)_{k \in \mathbf{Z} \setminus \{0\}} \cup (b_k)_{k \in \mathbf{Z} \setminus \{0\}} : ((a_k - b_k)\delta_k) \in \ell^2, (a_k + b_k) \in \ell^2\}.$$

Thus the topology obtained by making $(p_k) \cup (q_k)$ form an orthonormal basis must also be equivalent to the natural one given by (5.23). ∎

We can now easily prove the following result which describes the dependence of the observability upon the frequency.

PROPOSITION 5.7. *Assume (5.1) and let* $y = (u, v, z, \dot{u}, \dot{v}, \dot{z})^t$ *be a solution of (2.1) with*

$$y^0 = \sum_{k=-N, k \neq 0}^{N} c_k p_k + d_k q_k.$$

Then there exists $C > 0$ independent of N for which

$$(5.26) \qquad \left(\int_0^4 |v_x(1,t)|^2 dt \right)^{1/2} \geq \frac{C}{N} \|y^0\|_{\mathcal{H}}.$$

Furthermore, the bound is sharp in the sense that if $y^0 = q_N$ then the inequality in (5.26) is reversible for some larger value of C.

PROOF: First note that $(p_k) \cup (q_k)$ form and orthogonal system on $\mathcal{H}$. In particular,

$$(5.27) \qquad \|y^0\|_{\mathcal{H}}^2 \leq C_1 \sum_{k=-N, k \neq 0}^{N} |c_k|^2 + \frac{|d_k|^2}{\delta_k^2},$$

where C_1 is independent of N. By Proposition 5.5 and Lemma 5.6 there exists $C_2 > 0$, independent of N for which

$$(5.28) \qquad \int_0^4 |v_x(1,t)|^2 dt \geq C_2 \sum_{k=-N, k \neq 0}^{N} |c_k|^2 + |d_k|^2.$$

Let $\Delta_N = \min\{\delta_1^2, \delta_2^2, \ldots \delta_N^2\}$. Since

$$(5.29) \qquad \sum_{k=-N, k \neq 0}^{N} |c_k|^2 + |d_k|^2 \geq \Delta_N \sum_{k=-N, k \neq 0}^{N} |c_k|^2 + \frac{|d_k|^2}{\delta_k^2},$$

(5.26) follows from (5.6). To show the optimality, it is enough to see that all the inequalities are reversible when $y^0 = q_N$. For (5.27) this is obvious, for (5.28) the reversibility follows from Lemma 5.6, and for (5.29) we have equality for N sufficiently large by (5.6). Thus the reverse inequality of (5.26) holds for all N by making C sufficiently large. ∎

REMARK 5.7: In the previous proof, the constants C_1 and C_2 can be shown to be independent of M for $M \in (1, \infty)$. Thus (5.29) shows that the constant C in the statement of Proposition 5.7 is proportional to $\Delta_N^{1/2}$, i.e., to M^{-1} as $M \to \infty$. Thus for solutions of a fixed energy level, if the mass is doubled, the frequencies one can observe above a certain threshhold level are halved. As $M \to 0$, the numbers Δ_N in (5.29) approach 1 for all N, but not uniformly. Thus as $M \to 0$, for fixed N, we obtain the same observability constant as with the string without a point mass.

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6. Some results on boundary stabilization

In this section we examine the problem of stabilization by velocity feedback at one or both ends. Thus we will be interested in the decay properties of solutions of the system

$$(6.1) \quad \begin{cases} \rho_1 u_{tt} = \sigma_1 u_{xx}, & x \in \Omega_1, \ t > 0 \\ \rho_2 v_{tt} = \sigma_2 v_{xx}, & x \in \Omega_2, \ t > 0 \\ M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, & t > 0 \\ u(0,t) = v(0,t) = z(t), & t > 0 \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x), & x \in \Omega_1 \\ v(x,0) = v^0(x), \ v_t(x,0) = v^1(x), & x \in \Omega_2 \\ z(0) = z^0, \ z_t(0) = z^1 \end{cases}$$

under the following two types of boundary conditions:

$$(6.2) \quad \begin{cases} u(-\ell_1, t) = 0, & t > 0 \\ \sigma_2 v_x(\ell_2, t) + \gamma v_t(\ell_2, t) = 0, & t > 0 \end{cases}$$

and

$$(6.3) \quad \begin{cases} \sigma_1 u_x(-\ell_1, t) - \gamma u_t(-\ell_1, t) = 0, & t > 0 \\ \sigma_2 v_x(\ell_2, t) + \gamma v_t(\ell_2, t) = 0, & t > 0, \end{cases}$$

where γ is positive.

In (6.2) we are introducing some damping on the system at the extreme-point $x = \ell_2$ and in (6.3) at both extremes $x = -\ell_1, \ell_2$.

The effect of the damping can be seen by noting that (formally) the energy of solutions of (6.1)–(6.2) and (6.1),(6.3) satisfies

$$\frac{dE_M}{dt}(t) = -\gamma |v_t(\ell_2, t)|^2$$

and

$$\frac{dE_M}{dt}(t) = -\gamma [|u_t(-\ell_1, t)|^2 + |v_t(\ell_2, t)|^2]$$

respectively.

Standard semigroup theory allows us to prove the following two facts:

(i) If $y^0 = (u^0, v^0, z^0, u^1, v^1, z^1)^t$ and

$$(6.4) \quad \begin{cases} y^0 \in H^1(\Omega_1) \times H^1(\Omega_2) \times \mathbf{R} \times L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbf{R} \\ u^0(0) = v^0(0) = z^0 \quad (u^0(-\ell_1) = 0 \text{ in case of (6.2)}) \end{cases}$$

then (6.1)–(6.2) and (6.1),(6.3) have a unique solution in the class

$$(6.5) \quad \begin{cases} (u, v, z) \in C([0, \infty); H^1(\Omega_1) \times H^1(\Omega_2) \times \mathbf{R}) \\ (u_t, v_t, z_t) \in C([0, \infty); L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbf{R}) \end{cases}$$

and the following energy-identities hold

$$
(6.6) \quad
\begin{cases}
(i)E_M(t_2) - E_M(t_1) = -\gamma \int_{t_1}^{t_2} |v_t(\ell_2,t)|^2 dt \quad \text{(for (6.1)-(6.2))} \\[2ex]
(ii)E_M(t_2) - E_M(t_1) = -\gamma \int_{t_1}^{t_2} [\,|u_t(-\ell_1,t)|^2 + |v_t(\ell_2,t)|^2] dt \quad \text{(for (6.1),(6.3))}.
\end{cases}
$$

(ii) If the initial data satisfy the following further regularity and compatibility conditions:

$$
(6.7) \quad
\begin{cases}
(u^0,v^0) \in H^2(\Omega_1) \times H^2(\Omega_2); \quad (u^1,v^1) \in H^1(\Omega_1) \times H^1(\Omega_2) \\
u^1(0) = v^1(0) = z^1 \\
\sigma_2 v_x^0(\ell_2) + \gamma v^1(\ell_2) = 0, \\
u^1(-\ell_1) = 0 \quad \text{(for (6.1)-(6.2))} \\
\sigma_1 u_x^0(-\ell_1) - \gamma u^1(-\ell_1) = 0 \quad \text{(for (6.1),(6.3))}
\end{cases}
$$

then the solutions have the following added regularity

$$
(6.8) \quad
\begin{cases}
(u,v) \in C([0,\infty); H^2(\Omega_1) \times H^2(\Omega_2)) \\
(u_t,v_t) \in C([0,\infty); H^1(\Omega_1) \times H^1(\Omega_2)) \\
z \in C^2([0,\infty); \mathbf{R})
\end{cases}
$$

In the sequel, solutions in the class (6.5) will be referred to as finite-energy solutions and those in (6.8) as smooth solutions.

There is an important difference between boundary conditions (6.2) and (6.3). For solutions of (6.1)-(6.2) the energy E_M is coercive and thus the only equilibrium configuration is the zero one. However in the system (6.1),(6.3) the energy is not coercive and for every real constant $k, (u,v,z) = (k,k,k)$ defines a solution.

In both systems we may expect the energy to decay to zero and this is the case. However, in system (6.1)-(6.3) additional work is required to show that every solution converges to an equilibrium.

There is another important difference between system (6.1)-(6.2) and (6.1),(6.3). In (6.1),(6.3) we will prove a uniform exponential decay of the energy; however we will see that this does not hold for the system (6.1)-(6.2) since the system possesses a sequence of eigenfrequencies that approach the imaginary axis.

We have divided this section in three parts. In the first one we show that the energy of solutions of (6.1)-(6.2) converges to zero. In the second one we prove the uniform exponential decay of energy of solutions of (6.1), (6.3) and the fact that every trajectory converges towards an equilibrium. In the last part we study the spectral properties of system (6.1)-(6.2) and prove the non-uniform decay of energy.

6.1 Strong convergence to the equilibrium for (6.1)-(6.2). We have the following result.

THEOREM 6.1. *Every trajectory associated with a finite-energy solution of (6.1)–(6.2) converges strongly to zero in the finite-energy space (6.4).*

PROOF: It is sufficient to prove that

$$(6.9) \qquad E_M(t) \to 0 \text{ as } t \to \infty.$$

First observe that due to the density of the data satisfying (6.7) in the finite-energy space (6.4), the decreasing character of the energy (as in (6.6)) and the linearity of the system, if (6.9) holds for all smooth solutions then it also holds for all finite-energy solutions. Thus we will only need to consider smooth solutions.

Next we observe that $(u_t, v_t, z_t, u_{tt}, v_{tt}, z_{tt})$ is a solution (6.1)–(6.2) with the initial data $(u^1, v^1, z^1, \rho_1^{-1}\sigma_1 u_{xx}^0, \rho_2^{-1}\sigma_2 v_{xx}^0, M^{-1}(\sigma_2 v_x^0 - \sigma_1 u_x^0))$ which satisfies the compatibility condition $u^1(0) = v^1(0) = z^1$. Thus the trajectory $(u_t, v_t, z_t, u_{tt}, v_{tt}, z_{tt})$ is a finite energy solution and hence due to the fact that this energy is nonincreasing we deduce that

$$(u_t, v_t, z_t) \in L^\infty(0, \infty; H^1(\Omega_1) \times H^1(\Omega_2) \times \mathbf{R})$$
$$(u_{tt}, v_{tt}, z_{tt}) \in L^\infty(0, \infty; L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbf{R})$$

and then by elliptic regularity that

$$(u, v) \in L^\infty(0, \infty; H^2(\Omega_1) \times H^2(\Omega_2)).$$

This shows that trajectories

$$\{(u(t), v(t), z(t), u_t(t), v_t(t), z_t(t))\}_{t \geq 0}$$

are relatively compact in the finite-energy space. Let ω be its ω-limit set with respect to the strong topology of the finite-energy space.

Since the energy E_M is a Lyapunov function, by LaSalle's invariance principle we deduce that ω is reduced to states for which the corresponding solution has constant energy for all $t > 0$. It is easy to see that the only solution satisfying this property is the zero solution. This shows that $\omega = \{0\}$ and concludes the proof of the Theorem. ∎

6.2. Uniform exponential decay for (6.1),(6.3). We have the following result:

THEOREM 6.2. *There exist $C > 0$ and $\omega_0 > 0$ such that*

$$(6.10) \qquad E_M(t) \leq C E_M(0) e^{-\omega_0 t}, \quad \forall t > 0$$

holds for every finite-energy solution of (6.1),(6.3).
Moreover,

$$(6.11) \quad \begin{cases} \|(u(\cdot,t) - k)\|_{H^1(\Omega_1)}^2 + \|(v(\cdot,t) - k)\|_{H^1(\Omega_2)}^2 + |z(t) - k|^2 \leq C E_M(0) e^{-\omega_0 t} \\[2mm] \text{with } k \in \mathbf{R} \text{ such that } 2\gamma k = \int\limits_{-\ell_1}^{0} u^1 dx + \int\limits_{0}^{\ell_2} v^1 dx + M z^1 + \gamma u^0(-\ell_1) + \gamma v^0(\ell_2). \end{cases}$$

PROOF: It is sufficient to prove the result for smooth solutions. Let us first prove (6.10). We have

$$(6.12) \qquad E_M(T) - E_M(0) = -\gamma \int_0^T [\,|u_t(-\ell_1,t)|^2 + |v_t(\ell_2,t)|^2]dt, \quad \forall T > 0.$$

Therefore it is sufficient to prove the existence of $C > 0$ and $T > 0$ such that

$$(6.13) \qquad E_M(T) \le C \int_0^T [\,|u_t(-\ell_1,t)|^2 + |v_t(\ell_2,t)|^2]dt$$

for every smooth solution. Indeed from (6.12) and (6.13), (6.10) holds easily by using the semigroup property.

On the other hand (6.13) can be proved exactly in the same way as we proved inequality (3.7) in Proposition 3.3.

Let us now prove (6.11). By differentiating the quantity

$$(6.14) \qquad I(t) = \int_{-\ell_1}^0 \rho_1 u_t(x,t)dx + \int_0^{\ell_2} \rho_2 v_t(x,t)dx + M z_t(t) + \gamma u(-\ell_1,t) + \gamma v(\ell_2,t)$$

and using (6.1),(6.3), one sees that $I(t)$ remains constant in time.

Given a solution of (6.1),(6.3) we decompose it as follows:

$$(6.15) \qquad (u,v,z,u_t,v_t,z_t) = (\tilde{u},\tilde{v},\tilde{z},u_t,v_t,z_t) + (\bar{u},\bar{v},\bar{z},0,0,0)$$

where

$$(6.16) \qquad \bar{u} = \bar{v} = \bar{z} = k \text{ and } \tilde{u} = u - k, \quad \tilde{v} = v - k, \quad \tilde{z} = z - k$$

with k a constant such that

$$(6.17) \qquad 2\gamma k = I(0).$$

Clearly $(\bar{u},\bar{v},\bar{z})$ is a stationary solution of (6.1), (6.3). On the other hand, the quantity $I(t)$ associated to $(\tilde{u},\tilde{v},\tilde{z},u_t,v_t,z_t)$ is identically zero. It is easy to check that $E_M(\cdot)$ is coercive over the subspace of the finite-energy space in which $I = 0$. From (6.10) we then deduce that the component $(\tilde{u},\tilde{v},\tilde{z},u_t,v_t,z_t)$ of our solutions decays exponentially to zero in the energy space, i.e. (6.11) holds. ∎

6.3. Non-uniform energy decay. Let $\mathcal{A}$ be the differential operator in (2.1), however on the space

$$(6.18) \qquad \widetilde{\mathcal{H}} = \tilde{W}_1 \times W_0,$$

where

$$(6.19) \qquad \tilde{W}_1 = \{(u,v,z) \in H^1(\Omega_1) \times H^1(\Omega_2) \times \mathbf{R} : u(-\ell_1) = 0, \ u(0) = v(0) = z\}$$

with domain

$$(6.20) \quad \widetilde{\mathcal{D}}(\mathcal{A}) = \{y \in \widetilde{\mathcal{H}} : u \in H^2(\Omega_1), v \in H^2(\Omega_2), (\dot{u},\dot{v},\dot{z}) \in \tilde{W}_1 : \sigma_2 v_x(\ell_2) + \gamma \dot{v}(\ell_2) = 0\}.$$

We have the following result which describes the amount of damping in (6.1)–(6.2), in terms of $\sigma(\mathcal{A})$.

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THEOREM 6.3. *Let $\mathcal{A}$ be defined by (6.18)–(6.20), with $\gamma > 0$. Then there exists a sequence $(s_k)_{k\in\mathbb{Z}} \subset \sigma(\mathcal{A})$ for which*

(i) $\left| \sqrt{\dfrac{\sigma_1}{\rho_1}} \dfrac{k\pi}{\ell_1} - \operatorname{Im} s_k \right| \to 0 \text{ as } |k| \to \infty$

(ii)   *there exists $c_1, c_2 > 0$ for which* $-\dfrac{c_1}{k^2} > \operatorname{Re} s_k > -\dfrac{c_2}{|k|} \quad \forall k \in \mathbb{Z}$.

If $\lambda \in \sigma(\mathcal{A}) \setminus \{(s_k)\}$ then there exists $c_3 > 0$ for which $\operatorname{Re} \lambda < -c_3$.

REMARK 6.3: The second inequality of (ii) implies that the energy of solutions does not decay uniformly to zero in bounded sets of $\mathcal{H}$. But as a consequence of a general result due to W. Littman and L. Markus [8], in view of the first inequality of (ii), one can obtain uniform polynomial decay rates of energy for solutions with initial data belonging to a bounded set of the domain of some power of $\mathcal{A}$.

PROOF: We assume without loss of generality (by rescaling) that $\ell_1 = \ell_2 = 1$.
We seek nontrivial solutions of

(6.21) $\mathcal{A}y = i\omega y$.

Let

$$\alpha = \sqrt{\frac{\sigma_1}{\rho_1}}, \quad \beta = \sqrt{\frac{\sigma_2}{\rho_2}}.$$

Using (6.21) and the boundary conditions, one can easily calculate

(6.22) $\begin{cases} u = a \sin \dfrac{\omega}{\alpha}(x+1) \\[2mm] v = b\psi_\omega(x); \quad \psi_\omega(x) = \dfrac{\gamma}{\rho_2} \sin \dfrac{\omega}{\beta}(x-1) + i\beta \cos \dfrac{\omega}{\beta}(x-1) \\[2mm] z = \dfrac{b\sigma_2}{M\omega^2} \dfrac{d\psi_\omega}{dx}(0) - \dfrac{a\sigma_1}{M\alpha\omega} \cos \dfrac{\omega}{\alpha} \end{cases}$

where a, b, ω are to be determined. The boundary conditions in (6.19) imply

(6.23) $a \sin \dfrac{\omega}{\alpha} = b\psi_\omega(0) = z.$

The possibility that $z = 0$ leads to only trivial solutions. Thus we may set $a = \psi_\omega(0)$, $b = \sin \omega/\alpha$ and obtain from (6.23)

(6.24) $M\omega = h(\omega) - \dfrac{\sigma_1 \cos \omega/\alpha}{\alpha \sin \omega/\alpha}; \quad h(\omega) = \dfrac{\sigma_2}{\omega\psi_\omega(0)} \dfrac{d\psi_\omega}{dx}(0).$

Now there are three separate cases to consider:

$$\gamma < \sqrt{\rho_2\sigma_2}, \quad \gamma = \sqrt{\rho_2\sigma_2}, \quad \gamma > \sqrt{\rho_2\sigma_2}.$$

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In each case the result is the same, and the idea is the same. Thus let us consider only the first case $\gamma < \sqrt{\rho_2 \sigma_2}$.

Let $\mathbf{K}$ denote the roots of $\psi_\omega(0) = 0$. A simple calculation shows

$$(6.25) \qquad \mathbf{K} = \left\{ \beta \left(k\pi + \frac{\pi}{2} + i \tanh^{-1} \frac{\gamma}{\beta \rho_2} \right) \right\}_{k \in \mathbf{Z}}.$$

For $S \subset \mathbf{C}$, define

$$N_\delta(S) = \{ \lambda \in \mathbf{C} : \text{dist}(\lambda, S) < \delta \}.$$

From (6.25) it follows that $N_{\delta_0}(\mathbf{K}) \cap N_{\delta_0}(\alpha \pi \mathbf{Z}) = \emptyset$ for δ_0 small enough. Furthermore, since $h(\omega)$ and $\cot \omega / \alpha$ are periodic and analytic outside $N_{\delta_0}(\mathbf{K})$ and respectively $N_{\delta_0}(\alpha \pi \mathbf{Z})$, $h(\omega)$ is uniformly bounded in $N_{\delta_0}(\alpha \pi \mathbf{Z})$ and $\cot \omega / \alpha$ is uniformly bounded in $N_{\delta_0}(\mathbf{K})$. It follows that for any $\varepsilon > 0$, only finitely many eigenvalues lie outside $N_\varepsilon(\alpha \pi \mathbf{Z}) \cup N_\varepsilon(\mathbf{K})$ in order that the moduli of both sides of (6.24) match. Rouché's theorem can be used to show that for any $\varepsilon > 0$ if $|\omega|$ is sufficiently large, there is a unique root of (6.24) in each component of $N_\varepsilon(\alpha \pi \mathbf{Z})$ and $N_\varepsilon(\mathbf{K})$. (See [8] for an example of this calculation.)

A simple calculation shows that $\text{Im}\, h(\omega) < -a < 0$ for all $\omega \in \mathbf{R}$. Since $h(\omega)$ is periodic and analytic in a neighborhood of the real axis, we may assume there exist positive numbers a_0 and l for which

$$(6.26) \qquad \text{Im}\, h(\omega) < -a_0 \quad \forall \omega \in \mathbf{C} : |\text{Im}\, \omega| \leq l.$$

Let $\alpha k\pi + s + i\tau$ denote a root of (6.24) in $N_\varepsilon(\alpha \pi \mathbf{Z})$. By setting equal the imaginary parts of (6.24) and then utilizing (6.26) we find that for ε sufficiently small (and hence $|k|$ sufficiently large) we have

$$(6.27) \qquad \left(\tau - \frac{\sigma_1}{2a_0} \right)^2 + s^2 < \frac{\sigma_1^2}{4a_0^2}.$$

Thus for $|\omega|$ sufficiently large, these roots lie in disks of radii $\sigma_1 / 2a_0$ tangent to the real axis at the points in $\{\alpha k \mathbf{Z}\}$.

Due to the upperbound on $|h(\omega)|$ in $N_{\delta_0}(\alpha \pi \mathbf{Z})$, in order that the moduli of both sides of (6.24) agree, for $|\omega|$ sufficiently large we must have

$$(6.28) \qquad \frac{M}{2} |\omega| < \left| \frac{\sigma_1}{\alpha} \cot \frac{\omega}{\alpha} \right| < 2M |\omega|.$$

Since $\cot \omega / \alpha$ is periodic and has a first order pole at 0, (6.28) implies there exists c_1, $c_2 > 0$ and $k \in \mathbf{Z}$ for which

$$(6.29) \qquad \frac{c_1}{|\omega|} < |\omega - \alpha k\pi| < \frac{c_2}{|\omega|}, \quad |\omega| \text{ sufficiently large}.$$

Intersecting the sets described by (6.26) and (6.29) and recalling (6.21), we obtain (i) and (ii). For any other roots of (6.24), either $\omega \in N_{\delta_0}(\mathbf{K})$ or ω is one of the finitely many outside of both $N_{\delta_0}(\mathbf{K})$ and $N_\varepsilon(\alpha k \mathbf{Z})$. In either case, by (6.25) and the dissipativity of $\mathcal{A}$ we have $\text{Im}\, \omega > \text{const} > 0$. Thus $\text{Re}\, \lambda = \text{Re}\, i\omega < -c_3 < 0$. $\blacksquare$

7. Extensions

In this final section we discuss two important generalizations of the results in the previous sections. First we consider a string with n point masses and describe the manner in which the previous results extend to this case. Second we analyze again the one-mass problem, but this time with a string having spatially varying coefficients.

7.1. n masses. All the regularity results carry over to strings with n masses due to the local nature of these results. In particular Proposition 2.5 and the remark which follows it imply that the degree of regularity of a travelling wave solution increases exactly one order as it crosses each mass. (Of course, part of the wave is *reflected* at each mass point with no increase in regularity.) Furthermore, the method used to prove controllability (Theorem 4.1) relies only upon the regularity result (Proposition 2.7) and the use of characteristics (as in construction of the energy functions e_1 and e_2 in (3.7) and (3.9)) and hence applies equally well to strings with n masses. As such, analogous controllability and observability results hold for strings with n masses. For example, if we consider the problem of Dirichlet control of the right-most end of the string, which has $n-1$ masses (and n intervals), the control space one obtains is the same on the first (farthest right) interval as in Theorem 4.1. For each successive interval the regularity increases one order, i.e., on the kth interval I_k (from the right), the position of the string (resp. velocity) is in $H^{k-1}(I_k)$ (resp. $H^{k-2}(I_k)$). In addition, boundary conditions and compatibility conditions up to the proper order for that interval must be imposed to obtain a well-posed system. A detailed examination of this problem however is beyond the scope of this paper.

7.2. Variable coefficients. All the results of the previous sections remain valid for strings with spatially variable coefficients. To see this, let us consider the following system.

$$(7.1) \quad \begin{cases} \rho(x)w_{tt}(x,t) - \frac{\partial}{\partial x}(\sigma(x)w_x(x,t)) = 0 & x \in \Omega, \quad t > 0 \\ Mz_{tt}(t) = \sigma(0^+)w_x(0^+,t) - \sigma(0^-)w_x(0^-,t) & t > 0 \\ w(0^-,t) = w(0^+,t) = z(t) & t > 0 \\ w(-1,t) = 0, \quad w(1,t) = q(t) & t > 0 \\ w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x) & x \in \Omega \\ z(0) = z^0, \quad z_t(0) = z^1, \end{cases}$$

where $\Omega = (-1,0) \cup (0,1)$ and the notation 0^+, 0^- refers to right and left-hand limits at 0.

We have the following result:

THEOREM 7.1. *Suppose that $\rho, \sigma \in H^2((-1,0) \cup (0,1))$ and that both functions are bounded below by a positive constant. Then, if*

$$T > 2 \int_{-1}^{1} \left(\frac{\rho(\tau)}{\sigma(\tau)} \right)^{1/2} d\tau$$

the conclusion of Theorem 4.1 holds.

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SKETCH OF PROOF: Let us introduce the following change of variables:

$$s = \int_0^x \rho(\tau)^{1/2} \sigma(\tau)^{-1/2} d\tau$$

$$\widetilde{\Omega} = (-\ell_1, 0) \cup (0, \ell_2); \quad \ell_1 = -s(-1), \quad \ell_2 = s(1)$$

$$b(s) = \rho(x)^{-1} \frac{d}{dx} \sqrt{\rho(x)\sigma(x)} \,\Big|_{x=x(s)}$$

$$\tilde{w}(s,t) = \exp\left(\int_0^s b(r)/2\, dr\right) w(x(s), t)$$

$$a = \sqrt{\rho(0^+)\sigma(0^+)}\frac{b(0^+)}{2} - \sqrt{\rho(0^-)\sigma(0^-)}\frac{b(0^-)}{2}.$$

Then (7.1), in the absence of control ($q = 0$) becomes
(7.2)

$$\begin{cases} \tilde{w}_{tt} - \left[\tilde{w}_{ss} - \left(\frac{b'(s)}{2} + \frac{b^2(s)}{4}\right)\tilde{w}\right] = 0 & s \in \widetilde{\Omega},\ t > 0 \\ M z_{tt}(t) + az(t) = \sqrt{\rho(0^+)\sigma(0^+)}\tilde{w}_s(0^+, t) - \sqrt{\rho(0^-)\sigma(0^-)}\tilde{w}_s(0^-, t) & t > 0 \\ \tilde{w}(0^-, t) = \tilde{w}(0^+, t) = z(t) & t > 0 \\ \tilde{w}(-\ell_1, t) = 0 = \tilde{w}(\ell_2, t) & t > 0 \\ \tilde{w}(s, 0) = e^{\int_0^s b(r)/2\, dr} w^0(x(s)), \quad \tilde{w}_t(s, 0) = e^{\int_0^s b(r)/2\, dr} w^1(x(s)) & s \in \widetilde{\Omega} \\ z(0) = z^0, \quad z_t(0) = z^1. \end{cases}$$

Therefore, it is sufficient to prove the controllability of this system at time $T > 2(\ell_1 + \ell_2)$ with a control at the right end $x = \ell_2$.

First let us examine the system (7.2), but without the potential term. This is equivalent to

(7.3)

$$\begin{cases} w_{tt} - w_{xx} = 0 & x \in \widetilde{\Omega},\ t > 0 \\ M z_{tt}(t) + az(t) = \gamma w_x(0^+, t) - w_x(0^-, t) & t > 0 \\ w(0^-, t) = w(0^+, t) = z(t) & t > 0 \\ w(-\ell_1, t) = 0 = w(\ell_2, t) & t > 0 \\ w(x, 0) = w^0(x), \quad w_t(x, 0) = w^1(x) & x \in \widetilde{\Omega} \\ z(0) = z^0, \quad z_t(0) = z^1. \end{cases}$$

We assume $M, \gamma, \ell_1, \ell_2$ to be arbitrary positive numbers, and a is only assumed to be real. One can check that the proof of Proposition 2.5 remains valid for the system (7.3); when $M z_{tt} + az$ replaces $M z_{tt}$ in (2.14) one still obtains (2.18). It hence follows that all regularity results, in particular Proposition 2.7 remain valid for (7.3).

Finally, the assumptions on the coefficients imply the potential $\left(\frac{b'(s)}{2} + \frac{b^2(s)}{4}\right)$ remains in $L^2(\widetilde{\Omega})$. We can thus write the solution of (7.2) in terms of an integral equation given by

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the variation of constants formula for (7.3) and for small enough t determine a fixed point having the same regularity as the solution of (7.3) with initial data as in the hypothesis of Proposition 2.7. The remaining parts of the proof of Theorem 4.1 rely only upon the use of characteristics, and hence can be done directly for the variable coefficient problem (7.1). ∎

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