Exact Controllability and Stabilization. The Multiplier Method

Vilmos Komornik

Université Louis Pasteur, Strasbourg

Icának, Timinek és Zsoltnak

PREFACE

This book grew out of a series of lectures given over the past four years in France, Hungary and the USA.

In the first part exact boundary controllability problems are studied by the Hilbert Uniqueness Method. This approach, introduced by LIONS [3] in 1986, is based on uniqueness theorems leading to the construction of suitable Hilbert spaces of the controllable spaces. It is closely related to a duality theory of DOLECKI AND RUSSELL [1]. Following HO [1] and LIONS [2], these spaces may often be characterized by using the multiplier method. In chapters 2 to 4 we reproduce some results of LIONS [4], [5] with certain changes :

– some compactness–uniqueness arguments are replaced by constructive proofs ;

– equations containing lower-order terms are also considered ;

– more general boundary conditions are used which in fact *simplify* the theory.

The results of chapters 5 and 6 were obtained after the publication of Lions' monography. In chapter 5 we develop a general and constructive approach to improve the usual estimates of the exact controllability time. It was inspired by a new estimation method of HARAUX [3]. Using this approach, in chapter 6 we improve most of the results obtained in chapters 3 and 4. We also give elementary and constructive proofs for certain results of ZUAZUA [1], obtained earlier by indirect arguments.

The second part of the book is devoted to stabilizability. In chapters 8 and 9 strong and uniform boundary stabilization theorems are proved. Our method is a modified and simplified version of a Liapunov type approach introduced in KOMORNIK AND ZUAZUA [1]. We also present here a classical principle of RUSSELL [2] connecting the exact controllability to the stabilizability, and some recent results of CONRAD AND RAO [1].

For the sake of brevity in the first nine chapters we consider only the wave equation, Maxwell's equations and very simple plate models. In the last chapter we consider the internal stabilization of the Korteweg–de Vries equation : we prove a special case of a theorem in KOMORNIK, RUSSELL AND ZHANG [2].

The multiplier method, applied systematically in this book, is remarkably elementary and efficient. In the bibliography we have included some references which use other approaches : see in particular the work of Bardos et al., Littmann, Russell, Joó and their references. We have also included some material concerning other equations.

I wish to express my gratitude to

– J.-L. Lions for his advice and numerous suggestions concerning the subject of this book, for his valuable remarks on a preliminary version of these notes and for his proposal to publish it in the collection RMA ;

– P. G. Ciarlet who also proposed the publication of this book in this collection and who gave me useful advice concerning the presentation of the material ;

– F. Conrad, A. Haraux, J. Lagnese, B. Rao, D. L. Russell and E. Zuazua for many fruitful discussions ;

– the Mathematical Departments of Virginia Tech, the Universities of Strasbourg I, Nancy I, the Eötvös University of Budapest and the Institute of Mathematics and its Applications at the University of Minnesota, where my lectures were given and/or part of this book was written ;

– the students and colleagues following my lectures, in particular S. Kouémou-Patcheu who read the manuscript and made a number of useful remarks.

Strasbourg, November 29, 1993

TABLE OF CONTENTS

Preface . . . . **v**

0. Introduction. Vibrating strings **1**

1. Linear evolutionary problems **7**

1.1. The diagram $V \subset H = H' \subset V'$ 7
1.2. The equation $u'' + Au = 0$ 10
1.3. The wave equation 11
1.4. A Petrovsky system 13
1.5. Another Petrovsky system 15

2. Hidden regularity. Weak solutions **18**

2.1. A special vector field 18
2.2. The wave equation. Multiplier method 19
2.3. The first Petrovsky system 24
2.4. The second Petrovsky system 29

3. Uniqueness theorems **35**

3.1. The wave equation. Dirichlet condition 35
3.2. The first Petrovsky system 40
3.3. The second Petrovsky system 44
3.4. The wave equation. Mixed boundary conditions 47

4. Exact controllability. Hilbert uniqueness method **53**

4.1. The wave equation. Dirichlet control 53
4.2. The first Petrovsky system 57
4.3. The wave equation. Neumann or Robin control 59

5. Norm inequalities **63**

5.1. Riesz sequences 63
5.2. Formulation of the results 65
5.3. Proof of theorem 5.3 69
5.4. Proof of theorem 5.5 73

6. New uniqueness and exact controllability results **75**

6.1. A unique continuation theorem 75
6.2. The wave equation. Dirichlet condition 77
6.3. The first Petrovsky system 79

6.4. The second Petrovsky system. Uniqueness theorems 81
6.5. The second Petrovsky system. Exact controllability 83
6.6. The wave equation. Neumann or Robin condition 85

7. Dissipative evolutionary systems 90

7.1. Maximal monotone operators 90
7.2. The wave equation 91
7.3. Kirchhoff plates 97

8. Linear stabilization 103

8.1. An integral inequality 103
8.2. Uniform stabilization of the wave equation I 104
8.3. Links with the exact controllability. Russell's principle 108
8.4. Uniform stabilization of the wave equation II 111
8.5. Strong stabilization. LaSalle's principle 114
8.6. Uniform stabilization of the wave equation III 116
8.7. Uniform stabilization of Maxwell's equations 118

9. Nonlinear stabilization 124

9.1. A nonlinear integral inequality 124
9.2. Uniform stabilization of the wave equation I 126
9.3. Uniform stabilization of the wave equation II 134
9.4. Uniform stabilization of Kirchhoff plates 135

10. Internal stabilization of the Korteweg–de Vries equation . 144

10.1. Well-posedness and conservation laws 144
10.2. Uniform stabilization by linear feedbacks 146

References 152

0. Introduction. Vibrating strings

Let $I = (a, b)$ be a bounded interval, T a positive number and consider the following problem, modelling among other things the small transversal vibrations of a string :

$$(u_{tt} - u_{xx})(x, t) = 0, \quad (x, t) \in I \times (0, T), \tag{1}$$

$$u(a, t) = v_a(t) \quad \text{and} \quad u(b, t) = v_b(t), \quad t \in [0, T], \tag{2}$$

$$u(x, 0) = u^0(x) \quad \text{and} \quad u_t(x, 0) = u^1(x), \quad x \in I. \tag{3}$$

The problem (1)–(3) is said to be exactly controllable if for "arbitrarily" given initial "state" (u^0, u^1) there exist suitable "control" functions v_a and v_b such that the solution of (1)–(3) satisfies

$$u(x, T) = u_t(x, T) = 0, \quad x \in I. \tag{4}$$

We say that the controls v_a and v_b drive the system to rest in time T.

Naturally, we have to specify the functional spaces of the initial states and of the controls; the results depend on these choices.

The solution of (1)–(3) is by definition a function

$$u \in C([0, T]; H^1(I)) \cap C^1([0, T]; L^2(I)) \tag{5}$$

satisfying (1) in the distributional sense, the equalities (2) pointwise, and the equalities (3) almost everywhere. (As for the usual properties of the Sobolev spaces applied in this book we refer e.g. to Lions and Magenes [1].)

We have the following result :

Theorem 0.1. — *Let $T = b - a$ and let $(u^0, u^1) \in H^1(I) \times L^2(I)$ be such that*

$$u^0(a) + u^0(b) + \int_a^b u^1(s)\ \mathrm{ds} = 0. \tag{6}$$

Then there is a unique choice of functions

$$v_a, v_b \in H^1(0, T) \tag{7}$$

such that the solution of (1)–(3) satisfies (4).

Moreover, v_a and v_b are given by the formulae

$$2v_a(t) = u^0(a + t) + u^0(a) + \int_a^{a+t} u^1(s)\ \mathrm{ds}, \tag{8}$$

$$2v_b(t) = u^0(b-t) + u^0(b) + \int_{b-t}^{b} u^1(s)\ \mathrm{ds} \tag{9}$$

and the solution u has the following supplementary property :

$$u(a,t) + u(b,t) + \int_a^b u_t(s,t)\ \mathrm{ds} = 0, \quad \forall t \in [0,T]. \tag{10}$$

Proof. — Applying d'Alembert's formula the solutions of (1) may be written in the form

$$u(x,t) \equiv f(x+t) + g(x-t) \tag{11}$$

with suitable functions $f : (a, b+T) \to \mathbb{R}$ and $g : (a-T, b) \to \mathbb{R}$. Using (2) and (3) we obtain that

$$f(a+t) + g(a-t) = v_a(t), \quad t \in (0,T), \tag{12}$$

$$f(b+t) + g(b-t) = v_b(t), \quad t \in (0,T), \tag{13}$$

$$f(x) + g(x) = u^0(x), \quad x \in I, \tag{14}$$

$$f'(x) - g'(x) = u^1(x), \quad x \in I. \tag{15}$$

We deduce from (14), (15) that

$$2f(x) = u^0(x) + U^1(x), \quad x \in I \tag{16}$$

and

$$2g(x) = u^0(x) - U^1(x), \quad x \in I, \tag{17}$$

where U^1 is a suitable primitive of u^1.

We conclude from (12) and (16) that

$$2g(a-t) = 2v_a(t) - u^0(a+t) - U^1(a+t), \quad 0 < t < b-a;$$

similarly, we conclude from (13) and (17) that

$$2f(b+t) = 2v_b(t) - u^0(b-t) + U^1(b-t), \quad 0 < t < b-a;$$

we can rewrite these relations in the following form :

$$2g(s) = 2v_a(a-s) - (U^1 + u^0)(2a-s), \quad 2a-b < s < a, \tag{18}$$

$$2f(s) = 2v_b(s-b) + (U^1 - u^0)(2b-s), \quad b < s < 2b-a. \tag{19}$$

We deduce from (11), (18) and (19) that for

$$a < x < b \quad \text{and} \quad \max\{x-a, b-x\} < t < b-a \tag{20}$$

we have

$$\begin{gathered} 2u(x,t) = 2v_a(a+t-x) + 2v_b(x+t-b) \\ -u^0(2a-x+t) - u^0(2b-x-t) + \int_{2a-x+t}^{2b-x-t} u^1(s)\ \mathrm{ds}. \end{gathered} \tag{21}$$

We have in particular

$$u(x,b-a) = v_a(b-x) + v_b(x-a) - u^0(a+b-x), \quad x \in I$$

and

$$u_t(x,b-a) = v_a'(b-x) + v_b'(x-a) - u^1(a+b-x), \quad x \in I.$$

Thus the final conditions (4) are equivalent to

$$v_a(b-x) + v_b(x-a) = u^0(a+b-x), \quad x \in I$$

and

$$-v_a(b-x) + v_b(x-a) = -V^1(a+b-x), \quad x \in I$$

with an arbitrary primitive V^1 of u^1 whence

$$\begin{aligned} 2v_a(b-x) &= (u^0+V^1)(a+b-x), \quad x \in I, \\ 2v_b(x-a) &= (u^0-V^1)(a+b-x), \quad x \in I, \end{aligned}$$

or

$$2v_a(t) = (u^0+V^1)(a+t), \quad 0 < t < b-a, \tag{22}$$

$$2v_b(t) = (u^0-V^1)(b-t), \quad 0 < t < b-a. \tag{23}$$

Let us introduce the subsets U, R, D, L of $I \times (0,T)$ (the letters mean "Up", "Right", "Down", "Left") defined by

$$\begin{aligned} U &:= \{(x,t) \in I \times (0,T) : t > x-a \quad \text{et} \quad t > b-x\}, \\ R &:= \{(x,t) \in I \times (0,T) : b-x < t < x-a\}, \\ D &:= \{(x,t) \in I \times (0,T) : t < x-a \quad \text{et} \quad t < b-x\}, \\ L &:= \{(x,t) \in I \times (0,T) : x-a < t < b-x\}. \end{aligned}$$

One can easily deduce from (11), (16)–(19), (22) and (23) the following formulae :

$$2u(x,t) = \begin{cases} 0, & \text{if } (x,t) \in U; \\ (u^0 - V^1)(x-t), & \text{if } (x,t) \in R; \\ u^0(x+t) + u^0(x-t) + \int_{x-t}^{x+t} u^1(s)\,\mathrm{ds}, & \text{if } (x,t) \in D; \\ (u^0 + V^1)(x+t), & \text{if } (x,t) \in L. \end{cases} \tag{24}$$

Comparing the limits of these formulae at $t = x - a$ and $t = b - x$ we obtain that for any fixed $0 < t < T$, $t \neq T/2$, the function $x \mapsto u(x,t)$ is continuous only if

$$V^1(a) = u^0(a) \quad \text{and} \quad V^1(b) = -u^0(b). \tag{25}$$

By hypothesis (6) u^1 has a unique primitive V^1 with this property. Choosing V^1 in this way, (22), (23) follow from (8), (9), and we deduce from (24) that

$$2u(x,t) = \begin{cases} 0, & \text{if } (x,t) \in U; \\ u^0(x-t) + u^0(b) + \int_{x-t}^{b} u^1(s)\,\mathrm{ds}, & \text{if } (x,t) \in R; \\ u^0(x+t) + u^0(x-t) + \int_{x-t}^{x+t} u^1(s)\,\mathrm{ds}, & \text{if } (x,t) \in D; \\ u^0(x+t) + u^0(a) + \int_{a}^{x+t} u^1(s)\,\mathrm{ds}, & \text{if } (x,t) \in L; \end{cases} \tag{26}$$

using these formulae one can directly verify (5) and (10).

Remark 0.2. — The problem (1)–(3) is also exactly controllable if $T > b - a$. Indeed, it is sufficient to drive the system to rest in time $b - a$ and then to extend the control functions v_a, v_b by zero for $b - a < t < T$. Since $v_a(b-a) = v_b(b-a) = 0$ by (6), (8) and (9), the property (7) remains valid.

Remark 0.3. — The problem (1)–(3) is *not* exactly controllable if $T < b-a$. Indeed, assume that the system may be driven to rest in time T from some given initial data u^0, u^1. Extending the corresponding controls v_a, v_b by zero for $T < t < b - a$, we obtain that the system may be driven from the initial state (u^0, u^1) to rest in time $b-a$ with controls $v_a, v_b \in L^2(0, b-a)$ satisfying

$$v_a = v_b \equiv 0 \quad \text{in some left neighbourhood of} \quad b - a. \tag{27}$$

On the other hand, the proof of theorem 0.1 (see (22) and (23)) shows that

$$2v_a(t) + 2v_b(t) = u^0(a+t) + u^0(b-t) + \int_{b-t}^{a+t} u^1(s)\,\mathrm{ds}, \quad 0 < t < b-a. \tag{28}$$

But (27) and (28) are not satisfied simultaneously for all initial data $(u^0, u^1) \in H^1(I) \times L^2(I)$ satisfying (6).

The following result improves theorem 0.1 by driving the system to general final states :

THEOREM 0.4. — *Let $T = b - a$, $(u^0, u^1), (u^0_T, u^1_T) \in H^1(I) \times L^2(I)$ and assume that*

$$u^0(a) + u^0(b) + \int_a^b u^1(s)\,\mathrm{ds} = 0 \quad \textit{and} \quad u^0_T(a) + u^0_T(b) + \int_a^b u^1_T(s)\,\mathrm{ds} = 0.$$

Then there exist unique functions $v_a, v_b \in H^1(0,T)$ such that the solution of (1)–(3) *satisfies*

$$u(x,T) = u^0_T(x) \quad \textit{and} \quad u_t(x,T) = u^1_T(x), \quad x \in I. \tag{29}$$

PROOF. — First we solve the problem

$$\begin{gathered}(z_{tt} - z_{xx})(x,t) = 0, \quad (x,t) \in I \times (0,T),\\ z(a,t) = z(b,t) = 0, \quad t \in (0,T),\\ z(x,T) = u^0_T(x) \quad \text{and} \quad z_t(x,T) = u^1_T(x), \quad x \in I,\end{gathered}$$

and then we choose (using theorem 0.1) v_a and v_b such that the solution of the problem

$$\begin{gathered}(y_{tt} - y_{xx})(x,t) = 0, \quad (x,t) \in I \times (0,T),\\ y(a,t) = v_a(t), \quad y(b,t) = v_b(t), \quad t \in (0,T),\\ y(x,0) = u^0(x) - z(x,0) \quad \text{and} \quad y_t(x,0) = u^1(x) - z_t(x,0), \quad x \in I\end{gathered}$$

satisfies

$$y(x,T) = y_t(x,T) = 0, \quad x \in I.$$

Then $u := y + z$ is *the* solution of (1)–(3) and it satisfies (29). ∎

Let us return to theorem 0.1. From the point of view of applications it would be useful to find controls v_a, v_b defined by some "feedback law" $v_a = F_a(u)$, $v_b = F_b(u)$ with explicitly given functions F_a, F_b : this would realize an "automatic" control of the system. For the problem (1)–(3) such feedbacks can be found easily : let us consider the problem

$$(u_{tt} - u_{xx})(x,t) = 0, \quad (x,t) \in I \times \mathbb{R}_+, \tag{30}$$

$$(u_x - u_t)(a,t) = (u_x + u_t)(b,t) = 0, \quad t \in \mathbb{R}_+, \tag{31}$$

$$u(x,0) = u^0(x) \quad \text{and} \quad u_t(x,0) = u^1(x), \quad x \in I. \tag{32}$$

By definition, a solution of this problem is a function

$$u \in C([0,+\infty]; H^1(I)) \cap C^1([0,+\infty]; L^2(I))$$

satisfying (30) in the distributional sense, (32) almost everywhere and (31) in the following sense : in some neighbourhood of $\{a\} \times \mathbb{R}_+$ $u(x,t)$ is a function of $x+t$ and in some neighbourhood of $\{b\} \times \mathbb{R}_+$ it is a function of $x-t$.

Theorem 0.5. — *Let $(u^0, u^1) \in H^1(I) \times L^2(I)$ be such that*

$$u^0(a) + u^0(b) + \int_a^b u^1(s)\ ds = 0. \tag{33}$$

Then the solution of (30)–(32) *satisfies*

$$u(x,t) = 0, \quad \forall (x,t) \in I \times [b-a, +\infty). \tag{34}$$

Proof. — Adapting the proof of theorem 0.1 we obtain that the solution of (30)–(32) has again the form (11) with suitable functions $f : (a, +\infty) \to \mathbb{R}$ et $g : (-\infty, b) \to \mathbb{R}$ satisfying (14), (15) and

$$g'(a-t) = 0, \quad t \in \mathbb{R}_+, \tag{35}$$

$$f'(b+t) = 0, \quad t \in \mathbb{R}_+. \tag{36}$$

It follows that f, g satisfy (16), (17) with a suitable primitive U^1 of u^1, and that

$$g(s) = g(a), \quad s \le a, \tag{37}$$

$$f(s) = f(b), \quad s \ge b. \tag{38}$$

Introducing the subsets U, R, D, L as before, but replacing T by $+\infty$, we deduce from (11), (16), (17), (37) and (38) the following formulae :

$$2u(x,t) = \begin{cases} u^0(a) + u^0(b) + \int_a^b u^1(s)\ \mathrm{ds}, & \text{if } (x,t) \in U\,; \\ u^0(x-t) + u^0(b) + \int_{x-t}^b u^1(s)\ \mathrm{ds}, & \text{if } (x,t) \in R\,; \\ u^0(x+t) + u^0(x-t) + \int_{x-t}^{x+t} u^1(s)\ \mathrm{ds}, & \text{if } (x,t) \in D\,; \\ u^0(x+t) + u^0(a) + \int_a^{x+t} u^1(s)\ \mathrm{ds}, & \text{if } (x,t) \in L. \end{cases} \tag{39}$$

These show in particular the uniqueness of the solution of (30)–(32).

One can verify by a direct computation that the function defined by (39) is indeed a solution of (30)–(32). Finally, (34) follows from (33) and (39). $\square$

1. Linear evolutionary problems

The results of this chapter are standard; see e.g. Lions and Magenes [1] for proof.

1.1. The diagram $\mathbf{V} \subset \mathbf{H} = \mathbf{H}' \subset \mathbf{V}'$

Let V be an infinite-dimensional, separable, real or complex Hilbert space and introduce the duality mapping $A : V \to V'$ defined by

$$\langle Au, v\rangle_{V',V} := (u, v)_V, \ u, v \in V. \tag{1}$$

By the Riesz-Fréchet representation theorem A is an isometric isomorphism of V onto V'.

Let H be another Hilbert space with a dense and compact imbedding $V \subset H$. (The compactness means that every bounded subset of V is precompact in H.) Then in addition H is of infinite dimension, separable, and the imbedding $H' \subset V'$ is also dense and compact. Identifying H with H', we obtain the diagram

$$V \subset H = H' \subset V'. \tag{2}$$

We deduce from (1) that

$$(Au, v)_H = (u, v)_V, \ \forall u \in V \quad \text{such that} \quad Au \in H, \ \forall v \in V. \tag{3}$$

Denoting by i the compact imbedding of V into V', the linear map $T := A^{-1} \circ i : V \to V$ is also compact. Moreover, it is selfadjoint. Indeed, given $u, v \in V$ arbitrarily, we deduce from (3) that

$$(Tu, v)_V = (A^{-1}u, v)_V = (u, v)_H$$

and

$$(u, Tv)_V = (u, A^{-1}v)_V = (u, v)_H,$$

whence

$$(Tu, v)_V = (u, Tv)_V.$$

Applying the spectral theorem to $T = A^{-1} \circ i$ we conclude that there exists a sequence λ_1, λ_2,... of distinct real numbers (the eigenvalues of A) and a sequence Z_1, Z_2,... of subspaces of V such that

$$|\lambda_k| \to +\infty,$$

$$Az = \lambda_k z, \ \forall z \in Z_k, \ \forall k \geq 1, \tag{4}$$

$$\dim Z_k < +\infty, \ \forall k \geq 1, \tag{5}$$

$$Z_k \perp Z_l \quad \text{in} \quad V \quad \text{if} \quad k \neq l, \tag{6}$$

and

$$\text{the vector space } Z \text{ generated by } \cup Z_k \text{ is dense in } V. \tag{7}$$

It follows from (3) and (4) that the eigenvalues are positive. We may thus assume that

$$0 < \lambda_1 < \lambda_2 < \cdots \quad \text{and} \quad \lambda_k \to +\infty. \tag{8}$$

It follows from (6) and (7) that every $v \in V$ has a unique orthogonal expansion

$$v = \sum v_k, \ v_k \in Z_k, \ \forall k \geq 1, \tag{9}$$

converging in V; furthermore, we deduce from (6), (9), (3) and (4) that

$$\|v\|_V^2 = \sum \|v_k\|_V^2 = \sum \lambda_k \|v_k\|_H^2. \tag{10}$$

Now observe that

$$Z_k \perp Z_l \quad \text{in} \quad H \quad \text{and also in} \quad V' \quad \text{if} \quad k \neq l. \tag{11}$$

Indeed, the orthogonality in H follows from (3), (4), (6) and (8), while the orthogonality in V' follows from (4) and (6), using the isometric property of A : for $u \in Z_k$ and $v \in Z_l$, $k \neq l$, we have

$$(u, v)_{V'} = (A^{-1}u, A^{-1}v)_V = \lambda_k^{-1}\lambda_l^{-1}(u, v)_V = 0.$$

Next we deduce from (9) and from the density of the imbeddings in (2) that Z is also dense in H and in V'. Therefore every $v \in H$ (resp. every $v \in V'$)

has a unique orthogonal expansion of the form (9), converging in H (resp. in V'), and we have

$$\|v\|_H^2 = \sum \|v_k\|_H^2 \tag{12}$$

(resp.

$$\|v\|_{V'}^2 = \sum \|v_k\|_{V'}^2 = \sum \lambda_k^{-1} \|v_k\|_H^2). \tag{13}$$

The last equality of (13) follows from (10) :

$$\|v_k\|_{V'}^2 = \|A^{-1}v_k\|_V^2 = \lambda_k^{-2}\|v_k\|_V^2 = \lambda_k^{-1}\|v_k\|_H^2.$$

Let $\alpha \in \mathbb{R}$ and define a euclidean norm $\|\cdot\|_\alpha$ on Z by putting

$$\|v\|_\alpha := \Big(\sum \lambda_k^{2\alpha}\|v_k\|_H^2\Big)^{1/2}$$

(here we use the expansion (9) of v). Completing Z with respect to this norm we obtain a Hilbert space which will be denoted by D_α. It is easy to verify the orthogonality relations

$$Z_k \perp Z_l \quad \text{dans} \quad D_\alpha \quad \text{si} \quad k \neq l.$$

It is clear that for $\alpha > \beta$ the norm $\|\cdot\|_\alpha$ is stronger than $\|\cdot\|_\beta$. Thus we may assume that

$$D_\alpha \subset D_\beta \quad \text{if} \quad \alpha > \beta,$$

with a dense and continuous imbedding. (One can readily verify that these imbeddings are in fact compact.) Set

$$D_{-\infty} := \cup_\alpha D_\alpha.$$

For each fixed real number α let us introduce a linear mapping $A^\alpha : D_{-\infty} \to D_{-\infty}$ in the following way : first, for $v \in Z$ given by (9) we set

$$A^\alpha v := \sum \lambda_k^\alpha v_k. \tag{14}$$

Then for any given $v \in D_{-\infty}$ we choose $\beta \in \mathbb{R}$ such that $v \in D_\beta$ and then we choose a sequence (v_j) in Z such that $\|v - v_j\|_\beta \to 0$. One can readily verify that $A^\alpha v_j$ is a Cauchy sequence in $D_{\beta-\alpha}$, hence it converges to a certain $w \in D_{\beta-\alpha}$. Furthermore, it is easy to show that the limit w is independent of the particular choice of β and of the sequence v_j. Define $A^\alpha v := w$. It is clear that for $v \in Z$ this definition reduces to (14).

The following properties are easy to verify :

Given $\alpha, \beta \in \mathbb{R}$ arbitrarily, the restriction of A^α to D_β is an isometric isomorphism of D_β onto $D_{\beta-\alpha}$.

We have $A^\alpha A^\beta = A^{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{R}$.

We have

$$\|v\|_{1/2} = \|v\|_V, \quad \|v\|_0 = \|v\|_H \quad \textit{and} \quad \|v\|_{-1/2} = \|v\|_{V'}$$

for every $v \in Z$.

As a consequence, using also the density of Z in V, H and in V', we may identify $D_{1/2}$ with V, D_0 with H and $D_{-1/2}$ with V'. Then A^1 is an extension of A onto $D_{-\infty}$. In the sequel we shall also denote this extension by A i.e. we shall write A instead of A^1.

1.2. The equation $\mathbf{u'' + Au = 0}$

Let us first consider the homogeneous evolutionary problem

$$u'' + Au = 0 \quad \text{dans} \quad \mathbb{R}, \quad u(0) = u^0 \text{ et } u'(0) = u^1 \tag{15}$$

with arbitrarily given initial data $u^0, u^1 \in D_{-\infty}$. We shall use the orthogonal expansion of u^0 and u^1 :

$$u^0 = \sum_{k=1}^{\infty} u_k^0 \text{ and } u^1 = \sum_{k=1}^{\infty} u_k^1, \quad u_k^0,\ u_k^1 \in Z_k, \quad k = 1, 2, \ldots \tag{16}$$

Theorem 1.1. — *Let $\alpha \in \mathbb{R}$ and $(u^0, u^1) \in D_{\alpha+1/2} \times D_\alpha$. Then the problem* (15) *has a unique solution such that*

$$u \in C(\mathbb{R}; D_{\alpha+1/2}) \cap C^1(\mathbb{R}; D_\alpha) \cap C^2(\mathbb{R}; D_{\alpha-1/2}); \tag{17}$$

it is given by the series

$$u(t) = \sum_{k=1}^{\infty} u_k^0 \cos(\sqrt{\lambda_k} t) + u_k^1 \frac{\sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}}, \ t \in \mathbb{R}. \tag{18}$$

The energy $E_\alpha : \mathbb{R} \to \mathbb{R}_+$ of the solution, defined by

$$E_\alpha(t) := \tfrac{1}{2}\|u(t)\|_{\alpha+1/2}^2 + \tfrac{1}{2}\|u'(t)\|_\alpha^2, \tag{19}$$

is in fact independent of $t \in \mathbb{R}$.

The linear mapping $(u^0, u^1) \mapsto u$ is continuous from $D_{\alpha+1/2} \times D_\alpha$ into

$$C_b(\mathbb{R}; D_{\alpha+1/2}) \cap C_b^1(\mathbb{R}; D_\alpha) \cap C_b^2(\mathbb{R}; D_{\alpha-1/2}).$$

Since the energy is independent of $t \in \mathbb{R}$, we shall often write $E_\alpha(u)$ instead of $E_\alpha(u^0, u^1)$.

Let us recall that for $\alpha = 0$ we have $D_{\alpha+1/2} = V$, $D_\alpha = H$ and $D_{\alpha-1/2} = V'$. In this case we shall write E instead of E_0.

Remark 1.2. — The formula (18) shows that if $u^0 \perp Z_k$ and $u^1 \perp Z_k$, then $u(t) \perp Z_k$ and $u'(t) \perp Z_k$ for all $t \in \mathbb{R}$. Since the equation (15) is autonomous, we conclude more generally that if $u(T) \perp Z_k$ and $u'(T) \perp Z_k$ for some $T \in \mathbb{R}$, then necessarily $u(t) \perp Z_k$ and $u'(t) \perp Z_k$ for all $t \in \mathbb{R}$.

Remark 1.3. — One can readily verify that if $(u^0, u^1) \in D_{\alpha+1/2} \times D_\alpha$ for some $\alpha \in \mathbb{R}$ and if u is the corresponding solution of (15), then for any fixed $\beta \in \mathbb{R}$, $A^\beta u$ is the solution of (15) with (u^0, u^1) replaced by

$$(A^\beta u^0, A^\beta u^1) \in \times D_{\alpha-\beta+1/2} \times D_{\alpha-\beta}.$$

1.3. The wave equation

Let Ω be a bounded domain (that is, a non-empty open connected set) of class C^2 in $\mathbb{R}^n$; we denote by ν the outward unit normal vector to its boundary Γ. Let $\{\Gamma_0, \Gamma_1\}$ be a partition of Γ (the cases $\Gamma_0 = \emptyset$ or $\Gamma_1 = \emptyset$ are not excluded) and let $q : \Omega \to \mathbb{R}$, $a : \Gamma_1 \to \mathbb{R}$ be two given *nonnegative* functions.

We consider the problem

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{20}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}, \tag{21}$$

$$\partial_\nu u + au = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}, \tag{22}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{23}$$

In order to avoid some difficulties (studied in detail by Grisvard [1]), we shall assume throughout this book that

$$\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset \tag{24}$$

and that

$$q \in L^\infty(\Omega), \quad a \in C^1(\Gamma_1). \tag{25}$$

We introduce two real or complex Hilbert spaces H and V by the formulae

$$\begin{aligned} H &:= \{v \in L^2(\Omega) : \int_\Omega v \ \mathrm{dx} = 0, \\ V &:= \{v \in H^1(\Omega) : \int_\Omega v \ \mathrm{dx} = 0\} \end{aligned} \tag{26}$$

if $\Gamma_0 = \emptyset$, $q \equiv 0$ and $a \equiv 0$, and

$$H := L^2(\Omega) \quad \text{and} \quad V := H^1(\Omega) \tag{27}$$

otherwise; in both cases we define their norms by

$$\|v\|_H := \Big(\int_\Omega |v|^2 \ \mathrm{dx}\Big)^{1/2} \tag{28}$$

and

$$\|v\|_V := \Big(\int_\Omega |\nabla v|^2 + q|v|^2 \ \mathrm{dx} + \int_{\Gamma_1} a|v|^2 \ \ \mathrm{d}\Gamma\Big)^{1/2}. \tag{29}$$

One can readily verify that the seminorm $\|\cdot\|_V$ is indeed a *norm* and that it is equivalent to the norm induced by $H^1(\Omega)$. Applying Rellich's theorem it follows that the imbedding $V \subset H$ is dense and compact.

We introduce the corresponding linear map A and we *define* the solution of (20)-(23) as the (unique) solution of the problem

$$u'' + Au = 0 \quad \text{in} \quad \mathbb{R}, \quad u(0) = u^0, \quad u'(0) = u^1 \tag{30}$$

in the sense of theorem 1.1. We justify this definition by showing that every sufficiently smooth classical solution of (20)-(23) (for example of class C^2 on $\overline{\Omega} \times \mathbb{R}$) is also a solution of (30).

Fixing $t \in \mathbb{R}$ and $v \in V$ arbitrarily, we have

$$\begin{aligned} 0 &= \int_\Omega (u'' - \Delta u + qu)\bar{v} \ \ \mathrm{dx} \\ &= \int_\Omega u''\bar{v} + \nabla u \nabla \bar{v} + qu\bar{v} \ \ \mathrm{dx} - \int_\Gamma \partial_\nu u\bar{v} \ \ \mathrm{d}\Gamma \\ &= \int_\Omega (u'' + qu)\bar{v} + \nabla u \nabla \bar{v} \ \mathrm{dx} + \int_{\Gamma_1} au\bar{v} \ \mathrm{d}\Gamma \\ &= \langle u'', v\rangle_{V',V} + (u, v)_V = \langle u'' + Au, v\rangle_{V',V} \end{aligned}$$

on $\mathbb{R}$; hence $u''(t) + Au(t) = 0$ (as an element of V') for every $t \in \mathbb{R}$.

Let us note that $Au = -\Delta u + qu$ for every $u \in V \cap H^2(\Omega)$. Indeed, we have for every $v \in V$ the following equality :

$$\begin{aligned}\int_\Omega (-\Delta u + qu)\bar{v} \text{ dx} &= \int_\Omega \nabla u \nabla \bar{v} + qu\bar{v} \text{ dx} - \int_\Gamma \partial_\nu u \bar{v} \text{ d}\Gamma \\ &= \int_\Omega \nabla u \nabla \bar{v} + qu\bar{v} \text{ dx} + \int_{\Gamma_1} au\bar{v} \text{ d}\Gamma = \langle Au, v\rangle_{V',V}.\end{aligned}$$

We remark that for this choice of H, V and A the energy of the solutions of (30) is given by the formula

$$E(t) := \tfrac{1}{2}\int_\Omega |u'(t)|^2 + |\nabla u(t)|^2 + q|u(t)|^2 \text{ dx} + \tfrac{1}{2}\int_{\Gamma_1} a|u(t)|^2 \text{ d}\Gamma. \tag{31}$$

Remark 1.4. — We recall from the elliptic regularity theory that under conditions (24) and (25) for any given $g \in L^2(\Omega)$ the solution $v \in V$ of the problem

$$\begin{aligned}-\Delta v + qv = g \quad &\text{in} \quad \Omega, \\ v = 0 \quad &\text{on} \quad \Gamma_0, \\ \partial_\nu v + av = 0 \quad &\text{on} \quad \Gamma_1\end{aligned}$$

belongs to $H^2(\Omega)$ and that we have the estimate

$$\|v\|_{H^2(\Omega)} \le c\|g\|_{L^2(\Omega)}$$

with a constant c, independent of the choice of g. It follows that the eigenfunctions of A belong to $H^2(\Omega)$.

If $u^0, u^1 \in Z$, then it follows easily from formula (18) (which now reduces to a finite sum) that the solution of (20)-(23) satisfies

$$u \in C^\infty(\mathbb{R}; H^2(\Omega)). \tag{32}$$

1.4. A Petrovsky system

Let Ω be a bounded domain in $\mathbb{R}^n$ having a boundary Γ of class C^4 and consider the problem

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{33}$$

$$u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}, \tag{34}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{35}$$

We choose $H = L^2(\Omega)$ with its usual norm and $V = H_0^2(\Omega)$ endowed with the norm $\|v\|_V := \|\Delta v\|_{L^2(\Omega)}$; using remark 1.4 with $\Gamma_0 = \Gamma$ and $q = 0$ we conclude that this is indeed a *norm* on V (and even on $H^2(\Omega) \cap H_0^1(\Omega)$), and that this norm is equivalent to the norm induced by $H^2(\Omega)$.

It follows from Rellich's theorem that the imbedding $V \subset H$ is dense and compact. Introducing the corresponding operator A we *define* the solution of (33)-(35) as the solution of the problem

$$u'' + Au = 0 \quad \text{in} \quad \mathbb{R}, \quad u(0) = u^0, \quad u'(0) = u^1 \tag{36}$$

in the sense of theorem 1.1. To justify this definition we show that every sufficiently smooth (for example of class C^4 on $\overline{\Omega} \times \mathbb{R}$) classical solution of (33)-(35) is also a solution of (36). The only nontrivial property is the equality

$$u'' + Au = 0 \quad \text{in} \quad \mathbb{R}. \tag{37}$$

Fixing $t \in \mathbb{R}$ and $v \in C_c^\infty(\Omega)$ arbitrarily, we deduce from (33)-(35) that

$$\begin{aligned} 0 &= \int_\Omega (u'' + \Delta^2 u)\overline{v} \, \mathrm{dx} = \int_\Omega u''\overline{v} + \Delta u \Delta \overline{v} \, \mathrm{dx} \\ &= \langle u'', v \rangle_{V',V} + (u, v)_V = \langle u'' + Au, v \rangle_{V',V}. \end{aligned}$$

Using the density of $C_c^\infty(\Omega)$ in V hence (37) follows.

One can show similarly (as in the preceding section) that

$$Av = \Delta^2 v, \quad \forall v \in H^4(\Omega) \cap V.$$

For this choice of H, V and A the energy of the solutions of (36) is given by the formula

$$E(t) := \tfrac{1}{2} \int_\Omega |u'(t)|^2 + |\Delta u(t)|^2 \, \mathrm{dx}. \tag{38}$$

Remark 1.5. — We recall from the elliptic regularity theory that for $g \in L^2(\Omega)$ the solution $v \in V$ of the problem

$$\begin{aligned} \Delta^2 v = g \quad &\text{dans} \quad \Omega, \\ v = \partial_\nu v = 0 \quad &\text{sur} \quad \Gamma \end{aligned}$$

belongs to $H^4(\Omega)$; moreover, we have the estimate

$$\|v\|_{H^4(\Omega)} \leq c\|g\|_{L^2(\Omega)}$$

with a constant c, independent of the choice of g. In particular, the eigenfunctions of A belong to $H^4(\Omega)$.

As in remark 1.4, hence we conclude that for $u^0, u^1 \in Z$ the solution of (33)-(35) satisfies

$$u \in C^\infty(\mathbb{R}; H^4(\Omega)). \tag{39}$$

1.5. Another Petrovsky system

Let Ω be again a bounded domain in $\mathbb{R}^n$ having a boundary Γ of class C^4, and consider the problem

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{40}$$

$$u = \Delta u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}, \tag{41}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{42}$$

Set $H = L^2(\Omega)$, $V = H^2(\Omega) \cap H_0^1(\Omega)$ and $\|v\|_V := \|\Delta v\|_{L^2(\Omega)}$. It follows from remark 1.4 that the latter norm is equivalent to the norm induced by $H^2(\Omega)$ on V.

It follows from Rellich's theorem that the imbedding $V \subset H$ is dense and compact. Introducing the corresponding operator A we *define* the solution of (40)-(42) as the solution of the problem

$$u'' + Au = 0 \quad \text{in} \quad \mathbb{R}, \quad u(0) = u^0, \quad u'(0) = u^1 \tag{43}$$

in the sense of theorem 1.1. We justify this definition by showing that every sufficiently smooth (say of class C^4) classical solution of (40)–(42) is also a solution of (43). The only nontrivial property is the equality

$$u'' + Au = 0 \quad \text{in} \quad \mathbb{R}. \tag{44}$$

Fixing $t \in \mathbb{R}$ and $v \in V$ arbitrarily, we deduce from (40)-(42) (using also the relations $\Delta u = v = 0$ on Γ) that

$$\begin{aligned} 0 &= \int_\Omega (u'' + \Delta^2 u)\overline{v} \text{ dx} = \int_\Omega u''\overline{v} + \Delta u \Delta \overline{v} \text{ dx} \\ &= \langle u'', v\rangle_{V',V} + (u, v)_V = \langle u'' + Au, v\rangle_{V',V}, \end{aligned}$$

proving (44).

One can show similarly that

$$Av = \Delta^2 v, \quad \forall v \in H^4(\Omega) \cap V.$$

In this case the energy of the solutions of (44) is given by the formula

$$E(t) := \tfrac{1}{2}\int_\Omega |u'(t)|^2 + |\Delta u(t)|^2 \ \mathrm{dx}. \tag{45}$$

Remark 1.6. — It follows from the elliptic regularity theory that for any given $g \in L^2(\Omega)$ the solution $v \in V$ of the problem

$$\begin{gathered} \Delta^2 v = g \quad \text{in} \quad \Omega, \\ v = \Delta v = 0 \quad \text{on} \quad \Gamma \end{gathered}$$

belongs to $H^4(\Omega)$ and that the estimate

$$\|v\|_{H^4(\Omega)} \le c\|g\|_{L^2(\Omega)}$$

holds true with a constant c, independent of g. In particular, the eigenfunctions of A belong to $H^4(\Omega)$.

Using formula (18) hence we conclude that if $u^0, u^1 \in Z$, then the solution of (40)-(42) satisfies

$$u \in C^\infty(\mathbb{R}; H^4(\Omega)). \tag{46}$$

We end this section with a technical result; it shows in particular that the operator $A^{1/2}$ coincides with the operator A corresponding to the choice $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ (the latter being the particular case $\Gamma_0 = \Gamma$ of the situation studied in section 1.3).

Lemma 1.7. — *We have*

$$\begin{gathered} D_{1/4} = H_0^1(\Omega), \quad \|v\|_{1/4} = \|\nabla v\|_{L^2(\Omega)}, \\ D_{3/4} = \{v \in H^3(\Omega) : v = \Delta v = 0 \quad \mathit{on} \quad \Gamma\}, \quad \|v\|_{3/4} = \|\nabla \Delta v\|_{L^2(\Omega)}, \end{gathered}$$

and $D_{-1/4} = H^{-1}(\Omega)$ *with the corresponding dual norm. Moreover, we have*

$$A^{1/2}v = -\Delta v, \quad \forall v \in Z. \tag{47}$$

Proof. — Let us denote by $Z_1', Z_2', \dots$ the sequence of the eigenspaces of $-\Delta$ in $H_0^1(\Omega)$, by $(0 <)\lambda_1' < \lambda_2' < \cdots$ the sequence of the corresponding eigenvalues and by Z' the linear hull of $\cup Z_k'$. (This is the particular case $\Gamma_0 = \Gamma$ of the situation studied in §1.3.) If $v \in Z_k'$ for some k, then

$$\begin{aligned} (v, w)_V = (\Delta v, \Delta w)_H &= -\int_\Omega \lambda_k' v(\Delta w) \ \mathrm{dx} \\ = \lambda_k' &\int_\Omega (-\Delta v) w \ \mathrm{dx} = (\lambda_k')^2 (v, w)_H, \quad \forall w \in V \end{aligned}$$

whence

$$Av = (\lambda_k')^2 v, \quad \forall v \in Z_k', \quad k = 1, 2, \dots \tag{48}$$

Thus the eigenfunctions of $-\Delta|_{H_0^1(\Omega)}$ are also eigenfunctions of A and $\lambda_k = (\lambda_k')^2$. Since Z' is dense in $L^2(\Omega)$, A cannot have other eigenfunctions. Therefore $Z_k' = Z_k$, $\forall k$ and $Z' = Z$. Then (47) follows from (48).

For any $v = \sum v_k \in Z$ we have

$$\|v\|_{1/4}^2 = \|\sum v_k\|_{1/4}^2 = \sum \lambda_k^{1/2} \|v_k\|_H^2 = \sum \lambda_k' \|v_k\|_H^2 = \|v\|_{H_0^1(\Omega)}^2;$$

hence $D_{1/4} = H_0^1(\Omega)$ because $Z = Z'$ is dense in $H_0^1(\Omega)$.

Since $A^{1/2}$ is an isometric isomorphism of $D_{1/4}$ onto $D_{-1/4}$ and $-\Delta$ is an isometric isomorphism of $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$, using (47) hence we deduce that $H^{-1}(\Omega) = D_{-1/4}$.

Using (47) we obtain also that $D_{3/4}$ is the set of solutions of the problem

$$-\Delta v = g \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \Gamma$$

where g runs over $D_{1/4} = H_0^1(\Omega)$. It follows that

$$\{v \in H^3(\Omega) : v = \Delta v = 0 \quad \text{sur} \quad \Gamma\} \subset D_{3/4};$$

the inverse inclusion follows from remark 1.4.

The norm equalities in the formulation of the lemma follow from the above computations. $\square$

2. Hidden regularity. Weak solutions

The results of §§ 2.2–2.4 are due essentially to Lions [2] ; see also Lasiecka and Triggiani [1]. We consider the real case only : the complex case then follows easily by considering separately the real and imaginary parts of the solutions of the corresponding equations.

2.1. A special vector field

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain of class C^k, $k \geq 1$ and let $\{\Gamma_0, \Gamma_1\}$ be a partition of its boundary Γ such that

$$\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset.$$

Thus, we have in particular $\overline{\Gamma_0} = \Gamma_0$ and $\overline{\Gamma_1} = \Gamma_1$.

We recall for the reader's convenience the following standard construction :

Lemma 2.1. — *There exists a vector field $h : \overline{\Omega} \to \mathbb{R}^n$ of class C^{k-1} such that*

$$h = \nu \quad \textit{on} \quad \Gamma_0 \quad \textit{and} \quad h = 0 \quad \textit{on} \quad \Gamma_1.$$

Proof. — Since Ω is of class C^k, for every fixed $x^0 \in \Gamma_0$ there is an open neighbourhood V of x^0 in $\mathbb{R}^n$ and a function $\varphi : V \to \mathbb{R}$ of class C^k such that

$$\nabla\varphi(x) \neq 0, \quad \forall x \in V$$

and

$$\varphi(x) = 0 \Leftrightarrow x \in V \cap \Gamma.$$

Replacing φ by $-\varphi$ if needed, we may assume that

$$\nu(x^0) \cdot \nabla\varphi(x^0) > 0.$$

Choosing V sufficiently small we may assume also that

$$\overline{V} \cap \Gamma_1 = \emptyset$$

and that

$$V \cap \Gamma \quad \text{is connected.}$$

Then the function $\psi : V \to \mathbb{R}^n$ defined by $\psi := \nabla\varphi/|\nabla\varphi|$ is of class C^{k-1} and $\psi = \nu$ on $V \cap \Gamma$.

Since Ω is bounded, Γ_0 is compact; therefore it can be covered with a finite number of neighbourhoods $V_1, \ldots, V_m$ of this type. Denoting by $\psi_1, \ldots, \psi_m$ the corresponding functions we have

$$\Gamma_0 \subset V_1 \cup \ldots \cup V_m,$$

$$(V_1 \cup \ldots \cup V_m) \cap \Gamma_1 = \emptyset$$

and

$$\psi_i = \nu \quad \text{on} \quad V_i \cap \Gamma_0, \quad i = 1, \ldots, m.$$

Then we fix an open set V_0 in $\mathbb{R}^n$ such that

$$\overline{\Omega} \subset V_0 \cap \ldots \cap V_m,$$

$$V_0 \cap \Gamma_0 = \emptyset,$$

and we define $\psi_0 : V_0 \to \mathbb{R}^n$ by $\psi_0(x) = 0$, $\forall x \in V_0$.

Let $\theta_0, \ldots, \theta_m$ be a partition of unity of class C^k, corresponding to the covering $V_0, \ldots, V_m$ of $\overline{\Omega}$:

$$\theta_i \in C_c^k(V_i) \quad \text{and} \quad 0 \le \theta_i \le 1, \quad i = 0, \ldots, m$$

and

$$\theta_0 + \cdots + \theta_m = 1 \quad \text{on} \quad \overline{\Omega}.$$

One can readily verify that the vector field h defined by

$$h := \Big(\sum_{i=0}^{m} \theta_i \psi_i\Big)|_{\bar{\Omega}}$$

has the desired properties. $\square$

2.2. The wave equation. Multiplier method

We consider the problem

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R} \tag{1}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R} \tag{2}$$

$$\partial_\nu u + au = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R} \tag{3}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \tag{4}$$

introduced in § 1.3. We recall (see remark 1.4) that for arbitrarily given $u^0,\ u^1 \in Z$ the solution of (1)–(4) satisfies

$$u \in C^\infty(\mathbb{R}; H^2(\Omega)). \tag{5}$$

In particular the normal derivative of the solution is well-defined.

The following result will permit us to define the normal derivative of less regular solutions, too.

Theorem 2.2. — *Let $T > 0$. There is a constant $c = c(T) > 0$ such that for every $u^0,\ u^1 \in Z$ the solution of* (1)–(4) *satisfies the inequality*

$$\int_{-T}^{T} \int_{\Gamma_0} |\partial_\nu u|^2 \mathrm{d}\Gamma\ \mathrm{dt} \le c(\|u^0\|_V^2 + \|u^1\|_H^2). \tag{6}$$

Consequently, there is a unique continuous linear map

$$L : V \times H \to L^2_{loc}(\mathbb{R}; L^2(\Gamma_0))$$

such that

$$L(u^0, u^1) = \partial_\nu u,\ \forall (u^0, u^1) \in Z \times Z.$$

For the proof we need the following identity :

Lemma 2.3. — *Let $u \in H^2_{loc}(\mathbb{R}; H^2(\Omega))$ be a function satisfying* (1) *and let $h : \overline{\Omega} \to \mathbb{R}^n$ be a vector field of class C^1. Then for any fixed $-\infty < S < T < \infty$ the following identity holds true* :

$$\begin{aligned}\int_S^T \int_\Gamma 2(\partial_\nu u) h \cdot \nabla u + (h \cdot \nu)(u')^2 - (h \cdot \nu)|\nabla u|^2 \ \mathrm{d}\Gamma\ \mathrm{dt} \\ = \Big[\int_\Omega 2u' h \cdot \nabla u \ \mathrm{dx}\Big]_S^T \\ + \int_S^T \int_\Omega (\mathrm{div}\ h)\big((u')^2 - |\nabla u|^2\big) + 2quh \cdot \nabla u + \sum_{i,j=1}^n 2(\partial_i h_j)(\partial_i u)(\partial_j u) \mathrm{dx}\ \mathrm{dt}.\end{aligned} \tag{7}$$

(The dot denotes the usual scalar product in $\mathbb{R}^n$.)

Proof. — We multiply (1) by $2h \cdot \nabla u = 2 \sum_{j=1}^n h_j \partial_j u$ and we integrate by parts as follows :

$$\int_S^T \int_\Omega -2quh \cdot \nabla u \ \mathrm{dx}\ \mathrm{dt} = \int_S^T \int_\Omega 2(u'' - \Delta u) h \cdot \nabla u \ \mathrm{dx}\ \mathrm{dt}$$

$$= \Big[\int_\Omega 2u' h \cdot \nabla u \ \mathrm{dx}\Big]_S^T + \int_S^T \int_\Omega -2u' h \cdot \nabla u' - 2(\Delta u) h \cdot \nabla u \ \mathrm{dx}\ \mathrm{dt}$$

$$= \Big[\int_\Omega 2u'h\cdot\nabla u \,\mathrm{dx}\Big]_S^T - \int_S^T\int_\Gamma 2\partial_\nu u h\cdot\nabla u \,\mathrm{d}\Gamma\,\mathrm{dt}$$
$$+ \int_S^T\int_\Omega -h\cdot\nabla(u')^2 + 2\nabla u\cdot\nabla(h\cdot\nabla u)\,\mathrm{dx}\,\mathrm{dt}$$
$$= \Big[\int_\Omega 2u'h\cdot\nabla u \,\mathrm{dx}\Big]_S^T - \int_S^T\int_\Gamma 2\partial_\nu u h\cdot\nabla u + (h\cdot\nu)(u')^2\,\mathrm{d}\Gamma\,\mathrm{dt}$$
$$+ \int_S^T\int_\Omega (\mathrm{div}\ h)(u')^2 + 2\sum_{i,j=1}^n \partial_i u\partial_i(h_j\partial_j u)\,\mathrm{dx}\,\mathrm{dt}$$
$$= \Big[\int_\Omega 2u'h\cdot\nabla u \,\mathrm{dx}\Big]_S^T - \int_S^T\int_\Gamma 2\partial_\nu u h\cdot\nabla u + (h\cdot\nu)(u')^2\,\mathrm{d}\Gamma\,\mathrm{dt}$$
$$+ \int_S^T\int_\Omega (\mathrm{div}\ h)(u')^2 + \sum_{i,j=1}^n 2(\partial_i h_j)(\partial_i u)(\partial_j u) + h_j\partial_j((\partial_i u)^2)\,\mathrm{dx}\,\mathrm{dt}$$
$$= \Big[\int_\Omega 2u'h\cdot\nabla u \,\mathrm{dx}\Big]_S^T - \int_S^T\int_\Gamma 2\partial_\nu u h\cdot\nabla u + (h\cdot\nu)((u')^2-|\nabla u|^2)\,\mathrm{d}\Gamma\,\mathrm{dt}$$
$$+ \int_S^T\int_\Omega (\mathrm{div}\ h)((u')^2-|\nabla u|^2) + \sum_{i,j=1}^n 2(\partial_i h_j)(\partial_i u)(\partial_j u)\,\mathrm{dx}\,\mathrm{dt}. \quad \square$$

Proof of theorem 2.2. — Applying theorem 1.1 we have

$$\|u(t)\|_V^2 + \|u'(t)\|_H^2 = \|u^0\|_V^2 + \|u^1\|_H^2 \tag{8}$$

for every $t \in \mathbb{R}$. (The constant c_1 does not depend on the choice of u^0, u^1 and f.)

Applying the identity (7) with $S = -T$ and with the vector field h constructed in lemma 2.1 the left-hand side of (7) becomes

$$\int_{-T}^T\int_{\Gamma_0} |\partial_\nu u|^2\,\mathrm{d}\Gamma\,\mathrm{dt}.$$

(Note that $u' = 0$ and $\nabla u = (\partial_\nu u)\nu$ on $\Gamma_0 \times \mathbb{R}$ because of (2).)

Since h is of class C^1, there exists a constant c_1 such that

$$|h(x)| \le c_1 \quad \text{and} \quad \sum_{i,j=1}^n |\partial_i h_j(x)| \le c_1,\ \forall x \in \overline{\Omega}.$$

Using (8) and these inequalities to majorize the right-hand side of (7) we obtain easily the estimate (6) with a suitable constant c.

The last part of the theorem follows from the inequality (6), using the density of $Z \times Z$ in $V \times H$. $\square$

Remark 2.4. — Teorem 2.2 justifies the notation $\partial_\nu u$ or $\partial u/\partial\nu$ instead of $L(u^0,u^1)$ for any $(u^0,u^1)\in V\times H$. Then we have the following trace theorem :

$$(u^0,u^1)\in V\times H \Rightarrow \partial_\nu u\in L^2_{loc}(\mathbb{R};L^2(\Gamma_0)). \tag{9}$$

This result does not follow from the usual trace theorems of the Sobolev spaces : it is a *"hidden regularity"* result. The corresponding inequality (6) is often called a *direct inequality.*

Now we consider the following non-homogeneous problem :

$$y''-\Delta y+qy=0 \quad \text{in} \quad \Omega\times\mathbb{R} \tag{10}$$

$$y=v \quad \text{on} \quad \Gamma_0\times\mathbb{R} \tag{11}$$

$$\partial_\nu y+ay=0 \quad \text{on} \quad \Gamma_1\times\mathbb{R} \tag{12}$$

$$y(0)=y^0, \quad y'(0)=y^1 \quad \text{in} \quad \Omega. \tag{13}$$

In order to find a reasonably general definition of the solution we begin with a formal computation. Let us first assume that y is a sufficiently smooth function (say of class C^2 in $\overline{\Omega}\times\mathbb{R}$) satisfying (10)–(13). Fix $(u^0,u^1)\in V\times H$ arbitrarily and multiply the corresponding solution of (1)–(4) by y. Integrating by parts we obtain for every fixed $S\in\mathbb{R}$ the equality

$$\begin{aligned}
0=\int_0^S\int_\Omega(u''-\Delta u+qu)\,y\ \mathrm{dx}\ \mathrm{dt}&=\Big[\int_\Omega u'y-uy'\ \mathrm{dx}\Big]_0^S\\
&\quad-\int_0^S\int_\Gamma(\partial_\nu u)y-u(\partial_\nu y)\ \mathrm{d}\Gamma\ \mathrm{dt}\\
&\quad+\int_0^S\int_\Omega u(y''-\Delta y+qy)\ \mathrm{dx}\ \mathrm{dt}\\
=\int_\Omega u'(S)y(S)-u(S)y'(S)+u^0y^1-u^1y^0\ \mathrm{dx}&-\int_0^S\int_{\Gamma_0}(\partial_\nu u)v\ \mathrm{d}\Gamma\ \mathrm{dt}.
\end{aligned}$$

Putting

$$L_S(u^0,u^1):=\int_0^S\int_{\Gamma_0}(\partial_\nu u)v\ \mathrm{d}\Gamma\ \mathrm{dt}+\langle(-y^1,y^0),(u^0,u^1)\rangle_{V'\times H,V\times H}$$

we may rewrite this identity as

$$L_S(u^0,u^1)=\langle(-y'(S),y(S)),(u(S),u'(S))\rangle_{V'\times H,V\times H}. \tag{14}$$

This leads to the following *definition* :

Definition. — We say that (y, y') is a solution of (10)–(13) if $(y, y') \in C(\mathbb{R}; H \times V')$ and if (14) is satisfied for every $S \in \mathbb{R}$ and for every $(u^0, u^1) \in V \times H$.

This definition is justified by

Theorem 2.5. — *Given $(y^0, y^1) \in H \times V'$ and $v \in L^2_{loc}(\mathbb{R}; L^2(\Gamma_0))$ arbitrarily, the problem (10)–(13) has a unique solution.*

Furthermore, the linear map $(y^0, y^1, v) \mapsto (y, y')$ is continuous with respect to these topologies.

Proof. — It follows from theorem 2.2 that for every fixed $S \in \mathbb{R}$ the linear form L_S is bounded in $V \times H$. Furthermore, it follows from theorem 1.1 that the linear map

$$(u(S), u'(S)) \mapsto (u^0, u^1)$$

is an isomorphism of $V \times H$ onto itself. Hence the linear form

$$(u(S), u'(S)) \mapsto L_S(u^0, u^1)$$

is also bounded on $V \times H$. Therefore there is a unique pair $(y(S), y'(S)) \in H \times V'$ satisfying (14).

Next we show that the function

$$\mathbb{R} \ni S \mapsto \|(y(S), y'(S))\|_{H \times V'}$$

is bounded in every bounded interval. More precisely, for every bounded interval I and for all $S \in I$ we have the estimate

$$\|(y(S), y'(S))\|_{H \times V'} \le c(I)\big(\|v\|_{L^2(I; L^2(\Gamma_0))} + \|(y^0, y^1)\|_{H \times V'}\big) \tag{15}$$

where $c(I)$ is a constant independent of S, v, y^0 and y^1. Indeed, choose $(u^0, u^1) \in V \times H$ arbitrarily and write for brevity

$$Y = (-y', y), \quad Y^0 = (-y^1, y^0), \quad U = (u, u'), \quad U^0 = (u^0, u^1).$$

Then, using theorems 1.1 and 2.2 we have the following estimate (we denote by $\|\cdot\|_I$ the norm of $L^2(I; L^2(\Gamma_0))$ for brevity) :

$$\big|\langle Y(S), U(S)\rangle_{V' \times H, V \times H}\big| = \Big|\int_0^S \int_{\Gamma_0} (\partial_\nu u) v \,\mathrm{d}\Gamma\,\mathrm{dt} + \langle Y^0, U^0\rangle_{V' \times H, V \times H}\Big|$$
$$\le \|\partial_\nu u\|_I \|v\|_I + \|Y^0\|_{V' \times H}\|U^0\|_{V \times H} \le c(I)\big(\|v\|_I + \|Y^0\|_{V' \times H}\big)\|U^0\|_{V \times H}.$$

Hence (15) follows.

Next we recall (cf. e.g. Lions and Magenes [1]) that if $(y^0, y^1) \in V \times H$ and $v \in C_c^\infty(\mathbb{R}; H^{3/2}(\Gamma_0))$ are such that $v(0) = 0$, then (4)–(6) has a regular solution

$$y \in C(\mathbf{R}; \mathbf{V}) \cap \mathbf{C^1}(\mathbf{R}; \mathbf{H});$$

we have in particular $(y, y') \in C(\mathbb{R}; H \times V')$. Since the set of these data (y^0, y^1, v) is dense in $H \times V' \times L^2_{loc}(\mathbb{R}; L^2(\Gamma_0))$, using (15) hence we conclude that $(y, y') \in C(\mathbb{R}; H \times V')$ in the general case, too.

Finally, the continuous dependence of the solution on (y^0, y^1, v) also follows from (15). $\square$

2.3. The first Petrovsky system

Consider the problem introduced in § 1.4 :

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R} \tag{16}$$

$$u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R} \tag{17}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{18}$$

We recall (see remark 1.5) that for every $u^0,\ u^1 \in Z$ the solution of (16)–(18) satisfies

$$u \in C^\infty(\mathbb{R}; H^4(\Omega)).$$

In particular, Δu is well-defined on $\Gamma \times \mathbb{R}$.

The following result shows that this trace may be defined for weaker solutions as well.

Theorem 2.6. — *Given $T > 0$ arbitrarily, there exists a constant $c = c(T) > 0$ such that for every $(u^0, u^1) \in Z \times Z$ the solution of* (16)–(18) *satisfies the inequality*

$$\int_{-T}^{T} \int_\Gamma |\Delta u|^2 \ \mathrm{d}\Gamma \ \mathrm{d}t \le c(\|u^0\|^2_{H^2_0(\Omega)} + \|u^1\|^2_{L^2(\Omega)}). \tag{19}$$

Consequently, there is a unique continuous linear map

$$L : H^2_0(\Omega) \times L^2(\Omega) \to L^2_{loc}(\mathbb{R}; L^2(\Gamma))$$

such that

$$L(u^0, u^1) = \Delta u, \ \forall (u^0, u^1) \in Z \times Z.$$

We begin by establishing an identity analogous to that of lemma 2.3.

Lemma 2.7. — *Let $u \in H^2_{loc}(\mathbb{R}; H^4(\Omega))$ be a function satisfying* (16), (17), *and let $h : \overline{\Omega} \to \mathbb{R}^n$ be a vector field of class C^2. Then for any fixed $-\infty < S < T < \infty$ the following identity holds true :*

$$\begin{aligned}\int_S^T \int_\Gamma (h \cdot \nu)|\Delta u|^2 \, \mathrm{d}\Gamma \, \mathrm{dt} = \Big[\int_\Omega 2u' h \cdot \nabla u \, \mathrm{dx}\Big]_S^T \\ + \int_S^T \int_\Omega (\mathrm{div}\ h)((u')^2 - |\Delta u|^2) \\ +4 \sum_{i,j=1}^n (\partial_i h_j)(\Delta u)(\partial_i \partial_j u) + 2 \sum_{i=1}^n (\Delta h_i)(\Delta u)\partial_i u \, \mathrm{dx} \, \mathrm{dt}.\end{aligned} \tag{20}$$

Proof. — We multiply (16) by $2h \cdot \nabla u = 2\sum_{j=1}^n h_j \partial_j u$ and we integrate by parts :

$$\begin{aligned}0 &= \int_S^T \int_\Omega 2(u'' + \Delta^2 u) h \cdot \nabla u \, \mathrm{dx} \, \mathrm{dt} \\ &= \Big[\int_\Omega 2u' h \cdot \nabla u \, \mathrm{dx}\Big]_S^T + \int_S^T \int_\Omega -2u' h \cdot \nabla u' + 2(\Delta^2 u) h \cdot \nabla u \, \mathrm{dx} \, \mathrm{dt} \\ &= \Big[\int_\Omega 2u' h \cdot \nabla u \, \mathrm{dx}\Big]_S^T + 2\int_S^T \int_\Gamma (\partial_\nu \Delta u) h \cdot \nabla u - (\Delta u)\partial_\nu (h \cdot \nabla u) \, \mathrm{d}\Gamma \, \mathrm{dt} \\ &\quad + \int_S^T \int_\Omega -h \cdot \nabla (u')^2 + 2(\Delta u)\Delta(h \cdot \nabla u) \, \mathrm{dx} \, \mathrm{dt}.\end{aligned}$$

Remark that

$$\begin{aligned}\Delta(h \cdot \nabla u) &= \sum_{i,j=1}^n \partial_i^2 (h_j \partial_j u) \\ &= 2\sum_{i,j=1}^n (\partial_i h_j)(\partial_i \partial_j u) + \sum_{j=1}^n (\Delta h_j)(\partial_j u) + h_j \partial_j (\Delta u)\end{aligned}$$

whence

$$\begin{aligned}2(\Delta u)\Delta(h \cdot \nabla u) = h \cdot \nabla(\Delta u)^2 \\ + \\ 2\sum_{j=1}^n (\Delta h_j)(\partial_j u)(\Delta u) + 4 \sum_{i,j=1}^n (\partial_i h_j)(\partial_i \partial_j u)(\Delta u)\end{aligned}$$

and the above identity becomes

$$\begin{aligned}
0 = \Big[\int_\Omega 2u'h\cdot\nabla u\ \mathrm{dx}\Big]_S^T &+ 2\int_S^T\int_\Gamma (\partial_\nu\Delta u)h\cdot\nabla u - (\Delta u)\partial_\nu(h\cdot\nabla u)\ \mathrm{d}\Gamma\ \mathrm{dt}\\
&+\int_S^T\int_\Omega h\cdot\nabla((\Delta u)^2-(u')^2) + 2\sum_{j=1}^n(\Delta h_j)(\partial_j u)(\Delta u)\\
&+4\sum_{i,j=1}^n(\partial_i h_j)(\partial_i\partial_j u)(\Delta u)\ \mathrm{dx}\ \mathrm{dt}\\
&= \Big[\int_\Omega 2u'h\cdot\nabla u\ \mathrm{dx}\Big]_S^T\\
+\int_S^T\int_\Gamma 2(\partial_\nu\Delta u)h\cdot\nabla u &- 2(\Delta u)\partial_\nu(h\cdot\nabla u) + (h\cdot\nu)((\Delta u)^2-(u')^2)\ \mathrm{d}\Gamma\ \mathrm{dt}\\
&+\int_S^T\int_\Omega (\mathrm{div} h)((u')^2-(\Delta u)^2) + 2\sum_{j=1}^n(\Delta h_j)(\partial_j u)(\Delta u)\\
&+4\sum_{i,j=1}^n(\partial_i h_j)(\partial_i\partial_j u)(\Delta u)\ \mathrm{dx}\ \mathrm{dt}.
\end{aligned}$$

Comparing with (20) it remains to show that

$$\begin{aligned}
\int_S^T\int_\Gamma 2(\partial_\nu\Delta u)h\cdot\nabla u - 2(\Delta u)\partial_\nu(h\cdot\nabla u) + (h\cdot\nu)((\Delta u)^2-(u')^2)\ \mathrm{d}\Gamma\ \mathrm{dt}&\\
= -\int_S^T\int_\Gamma (h\cdot\nu)(\Delta u)^2\ \mathrm{d}\Gamma\ \mathrm{dt}.&
\end{aligned}\tag{21}$$

For this we need the boundary conditions (17).

First, (17) implies that $u' = |\nabla u| = 0$ on $\Gamma\times\mathbb{R}$ and therefore (21) reduces to

$$\int_S^T\int_\Gamma (h\cdot\nu)(\Delta u)^2\ \mathrm{d}\Gamma\ \mathrm{dt} = \int_S^T\int_\Gamma (\Delta u)\partial_\nu(h\cdot\nabla u)\ \mathrm{d}\Gamma\ \mathrm{dt}.\tag{22}$$

To compute $\partial_\nu(h\cdot\nabla u)$ at a fixed point $x\in\Gamma$, let us choose the coordinates such that $\nu(x) = (0,\dots,0,1)$. Since $\partial_1 u = \dots = \partial_n u = 0$ on Γ, we have

$$\nabla(\partial_j u) = (\partial_\nu\partial_j u)\nu \text{ sur } \Gamma,\ j = 1,\dots,n.$$

In particular, we have

$$\partial_i\partial_j u(x) = 0,\ i = 1,\dots,n-1,\ j = 1,\dots,n.$$

Since $\partial_i \partial_j u = \partial_j \partial_i u$, hence we conclude that

$$\partial_i \partial_j u(x) = 0 \text{ always, unless } i = j = n. \tag{23}$$

Using (23) and the property $\nu(x) = (0, \dots, 0, 1)$ we obtain that

$$\partial_\nu (h \cdot \nabla u)(x) = \Big(\sum_{j=1}^{n} (\partial_\nu h_j)(\partial_j u) + h_j \partial_n \partial_j u\Big)(x)$$

$$= \sum_{j=1}^{n} h_j \partial_n \partial_j u(x) = h_n \partial_n^2 u(x) = h_n \Delta u(x) = (h \cdot \nu) \Delta u(x).$$

Since $x \in \Gamma$ was chosen arbitrarily, we conclude that $\partial_\nu (h \cdot \nabla u) = (h \cdot \nu)\Delta u$ on Γ, proving (22). ∎

Proof of theorem 2.6. — As in § 2.2, it is sufficient to prove the inequalities (19). Let us apply the identity (20) with $S := -T$ and with the vector field of lemma 2.1, corresponding to the case $\Gamma_0 = \Gamma$. Then the left-hand side of (23) coincides with that of (22), while the right-hand side of (23) is easily majorized by

$$c_1 (\|u\|^2_{C([-T,T];H^2(\Omega))} + \|u'\|^2_{C([-T,T];L^2(\Omega))})$$

where c_1 is a constant depending on $\|h\|_{C^2(\bar{\Omega})}$ and T only. Applying theorem 1.1 this last expression is majorized by the right-hand side of (19) for a suitable constant c. ∎

Remark 2.8. — In the sequel we shall write Δu on Γ instead of $L(u^0, u^1)$ for *every* $(u^0, u^1) \in H_0^2(\Omega) \times L^2(\Omega)$. Then we have the following trace theorem :

$$(u^0, u^1) \in H_0^2(\Omega) \times L^2(\Omega) \Rightarrow \Delta u \in L^2_{loc}(\mathbb{R}; L^2(\Gamma)).$$

Now we consider the non-homogeneous boundary value problem

$$y'' + \Delta^2 y = 0 \quad \text{in} \quad \Omega \times \mathbb{R} \tag{24}$$

$$y = 0 \quad \text{and} \quad \partial_\nu y = v \quad \text{on} \quad \Gamma \times \mathbb{R} \tag{25}$$

$$y(0) = y^0, \quad y'(0) = y^1. \tag{26}$$

As usual, we begin with a formal computation. Let $y \in C^4(\overline{\Omega} \times [0, T])$ be a function satisfying (24)–(26). Fix $(u^0, u^1, f) \in V \times H = H_0^2(\Omega) \times L^2(\Omega)$ arbitrarily and multiply (16) by y. Using the boundary conditions (17) and

(25) we obtain for every fixed $S \in \mathbb{R}$ that

$$\begin{aligned}
0 &= \int_0^S \int_\Omega (u'' + \Delta^2 u)\, y \,\mathrm{dx}\,\mathrm{dt} \\
&= \Big[\int_\Omega u'y - uy' \,\mathrm{dx}\Big]_0^T + \int_0^S \int_\Omega u(y'' + \Delta^2 y)\,\mathrm{dx}\,\mathrm{dt} \\
&+ \int_0^S \int_\Gamma (\partial_\nu \Delta u)y - (\Delta u)(\partial_\nu y) + (\partial_\nu u)(\Delta y) - u(\partial_\nu \Delta y)\,\mathrm{d}\Gamma\,\mathrm{dt} \\
&= \int_\Omega u'(S)y(S) - u(S)y'(S) + u^0 y^1 - u^1 y^0 \,\mathrm{dx} - \int_0^S \int_\Gamma (\Delta u) v \,\mathrm{d}\Gamma\,\mathrm{dt}.
\end{aligned}$$

Introducing the linear form

$$L_S(u^0, u^1) := \int_0^S \int_\Gamma (\Delta u) v \,\mathrm{d}\Gamma\,\mathrm{dt} + \int_\Omega u^1 y^0 - u^0 y^1 \,\mathrm{dx}$$

we may rewrite this identity in the following form :

$$L_S(u^0, u^1) = \langle(-y'(S), y(S)), (u(S), u'(S))\rangle_{V' \times H, V \times H}. \tag{27}$$

(Note that $V = H_0^2(\Omega)$, $H = L^2(\Omega)$ and $V' = H^{-2}(\Omega)$.)

Definition. — We say that (y, y') is a solution of (24)–(26) if $(y, y') \in C(\mathbb{R}; H \times V')$ and if (27) is satisfied for all $S \in \mathbb{R}$ and for all $(u^0, u^1) \in V \times H$.

To justify this definition we prove the

Theorem 2.9. — *Given $(y^0, y^1) \in L^2(\Omega) \times H^{-2}(\Omega)$ and $v \in L^2_{loc}(\mathbb{R}; L^2(\Gamma))$ arbitrarily, the problem (24)–(26) has a unique solution.*

Furthermore, the linear map $(y^0, y^1, v) \mapsto (y, y')$ is continuous with respect to these topologies.

Proof. — It follows from theorem 2.6 that for every fixed $S \in \mathbb{R}$ the linear form L_S is bounded in $V \times H$. Furthermore, it follows from theorem 1.1 that the linear map

$$(u(S), u'(S)) \mapsto (u^0, u^1)$$

is an isomorphism of $V \times H$ onto itself. Hence the linear form

$$(u(S), u'(S)) \mapsto L_S(u^0, u^1)$$

is also bounded on $V \times H$. Therefore there is a unique pair $(y(S), y'(S)) \in H \times V'$ satisfying (27).

The rest of the theorem may be proved exactly as the corresponding part of theorem 2.5. $\square$

2.4. The second Petrovsky system

Here we consider the problem introduced in § 1.5 :

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R} \tag{28}$$

$$u = \Delta u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R} \tag{29}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{30}$$

If $u^0 \in H^2(\Omega) \cap H_0^1(\Omega)(= V)$ and $u^1 \in L^2(\Omega)(= H)$, then by theorem 1.1 this problem has a unique solution satisfying

$$u \in C(\mathbb{R}; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbb{R}; L^2(\Omega)).$$

In particular, $\partial_\nu u$ is well-defined. The following result permits us to define the normal derivative of weaker solutions, too.

Theorem 2.10. — *Given $T > 0$ arbitrarily, there exists a constant $c = c(T) > 0$ such that for every $(u^0, u^1) \in Z \times Z$ the solution of* (28)–(30) *satisfies the inequality*

$$\int_{-T}^{T} \int_\Gamma |\partial_\nu u|^2 \, \mathrm{d}\Gamma \, \mathrm{d}t \le c(\|u^0\|^2_{H_0^1(\Omega)} + \|u^1\|^2_{H^{-1}(\Omega)}). \tag{31}$$

Consequently, there is a unique continuous linear map

$$L : H_0^1(\Omega) \times H^{-1}(\Omega) \to L^2_{loc}(\mathbb{R}; L^2(\Gamma))$$

such that

$$L(u^0, u^1) = \partial_\nu u, \ \forall (u^0, u^1) \in Z \times Z.$$

Remark 2.11. — Let us recall from lemma 1.7 that $H_0^1(\Omega) = D_{1/4}$ and $H^{-1}(\Omega) = D_{-1/4}$. Thus for any given $(u^0, u^1) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ (28)–(30) has a unique solution such that

$$u \in C(\mathbb{R}; H_0^1(\Omega)) \cap C^1(\mathbb{R}; H^{-1}(\Omega)).$$

(See remark 1.3.) Therefore, writing $\partial_\nu u$ instead of $L(u^0, u^1)$ we will have the following hidden regularity result :

$$(u^0, u^1) \in H_0^1(\Omega) \times H^{-1}(\Omega) \Rightarrow \partial_\nu u \in L^2_{loc}(\mathbb{R}; L^2(\Gamma)).$$

Lemma 2.12. — *Let $u \in H^2_{loc}(\mathbb{R}; H^4(\Omega))$ be a function satisfying* (28), (29) *and let $h : \overline{\Omega} \to \mathbb{R}^n$ be a vector field of class C^2. Then for any given $-\infty < S < T < \infty$ the following identity holds :*

$$\begin{aligned}\int_S^T \int_\Gamma (h\cdot\nu)(|\partial_\nu u'|^2 + |\partial_\nu \Delta u|^2)\ \mathrm{d}\Gamma\ \mathrm{dt} = -\Big[\int_\Omega 2u' h\cdot\nabla\Delta u\ \mathrm{dx}\Big]_S^T \\ + \int_S^T \int_\Omega (\mathrm{div}\ h)(|\nabla u'|^2 - |\nabla\Delta u|^2) \qquad (32)\\ +2\sum_{i,j=1}^n (\partial_i h_j)(\partial_i u')(\partial_j u') + (\partial_i h_j)(\partial_i \Delta u)(\partial_j \Delta u) + 2\sum_{i=1}^n (\Delta h_i) u' \partial_i u'\ \mathrm{dx}\ \mathrm{dt}.\end{aligned}$$

Proof. — We multiply (28) by $2h\cdot\nabla\Delta u$ and we integrate by parts. We obtain

$$\begin{aligned} 0 &= \int_S^T \int_\Omega 2(u'' + \Delta^2 u) h\cdot\nabla\Delta u\ \mathrm{dx}\ \mathrm{dt}\\ &= \Big[\int_\Omega 2u'h\cdot\nabla\Delta u\ \mathrm{dx}\Big]_S^T + \int_S^T \int_\Omega -2u'h\cdot\nabla\Delta u' + 2(\Delta^2 u)h\cdot\nabla\Delta u\ \mathrm{dx}\ \mathrm{dt}\\ &= \Big[\int_\Omega 2u'h\cdot\nabla\Delta u\ \mathrm{dx}\Big]_S^T + 2\int_S^T \int_\Gamma (\partial_\nu\Delta u) h\cdot\nabla\Delta u\ \mathrm{d}\Gamma\ \mathrm{dt}\\ &\quad - \int_S^T \int_\Omega 2u'h\cdot\nabla\Delta u' + 2(\nabla\Delta u)\cdot\nabla(h\cdot\nabla\Delta u)\ \mathrm{dx}\ \mathrm{dt} \qquad (33)\\ &= \Big[\int_\Omega 2u'h\cdot\nabla\Delta u\ \mathrm{dx}\Big]_S^T + 2\int_S^T \int_\Gamma (h\cdot\nu)|\partial_\nu\Delta u|^2\ \mathrm{d}\Gamma\ \mathrm{dt}\\ &\quad - \int_S^T \int_\Omega 2u'h\cdot\nabla\Delta u' + 2(\nabla\Delta u)\cdot\nabla(h\cdot\nabla\Delta u)\ \mathrm{dx}\ \mathrm{dt}\end{aligned}$$

because

$$\Delta u = 0 \quad \text{on} \quad \Gamma \Rightarrow \nabla\Delta u = (\partial_\nu\Delta u)\nu \quad \text{on} \quad \Gamma.$$

Furthermore, we have

$$\int_\Omega -2u'h\cdot\nabla\Delta u'\ \mathrm{dx} = \sum_{i,j=1}^n \int_\Omega -2u'h_i\partial_i\partial_j^2 u'\ \mathrm{dx}$$

$$= -2\sum_{i,j=1}^{n}\int_\Gamma u' h_i \nu_j \partial_i\partial_j u' \ \mathrm{d}\Gamma + 2\sum_{i,j=1}^{n}\int_\Omega \partial_j u' h_i \partial_i \partial_j u' + (\partial_j h_i) u' (\partial_i \partial_j u') \ \mathrm{dx}$$

$$= -2\sum_{i,j=1}^{n}\int_\Gamma u' h_i \nu_j \partial_i\partial_j u' \ \mathrm{d}\Gamma + \int_\Omega h \cdot \nabla |\nabla u'|^2 + 2\sum_{i,j=1}^{n} (\partial_j h_i) u' (\partial_i \partial_j u') \ \mathrm{dx}$$

$$= -2\sum_{i,j=1}^{n}\int_\Gamma u' h_i \nu_j \partial_i\partial_j u' \ \mathrm{d}\Gamma + \int_\Gamma (h\cdot\nu)|\nabla u'|^2 \ \mathrm{d}\Gamma$$

$$- \int_\Omega (\mathrm{div} h)|\nabla u'|^2 \ \mathrm{dx} + 2\sum_{i,j=1}^{n}\int_\Gamma \nu_j (\partial_j h_i) u' (\partial_i u') \ \mathrm{d}\Gamma$$

$$-2\sum_{i=1}^{n}\int_\Omega (\Delta h_i) u' (\partial_i u') \ \mathrm{dx} - 2\sum_{i,j=1}^{n}\int_\Omega (\partial_j h_i)(\partial_j u')(\partial_i u') \ \mathrm{dx}.$$

Since the condition $u = 0$ on Γ implies that $u' = 0$ and $\nabla u' = (\partial_\nu u')\nu$ on Γ, the first and third integrals on Γ vanish, and the second one is equal to

$$\int_\Gamma (h\cdot\nu)|\partial_\nu u'|^2 \ \mathrm{d}\Gamma.$$

Therefore we have

$$\int_\Omega -2u' h\cdot\nabla\Delta u' \ \mathrm{dx} = \int_\Gamma (h\cdot\nu)|\partial_\nu u'|^2 \ \mathrm{d}\Gamma - \int_\Omega (\mathrm{div} h)|\nabla u'|^2 \ \mathrm{dx}$$

$$-2\int_\Omega \sum_{i=1}^{n}(\Delta h_i) u' (\partial_i u') + \sum_{i,j=1}^{n}(\partial_j h_i)(\partial_j u')(\partial_i u') \ \mathrm{dx}.$$

Substituting into (33) we obtain that

$$\begin{aligned} 0 = \Big[\int_\Omega 2u' h\cdot\nabla\Delta u \ \mathrm{dx}\Big]_S^T + \int_S^T \int_\Gamma 2(h\cdot\nu)|\partial_\nu\Delta u|^2 + (h\cdot\nu)|\partial_\nu u'|^2 \ \mathrm{d}\Gamma \ \mathrm{dt} \\ - \int_S^T \int_\Omega (\mathrm{div} h)|\nabla u'|^2 + 2\sum_{i,j=1}^{n}(\partial_j h_i)(\partial_j u')(\partial_i u') \\ +2\sum_{i=1}^{n}(\Delta h_i) u'(\partial_i u') + 2\nabla\Delta u\cdot\nabla(h\cdot\nabla\Delta u) \ \mathrm{dx} \ \mathrm{dt}. \end{aligned} \tag{34}$$

Let us consider the last term of (34). We have

$$- \int_\Omega 2\nabla\Delta u\cdot\nabla(h\cdot\nabla\Delta u) \ \mathrm{dx} = \sum_{i,j=1}^{n}\int_\Omega -2(\partial_j\Delta u)\partial_j(h_i\partial_i\Delta u) \ \mathrm{dx}$$

$$= \sum_{i,j=1}^{n}\int_\Omega -2(\partial_j h_i)(\partial_j\Delta u)(\partial_i\Delta u) - h_i\partial_i|\partial_j\Delta u|^2 \ \mathrm{dx}$$

$$= \sum_{i,j=1}^{n} \int_\Omega -2(\partial_j h_i)(\partial_j \Delta u)(\partial_i \Delta u) \ \mathrm{dx} + \int_\Omega (\mathrm{div} h)|\nabla \Delta u|^2 \ \mathrm{dx}$$

$$- \int_\Gamma (h \cdot \nu)|\nabla \Delta u|^2 \ \mathrm{d}\Gamma.$$

Since $\Delta u = 0$ on Γ, in the last integral we may write $|\partial_\nu \Delta u|$ instead of $|\nabla \Delta u|$. Using this equality we deduce from (34) that

$$0 = \Big[\int_\Omega 2u'h \cdot \nabla \Delta u \ \mathrm{dx}\Big]_S^T + \int_S^T \int_\Gamma (h \cdot \nu)|\partial_\nu \Delta u|^2 + (h \cdot \nu)|\partial_\nu u'|^2 \ \mathrm{d}\Gamma \ \mathrm{dt}$$

$$+ \int_S^T \int_\Omega (\mathrm{div} h)(|\nabla \Delta u|^2 - |\nabla u'|^2) \ \mathrm{dx} \ \mathrm{dt}$$

$$-2 \int_S^T \int_\Omega \sum_{i,j=1}^{n} (\partial_j h_i)((\partial_j u')(\partial_i u') + (\partial_j \Delta u)(\partial_i \Delta u)) \ \mathrm{dx} \ \mathrm{dt}$$

$$-2 \int_S^T \int_\Omega \sum_{i=1}^{n} u'(\Delta h_i)(\partial_i u') \ \mathrm{dx} \ \mathrm{dt},$$

and (32) follows. ∎

Theorem 2.13. — *Given $T > 0$ arbitrarily, there exists a constant $c = c(T) > 0$ such that for every $(u^0, u^1) \in Z \times Z$ the solution of* (28)–(30) *satisfies the inequality*

$$\int_{-T}^T \int_\Gamma |\partial_\nu u'|^2 + |\partial_\nu \Delta u|^2 \ \mathrm{d}\Gamma \ \mathrm{dt} \le c(\|\nabla \Delta u^0\|^2_{L^2(\Omega)} + \|\nabla u^1\|^2_{L^2(\Omega)}). \tag{35}$$

Proof. — Applying the identity (32) with $S = -T$ and with the vector field of lemma 2.1 corresponding to $\Gamma_0 = \Gamma$, we obtain the estimate

$$\int_{-T}^T \int_\Gamma |\partial_\nu u'|^2 + |\partial_\nu \Delta u|^2 \ \mathrm{d}\Gamma \ \mathrm{dt} \le c_1 \|(u, u')\|^2_{C([-T,T];H^3(\Omega)\times H^1(\Omega))} \tag{36}$$

for some constant c_1. Using lemma 1.7 and theorem 1.1 hence (35) follows. ∎

Proof of theorem 2.10. — It suffices to prove the inequalities (31). Given $(u^0, u^1) \in Z \times Z$ arbitrarily, let us apply the estimate (35) of theorem 2.13 with

$$(A^{-1/2}u^0, A^{-1/2}u^1)$$

instead of (u^0, u^1). We obtain that (see also remark 1.3)

$$\begin{aligned} &\int_{-T}^T \int_\Gamma |\partial_\nu A^{-1/2}u'|^2 + |\partial_\nu \Delta A^{-1/2}u|^2 \ \mathrm{d}\Gamma \ \mathrm{dt} \\ &\le c(\|\nabla \Delta A^{-1/2}u^0\|^2_{L^2(\Omega)} + \|\nabla A^{-1/2}u^1\|^2_{L^2(\Omega)}. \end{aligned} \tag{37}$$

Using lemma 1.7 we have

$$\Delta A^{-1/2}v = -v,$$

$$\|\nabla v\|_{L^2(\Omega)} = \|v\|_{1/4} = \|v\|_{H_0^1(\Omega)}$$

and

$$\|\nabla A^{-1/2}v\|_{L^2(\Omega)} = \|A^{-1/2}v\|_{1/4} = \|v\|_{-1/4} = \|v\|_{H^{-1}(\Omega)}$$

for every $v \in Z$. Therefore we deduce from (37) that

$$\int_{-T}^{T}\int_{\Gamma} |\partial_\nu A^{-1/2}u'|^2 + |\partial_\nu u|^2 \ \mathrm{d}\Gamma \ \mathrm{dt} \leq c\big(\|u^0\|^2_{H_0^1(\Omega)} + \|u^1\|^2_{H^{-1}(\Omega)}\big).$$

Omitting the term $|\partial_\nu A^{-1/2}u'|^2$ we find (31). $\square$

Now let us apply theorem 2.10 for the study of the non-homogeneous problem

$$y'' + \Delta^2 y = 0 \quad \text{in} \quad \Omega \times \mathbb{R} \tag{38}$$

$$y = 0 \quad \text{and} \quad \Delta y = v \quad \text{on} \quad \Gamma \times \mathbb{R} \tag{39}$$

$$y(0) = y^0, \quad y'(0) = y^1. \tag{40}$$

Let $(u^0, u^1) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ and consider the solution of (28)–(30). If $y \in C^4(\overline{\Omega} \times [0,T])$ satisfies (38)–(40), then multiplying (28) by y and integrating by parts we find for every fixed $S \in \mathbb{R}$ that

$$\begin{aligned} 0 &= \int_0^S \int_\Omega (u'' + \Delta^2 u) \ y \ \mathrm{dx} \ \mathrm{dt} \\ &= [\int_\Omega u'y - uy' \ \mathrm{dx}]_0^S + \int_0^S \int_\Omega u(y'' + \Delta^2 y) \ \mathrm{dx} \ \mathrm{dt} \\ &+ \int_0^S \int_\Gamma (\partial_\nu \Delta u)y - (\Delta u)(\partial_\nu y) + (\partial_\nu u)(\Delta y) - u(\partial_\nu \Delta y) \ \mathrm{d}\Gamma \ \mathrm{dt} \\ &= \int_\Omega u'(S)y(S) - u(T)y'(S) + u^0 y^1 - u^1 y^0 \ \mathrm{dx} + \int_0^S \int_\Gamma (\partial_\nu u)v \ \mathrm{d}\Gamma \ \mathrm{dt}. \end{aligned}$$

Setting

$$L_S(u^0, u^1) := \int_0^S \int_\Gamma (\partial_\nu u)v \ \mathrm{d}\Gamma \ \mathrm{dt} + \int_\Omega u^0 y^1 - u^1 y^0 \ \mathrm{dx}$$

and writing for brevity $X := H_0^1(\Omega) \times H^{-1}(\Omega)$ we may rewrite this identity in the following form :

$$L_S(u^0, u^1) = \langle (y'(S), -y(S)), (u(S), u'(S)) \rangle_{X',X}, \quad \forall (u^0, u^1) \in X. \qquad (41)$$

Definition. — We say that (y, y') is a solution of (38)–(40) if $(y, y') \in C(\mathbb{R}; H_0^1(\Omega) \times H^{-1}(\Omega))$ and if (41) is satisfied for all $S \in \mathbb{R}$.

Then we have

Theorem 2.14. — *Let $(y^0, y^1) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ and $v \in L^2_{loc}(\mathbb{R}; L^2(\Gamma))$ be given arbitrarily. Then the problem* (38)–(40) *has a unique solution.*

Furthermore, the linear map $(y^0, y^1, v) \mapsto (y, y')$ is continuous with respect to these topologies.

Proof. — It follows from theorem 2.10 that for every fixed $S \in \mathbb{R}$ the linear form L_S is bounded in $X = H_0^1(\Omega) \times H^{-1}(\Omega)$. Furthermore, it follows from theorem 1.1 that the linear map

$$(u(S), u'(S)) \mapsto (u^0, u^1)$$

is an isomorphism of X onto itself. Hence the linear form

$$(u(S), u'(S)) \mapsto L_S(u^0, u^1)$$

is also bounded on X. Therefore there is a unique pair $(y(S), y'(S)) \in X$ satisfying (41).

The proof now can be completed as that of theorem 2.5 before ; we omit the details. $\square$

3. Uniqueness theorems

The results of this chapter will serve as a basis for the Hilbert Uniqueness Method (HUM) of J.-L. Lions (see the following chapter). In control-theoretic terminology, we shall here establish observability theorems; see remark 3.5 below. These will be obtained (following Ho [1]) by using a special multiplier. The same multiplier has already been used by many authors for different reasons : see e.g. Rellich [1], Pohožaev [1], Lax, Morawetz and Phillips [1], Chen [1].

As in the preceding chapter, we consider the real case only : the complex case then follows easily by applying the results to the real and imaginary parts of the solutions.

We shall use the following notation : for any fixed $x^0 \in \mathbb{R}^n$ we set

$$m(x) := x - x^0 \quad (x \in \mathbb{R}^n), \tag{1}$$

$$R = R(x^0) := \sup_{x\in\Omega} |m(x)|, \tag{2}$$

$$\Gamma_+ := \{x \in \Gamma : m \cdot \nu > 0\}, \tag{3}$$

and we introduce on Γ the (signed) surface measure

$$\mathrm{d}\Gamma_m := (m \cdot \nu)\ \mathrm{d}\Gamma; \tag{4}$$

clearly, we have

$$|\mathrm{d}\Gamma_m| \le R\ \mathrm{d}\Gamma.$$

Let us note the following obvious relations :

$$\partial_i m_j = \delta_{ij} \text{ and } \operatorname{div} m \equiv n. \tag{5}$$

Here and in the rest of this book we shall always mean by an interval a *bounded* interval of strictly positive length.

3.1. The wave equation. Dirichlet condition

Consider the problem of sections 1.3 and 2.2 with $\Gamma_0 = \Gamma$ and $\Gamma_1 = \emptyset$:

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R} \tag{6}$$

$$u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R} \tag{7}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{8}$$

Recall (see theorems 1.1 and 2.2) that for every $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ this problem has a unique solution, whose energy

$$E := \tfrac{1}{2} \int_\Omega |u'|^2 + |\nabla u|^2 + q|u|^2 \, \mathrm{dx} \tag{9}$$

is conserved, and that for every interval I the following (so-called *direct*) inequality holds true :

$$\int_I \int_\Gamma |\partial_\nu u|^2 \, \mathrm{d}\Gamma \, \mathrm{dt} \le cE. \tag{10}$$

The aim of this section is to establish, under some further hypotheses, the *inverse* inequality

$$\int_I \int_\Gamma |\partial_\nu u|^2 \, \mathrm{d}\Gamma \, \mathrm{dt} \ge c'E. \tag{11}$$

Fix $x^0 \in \mathbb{R}^n$ arbitrarily and set $Q := \sup_\Omega q$ and

$$Q_1 := \begin{cases} 2RQ/\sqrt{\lambda_1}, & \text{if } n \ge 2\,; \\ 2RQ/\sqrt{\lambda_1} + Q/\lambda_1, & \text{if } n = 1. \end{cases}$$

(We recall from §1.3 that the function q is supposed to be nonnegative, bounded and measurable.) Let us recall that λ_1 is the biggest constant such that

$$\int_\Omega |\nabla v|^2 + q|v|^2 \, \mathrm{dx} \ge \lambda_1 \int_\Omega |v|^2 \, \mathrm{dx}, \ \forall v \in H_0^1(\Omega). \tag{12}$$

Theorem 3.1. — *Assume that*

$$Q_1 < 1 \tag{13}$$

and let I be an interval of length

$$|I| > 2R/(1 - Q_1). \tag{14}$$

Then there is a constant $c' > 0$ such that

$$\int_I \int_\Gamma |\partial_\nu u|^2 \mathrm{d}\Gamma_m \, \mathrm{dt} \ge c'E, \ \forall (u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega). \tag{15}$$

Clearly the estimate (15) implies (11) (with c'/R instead of c').

For $q \equiv 0$ this theorem was proved by Ho [1] under a stronger condition on the length of I; his condition was weakened by Lions [2], [3], using an indirect compactness-uniqueness argument. The following constructive proof, based on lemma 3.2 below, was first given in Komornik [1].

Proof. — It is sufficient to prove the estimate (15) for $u^0,\ u^1 \in Z$: the general case then follows by a density argument, using theorem 2.2.

Let us write $I = [S,T]$. Applying lemma 2.3 with $h = m$ and using (5) we obtain the identity

$$\begin{aligned}\int_S^T \int_\Gamma 2\partial_\nu u m \cdot \nabla u + (m\cdot\nu)((u')^2 - |\nabla u|^2)\ \mathrm{d}\Gamma\ \mathrm{dt} \\ = \Big[\int_\Omega 2u'm\cdot\nabla u\ \mathrm{dx}\Big]_S^T \\ + \int_S^T \int_\Omega n(u')^2 + (2-n)|\nabla u|^2 + 2qum\cdot\nabla u\ \mathrm{dx}\ \mathrm{dt}.\end{aligned} \tag{16}$$

Let us multiply the equation (6) by u; integrating by parts we obtain

$$\begin{aligned}0 = \int_S^T \int_\Omega u(u'' - \Delta u + qu)\ \mathrm{dx}\ \mathrm{dt} = \Big[\int_\Omega uu'\ \mathrm{dx}\Big]_S^T \\ - \int_S^T \int_\Gamma u\partial_\nu u\ \mathrm{d}\Gamma\ \mathrm{dt} + \int_S^T \int_\Omega -(u')^2 + |\nabla u|^2 + qu^2\ \mathrm{dx}\ \mathrm{dt},\end{aligned}$$

whence

$$\int_S^T \int_\Gamma u\partial_\nu u\ \mathrm{d}\Gamma\ \mathrm{dt} = \Big[\int_\Omega uu'\ \mathrm{dx}\Big]_S^T + \int_S^T \int_\Omega -(u')^2 + |\nabla u|^2 + qu^2\ \mathrm{dx}\ \mathrm{dt}. \tag{17}$$

Putting

$$Mu := 2m\cdot\nabla u + (n-1)u$$

we deduce from (16) and (17) that

$$\begin{aligned}\int_S^T \int_\Gamma (\partial_\nu u)Mu + (m\cdot\nu)((u')^2 - |\nabla u|^2)\ \mathrm{d}\Gamma\ \mathrm{dt} \\ = \Big[\int_\Omega u'Mu\ \mathrm{dx}\Big]_S^T \\ + \int_S^T \int_\Omega (u')^2 + |\nabla u|^2 + (n-1)qu^2 + 2qum\cdot\nabla u\ \mathrm{dx}\ \mathrm{dt}.\end{aligned} \tag{18}$$

Using (9) we may rewrite (18) in the following form :

$$\begin{aligned}\int_S^T \int_\Gamma (\partial_\nu u)Mu + (m\cdot\nu)((u')^2 - |\nabla u|^2)\ \mathrm{d}\Gamma\ \mathrm{dt} \\ = \Big[\int_\Omega u'Mu\ \mathrm{dx}\Big]_S^T + 2|I|E + \int_S^T \int_\Omega (n-2)qu^2 + 2qum\cdot\nabla u\ \mathrm{dx}\ \mathrm{dt}.\end{aligned} \tag{19}$$

By the boundary conditions (7) we have $u = u' = 0$ and $\nabla u = (\partial_\nu u)\nu$ on Γ; this permits to reduce the left-hand side of (19) to $\int_I \int_\Gamma |\partial_\nu u|^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt}$.

On the other hand, using (12) the last integral in (19) is minorized by $-2|I|Q_1 E$. Indeed, the case $q \equiv 0$ is trivial. If $q \not\equiv 0$ and $n \geq 2$, then we have

$$\begin{aligned}\int_\Omega (n-2)qu^2 + 2qum \cdot \nabla u \, \mathrm{dx} &\geq -2RQ\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\ &\geq -Q_1 \int_\Omega |\nabla u|^2 + qu^2 \, \mathrm{dx} \geq -2Q_1 E.\end{aligned}$$

If $q \not\equiv 0$ and $n = 1$, then we have

$$\int_\Omega (n-2)qu^2 + 2qum \cdot \nabla u \, \mathrm{dx} \geq \int_\Omega -Qu^2 + 2qum \cdot \nabla u \, \mathrm{dx};$$

repeating the above computation and using the definition of Q_1 for $n = 1$, this integral is again minorized by $-2Q_1 E$.

Thus we arrive at the following inequality :

$$\int_I \int_\Gamma |\partial_\nu u|^2 \, \mathrm{d}\Gamma_m \mathrm{dt} \geq 2|I|(1 - Q_1)E + \Big[\int_\Omega u' M u \, \mathrm{dx}\Big]_S^T. \tag{20}$$

To minorize the last term of (20) we need the

Lemma 3.2. — *The solution of* (6)–(8) *satisfies the estimate*

$$\Big|\int_\Omega u' M u \, dx\Big| \leq 2RE, \ \forall t \in \mathbb{R}. \tag{21}$$

Proof. — First we show that

$$\|Mu\|_{L^2(\Omega)} \leq \|2m \cdot \nabla u\|_{L^2(\Omega)}. \tag{22}$$

Indeed, the application of Green's formula gives

$$\begin{aligned}&\|Mu\|^2_{L^2(\Omega)} - \|2m \cdot \nabla u\|^2_{L^2(\Omega)} = \|2m \cdot \nabla u + (n-1)u\|^2_{L^2(\Omega)} - \|2m \cdot \nabla u\|^2_{L^2(\Omega)} \\ &= \int_\Omega |2m \cdot \nabla u + (n-1)u|^2 - |2m \cdot \nabla u|^2 \, \mathrm{dx} = \int_\Omega (n-1)^2 u^2 + 4(n-1)um \cdot \nabla u \, \mathrm{dx} \\ &= \int_\Omega (n-1)^2 u^2 + 2(n-1)m \cdot \nabla(u^2) \, \mathrm{dx} \\ &= 2(n-1)\int_\Gamma (m \cdot \nu)u^2 \, \mathrm{d}\Gamma + \int_\Omega (n-1)^2 u^2 - 2(n-1)(\mathrm{div}\, m)u^2 \, \mathrm{dx} \\ &= 2(n-1)\int_\Gamma (m \cdot \nu)u^2 \, \mathrm{d}\Gamma + (1-n^2)\int_\Omega u^2 \, \mathrm{dx} = (1-n^2)\int_\Omega u^2 \, \mathrm{dx} \leq 0.\end{aligned}$$

(We used here (5) and (7).)

It follows from (22) and from the definition of the energy that

$$\begin{aligned}
&\Big|\int_\Omega u' M u \ \mathrm{dx}\Big| \le \|u'\|_{L^2(\Omega)}\|Mu\|_{L^2(\Omega)}\\
&\le \|u'\|_{L^2(\Omega)}\|2m\cdot\nabla u\|_{L^2(\Omega)} \le R\|u'\|^2_{L^2(\Omega)} + \frac{1}{R}\|m\cdot\nabla u\|^2_{L^2(\Omega)}\\
&= \int_\Omega R|u'|^2 + R^{-1}|m\cdot\nabla u|^2 \ \mathrm{dx} \le R\int_\Omega |u'|^2 + |\nabla u|^2 \ \mathrm{dx} \le 2RE,
\end{aligned}$$

proving (21). □

Applying the lemma with $t = S$ and $t = T$ we deduce from (20) that

$$\int_I\int_\Gamma |\partial_\nu u|^2 \ \mathrm{d}\Gamma_m \ \mathrm{dt} \ge 2|I|(1-Q_1)E - 4RE,$$

and the theorem follows with

$$c' := 2|I|(1-Q_1) - 4R. \quad \square$$

Remark 3.3. — Let k an arbitrary positive integer. Let us replace λ_1 by λ_k in the definition of Q_1 and let us denote by Q_k the corresponding quantity. Assume that the conditions (13), (14) are fulfilled with Q_k instead of Q_1. (If I is an arbitrary interval of length $|I| > 2R$, then (13) and (14) are satisfied for a sufficiently large k, but not necessarily for $k = 1$.) Then the estimate (15) is valid (with some constant c' depending on k) for every couple (u^0, u^1) in $H_0^1(\Omega)\times L^2(\Omega)$ satisfying the orthogonality conditions $u^0, u^1 \perp Z_j$ for every $j < k$. To see this it suffices to repeat the proof of the theorem and to use for $v := u(t)$ instead of (12) the inequality

$$\int_\Omega |\nabla v|^2 + q|v|^2 \ \mathrm{dx} \ge \lambda_k \int_\Omega |v|^2 \ \mathrm{dx},$$

which is valid for every $v \in H_0^1(\Omega)$ such that $v \perp Z_j$ for every $j < k$. (This estimate follows easily from the identity (1.10).) Let us recall from remark 1.2 that $u(t)$ satisfies these orthogonality relations.

Remark 3.4. — Theorem 3.1 implies the following uniqueness result : *If the solution of* (6)–(8) *satisfies the condition* $\partial_\nu u = 0$ *on* $\Gamma_+ \times I$ *with* I *satisfying* (13) *and* (14), *then in fact* $u^0 = u^1 = 0$ *and hence* $u \equiv 0$ *on* $\Omega\times\mathbb{R}$.

Remark 3.5. — Theorem 3.1 shows that the "observation" of $\partial_\nu u = 0$ on $\Gamma_+ \times I$ permits one to distinguish the initial data provided I is sufficiently long. (Take the difference of two solutions corresponding to different initial data and apply the preceding remark.)

Remark 3.6. — The optimality of condition (14) in theorem 3.1 and analogous questions are studied by microlocal analysis in Bardos, Lebeau and Rauch [1], [2]; it is related to the finite propagation speed for the wave equation. In some cases the optimality of this condition can be shown easily. Consider for example the special case where Ω is an open ball of radious R, centered at the origin, and assume for simplicity that $q \equiv 0$ and that $I = (-R+\varepsilon, R-\varepsilon)$ for some $\varepsilon > 0$. Fix smooth initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and (following Cazenave [1]) set

$$F(t) := \int_{\varepsilon+|t|<|x|<R} \big((u')^2 + |\nabla u|^2\big)(t,x)\ \mathrm{dx}, \quad t \in I.$$

Then $F \geq 0$ and F has a global maximum at $t = 0$. Indeed, for almost every $t > 0$ we have

$$\begin{aligned} F'(t) &= 2\int_{\varepsilon+t<|x|<R} \big(uu' + \nabla u \cdot \nabla u'\big)(t,x)\ \mathrm{dx} - \int_{\varepsilon+t=|x|} \big((u')^2 + |\nabla u|^2\big)(t,x)\ \mathrm{dx} \\ &= \int_{\varepsilon+t=|x|} \big(2u'\partial_\nu u - (u')^2 - |\nabla u|^2\big)(t,x)\ \mathrm{dx} \leq 0 \end{aligned}$$

because $|\partial_\nu u| \leq |\nabla u|$. For $t < 0$ we obtain similarly that

$$F'(t) = \int_{\varepsilon-t=|x|} \big(2u'\partial_\nu u + (u')^2 + |\nabla u|^2\big)(t,x)\ \mathrm{dx} \geq 0.$$

Now choose non-zero initial data (u^0, u^1) supported by the ball $\{|x| < \varepsilon\}$. Then $F(0) = 0$ and therefore $F \equiv 0$ in I. It follows that $\partial_\nu u \equiv 0$ on $\Gamma \times I$ and therefore the estimate (15) of theorem 3.1 does not hold.

3.2. The first Petrovsky system

Consider the homogeneous problem

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R} \tag{23}$$

$$u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R} \tag{24}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \tag{25}$$

already studied in sections 1.4 and 2.3. Let us recall the theorem 2.6 : for every interval I there exists a constant $c > 0$ such that for every $(u^0, u^1) \in H_0^2(\Omega) \times L^2(\Omega)$ the solution of (23)–(25) satisfies

$$\int_I \int_\Gamma |\Delta u|^2\ \mathrm{d}\Gamma\ \mathrm{dt} \leq cE, \tag{26}$$

where the energy E of the solution is defined by

$$E = \tfrac{1}{2} \int_\Omega |\Delta u|^2 + |u'|^2 \text{ dx}. \tag{27}$$

We are going to prove the following inverse inequality :

Theorem 3.7. — *Let μ_1 denote the first eigenvalue of the problem*

$$\Delta^2 v = -\mu \Delta v, \quad v \in H_0^2(\Omega),$$

and let I be an interval satisfying

$$|I| > 2R\mu_1^{-1/2} \quad \textit{if} \quad n = 1 \quad \textit{and} \quad |I| > R\mu_1^{-1/2} \quad \textit{if} \quad n \geq 2. \tag{28}$$

Then there is a constant $c' > 0$ such that

$$\int_I \int_\Gamma |\Delta u|^2 \mathrm{d}\Gamma_m \text{ dt} \geq c' E, \ \forall (u^0, u^1) \in H_0^2(\Omega) \times L^2(\Omega). \tag{29}$$

Remark 3.8. — To be more precise, μ_1 is the smallest eigenvalue of the operator A associated to the spaces $H = H_0^1(\Omega)$, $\|v\|_H = \|\nabla v\|_{L^2(\Omega)}$ and $V = H_0^2(\Omega)$, $\|v\|_V = \|\Delta v\|_{L^2(\Omega)}$ in the sense of section 1.1. In particular, we have

$$\|\nabla v\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\mu_1}} \|\Delta v\|_{L^2(\Omega)}, \ \forall v \in H_0^2(\Omega) \tag{30}$$

(see (1.10)).

Teorem 3.7 was proved by Lions [3], [4] under a stronger condition on the length of I. His assumption was later weakened in Komornik [1].

Proof of theorem 3.7. — It suffices to prove (29) for $u^0, u^1 \in Z$.

Write $I = [S, T]$ and apply lemma 2.7 with $h = m$. Using (5) we obtain that

$$\begin{aligned} \int_S^T \int_\Gamma |\Delta u|^2 \text{ d}\Gamma_m \text{ dt} &= \Big[\int_\Omega 2u' m \cdot \nabla u \text{ dx} \Big]_S^T \\ &+ \int_S^T \int_\Omega n(u')^2 + (4-n)|\Delta u|^2 \text{ dx dt}. \end{aligned} \tag{31}$$

If $n = 1$, then the last integral is minorized by $2|I|E$, while using (30) and (27) we obtain for every $t \in \mathbb{R}$ that

$$\Big| \int_\Omega 2u' m \cdot \nabla u \text{ dx} \Big| \leq \frac{2R}{\sqrt{\mu_1}} \|u'\|_{L^2(\Omega)} \|\Delta v\|_{L^2(\Omega)} \leq \frac{2R}{\sqrt{\mu_1}} E.$$

Using the conservation of energy hence we conclude that

$$\int_S^T \int_\Gamma |\Delta u|^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt} \geq \Big(2|I| - \frac{4R}{\sqrt{\mu_1}}\Big) E,$$

and (29) follows with $c' = 2|I| - 4R/\sqrt{\mu_1}$.

Henceforth we assume that $n \geq 2$. We multiply the equation (23) by u and we integrate by parts :

$$\begin{aligned} 0 &= \int_S^T \int_\Omega u(u'' + \Delta^2 u) \, \mathrm{dx} \, \mathrm{dt} = \Big[\int_\Omega uu' \, \mathrm{dx}\Big]_S^T \\ &+ \int_S^T \int_\Gamma u(\partial_\nu \Delta u) - (\partial_\nu u)(\Delta u) \, \mathrm{d}\Gamma \, \mathrm{dt} + \int_S^T \int_\Omega -(u')^2 + |\Delta u|^2 \, \mathrm{dx} \, \mathrm{dt} \\ &= \Big[\int_\Omega uu' \, \mathrm{dx}\Big]_S^T + \int_S^T \int_\Omega -(u')^2 + |\Delta u|^2 \, \mathrm{dx} \, \mathrm{dt}. \end{aligned}$$

(We used the boundary conditions (24).) Multiplying this identity by $n-2$ and adding the result to (31) we obtain, putting

$$Mu := 2m \cdot \nabla u + (n-2)u$$

for brevity, that

$$\int_S^T \int_\Gamma |\Delta u|^2 \mathrm{d}\Gamma_m \, \mathrm{dt} = 4|I|E + \Big[\int_\Omega u' Mu \, \mathrm{dx}\Big]_S^T. \tag{32}$$

Lemma 3.9. — *If* $n \geq 2$, *then we have*

$$\Big|\int_\Omega u' Mu \mathrm{dx}\Big| \leq \frac{2R}{\sqrt{\mu_1}} E, \ \forall t \in \mathbb{R}. \tag{33}$$

Proof. — We have (compare with lemma 3.2)

$$\begin{aligned} &\|Mu\|^2_{L^2(\Omega)} - \|2m \cdot \nabla u\|^2_{L^2(\Omega)} = \|2m \cdot \nabla u + (n-2)u\|^2_{L^2(\Omega)} - \|2m \cdot \nabla u\|^2_{L^2(\Omega)} \\ &= \int_\Omega |2m \cdot \nabla u + (n-2)u|^2 - |2m \cdot \nabla u|^2 \, \mathrm{dx} = \int_\Omega (n-2)^2 u^2 + 4(n-2) u m \cdot \nabla u \, \mathrm{dx} \\ &= \int_\Omega (n-2)^2 u^2 + 2(n-2) m \cdot \nabla(u^2) \, \mathrm{dx} \\ &= 2(n-2) \int_\Gamma (m \cdot \nu) u^2 \, \mathrm{d}\Gamma + \int_\Omega (n-2)^2 u^2 - 2(n-2)(\operatorname{div} m) u^2 \, \mathrm{dx} \\ &= 2(n-2) \int_\Gamma (m \cdot \nu) u^2 \, \mathrm{d}\Gamma + (4-n^2) \int_\Omega u^2 \, \mathrm{dx} = (4-n^2) \int_\Omega u^2 \, \mathrm{dx} \leq 0. \end{aligned}$$

Using (27) and (30) hence we deduce that

$$\Big|\int_\Omega u'Mu \ \mathrm{dx}\Big| \le \|u'\|_{L^2(\Omega)}\|Mu\|_{L^2(\Omega)} \le \|u'\|_{L^2(\Omega)}\|2m\cdot\nabla u\|_{L^2(\Omega)}$$
$$\le 2R\|u'\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} \le \frac{2R}{\sqrt{\mu_1}}\|u'\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)} \le \frac{2R}{\sqrt{\mu_1}}E. \quad \square$$

We conclude from (32) and (33) that

$$\int_S^T \int_\Gamma |\Delta u|^2 \ \mathrm{d}\Gamma_m \ \mathrm{dt} \ge \Big(4|I| - \frac{4R}{\sqrt{\mu_1}}\Big)E \tag{34}$$

and (29) follows with $c' = 4|I| - 4R/\sqrt{\mu_1}$. $\square$

Remark 3.10. — Theorem 3.7 implies the following uniqueness result : *If the solution of* (23)–(25) *satisfies the condition* $\Delta u = 0$ *on* $\Gamma \times I$ *for some interval* I *satisfying* (28), *then in fact* $u^0 = u^1 = 0$ *and hence* $u \equiv 0$ *on* $\Omega \times \mathbb{R}$.

Remark 3.11. — Theorem 3.7 shows that the "observation" of $\Delta u = 0$ on $\Gamma \times I$ permits one to distinguish the initial data provided I is sufficiently long.

In connection with the inequality (30) let us note the following result :

Lemma 3.12. — *Let* k *be an arbitrary positive integer. If* $v \in Z$ *and* $v \perp Z_j$ *for every* $j < k$, *then*

$$\|\nabla v\|_{L^2(\Omega)} \le \lambda_k^{-1/4}\|\Delta v\|_{L^2(\Omega)}. \tag{35}$$

(Here the eigenvalues λ_j and the eigenspaces Z_j are those associated with $H = L^2(\Omega)$ and $V = H_0^2(\Omega)$.)

Proof. — Integrating by parts we obtain that

$$\|\nabla v\|^2_{L^2(\Omega)} = -\int_\Omega v\Delta v \ \mathrm{dx} \le \|v\|_{L^2(\Omega)}\|\Delta v\|_{L^2(\Omega)}.$$

The hypothesis on v implies that (we apply (1.10) with H and V as above)

$$\|v\|_{L^2(\Omega)} \le \lambda_k^{-1/2}\|v\|_{H_0^2(\Omega)} (= \lambda_k^{-1/2}\|\Delta v\|_{L^2(\Omega)}).$$

Hence

$$\|\nabla v\|^2_{L^2(\Omega)} \le \lambda_k^{-1/2}\|\Delta v\|^2_{L^2(\Omega)}$$

and (35) follows. $\square$

Remark 3.13. — Let k be a positive integer and assume that condition (28) is fulfilled with $\lambda_k^{-1/4}$ instead of $\mu_1^{-1/2}$. (Observe that every interval, of arbitrarily small length, satisfies this condition for a sufficiently large k.) Then the estimate (29) holds (with some constant c' depending on k) for every couple (u^0, u^1) in $H_0^2(\Omega) \times L^2(\Omega)$ satisfying the orthogonality conditions $u^0, u^1 \perp Z_j$ for every $j < k$. Indeed, it is sufficient to repeat the proof of theorem 3.7 and to apply for $v := u(t)$ the inequality (35) instead of (30). (We recall from remark 1.2 that $u(t)$ satisfies these orthogonality relations.)

3.3. The second Petrovsky system

We consider here the problem

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R} \tag{36}$$

$$u = \Delta u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R} \tag{37}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \tag{38}$$

studied earlier in sections 1.5 and 2.4. We recall from theorem 2.13 the following direct inequality :

$$\int_I \int_\Gamma |\partial_\nu u'|^2 + |\partial_\nu \Delta u|^2 \, \mathrm{d}\Gamma \, \mathrm{dt} \leq cE_{1/4}; \tag{39}$$

it is valid for every interval I and for every solution of (36)–(38) corresponding to the initial data

$$u^0 \in D_{3/4} = \{v \in H^3(\Omega) : v = \Delta v = 0 \quad \text{on} \quad \Gamma\}$$

and

$$u^1 \in D_{1/4} = H_0^1(\Omega).$$

(The constant c does not depend on I.)

Let us recall that the "increased" energy $E_{1/4}$ of the solution is defined by

$$E_{1/4} := \tfrac{1}{2}\|u\|_{3/4}^2 + \tfrac{1}{2}\|u'\|_{1/4}^2 (= \tfrac{1}{2} \int_\Omega |\nabla \Delta u|^2 + |\nabla u'|^2 \, \mathrm{dx}). \tag{40}$$

In order to formulate an inverse result let us fix a point $x^0 \in \mathbb{R}^n$ arbitrarily.

Theorem 3.14. — *Let λ_1 denote the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and let I be an interval of length*

$$|I| > \frac{R}{\sqrt{\lambda_1}}. \tag{41}$$

Then there exists a constant $c' > 0$ *such that*

$$\int_I \int_\Gamma |\partial_\nu u'|^2 + |\partial_\nu \Delta u|^2 \mathrm{d}\Gamma_m \mathrm{~dt} \geq c' E_{1/4}, \ \forall (u^0, u^1) \in D_{3/4} \times D_{1/4}. \tag{42}$$

Remark 3.15. — Here λ_1 is the smallest eigenvalue of the operator A associated to $H := L^2(\Omega)$ and $V := H_0^1(\Omega)$; we have

$$\|v\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\lambda_1}} \|\nabla v\|_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{43}$$

Theorem 3.14 was first proved, under a stronger condition on $|I|$, by Lions [3], [4]. His condition was later weakened in Komornik [1].

Proof of theorem 3.14. — It is sufficient to consider $u^0, u^1 \in Z$. Let us apply lemma 2.12 with $h = m$ where $I = [S, T]$. We obtain

$$\begin{aligned}\int_S^T \int_\Gamma |\partial_\nu u'|^2 + |\partial_\nu \Delta u|^2 \mathrm{~d}\Gamma_m \mathrm{~dt} &= -\Big[\int_\Omega 2u' m \cdot \nabla \Delta u \mathrm{~dx}\Big]_S^T \\ &+ \int_S^T \int_\Omega (2+n)|\nabla u'|^2 + (2-n)|\nabla \Delta u|^2 \mathrm{~dx~dt}.\end{aligned} \tag{44}$$

On the other hand, multiplying the equation (36) by Δu we obtain that

$$\begin{aligned} 0 &= \int_S^T \int_\Omega \Delta u(u'' + \Delta^2 u) \mathrm{~dx~dt} \\ &= \Big[\int_\Omega u' \Delta u \mathrm{~dx}\Big]_S^T + \int_S^T \int_\Gamma (\Delta u)(\partial_\nu \Delta u) \mathrm{~d}\Gamma \mathrm{~dt} \\ &\quad + \int_S^T \int_\Omega -u' \Delta u' - |\nabla \Delta u|^2 \mathrm{~dx~dt} \\ &= \Big[\int_\Omega u' \Delta u \mathrm{~dx}\Big]_S^T + \int_S^T \int_\Omega |\nabla u'|^2 - |\nabla \Delta u|^2 \mathrm{~dx~dt} \\ &\quad + \int_S^T \int_\Gamma (\Delta u)(\partial_\nu \Delta u) - u'(\partial_\nu u') \mathrm{~d}\Gamma \mathrm{~dt} \\ &= \Big[\int_\Omega u' \Delta u \mathrm{~dx}\Big]_S^T + \int_S^T \int_\Omega |\nabla u'|^2 - |\nabla \Delta u|^2 \mathrm{~dx~dt} \end{aligned}$$

because $u' = \Delta u = 0$ on $\Gamma \times \mathbb{R}$ by (37). Using the notation

$$Mu := 2m \cdot \nabla \Delta u + n \Delta u,$$

combining with (44) and using (40) we find that

$$\int_S^T \int_\Gamma |\partial_\nu u'|^2 + |\partial_\nu \Delta u|^2 \ \mathrm{d}\Gamma_m \ \mathrm{dt} = 4|I|E_{1/4} - \Big[\int_\Omega u' M u \ \mathrm{dx}\Big]_S^T. \tag{45}$$

Lemma 3.16. — *We have*

$$\Big|\int_\Omega u' M u \ \mathrm{dx}\Big| \le \frac{2R}{\sqrt{\lambda_1}} E_{1/4}, \quad \forall t \in \mathbb{R}. \tag{46}$$

Proof. — We proceed as in lemmas 3.2 and 3.9. First, we have

$$\|Mu\|_{L^2(\Omega)} \le \|2m \cdot \nabla \Delta u\|_{L^2(\Omega)} \tag{47}$$

because

$$\begin{aligned}
\|Mu\|^2_{L^2(\Omega)} - \|2m\cdot\nabla\Delta u\|^2_{L^2(\Omega)} &= \|2m\cdot\nabla\Delta u + n\Delta u\|^2_{L^2(\Omega)} - \|2m\cdot\nabla\Delta u\|^2_{L^2(\Omega)} \\
&= \int_\Omega |2m\cdot\nabla\Delta u + n\Delta u|^2 - |2m\cdot\nabla\Delta u|^2 \ \mathrm{dx} \\
= \int_\Omega n^2(\Delta u)^2 + 4n(\Delta u) m\cdot\nabla\Delta u \ \mathrm{dx} &= \int_\Omega n^2(\Delta u)^2 + 2nm\cdot\nabla(\Delta u)^2 \ \mathrm{dx} \\
&= 2n\int_\Gamma (m\cdot\nu)(\Delta u)^2 \ \mathrm{d}\Gamma + \int_\Omega (n^2 - 2n \mathrm{div}\ m)(\Delta u)^2 \ \mathrm{dx} \\
&= -n^2 \int_\Omega (\Delta u)^2 \ \mathrm{dx} \le 0.
\end{aligned}$$

Consequently, using (40) and (43) we obtain that

$$\begin{aligned}
\Big|\int_\Omega u' M u \ \mathrm{dx}\Big| &\le \|u'\|_{L^2(\Omega)} \|Mu\|_{L^2(\Omega)} \\
\le \|u'\|_{L^2(\Omega)} \|2m\cdot\nabla\Delta u\|_{L^2(\Omega)} &\le 2R\|u'\|_{L^2(\Omega)} \|\nabla\Delta u\|_{L^2(\Omega)} \\
\le (2R/\sqrt{\lambda_1}) \|\nabla u'\|_{L^2(\Omega)} \|\nabla\Delta u\|_{L^2(\Omega)} &\le (2R/\sqrt{\lambda_1}) E_{1/4}. \quad \square
\end{aligned}$$

We deduce from (45) and (46) the inequality

$$\int_S^T \int_\Gamma |\partial_\nu u'|^2 + |\partial_\nu \Delta u|^2 \ \mathrm{d}\Gamma_m \ \mathrm{dt} \ge 4|I|E_{1/4} - \frac{4R}{\sqrt{\lambda_1}} E_{1/4},$$

and the theorem follows with $c' = 4|I| - 4R/\sqrt{\lambda_1}$. $\square$

Remark 3.17. — Let k be a positive integer and assume that condition (41) is satisfied with λ_k instead of λ_1. (Observe that every interval, arbitrarily small, satisfies this condition for a sufficiently large k.) Then the estimate

(42) holds (with a constant c' depending on k) for every couple (u^0, u^1) satisfying the orthogonality conditions $u^0, u^1 \perp Z_j$ for all $j < k$. Indeed, it is sufficient to repeat the proof of the theorem and to apply for $v := u(t)$ the inequality

$$\|v\|_{L^2(\Omega)} \le \frac{1}{\sqrt{\lambda_k}} \|\nabla v\|_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega),\ v \perp Z_j,\ \forall j < k \tag{48}$$

instead of (43). (Compare with remarks 3.3 and 3.13.)

Remark 3.18. — As before, theorem 3.15 yields some uniqueness and observability results.

3.4. The wave equation. Mixed boundary conditions

Here we consider the problem of section 1.3 with $\Gamma_1 \ne \emptyset$:

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{49}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}, \tag{50}$$

$$\partial_\nu u + au = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}, \tag{51}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{52}$$

In order to avoid some difficulites we shall assume that

$$m \cdot \nu \le 0 \quad \text{on} \quad \Gamma_0, \quad m \cdot \nu \ge 0 \quad \text{on} \quad \Gamma_1, \tag{53}$$

and that a has the form

$$a = (m \cdot \nu) b, \quad b \in C^1(\Gamma_1), \quad b \ge 0 \text{ on } \Gamma_1. \tag{54}$$

We recall that for every couple $(u^0, u^1) \in V \times H$ the problem (49)–(52) has a unique solution whose energy

$$E := \tfrac{1}{2} \int_\Omega |u'|^2 + |\nabla u|^2 + q|u|^2 \,\mathrm{d}x + \tfrac{1}{2} \int_{\Gamma_1} a|u|^2 \,\mathrm{d}\Gamma \tag{55}$$

is conserved.

Remark 3.19. — The conditions (53) and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ (see (1.25)) together impose a very restrictive geometrical property on Ω. It is satisfied if Ω is star-shaped with respect to x^0 or if Ω has the form $\Omega_1 \backslash \overline{\Omega_0}$ with two open sets Ω_1 and Ω_0, both star-shaped with respect to x^0 and such that $\overline{\Omega_0} \subset \Omega_1$. In this case we may choose for Γ_0, Γ_1 the boundaries of Ω_0 and Ω_1, respectively. We shall return to this point later in remark 3.25.

Let us denote by R_1 the smallest positive constant such that

$$\begin{aligned} &4R^2 \int_\Omega |\nabla v|^2 \, \mathrm{dx} + (2n-2) \int_{\Gamma_1} (m\cdot\nu)|v|^2 \, \mathrm{d}\Gamma \\ &\leq 4R_1^2 \Big(\int_\Omega |\nabla v|^2 + q|v|^2 \, \mathrm{dx} + \int_{\Gamma_1} a|v|^2 \, \mathrm{d}\Gamma \Big), \quad \forall v \in V. \end{aligned} \tag{56}$$

Remark 3.20. — If

$$b \geq \frac{n-1}{2R^2} \quad \text{sur} \quad \Gamma_1, \tag{57}$$

then we may choose $R_1 := R$.

We define $Q := \sup_\Omega q$ and

$$Q_k := \begin{cases} 2RQ/\sqrt{\lambda_k}, & \text{si } n \geq 2\,; \\ 2RQ/\sqrt{\lambda_k} + Q/\lambda_k, & \text{si } n = 1 \end{cases} \tag{58}$$

for $k = 1, 2, \ldots$, as in section 3.1. Let us recall that λ_k is here the biggest constant such that

$$\int_\Omega |\nabla v|^2 + q|v|^2 \, \mathrm{dx} + \int_{\Gamma_1} a|v|^2 \, \mathrm{d}\Gamma \geq \lambda_k \int_\Omega |v|^2 \, \mathrm{dx} \tag{59}$$

for every $v \in V$ satisfying the orthogonality conditions $v \perp Z_j$, $\forall j < k$.

We are going to prove the

Proposition 3.21. — *Assume that*

$$Q_1 < 1 \tag{60}$$

and let I be an interval of length

$$|I| > 2R_1/(1-Q_1). \tag{61}$$

Then there exists a constant $c' > 0$ such that

$$\int_I \int_{\Gamma_1} |u|^2 + |u'|^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt} \geq c'E, \quad \forall (u^0, u^1) \in D_1 \times D_{1/2}. \tag{62}$$

Let us recall that $D_{1/2} = V \subset H^1(\Omega)$ and that $D_1 \subset H^2(\Omega) \cap V$ by remark 1.4.

Proof. — It is sufficient to consider the case $(u^0, u^1) \in Z \times Z$. Then the following computations are justified by remark 1.4.

We recall from the proof of theorem 3.1 the identity (18) :

$$\int_S^T \int_\Gamma (\partial_\nu u) Mu + (m \cdot \nu)((u')^2 - |\nabla u|^2) \, \mathrm{d}\Gamma \, \mathrm{dt}$$
$$= \Big[\int_\Omega u' Mu \, \mathrm{dx}\Big]_S^T + \int_S^T \int_\Omega (u')^2 + |\nabla u|^2 + (n-1)qu^2 + 2qum \cdot \nabla u \, \mathrm{dx} \, \mathrm{dt}$$

where

$$Mu = 2m \cdot \nabla u + (n-1)u;$$

using (55) hence we deduce that

$$\int_S^T \int_\Gamma (\partial_\nu u) Mu + (m \cdot \nu)((u')^2 - |\nabla u|^2) \, \mathrm{d}\Gamma \, \mathrm{dt} + \int_I \int_{\Gamma_1} au^2 \, \mathrm{d}\Gamma \, \mathrm{dt}$$
$$= \Big[\int_\Omega u' Mu \, \mathrm{dx}\Big]_S^T + 2|I|E + \int_S^T \int_\Omega (n-2)qu^2 + 2qum \cdot \nabla u \, \mathrm{dx} \, \mathrm{dt}.$$

Using (50), (51) and (54) we obtain

$$\begin{aligned}\int_I \int_{\Gamma_1} (u')^2 - |\nabla u|^2 - 2bu(m \cdot \nabla u) + (2-n)bu^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt} \\ + \int_I \int_{\Gamma_0} (\partial_\nu u)^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt} \\ = \Big[\int_\Omega u' Mu \, \mathrm{dx}\Big]_S^T + 2|I|E + \int_S^T \int_\Omega (n-2)qu^2 + 2qum \cdot \nabla u \, \mathrm{dx} \, \mathrm{dt}.\end{aligned} \tag{63}$$

By (53) the second integral is ≤ 0. Using the inequality

$$-2bu(m \cdot \nabla u) \le 2R|bu||\nabla u| \le |\nabla u|^2 + R^2 b^2 u^2$$

we may majorize the first integral (and therefore the left-hand side of (63)) by

$$\int_I \int_{\Gamma_1} (u')^2 + (R^2 b + 2 - n)bu^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt}.$$

Using (59) and (55) the last integral in (63) is minorized (as in section 3.1) by $-2|I|Q_1 E$. Thus we deduce from (63) the estimate

$$\begin{aligned}\int_I \int_{\Gamma_1} (u')^2 + (R^2 b + 2 - n)bu^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt} \\ \ge 2|I|(1 - Q_1)E + \Big[\int_\Omega u' Mu \, \mathrm{dx}\Big]_S^T.\end{aligned} \tag{64}$$

Lemma 3.22. — *We have the estimate*

$$\Big|\int_\Omega u' Mu\ dx\Big| \le 2R_1 E, \quad \forall t \in \mathbb{R}. \tag{65}$$

Proof. — We have

$$\begin{aligned}
&\int_\Omega |Mu|^2 \text{ dx}\\
&= \int_\Omega |2m\cdot\nabla u + (n-1)u|^2 \text{ dx}\\
&= \int_\Omega |2m\cdot\nabla u|^2 + (n-1)^2u^2 + 4(n-1)um\cdot\nabla u \text{ dx}\\
&= \int_\Omega |2m\cdot\nabla u|^2 + (n-1)^2u^2 + (2n-2)m\cdot\nabla(u^2) \text{ dx}\\
&= \int_\Omega |2m\cdot\nabla u|^2 + (n-1)^2u^2 - n(2n-2)u^2 \text{ dx} + (2n-2)\int_\Gamma (m\cdot\nu)u^2 \text{ d}\Gamma\\
&= \int_\Omega |2m\cdot\nabla u|^2 + (1-n^2)u^2 \text{ dx} + (2n-2)\int_{\Gamma_1} (m\cdot\nu)u^2 \text{ d}\Gamma\\
&\le 4R^2\int_\Omega |\nabla u|^2 \text{ dx} + (2n-2)\int_{\Gamma_1} (m\cdot\nu)u^2 \text{ d}\Gamma\\
&\le 4R_1^2(\int_\Omega |\nabla u|^2 + qv^2 \text{ dx} + \int_{\Gamma_1} (m\cdot\nu)bu^2 \text{ d}\Gamma)
\end{aligned}$$

whence

$$\begin{aligned}
&\int_\Omega |u'Mu| \text{ dx}\\
&\le R_1\int_\Omega (u')^2 \text{ dx} + \frac{4R_1^2}{4R_1}\Big(\int_\Omega |\nabla u|^2 + qu^2 \text{ dx} + \int_{\Gamma_1} (m\cdot\nu)bu^2 \text{ d}\Gamma\Big) = 2R_1E. \quad \square
\end{aligned}$$

Applying lemma 3.22 with $t = S$ and $t = T$ we conclude from (64) that

$$\int_I\int_{\Gamma_1} (u')^2 + (R^2b + 2 - n)bu^2 \text{ d}\Gamma_m \text{ dt} \ge 2|I|(1-Q_1)E - 4R_1E. \tag{66}$$

Let $c_1 \ge 1$ be a majorant of $R^2b^2 + (2-n)b$ on Γ_1, then (62) follows from (66) with

$$c' := \big(2|I|(1-Q_1) - 4R_1\big)/c_1. \quad \square$$

Remark 3.23. — If

$$R^2b^2 + (2-n)b \le 0 \quad \text{on} \quad \Gamma_1, \tag{67}$$

then we may deduce from (66) the stronger estimate

$$\int_I \int_{\Gamma_1} |u'|^2 \mathrm{d}\Gamma_m \ \mathrm{dt} \geq c' E, \quad \forall (u^0, u^1) \in D_1 \times D_{1/2} \tag{68}$$

with

$$c' := 2|I|(1 - Q_1) - 4R_1.$$

Remark 3.24. — Let us denote by R_k $(k = 1, 2 \ldots)$ the smallest positive constant such that

$$\begin{aligned} &4R^2 \int_\Omega |\nabla v|^2 \ \mathrm{dx} + (2n-2) \int_{\Gamma_1} (m \cdot \nu)|v|^2 \ \mathrm{d}\Gamma \\ &\leq 4R_k^2 \Big(\int_\Omega |\nabla v|^2 + q|v|^2 \ \mathrm{dx} + \int_{\Gamma_1} b(m \cdot \nu)|v|^2 \ \mathrm{d}\Gamma \Big) \end{aligned}$$

for every $v \in V$ satisfying the orthogonality conditions $v \perp Z_j$, $\forall j < k$. Let k be a positive integer and assume that the conditions (60) and (61) are satisfied with Q_k, R_k instead of Q_1, R_1. (One can readily verify that for every interval I of length $> 2R$ the conditions (60) and (61) are satified with a sufficiently large k.) Then (62) is valid (with a constant c' depending on k) for every couple (u^0, u^1) satisfying the orthogonality conditions $u^0, u^1 \perp Z_j$, $\forall j < k$. Indeed, it suffices to repeat the above proof and to observe that (by remark 1.2) we may replace Q_1, R_1 in the estimates by Q_k, R_k.

Remark 3.25. — Grisvard [1] proved by a delicate analysis of singularities that in dimension $n \leq 3$ the inequality (62) remains valid without the hypothesis $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. The difficulty comes from the fact that the identity of lemma 2.3 and the identity (18) do not hold any more even if $(u^0, u^1) \in Z \times Z$. (The solutions are not sufficiently smooth to justify the integrations by parts.) We only have inequalities instead of these identities; fortunately, these are still sufficient for the proof of the desired estimates.

Theorem 3.26. — *Assume that*

$$Q_1 < 1, \tag{69}$$

$$R^2 b^2 + (2-n) b \leq 0 \quad \textit{on} \quad \Gamma_1, \tag{70}$$

and let I be an interval of length

$$|I| > 2R_1/(1 - Q_1). \tag{71}$$

Then there is a constant $c' > 0$ *such that*

$$\int_I \int_{\Gamma_1} |u|^2 \mathrm{d}\Gamma_m \, \mathrm{dt} \geq c' E_{-1/2}, \quad \forall (u^0, u^1) \in V \times H.$$

We recall that

$$E_{-1/2} = \tfrac{1}{2}\|u^0\|_H^2 + \tfrac{1}{2}\|u^1\|_{V'}^2.$$

The case $q \equiv 0$ and $b \equiv 0$ of this result is a weakened version of a theorem of Lions [4, p. 200]. We shall prove a stronger result later (see theorem 6.15).

Proof. — It suffices to consider $u^0, u^1 \in Z$. Set $z^0 := -A^{-1}u^1$, $z^1 := u^0$, and apply the estimate (68) to the solution z of (49)–(52) corresponding to the initial data (z^0, z^1) instead of (u^0, u^1). We obtain that

$$\begin{aligned} 2\int_I \int_{\Gamma_1} |z'|^2 \, \mathrm{d}\Gamma \, \mathrm{dt} &\geq c'(\|z^0\|_V^2 + \|z^1\|_H^2) \\ = c'(\|A^{-1}u^1\|_V^2 + \|u^0\|_H^2) &= c'(\|u^0\|_H^2 + \|u^1\|_{V'}^2) = c' E_{-1/2}(u); \end{aligned}$$

it remains to verify that $z' \equiv u$ or, by the uniqueness of the solutions, that z' satisfies (49)–(52). First, (49)–(51) are easily obtained by differentiating with respect to t the analogous equations for z; the differentiation is permitted because $z \in C^\infty(\mathbb{R}; H^2(\Omega))$. Finally, (52) may be verified directly :

$$z'(0) = z^1 = u^0 \quad \text{and} \quad (z')'(0) = (\Delta z - qz)(0) = -Az^0 = u^1. \quad \square$$

4. Exact controllability. Hilbert Uniqueness Method

We present here the Hilbert Uniqueness Method (HUM), introduced by LIONS [3], [4], [5]. As in the preceding chapter, for any fixed $x^0 \in \mathbb{R}^n$ we shall use the notation

$$m(x) := x - x^0, \quad x \in \mathbb{R}^n, \tag{1}$$

$$R = R(x^0) := \sup\{|x - x^0| : x \in \Omega\}, \tag{2}$$

$$\mathrm{d}\Gamma_m := (m \cdot \nu)\ \mathrm{d}\Gamma, \tag{3}$$

$$\Gamma_+ := \{x \in \Gamma : m(x) \cdot \nu(x) > 0\}, \tag{4}$$

and we set

$$\Gamma_- := \{x \in \Gamma : m(x) \cdot \nu(x) \le 0\}. \tag{5}$$

As usual, we consider the real case only ; the complex case then follows easily.

4.1. The wave equation. Dirichlet control

Fix $T > 0$ and consider the problem

$$y'' - \Delta y + qy = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{6}$$

$$y = v \quad \text{on} \quad \Gamma \times (0, T), \tag{7}$$

$$y(0) = y^0 \quad \text{and} \quad y'(0) = y^1. \tag{8}$$

It follows from theorem 2.5 that for any given $y^0 \in L^2(\Omega)$, $y^1 \in H^{-1}(\Omega)$ and $v \in L^2(0, T; L^2(\Gamma))$ the problem (6)–(8) has a unique solution $(y, y') \in C([0, T]; L^2(\Omega) \times H^{-1}(\Omega))$. (Observe that here $\Gamma_0 = \Gamma$ whence $V = H^{-1}(\Omega)$).

DEFINITION. — The problem (6)–(8) is *exactly controllable* if for any given (y^0, y^1), $(y_T^0, y_T^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists $v \in L^2(0, T; L^2(\Gamma))$ such that the solution of (6)–(8) satisfies

$$y(T) = y_T^0 \quad \text{and} \quad y'(T) = y_T^1. \tag{9}$$

As in section 3.1, fix $x^0 \in \mathbb{R}^n$ arbitrarily and set $Q := \sup_\Omega q$,

$$Q_1 := \begin{cases} 2RQ/\sqrt{\lambda_1}, & \text{if } n \ge 2\,; \\ 2RQ/\sqrt{\lambda_1} + Q/\lambda_1, & \text{if } n = 1. \end{cases}$$

Theorem 4.1. — *Assume that*

$$Q_1 < 1, \tag{10}$$

and let

$$T > 2R/(1 - Q_1). \tag{11}$$

Then for any given $(y^0, y^1), (y_T^0, y_T^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *there exists* $v \in L^2(0, T; L^2(\Gamma))$ *such that*

$$v = 0 \quad \text{a.e. on} \quad \Gamma_- \times (0, T) \tag{12}$$

and the solution of (6)–(8) *satisfies*

$$y(T) = y_T^0 \quad \text{and} \quad y'(T) = y_T^1. \tag{13}$$

This result shows that if Ω is contained in a ball of center x^0 and diameter $< T$, then the problem (6)–(8) is exactly controllable, even if we act on Γ_+ only.

Theorem 4.1 is due (for $q \equiv 0$) to Lions [4], [5].

Remark 4.2. — As for the choice of T and Γ_+ very precise conditions were given in Bardos, Lebeau and Rauch [1], using microlocal analysis. See also Cazenave [1], Graham and Russell [1], Joó [1] and Komornik [11] for estimates of T.

Let us consider the solution of the problem

$$\begin{gathered} y_1'' - \Delta y_1 + q y_1 = 0 \quad \text{in} \quad \Omega \times (0, T) \\ y_1 = 0 \quad \text{on} \quad \Gamma \times (0, T) \\ y_1(T) = y_T^0 \quad \text{and} \quad y_1'(T) = y_T^1 \end{gathered}$$

and assume that there exists a unique function $v \in L^2(0, T; L^2(\Gamma))$ satisfying (12) and such that the solution of the problem

$$\begin{gathered} y_2'' - \Delta y_2 + q y_2 = 0 \quad \text{in} \quad \Omega \times (0, T) \\ y_2 = v \quad \text{on} \quad \Gamma \times (0, T) \\ y_2(0) = y^0 - y_1(0) \quad \text{and} \quad y_2'(0) = y^1 - y_1'(0) \end{gathered}$$

satisfies $y_2(T) = y_2'(T) = 0$. Then $y := y_1 + y_2$ is a/the solution of (6)–(8) and it satisfies (13). In view of this remark it is sufficient to prove theorem 4.1 in the special case where $y_T^0 = y_T^1 = 0$.

Henceforth we shall assume that $y_T^0 = y_T^1 = 0$.

The first idea of HUM is to seek a control v in the special form $v = \partial_\nu u$ where u is the solution of the homogeneous problem

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times (0,T) \tag{14}$$

$$u = 0 \quad \text{on} \quad \Gamma \times (0,T) \tag{15}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \tag{16}$$

for a suitable choice of $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. Let us recall (see theorems 1.1 and 2.2) that for any given $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ the problem (14)–(16) has a unique solution, that $\partial_\nu u \in L^2(0,T; L^2(\Gamma))$, and that the linear map $(u^0, u^1) \mapsto \partial_\nu u$ is continuous from $H_0^1(\Omega) \times L^2(\Omega)$ into $L^2(0,T; L^2(\Gamma))$.

Using theorem 2.5 hence we deduce that the second problem

$$y'' - \Delta y + qy = 0 \quad \text{in} \quad \Omega \times (0,T) \tag{17}$$

$$y = \partial_\nu u \quad \text{on} \quad \Gamma_+ \times (0,T) \tag{18}$$

$$y = 0 \quad \text{on} \quad \Gamma_- \times (0,T) \tag{19}$$

$$y(T) = y'(T) = 0 \tag{20}$$

has a unique solution satisfying $(y(0), y'(0)) \in L^2(\Omega) \times H^{-1}(\Omega)$ and that the linear map $(u^0, u^1) \mapsto (y(0), y'(0))$ is continuous from $H_0^1(\Omega) \times L^2(\Omega)$ into $L^2(\Omega) \times H^{-1}(\Omega)$.

If (u^0, u^1) is such that $(y(0), y'(0)) = (y^0, y^1)$, then the control $v := \partial_\nu u$ on Γ_+ and $v = 0$ on Γ_- drives the system (6)–(8) in rest. Thus theorem 4.1 will be proved if we show the *surjectivity* of the map

$$H_0^1(\Omega) \times L^2(\Omega) \ni (u^0, u^1) \mapsto (y(0), y'(0)) \in L^2(\Omega) \times H^{-1}(\Omega).$$

For some technical reasons it is more convenient to study the surjectivity of the map

$$\Lambda : H_0^1(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega) \times L^2(\Omega)$$

defined by

$$\Lambda(u^0, u^1) := (y'(0), -y(0)).$$

Clearly, the two maps are surjective at the same time.

In fact, we shall prove a stronger result :

Lemma 4.3. — *Assume* (10) *and* (11). *Then Λ is an isomorphism of $H_0^1(\Omega) \times L^2(\Omega)$ onto $H^{-1}(\Omega) \times L^2(\Omega)$.*

Proof. — Clearly Λ is a bounded linear map. Applying the Lax-Milgram theorem (see Brézis [2]), it suffices to show the existence of a constant $c > 0$ such that, putting for brevity

$$F := H_0^1(\Omega) \times L^2(\Omega)$$

we have

$$\langle \Lambda(u^0, u^1), (u^0, u^1) \rangle_{F',F} \geq c \|(u^0, u^1)\|_F^2 \tag{21}$$

for every $(u^0, u^1) \in F$. Since $\Lambda : F \to F'$ is continuous and $Z \times Z$ is dense in F, it is sufficient to prove this inequality for $u^0, u^1 \in Z$.

Multiplying the equation (17) by u and integrating by parts we obtain

$$\begin{aligned} 0 = \int_0^T \int_\Omega u(y'' - \Delta y + qy) \, \mathrm{dx} \, \mathrm{dt} &= \Big[\int_\Omega uy' - u'y \, \mathrm{dx} \Big]_0^T \\ + \int_0^T \int_\Omega (u'' - \Delta u + qu) y \, \mathrm{dx} \, \mathrm{dt} &+ \int_0^T \int_\Gamma -u \partial_\nu y + y \partial_\nu u \, \mathrm{d}\Gamma \, \mathrm{dt} \\ = \int_\Omega -u^0 y'(0) + u^1 y(0) \, \mathrm{dx} &+ \int_0^T \int_{\Gamma_+} |\partial_\nu u|^2 \, \mathrm{d}\Gamma \, \mathrm{dt} \end{aligned}$$

whence

$$\langle \Lambda(u^0, u^1), (u^0, u^1) \rangle_{F',F} = \int_0^T \int_{\Gamma_+} |\partial_\nu u|^2 \, \mathrm{d}\Gamma \, \mathrm{dt}.$$

By (10) and (11) we may apply theorem 3.1. We obtain the estimate

$$\langle \Lambda(u^0, u^1), (u^0, u^1) \rangle_{F',F} \geq c' E$$

with a positive constant $c' = c'(T)$. Using the definition (3.5) of the energy hence we deduce (21) with $c := c'/2$. $\square$

Remark 4.4. — HUM is based on the idea that the observability of the homogeneous problem (14)–(16) is *sufficient* for the exact controllability of the non-homogeneous problem (6)–(8). It is useful to observe that the observability is also a *necessary* condition. Indeed, assume that the problem (14)–(16) is *not* observable and fix non-zero initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ such that $\partial_\nu u \equiv 0$ on $\Gamma \times (0, T)$. Choose $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ such that

$$\int_\Omega u^0 y^1 - u^1 y^0 \, \mathrm{dx} \neq 0.$$

Then there is no control $v \in L^2(0,T;L^2(\Gamma))$ such that the solution of (6)–(8) satisfies $y(T) = y'(T) = 0$ on Ω. To show this, fix $v \in L^2(0,T;L^2(\Gamma))$ arbitrarily and multiply (6) by the solution of (14)–(16). Integrating by parts and using (14), (15) and the property $\partial_\nu u \equiv 0$ on $\Gamma \times (0,T)$, we obtain that

$$\begin{aligned} 0 &= \int_0^T \int_\Omega u(y'' - \Delta y + qy) \text{ dx dt} \\ &= \Big[\int_\Omega uy' - u'y \text{ dx}\Big]_0^T - \int_0^T \int_\Gamma u\partial_\nu y - (\partial_\nu u)y \text{ d}\Gamma \text{ dt} \\ &\quad + \int_0^T \int_\Omega (u'' - \Delta u + qu)y \text{ dx dt} \\ &= \Big[\int_\Omega uy' - u'y \text{ dx}\Big]_0^T. \end{aligned}$$

Hence

$$\int_\Omega u(T)y'(T) - u'(T)y(T) \text{ dx} = \int_\Omega u^0y^1 - u^1y^0 \text{ dx} \neq 0$$

and therefore we cannot have $y(T) = y'(T) = 0$ on Ω.

4.2. The first Petrovsky system

Fix $T > 0$ and consider the problem

$$y'' + \Delta^2 y = 0 \quad \text{in} \quad \Omega \times (0,T) \tag{22}$$

$$y = 0 \quad \text{and} \quad \partial_\nu y = v \quad \text{on} \quad \Gamma \times (0,T) \tag{23}$$

$$y(0) = y^0 \quad \text{and} \quad y'(0) = y^1. \tag{24}$$

It follows from theorem 2.9 that for every $(y^0, y^1, v) \in L^2(\Omega) \times H^{-2}(\Omega) \times L^2(0,T;L^2(\Gamma))$ the problem (22)–(24) has a unique solution $(y, y') \in C([0,T]; L^2(\Omega) \times H^{-2}(\Omega))$.

Definition. — The problem (22)–(24) is exactly controllable if for any given (y^0, y^1) and $(y_T^0, y_T^1) \in L^2(\Omega) \times H^{-2}(\Omega)$ there exists $v \in L^2(0,T;L^2(\Gamma))$ such that the solution of (22)–(24) satisfies

$$y(T) = y_T^0 \quad \text{and} \quad y'(T) = y_T^1.$$

Let us denote by μ_1 the first eigenvalue of the problem

$$\Delta^2 v = -\mu\Delta v, \quad v \in H_0^2(\Omega).$$

Fix $x^0 \in \mathbb{R}^n$ arbitrarily.

Theorem 4.5. — *If $T > 2R/\sqrt{\mu_1}$, then for any given (y^0, y^1) and $(y_T^0, y_T^1) \in L^2(\Omega) \times H^{-2}(\Omega)$ there exists $v \in L^2(0,T; L^2(\Gamma))$ such that*

$$v = 0 \quad a.e.\ on \quad \Gamma_- \times (0,T) \tag{25}$$

and that the solution of (22)–(24) *satisfies*

$$y(T) = y_T^0 \quad and \quad y'(T) = y_T^1.$$

In particular, if Ω is contained in some ball of diameter $< T\sqrt{\mu_1}$, then the problem (22)–(24) is exactly controllable.

Theorem 4.5 was first proved by Lions [4], [5] under a stronger assumption on T; this assumption was weakened in Komornik [1].

Using an indirect compactness-uniqueness argument, Zuazua [1] later proved that these results remain valid in fact for arbitrarily small $T > 0$. We shall prove his results in a constructive way in chapter 6 (see theorem 6.8).

Proof of theorem 4.5. — Using the same argument as in the preceding section, we may assume that $y_T^0 = y_T^1 = 0$.

Fix $(u^0, u^1) \in H_0^2(\Omega) \times L^2(\Omega)$ arbitrarily. Solve the problem

$$\begin{gathered}
u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times (0,T),\\
u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma \times (0,T),\\
u(0) = u^0 \quad \text{and} \quad u'(0) = u^1,
\end{gathered}$$

and then the problem

$$\begin{gathered}
y'' + \Delta^2 y = 0 \quad \text{in} \quad \Omega \times (0,T),\\
y = 0 \quad \text{and} \quad \partial_\nu y = \Delta u \quad \text{on} \quad \Gamma_+ \times (0,T),\\
y = \partial_\nu y = 0 \quad \text{on} \quad \Gamma_- \times (0,T),\\
y(T) = y'(T) = 0.
\end{gathered}$$

It follows from theorems 1.1, 2.6 and 2.9 that the formula

$$\Lambda(u^0, u^1) := (y'(0), -y(0))$$

defines a linear and continuous map of $F := H_0^2(\Omega) \times L^2(\Omega)$ into F'. It is sufficient to prove that Λ is surjective. We establish a stronger result :

Lemma 4.6. — *If $T > 2R/\sqrt{\mu_1}$, Then Λ is an isomorphism of F onto F'.*

PROOF. — Applying the Lax-Milgram theorem, using the continuity of Λ and the density of $Z \times Z$ in F it suffices to prove the estimate

$$\langle \Lambda(u^0, u^1), (u^0, u^1) \rangle_{F',F} \geq c \|(u^0, u^1)\|_F^2 \tag{26}$$

for every $(u^0, u^1) \in Z \times Z$, with a suitable positive constant c.

First we observe that the right-hand side of (26) is equal to $2cE$. Furthermore, the left-hand side of (26) equals $\int_0^T \int_{\Gamma_+} |\Delta u|^2 \, \mathrm{d}\Gamma \, \mathrm{dt}$. Indeed, we have

$$\begin{aligned} 0 &= \int_0^T \int_\Omega u(y'' + \Delta^2 y) \, \mathrm{dx} \, \mathrm{dt} \\ &= \int_0^T \int_\Omega (u'' + \Delta^2 u) y \, \mathrm{dx} \, \mathrm{dt} + \Big[\int_\Omega u y' - u' y \, \mathrm{dx} \Big]_0^T \\ &\quad + \int_0^T \int_\Gamma u(\partial_\nu \Delta y) - (\partial_\nu u)(\Delta y) + (\Delta u)(\partial_\nu y) - (\partial_\nu \Delta u) y \, \mathrm{d}\Gamma \, \mathrm{dt} \\ &= \int_\Omega -u^0 y'(0) + u^1 y(0) \, \mathrm{dx} + \int_0^T \int_{\Gamma_+} (\Delta u)^2 \, \mathrm{d}\Gamma \, \mathrm{dt} \end{aligned}$$

whence

$$\langle \Lambda(u^0, u^1), (u^0, u^1) \rangle_{F',F} = \int_0^T \int_{\Gamma_+} (\Delta u)^2 \, \mathrm{d}\Gamma \, \mathrm{dt}.$$

It remains to prove the estimate

$$\int_0^T \int_{\Gamma_+} (\Delta u)^2 \, \mathrm{d}\Gamma \, \mathrm{dt} \geq 2cE, \tag{27}$$

and this follows from theorem 3.7 and from the hypothesis $T > 2R/\sqrt{\mu_1}$. $\square$

4.3. The wave equation. Neumann or Robin control

Consider the problem

$$y'' - \Delta y + qy = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{28}$$

$$y = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}, \tag{29}$$

$$\partial_\nu y + ay = v \quad \text{on} \quad \Gamma_1 \times \mathbb{R}, \tag{30}$$

$$y(0) = y^0 \quad \text{and} \quad y'(0) = y^1 \quad \text{in} \quad \Omega \tag{31}$$

with $\Gamma_1 \neq \emptyset$.

We begin by *defininig* the solution of (28)–(31). Now we do not have suitable hidden regularity results leading to optimal existence and uniqueness results expressed in terms of the usual Sobolev spaces; therefore the transposition (or duality) method will provide less precise existence results for the weak solutions.

As usual, we begin with a formal computation. Fix $(u^0, u^1) \in V \times H$ arbitrarily, solve the problem (cf. section 1.3)

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{32}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}, \tag{33}$$

$$\partial_\nu u + au = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}, \tag{34}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{in} \quad \Omega, \tag{35}$$

and multiply (32) by the solution of (28)–(31). We obtain for every fixed $S \in \mathbb{R}$ the equality

$$\begin{aligned} 0 &= \int_0^S \int_\Omega (u'' - \Delta u + qu)\, y \text{ dx dt} = \Big[\int_\Omega u'y - uy' \text{ dx} \Big]_0^S \\ &- \int_0^S \int_\Gamma (\partial_\nu u)y - u(\partial_\nu y) \text{ d}\Gamma \text{ dt} + \int_0^S \int_\Omega u(y'' - \Delta y + qy) \text{ dx dt} \\ &= \int_\Omega u'(S)y(S) - u(S)y'(S) + u^0 y^1 - u^1 y^0 \text{ dx} + \int_0^T \int_{\Gamma_1} uv \text{ d}\Gamma \text{ dt}. \end{aligned}$$

Putting

$$L_S(u^0, u^1) := \int_0^S \int_{\Gamma_1} uv \text{ d}\Gamma \text{ dt} + \langle (y^1, -y^0), (u^0, u^1) \rangle_{V' \times H, V \times H}$$

we may rewrite this identity as

$$L_S(u^0, u^1) = \langle (y'(S), -y(S)), (u(S), u'(S)) \rangle_{V' \times H, V \times H}, \quad \forall (u^0, u^1) \in V \times H. \tag{36}$$

This leads to the following *definition* :

Definition. — We say that (y, y') is a solution of (28)–(31) if $(y, y') \in C(\mathbb{R}; H \times V')$ and if (36) is satisfied for every $S \in \mathbb{R}$.

We have the

Theorem 4.7. — *Given $(y^0, y^1) \in H \times V'$ and $v \in L^2_{loc}(\mathbb{R}; L^2(\Gamma_1))$ arbitrarily, the problem (28)–(31) has a unique solution $(y, y') \in C(\mathbb{R}; H \times V')$.*

Furthermore, the linear map $(y^0, y^1, v) \mapsto (y, y')$ is continuous with respect to these topologies.

PROOF. — We may easily adapt the proof of theorem 2.5 with one modification : to show the continuity of the linear form L_S now we apply the standard trace theorem $(V \subset) H^1(\Omega) \hookrightarrow L^2(\Gamma)$. $\square$

Now assume, as in section 3.4, that there exists $x^0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \leq 0 \quad \text{on} \quad \Gamma_0, \quad m \cdot \nu \geq 0 \quad \text{on} \quad \Gamma_1, \tag{37}$$

that a has the form

$$a = (m \cdot \nu)b, \quad b \in C^1(\Gamma_1), \quad b \geq 0 \quad \text{on} \quad \Gamma_1, \tag{38}$$

and that

$$R^2 b^2 + (2-n)b \leq 0 \quad \text{on} \quad \Gamma_1. \tag{39}$$

Let us introduce the constants R_1 and Q_1 as in section 3.4 and assume that

$$Q_1 < 1. \tag{40}$$

THEOREM 4.8. — *Assume* (37)–(40) *and let*

$$T > 2R_1/(1-Q_1). \tag{41}$$

Then for any given $(y^0, y^1), (y_T^0, y_T^1) \in V \times H$ *there exists a control* $v \in L^2(0,T;L^2(\Gamma_1))$ *such that* (*extending* v *by zero outside* $(0,T)$) *the solution of* (28)–(31) *satisfies*

$$y(T) = y_T^0 \quad \textit{and} \quad y'(T) = y_T^1.$$

For $\Gamma_0 \neq \emptyset$ and $q \equiv 0$ we obtain a weakened version of a theorem of LIONS [4; p. 203] : he proved this result (using indirect compactness–uniqueness arguments) with R_1 replaced by R in (41). Theorem 4.8 will be improved later (see theorems 6.20 and 8.10).

PROOF. — As usual, we may assume that $y_T^0 = y_T^1 \equiv 0$. We apply HUM.

Consider on $Z \times Z$ the seminorm defined by

$$\|(u^0, u^1)\|_F := \|u\|_{L^2(0,T;L^2(\Gamma_1))}$$

where u is the solution of (32)–(35). By theorem 3.26 it is a *norm*.

Completing $Z \times Z$ with respect to this norm we obtain a Hilbert space F and we deduce from the trace theorem $V \hookrightarrow L^2(\Gamma)$ and from theorem 3.26 the algebraical and topological inclusions

$$V \times H \subset F \subset H \times V'. \tag{42}$$

Given $(u^0, u^1) \in F$ arbitrarily, we solve (32)–(35), then we solve

$$y'' - \Delta y + qy = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{43}$$

$$y = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}, \tag{44}$$

$$\partial_\nu y + ay = -u \quad \text{on} \quad \Gamma_1 \times \mathbb{R}, \tag{45}$$

$$y(T) = y'(T) = 0 \tag{46}$$

and we set

$$\Lambda(u^0, u^1) = (y'(0), -y(0)).$$

By the definition of F and by theorem 4.7 Λ is a bounded linear map of F into $H \times V'$. The proof will be completed if we show that F' is contained in the range of Λ. Indeed, then we will conclude from (42) that $H \times V \subset F'$. In fact, the following stronger result holds true :

Lemma 4.9. — *Λ is an isomorphism of F onto F'.*

Proof. — Compute

$$\langle \Lambda(u^0, u^1), (v^0, v^1) \rangle_{V' \times H, V \times H}$$

for $(u^0, u^1), (v^0, v^1) \in Z \times Z$. Denoting by u, v the corresponding solutions of (32)–(35) and considering the corresponding solution y of (43)–(46), we have

$$0 = \int_0^T \int_\Omega (y'' - \Delta y + qy) v \text{ dx dt} = \Big[\int_\Omega y'v - yv' \text{ dx} \Big]_0^T$$

$$- \int_0^T \int_\Gamma (\partial_\nu y) v - y \partial_\nu v \text{ d}\Gamma \text{ dt} + \int_0^T \int_\Omega y(v'' - \Delta v + qv) \text{ dx dt}$$

$$= \int_\Omega y'(T)v(T) - y(T)v'(T) - y'(0)v(0) + y(0)v('0) \text{ dx} + \int_0^T \int_{\Gamma_1} uv \text{ d}\Gamma \text{ dt}$$

$$= \int_\Omega -y'(0)v(0) + y(0)v'(0) \text{ dx} + \int_0^T \int_{\Gamma_1} uv \text{ d}\Gamma \text{ dt}$$

whence

$$\langle \Lambda(u^0, u^1), (v^0, v^1) \rangle_{V' \times H, V \times H} = (u, v)_{L^2(0,T;L^2(\Gamma_1))},$$

i.e.

$$\langle \Lambda(u^0, u^1), (v^0, v^1) \rangle_{V' \times H, V \times H} = ((u^0, u^1), (v^0, v^1))_F. \tag{47}$$

We conclude from (47) that $\Lambda(u^0, u^1) \in F'$ and that

$$\| \Lambda(u^0, u^1) \|_{F'} = \| (u^0, u^1) \|_F.$$

Consequently, Λ is a bounded linear map of F into F'. By (47) we may apply the Lax-Milgram theorem and the proof is completed. □

Remark 4.10. — Let us recall from remark 3.20 that in certain cases we have $R_1 \leq R$.

5. Norm inequalities

The aim of this chapter is to introduce a general method, which will permit us in the following chapter to improve and complete the uniqueness and exact controllability results obtained earlier, by weakening the hypotheses on the length of the intervals of uniqueness and on the sufficient time of exact controllability.

This method is closely related to an estimation method set out by Haraux [3]. In section 5.4 we shall also apply an idea of Lebeau [1].

Throughout this chapter all spaces are assumed to be *complex*.

In section 5.1 we outline the main ideas in the case of the first Petrovsky system. The precise results will be formulated and proved in sections 5.2–5.4.

5.1. Riesz sequences

Let $(\omega_j)_{j\ge1}$ be a sequence of distinct real numbers and consider the functions of the form

$$u(t) := \sum_{j=1}^{\infty} z_j e^{i\omega_j t}, \quad t \in \mathbb{R} \tag{1}$$

with complex scalar coefficients z_j. We say that $\left(e^{i\omega_j t}\right)_{j\ge1}$ is a *Riesz sequence* on $(0,T)$ if there exist two positive constants $c_1(T)$ and $c_2(T)$ such that

$$c_1(T)\sum_{j=1}^{\infty}|z_j|^2 \le \int_0^T |u(t)|^2 \,\mathrm{dt} \le c_2(T)\sum_{j=1}^{\infty}|z_j|^2 \tag{2}$$

for every function u of the form (1).

It is clear that if $\left(e^{i\omega_j t}\right)_{j\ge1}$ is a *Riesz sequence* on $(0,T)$, then for every $k>1$ the subsequence $\left(e^{i\omega_j t}\right)_{j\ge k}$ is also a Riesz sequence on $(0,T)$. Conversely, the following standard result holds true (see e.g. Ball and Slemrod [1] or Haraux [3]) :

Let $k>1$ and assume that the subsequence $\left(e^{i\omega_j t}\right)_{j\ge k}$ is a Riesz sequence on some interval $(0,T_k)$. Then the entire sequence $\left(e^{i\omega_j t}\right)_{j\ge1}$ is a Riesz sequence on $(0,T)$ for all $T>T_k$, arbitrarily close to T_k.

Now consider the problem

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega\times\mathbb{R} \tag{3}$$

$$u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma\times\mathbb{R} \tag{4}$$

$$u(0)=u^0 \quad \text{and} \quad u'(0)=u^1; \tag{5}$$

the energy E of the solution is defined by

$$E = \tfrac{1}{2} \int_\Omega |\Delta u|^2 + |u'|^2 \ \mathrm{dx}. \tag{6}$$

Given $T > 2R\lambda_1^{-1/4}$ arbitrarily, by the results of sections 2.3 and 3.2 (see theorems 2.6, 3.6 and remark 3.12) there exist two constants $c_1(T)$, $c_2(T)$ such that for every $u^0, u^1 \in Z$ the solution of (3)–(5) satisfies the estimates

$$c_1(T)E \le \int_0^T \int_{\Gamma_+} |\Delta u|^2 \ \mathrm{d}\Gamma \ \mathrm{dt} \le c_2(T)E. \tag{7}$$

Let us recall from section 4.2 that the first inequality in (7) implies the exact controllability of the first Petrovsky system in time T.

In fact, the estimates (7) hold for arbitrarily small $T > 0$. To convince ourselves, set

$$\omega_{2j-1} = \sqrt{\lambda_j},\ \omega_{2j} = -\sqrt{\lambda_j},\ \widetilde{Z}_{2j-1} = \widetilde{Z}_{2j} = Z_j,\ j \ge 1,$$

and introduce in Z the norm

$$\|v\| := \Big(\int_\Omega |\Delta v|^2 \ \mathrm{dx} \Big)^{1/2}$$

and the *semi-norm*

$$|v| := \Big(\int_{\Gamma_+} |\Delta v|^2 \ \mathrm{dx} \Big)^{1/2}.$$

Then the solution of (3)–(5) may be written in the form (1) (see theorem 1.3) with *vector* coefficients $z_j \in \widetilde{Z}_j$, and one can readily verify that

$$E = \sum_{j=1}^{\infty} \|z_j\|^2. \tag{8}$$

Consequently, we may rewrite the estimates (7) in a form analogous to (2) :

$$c_1(T) \sum_{j=1}^{\infty} \|z_j\|^2 \le \int_0^T |u(t)|^2 \ \mathrm{dt} \le c_2(T) \sum_{j=1}^{\infty} \|z_j\|^2. \tag{9}$$

By analogy with the scalar case, let us say that $\big(e^{i\omega_j t}, \widetilde{Z}_j\big)_{j\ge1}$ is a *vector Riesz sequence* on $(0, T)$ if the estimates (9) are satisfied.

Now fix $T > 0$ arbitrarily. Choose $k > 1$ such that $T > 2R\lambda_k^{-1/4}$ and then choose T_k such that $T > T_k > 2R\lambda_k^{-1/4}$. By theorems 2.6, 3.6 and remark 3.12 the subsequence $\left(e^{i\omega_j t}, \widetilde{Z}_j\right)_{j\geq 2k-1}$ is a vector Riesz sequence on $(0, T_k)$: observe that

$$z_1 = \cdots = z_{2k-2} = 0 \Longleftrightarrow u^0, u^1 \perp Z_1, \cdots, Z_{k-1}. \tag{10}$$

Applying a generalization of the above mentioned scalar result, hence we will conclude that the entire sequence $\left(e^{i\omega_j t}\right)_{j\geq 1}$, $(\widetilde{Z}_j)_{j\geq 1}$ is a vector Riesz sequence on $(0, T)$ (see theorem 5.2 below) ; in other words, the estimates (7) hold for arbitrarily small $T > 0$.

5.2. Formulation of the results

Let $\mathcal{A}$ be a linear operator in an infinite-dimensional *complex* Hilbert space $\mathcal{H}$. Assume that $\mathcal{A}$ has an infinite sequence of purely imaginary eigenvalues $i\omega_j$ ($\omega_j \in \mathbb{R}$, $j = 1, 2, \ldots$) satisfying

$$|\omega_j| \to +\infty \quad \text{as} \quad j \to +\infty \tag{11}$$

and a corresponding sequence of finite-dimensional, pairwise orthogonal eigenspaces $\mathcal{Z}_j$ whose linear hull $\mathcal{Z}$ is dense in $\mathcal{H}$. Then for any given $U^0 \in \mathcal{Z}$ the initial value problem

$$U' = \mathcal{A}U \quad \text{in} \quad \mathbb{R}, \quad U(0) = U^0 \tag{12}$$

has a unique solution $U \in C^\infty(\mathbb{R}; \mathcal{H})$ and this solution has a unique expansion of the form

$$U(t) = \sum_j U_j e^{i\omega_j t}, \quad U_j \in \mathcal{Z}_j \tag{13}$$

where finitely many coefficients U_j are different from zero only. It follows that

$$\|U(t)\|_{\mathcal{H}} = \|U^0\|_{\mathcal{H}}, \quad \forall t \in \mathbb{R}. \tag{14}$$

(Our hypotheses mean that $\mathcal{A}$ has a skew-adjoint extension (i.e. $i\mathcal{A}$ is self-adjoint) in $\mathcal{H}$, having a compact resolvent.)

Example 5.1. — Consider the abstract problem

$$u'' + Au = 0 \quad \text{in} \quad \mathbb{R}, \quad u(0) = u^0, \quad u'(0) = u^1 \tag{15}$$

introduced in section 1.2. We can rewrite it in the form (12) by setting

$$\mathcal{A}(u,v) = (v, -Au), \quad U = (u, u'), \quad \text{and} \quad U^0 = (u^0, u^1).$$

Introducing the Hilbert space $\mathcal{H} = V \times H$ and the sequences (ω_j), $(\mathcal{Z}_j)$ by

$$\omega_{2k-1} = \sqrt{\lambda_k}, \quad \mathcal{Z}_{2k-1} = \{(v, i\omega_{2k-1}v) : v \in Z_k\}$$

and

$$\omega_{2k} = -\sqrt{\lambda_k}, \quad \mathcal{Z}_{2k} = \{(v, i\omega_{2k}v) : v \in Z_k\}$$

for $k = 1, 2, \ldots$, the above mentioned conditions can be easily verified. Furthermore, the energy of the solutions of (15) is closely related to the norm of the "energy space" $\mathcal{H}$:

$$\|U^0\|_{\mathcal{H}} = \|(u^0, u^1)\|_{\mathcal{H}} = 2E.$$

Let us observe that for any fixed $U^0 = (u^0, u^1) \in \mathcal{Z}$ and for any positive integer n the following properties are equivalent :

$$U^0 \perp \mathcal{Z}_j \quad \text{in } \mathcal{H} \text{ for} \quad j = 1, \ldots, 2n-2,$$
$$u^0, u^1 \perp Z_j \quad \text{in } H \text{ (or in } V) \text{ for} \quad j = 1, \ldots, n-1.$$

Indeed, writing

$$U^0 = (u^0, u^1) = \sum_k (v_k, i\omega_{2k-1}v_k) + (w_k, i\omega_{2k}w_k), \quad v_k, w_k \in Z_k,$$

for every fixed $m \geq 1$ we have the following equivalences :

$$u^0, u^1 \perp Z_m \Longleftrightarrow v_m + w_m = v_m - w_m = 0 \Longleftrightarrow v_m = w_m = 0,$$
$$U^0 \perp \mathcal{Z}_{2m-1}, Z_{2m} \Longleftrightarrow v_m = w_m = 0.$$

Now let p be a *semi-norm* in $\mathcal{Z}$. The following theorem will play a crucial role in the following chapter.

Theorem 5.2. — *Assume that p is a* norm *in each of the eigenspaces $\mathcal{Z}_j$:*

$$U \in \mathcal{Z}_j \quad \textit{and} \quad p(U) = 0 \Longrightarrow U = 0 \quad (j = 1, 2, \ldots). \tag{16}$$

Assume that there exist an integer $k > 1$, two intervals I_1, I_2 and two positive constants c_1, c_2 such that the solutions of (12) *satisfy the inequalities*

$$\int_{I_1} p(U(t))^2 \,\mathrm{dt} \geq c_1 \|U^0\|^2_{\mathcal{H}} \tag{17}$$

and

$$\int_{I_2} p(U(t))^2 \ \mathrm{dt} \le c_2 \|U^0\|_{\mathcal{H}}^2 \tag{18}$$

whenever

$$U^0 \perp \mathcal{Z}_1, \ldots, \mathcal{Z}_{k-1}. \tag{19}$$

Then for every interval I of length $|I| > |I_1|$ there exist two positive constants c_3 and c_4 such that

$$c_3 \|U^0\|_{\mathcal{H}}^2 \le \int_I p(U(t))^2 \ \mathrm{dt} \le c_4 \|U^0\|_{\mathcal{H}}^2 \tag{20}$$

for all $U^0 \in \mathcal{Z}$.

In fact, we shall prove the following more general result :

Theorem 5.3. — *Assume that there exist an integer $k > 1$, two intervals I_1, I_2, a semi-norm q in $\mathcal{Z}$ and three positive constants c_0, c_1, c_2 such that*

$$U \in \mathcal{Z}_j \quad \text{and} \quad q(U) = 0 \Longrightarrow U = 0 \quad (j = 1, 2, \ldots), \tag{21}$$

$$q \le c_0 p \quad \text{in} \quad \mathcal{Z} \tag{22}$$

and that the solutions of (12) *satisfy the inequalities*

$$\int_{I_1} p(U(t))^2 \ \mathrm{dt} \ge c_1 \|U^0\|_{\mathcal{H}}^2 \tag{23}$$

and

$$\int_{I_2} q(U(t))^2 \ \mathrm{dt} \le c_2 \|U^0\|_{\mathcal{H}}^2 \tag{24}$$

whenever

$$U^0 \perp \mathcal{Z}_1, \ldots, \mathcal{Z}_{k-1}. \tag{25}$$

Then for every interval I of length $|I| > |I_1|$ there exists a positive constant c_3 such that

$$\int_I p(U(t))^2 \ \mathrm{dt} \ge c_3 \|U^0\|_{\mathcal{H}}^2 \tag{26}$$

and for every interval I there exists a positive constant c_4 such that

$$\int_I q(U(t))^2 \ \mathrm{dt} \le c_4 \|U^0\|_{\mathcal{H}}^2 \tag{27}$$

for all $U^0 \in \mathcal{Z}$.

For $q = p$ this theorem reduces to theorem 5.2.

Remark 5.4. — The following remark is often useful to verify the condition (21) or (16). Assume that for some given j there is an interval I' and a positive constant c' such that

$$\int_{I'} q(U(t))^2 \ \mathrm{dt} \geq c' \|U^0\|^2_{\mathcal{H}} \tag{28}$$

for all $U^0 \in \mathcal{Z}_j$. Then q is a *norm* in $\mathcal{Z}_j$. Indeed, if $U^0 \in \mathcal{Z}_j$ then the corresponding solution of (12) is

$$U(t) = U^0 e^{i\omega_j t}$$

and therefore

$$|I'| q(U^0)^2 = \int_{I'} q(U(t))^2 \ \mathrm{dt} \geq c' \|U^0\|^2_{\mathcal{H}}.$$

Hence $U^0 \neq 0$ implies $q(U^0) \neq 0$.

Sometimes we shall find I' and c' satisfying (28) for *all* $U^0 \in \mathcal{H}$. However, the interval I' will be much longer than I_1 in theorems 5.2 and 5.3, hence it will not replace (17) or (23).

We shall also prove a euclidean version of theorem 5.3. Let p be a *euclidean* semi-norm in $\mathcal{Z}$ (i.e. defined by a positive semi-definite hermitian bilinear form $p(\cdot,\cdot)$ in $\mathcal{Z}$ such that $p(U) = p(U,U)^{1/2}$ for all $U \in \mathcal{Z}$) and assume that there exist two positive constants α and c_α such that

$$p(U) \leq c_\alpha |\omega_j|^\alpha \|U\|_{\mathcal{H}}, \quad \forall U \in \mathcal{Z}_j, \quad \forall \omega_j \neq 0. \tag{29}$$

We also need the following assumption on the spectrum of $\mathcal{A}$: there is a positive constant d satisfying

$$\sum_{j=1}^{\infty} |\omega_j|^{-d} < +\infty. \tag{30}$$

Let us denote by $\mathcal{Z}_+$ (resp. by $\mathcal{Z}_-$) the linear hull of the eigenspaces $\mathcal{Z}_j$ corresponding to the eigenvalues $i\omega_j$ with $\omega_j \geq 0$ (resp. $\omega_j < 0$). Clearly we have

$$\mathcal{Z}_+ \perp \mathcal{Z}_- \quad \text{and} \quad \mathcal{Z} = \mathcal{Z}_+ + \mathcal{Z}_-. \tag{31}$$

Theorem 5.5. — *Assume* (30) *and let p be a euclidean semi-norm in $\mathcal{Z}$ satisfying* (29). *Assume that there exist an integer $l > 1$, two intervals I_1', I_2, a semi-norm q in $\mathcal{Z}$ and three positive constants c_0, c_1' and c_2' satisfying* (21), (22) *and such that the solutions of* (12) *satisfy the inequalities*

$$\int_{I_1'} p(U(t))^2 \, \mathrm{dt} \geq c_1' \|U^0\|_{\mathcal{H}}^2 \tag{32}$$

and

$$\int_{I_2} q(U(t))^2 \, \mathrm{dt} \leq c_2' \|U^0\|_{\mathcal{H}}^2 \tag{33}$$

whenever

$$U^0 \perp \mathcal{Z}_1, \ldots, \mathcal{Z}_{l-1} \tag{34}$$

and

$$\textit{either} \quad U^0 \in \mathcal{Z}_+ \quad \textit{or} \quad U^0 \in \mathcal{Z}_-. \tag{35}$$

Then for every interval I of length $|I| > |I_1'|$ there exist two positive constants c_3 and c_4 such that

$$\int_I p(U(t))^2 \, \mathrm{dt} \geq c_3 \|U^0\|_{\mathcal{H}}^2 \tag{36}$$

and

$$\int_I q(U(t))^2 \, \mathrm{dt} \leq c_4 \|U^0\|_{\mathcal{H}}^2 \tag{37}$$

for all $U^0 \in \mathcal{Z}$.

The rest of this chapter is devoted to the proof of theorems 5.3 and 5.5.

5.3. Proof of theorem 5.3

First of all, we may assume by an obvious induction argument that $k = 2$. Furthermore, we may assume that $\omega_1 = 0$. To see this let us consider the initial value problem

$$V' = (\mathcal{A} - i\omega_1 \mathcal{I})V \quad \text{in} \quad \mathbb{R}, \quad V(0) = U^0. \tag{38}$$

The solutions of (12) and (38) are clearly connected by the relation

$$V(t) \equiv U(t) e^{-i\omega_1 t}.$$

Since

$$p(V(t)) \equiv p(U(t)) \quad \text{and} \quad q(V(t)) \equiv q(U(t))$$

the inequalities (23), (24), (26), (27) of theorem 5.3 are the same for $U(t)$ and for $V(t)$. Furthermore, the eigenvalues of $\mathcal{A} - i\omega_1\mathcal{I}$ are those of $\mathcal{A}$ shifted by $-i\omega_1$; in particular the eigenvalue $i\omega_1$ of $\mathcal{A}$ corresponds to the zero eigenvalue of $\mathcal{A} - i\omega_1\mathcal{I}$.

Step 1. — First we establish a weakened version of inequality (26) : there is a positive constant c' such that

$$\int_I p(U(t))^2 \, \mathrm{dt} \geq c' \|U^0 - U_1\|_{\mathcal{H}}^2 \tag{39}$$

for all $U^0 \in \mathcal{Z}$ where U_1 denotes the orthogonal projection of U^0 onto $\mathcal{Z}_1$. (Since $\omega_1 = 0$, U_1 is also the orthogonal projection of $U(t)$ onto $\mathcal{Z}_1$ for every $t \in \mathbb{R}$. In other words, U_1 is the constant part of $U(t)$, cf. (13).)

Let us first note that inequality (23) remains true (with the same constant c_1) for every translate $I_1 + \tau$ of I_1. Indeed, set

$$V(t) := U(t+\tau), \quad t \in \mathbb{R}, \tag{40}$$

then V is the solution of (12) with U^0 replaced by $U(\tau)$. Using (23) and (14) we have

$$\int_{I_1+\tau} p(U(t))^2 \, \mathrm{dt} = \int_{I_1} p(V(t))^2 \, \mathrm{dt} \geq c_1 \|V(0)\|_{\mathcal{H}}^2 = c_1 \|U(\tau)\|_{\mathcal{H}}^2 = c_1 \|U^0\|_{\mathcal{H}}^2.$$

Since I is longer than I_1, we may therefore assume that I contains the closure of I_1 in its interior, say

$$I_1 = (a, b) \quad \text{and} \quad (a-\varepsilon, b+\varepsilon) \subset I, \quad \varepsilon > 0. \tag{41}$$

Now fix $U^0 \in \mathcal{Z}$ arbitrarily, solve (12) and (following Haraux [3]) set

$$V(t) := U(t) - \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} U(t+s) \, \mathrm{ds}, \; t \in \mathbb{R}. \tag{42}$$

Observe that

$$\int_{I_1} p(V(t))^2 \, \mathrm{dt} \leq 4 \int_I p(U(t))^2 \, \mathrm{dt}. \tag{43}$$

Indeed, we have

$$\begin{aligned} p(V(t))^2 &\leq 2p(U(t))^2 + 2p\Big(\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} U(t+s) \, \mathrm{ds}\Big)^2 \\ &\leq 2p(U(t))^2 + \frac{1}{2\varepsilon^2} \Big(\int_{-\varepsilon}^{\varepsilon} p(U(t+s)) \, \mathrm{ds}\Big)^2 \\ &\leq 2p(U(t))^2 + \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} p(U(t+s))^2 \, \mathrm{ds} \end{aligned}$$

for every $t \in \mathbb{R}$, whence, using (41) and the Fubini theorem,

$$\int_{I_1} p(V(t))^2 \ \mathrm{dt} \leq 2 \int_{I_1} p(U(t))^2 \ \mathrm{dt} + \frac{1}{\varepsilon} \int_{I_1} \int_{-\varepsilon}^{\varepsilon} p(U(t+s))^2 \ \mathrm{ds\ dt}$$
$$\leq 2 \int_{I_1} p(U(t))^2 \ \mathrm{dt} + 2 \int_{a-\varepsilon}^{b+\varepsilon} p(U(t'))^2 \ \mathrm{dt'} \leq 4 \int_{I} p(U(t))^2 \ \mathrm{dt}.$$

Using the expansion (13) of $U(t)$ one computes easily that

$$V(t) = \sum_j V_j e^{i\omega_j t}, \quad V_j \in \mathcal{Z}_j$$

with

$$V_1 = 0 \quad \text{and} \quad V_j = \Big(1 - \frac{\sin \omega_j \varepsilon}{\omega_j \varepsilon}\Big) U_j \quad \text{if} \quad j > 1. \tag{44}$$

Hence V is the solution of (12) with U^0 replaced by $V^0 := \sum_j V_j$, which satisfies the orthogonality condition (25). Applying (23) we obtain that

$$\int_{I_1} p(V(t))^2 \ \mathrm{dt} \geq c_1 \|V^0\|^2_{\mathcal{H}}. \tag{45}$$

Since $\omega_j \neq 0 (= \omega_1)$ for $j \neq 1$ and since $|\omega_j| \to +\infty$ as $j \to +\infty$ (cf. (11)), there exists a positive constant α such that

$$1 - \frac{\sin \omega_j \varepsilon}{\omega_j \varepsilon} \geq \alpha, \quad \forall j \geq 2. \tag{46}$$

Using the orthogonality of the eigenspaces $\mathcal{Z}_j$ we deduce from (44) and (46) that

$$\|V^0\|_{\mathcal{H}} \geq \alpha \|U^0 - U_1\|_{\mathcal{H}} \tag{47}$$

and (39) follows from (43), (45) and (47) with $c' := \alpha^2 c_1/4$.

Step 2. — Next we prove the (easy) estimate (27) for an *arbitrary* interval I. (Its length does not play any role here.) Let us first note that the estimate (24) remains true (with another constant) without the orthogonality assumption (25). Indeed, since the square root of the left hand side of (24) defines a semi-norm of U^0 and since this semi-norm is obviously majorized by the norm of $\mathcal{H}$ on the *finite-dimensional* vector space $\mathcal{Z}_1$, there exists a positive constant c_2' such that

$$\int_{I_2} q(U_1)^2 \ \mathrm{dt} \leq c_2' \|U_1\|^2_{\mathcal{H}}, \quad \forall U_1 \in \mathcal{Z}_1.$$

We may assume that $c_2' \geq c_2$. Now, given $U^0 \in \mathcal{Z}$ arbitrarily, the (constant) solution U_1 of (12) with the initial value U_1 (cf. (13)) satisfies the above estimate while the solution $U(t) - U_1$ of (12) with the initial value $U^0 - U_1$ satisfies the estimate (24) by assumption. Hence, using the triangle inequality and then the orthogonality of the eigenspaces, we have

$$\begin{aligned}\int_{I_2} q(U(t))^2 \, \mathrm{dt} &\leq 2 \int_{I_2} q(U_1)^2 \, \mathrm{dt} + 2 \int_{I_2} q(U(t) - U_1)^2 \, \mathrm{dt} \\ &\leq 2c_2' \|U_1\|_{\mathcal{H}}^2 + 2c_2 \|U^0 - U_1\|_{\mathcal{H}}^2 \leq 2c_2' \|U^0\|_{\mathcal{H}}^2.\end{aligned}$$

Therefore (24) is satisfied for all $U^0 \in \mathcal{Z}$ if we replace c_2 by $2c_2'$.

If I is a translate of I_2, say $I = I_2 + \tau$, then (27) is true with $c_4 = c_2$. Indeed, set

$$V(t) := U(t + \tau), \quad t \in \mathbb{R},$$

then V is the solution of (12) with U^0 replaced by $U(\tau)$. Using (24) and (14) we have

$$\int_I q(U(t))^2 \, \mathrm{dt} = \int_{I_2} q(V(t))^2 \, \mathrm{dt} \leq c_2 \|V(0)\|_{\mathcal{H}}^2 = c_2 \|U(\tau)\|_{\mathcal{H}}^2 = c_2 \|U^0\|_{\mathcal{H}}^2.$$

In the general case let us cover I with a finite number of translates of I_2, say

$$I \subset \cup_{k=1}^m (I_2 + \tau_k).$$

Then we have

$$\int_I q(U(t))^2 \, \mathrm{dt} \leq \sum_{k=1}^m \int_{I_2 + \tau_k} q(U(t))^2 \, \mathrm{dt} \leq \sum_{k=1}^m c_2 \|U^0\|_{\mathcal{H}}^2 = mc_2 \|U^0\|_{\mathcal{H}}^2$$

i.e. (27) is satisfied with $c_4 = mc_2$.

Step 3. — Now we establish the estimate (26). Since $\mathcal{Z}_1$ is finite-dimensional, by assumption (21) there exists a positive constant c satisfying

$$\|U_1\|_{\mathcal{H}}^2 \leq c \int_I q(U_1)^2 \, \mathrm{dt} (= c|I| q(U_1)^2), \quad \forall U_1 \in \mathcal{Z}_1.$$

Using this inequality we have for any given $U^0 \in \mathcal{Z}$ the following estimate :

$$\begin{aligned}\|U^0\|_{\mathcal{H}}^2 &= \|U_1\|_{\mathcal{H}}^2 + \|U^0 - U_1\|_{\mathcal{H}}^2 \leq c \int_I q(U_1)^2 \, \mathrm{dt} + \|U^0 - U_1\|_{\mathcal{H}}^2 \\ &\leq 2c \int_I q(U(t))^2 \, \mathrm{dt} + 2c \int_I q(U(t) - U_1)^2 \, \mathrm{dt} + \|U^0 - U_1\|_{\mathcal{H}}^2.\end{aligned}$$

Using (22), (27) and (39) to majorize the right-hand side of this estimate we obtain (26) :

$$\begin{aligned}\|U^0\|_{\mathcal{H}}^2 &\le 2cc_0^2 \int_I p(U(t))^2 \ \mathrm{dt} + (1+2cc_2)\|U^0 - U_1\|_{\mathcal{H}}^2 \\ &\le \left(2cc_0^2 + (1+2cc_4)/c'\right) \int_I p(U(t))^2 \ \mathrm{dt}. \quad \square\end{aligned}$$

5.4. Proof of theorem 5.5

Let us fix an interval I_1 containing the closure of I_1' in its interior and such that $|I_1'| < |I_1|$. In view of theorem 5.3 it is sufficient to prove that conditions (23), (24) of theorem 5.3 are satisfied with a suitable integer k and with suitable constants c_1, c_2. Let us fix an integer $k \ge l$, to be chosen later.

Condition (24) follows easily from (33). Indeed, given $U^0 \in \mathcal{Z}$ satisfying (25), let us denote by U_+^0 (resp. by U_-^0) its orthogonal projection onto $\mathcal{Z}_+$ (resp. onto $\mathcal{Z}_-$). Then U_+^0 and U_-^0 satisfy (34) and (35). Observing that the corresponding solutions U_+, U_- of (12) satisfy $U = U_+ + U_-$ and applying inequality (33) for U_+, U_- we obtain easily the estimate (24) with $c_2 = 2c_2'$:

$$\begin{aligned}\int_{I_2} q(U(t))^2 \ \mathrm{dt} &\le 2\int_{I_2} q(U_+(t))^2 \ \mathrm{dt} + 2\int_{I_2} q(U_-(t))^2 \ \mathrm{dt} \\ &\le 2c_2'\left(\|U_+^0\|_{\mathcal{H}}^2 + \|U_-^0\|_{\mathcal{H}}^2\right) = 2c_2'\|U^0\|_{\mathcal{H}}^2.\end{aligned}$$

Turning to the proof of (23) let us choose , following Lebeau [1], an even function $\varphi : \mathbb{R} \to \mathbb{R}$ of class C^∞, satisfying the following conditions :

$$0 \le \varphi \le 1 \quad \text{in} \quad \mathbb{R}, \ \varphi = 1 \quad \text{in} \quad I_1', \ \varphi = 0 \quad \text{in} \quad \mathbb{R}\setminus I_1. \tag{48}$$

Then $\varphi \in \mathcal{S}(\mathbb{R})$ and its Fourier transform

$$\hat{\varphi}(x) = \int_{-\infty}^{+\infty} \varphi(t) e^{ixt} \ \mathrm{dt}, \ x \in \mathbb{R}$$

is also an even function belonging to $\mathcal{S}(\mathbb{R})$. Hence there is a constant $c_5 > 0$ such that

$$|\hat{\varphi}(x)| = |\hat{\varphi}(-x)| \le c_5|x|^{-2\alpha-d}, \ \forall x \in \mathbb{R}\setminus\{0\}. \tag{49}$$

Now given $U^0 \in \mathcal{Z}$ satisfying (25), let us introduce U_+, U_-, $U_+(t)$ and $U_-(t)$ as above. Using (48), (32) and (29) we have

$$\begin{aligned}
\int_{I_1} p(U(t))^2 \ \mathrm{dt} &\ge \int_{-\infty}^{+\infty} \varphi(t) p(U(t))^2 \ \mathrm{dt} \\
&= \int_{-\infty}^{+\infty} \varphi(t) p(U_+(t))^2 \ \mathrm{dt} + \int_{-\infty}^{+\infty} \varphi(t) p(U_-(t))^2 \ \mathrm{dt} \\
&+ \int_{-\infty}^{+\infty} \varphi(t) p(U_+(t), U_-(t)) \ \mathrm{dt} + \int_{-\infty}^{+\infty} \varphi(t) p(U_-(t), U_+(t)) \ \mathrm{dt} \\
&\ge c_1' \|U_+^0\|_{\mathcal{H}}^2 + c_1' \|U_-^0\|_{\mathcal{H}}^2 + \sum_{\omega_j \ge 0} \sum_{\omega_i < 0} \hat{\varphi}(\omega_j - \omega_i) \Big(p(U_j, U_i) + p(U_i, U_j) \Big) \\
&\ge c_1' \|U^0\|_{\mathcal{H}}^2 - \sum_{\omega_j \ge 0} \sum_{\omega_i < 0} |\hat{\varphi}(\omega_j - \omega_i)| \Big(p(U_j)^2 + p(U_i)^2 \Big) \\
&\ge c_1' \|U^0\|_{\mathcal{H}}^2 - c_\alpha^2 \sum_{\omega_j \ge 0} \sum_{\omega_i < 0} |\hat{\varphi}(\omega_j - \omega_i)| \Big(|\omega_j|^{2\alpha} \|U_j\|_{\mathcal{H}}^2 + |\omega_i|^{2\alpha} \|U_i\|_{\mathcal{H}}^2 \Big)
\end{aligned}$$

yielding

$$\begin{aligned}
\int_{I_1} p(U(t))^2 \ \mathrm{dt} \ge &\sum_{\omega_j \ge 0} \Big(c_1' - c_\alpha^2 |\omega_j|^{2\alpha} \sum_{\omega_i < 0} |\hat{\varphi}(\omega_j - \omega_i)| \Big) \|U_j\|_{\mathcal{H}}^2 \\
&+ \sum_{\omega_i < 0} \Big(c_1' - c_\alpha^2 |\omega_i|^{2\alpha} \sum_{\omega_j \ge 0} |\hat{\varphi}(\omega_j - \omega_i)| \Big) \|U_i\|_{\mathcal{H}}^2 .
\end{aligned} \tag{50}$$

Using (49) we have

$$\begin{aligned}
&|\omega_j|^{2\alpha} \sum_{\omega_i < 0, i \ge k} |\hat{\varphi}(\omega_j - \omega_i)| \\
&\le c_5 |\omega_j|^{2\alpha} \sum_{\omega_i < 0, i \ge k} (|\omega_j| + |\omega_i|)^{-2\alpha - d} \le c_5 \sum_{i \ge k} |\omega_i|^{-d}
\end{aligned}$$

whenever $\omega_j \ge 0$ and analogously

$$|\omega_i|^{2\alpha} \sum_{\omega_j > 0, i \ge k} |\hat{\varphi}(\omega_j - \omega_i)| \le c_5 \sum_{j \ge k} |\omega_j|^{-d}$$

whenever $\omega_i < 0$. Substituting these inequalities into (50) we find that

$$\int_{I_1} p(U(t))^2 \ \mathrm{dt} \ge \Big(c_1' - c_\alpha^2 c_5 \sum_{j \ge k} |\omega_j|^{-d} \Big) \|U^0\|_{\mathcal{H}}^2 =: c_1 \|U^0\|_{\mathcal{H}}^2 . \tag{51}$$

By (30) the coefficient c_1 in (51) is positive if we choose a sufficiently large integer k. Then (23) follows from (51). $\square$

6. New uniqueness and exact controllability results

The aim of this chapter is to complete and improve the results of chapters 3 and 4 by applying the theorems of the preceding chapter. In particular, we give simple and constructive proofs of certain theorems of LIONS [4] and ZUAZUA [1], originally proved by applying compactness arguments combined with Holmgren type unique continuation theorems. Our approach is general and may be applied in cases where Holmgren type theorems are not available, e.g. for equations with non-analytic coefficients. In other cases we only need *elliptic* unique continuation theorems but not deeper ones concerning evolutionary problems.

6.1. A unique continuation theorem

We begin by recalling the following standard result of CARLEMAN [1] :

THEOREM 6.1. — *Let G be an open domain in $\mathbb{R}^n$, $V \in L^\infty(G)$, $u \in H^2(G)$, and assume that*

$$-\Delta u + Vu = 0 \quad in \quad G. \tag{1}$$

Assume that $u \equiv 0$ in some neighbourhood of a point $x' \in G$. Then $u \equiv 0$ in G.

See e.g. GAROFALO ET LIN [1], [2] for a short proof based on the multiplier method.

COROLLARY 6.2. — *Let Ω be a bounded open domain in $\mathbb{R}^n$ with a boundary Γ of class C^2 and let B be an (arbitrarily small) open ball such that*

$$\Gamma \cap B \neq \emptyset. \tag{2}$$

Let $V \in L^\infty(\Omega)$, $u \in H^2(\Omega)$ and assume that

$$-\Delta u + Vu = 0 \quad in \quad \Omega$$

and

$$u = \partial_\nu u = 0 \quad on \quad \Gamma \cap B. \tag{3}$$

Then $u \equiv 0$ in Ω.

Proof. — Set $G := \Omega \cup B$ and define $V(x) = u(x) = 0$ for $x \in B \backslash \Omega$. Clearly, we have $V \in L^\infty(G)$; it is sufficient to verify that $u \in H^2(G)$. Indeed, the we can conclude by applying theorem 6.1 with an arbitrary point $x' \in B \backslash \overline{\Omega}$.

Let us denote by g_i, g_{ij} the extensions by zero to G of the functions $\partial_i u$, $\partial_i \partial_j u$, $i, j = 1, \ldots, n$. Then $g_i, g_{ij} \in L^2(G)$ and it suffices the show that

$$\int_G u(\partial_j \varphi) \mathrm{\,dx} = -\int_G g_j \varphi \mathrm{\,dx}, \quad \forall \varphi \in \mathcal{D}(G)$$

and

$$\int_G (\partial_i u)(\partial_j \varphi) \mathrm{\,dx} = -\int_G g_{ij} \varphi \mathrm{\,dx}, \quad \forall \varphi \in \mathcal{D}(G).$$

These are obtained easily by integration by parts and by using the following properties : $\partial_i u \equiv 0$ and $\partial_i \partial_j u \equiv 0$ outside of Ω, $\varphi \equiv 0$ on $\Gamma \backslash (\Gamma \cap B) (\subset \partial G)$, and $u = \partial_i u \equiv 0$ on $\Gamma \cap B$ (by (3)). We have

$$\begin{aligned} \int_G u(\partial_j \varphi) \mathrm{\,dx} &= \int_\Omega u(\partial_j \varphi) \mathrm{\,dx} = \int_\Gamma u \varphi \nu_j \mathrm{\,d}\Gamma - \int_\Omega (\partial_j u) \varphi \mathrm{\,dx} \\ &= \int_{\Gamma \cap B} u \varphi \nu_j \mathrm{\,d}\Gamma - \int_\Omega (\partial_j u) \varphi \mathrm{\,dx} = -\int_\Omega (\partial_j u) \varphi \mathrm{\,dx} = -\int_G g_j \varphi \mathrm{\,dx} \end{aligned}$$

and

$$\begin{aligned} \int_G (\partial_i u)(\partial_j \varphi) \mathrm{\,dx} &= \int_\Omega (\partial_i u)(\partial_j \varphi) \mathrm{\,dx} = \int_\Gamma (\partial_i u) \varphi \nu_j \mathrm{\,d}\Gamma - \int_\Omega (\partial_j \partial_i u) \varphi \mathrm{\,dx} \\ &= \int_{\Gamma \cap B} (\partial_i u) \varphi \nu_j \mathrm{\,d}\Gamma - \int_\Omega (\partial_j \partial_i u) \varphi \mathrm{\,dx} = -\int_\Omega (\partial_j \partial_i u) \varphi \mathrm{\,dx} = -\int_G g_{ij} \varphi \mathrm{\,dx}. \quad \square \end{aligned}$$

Corollary 6.3. — *Let Ω be a bounded open domain with a boundary Γ of class C^2 and let $V \in L^\infty(\Omega)$. Fix $x^0 \in \mathbb{R}^n$ arbitrarily and set*

$$\Gamma_+ := \{x \in \Gamma : m(x) \cdot \nu(x) > 0\}$$

as usual. Let $u \in H^2(\Omega)$ and assume that

$$-\Delta u + Vu = 0 \quad in \quad \Omega$$

and

$$u = \partial_\nu u = 0 \quad on \quad \Gamma_+.$$

Then $u \equiv 0$ in Ω.

Proof. — In view of the preceding corollary it is sufficient to find a ball B satisfying $\Gamma \cap B \subset \Gamma_+$.

By the compactness of Γ there exists a point $x^1 \in \Gamma$ such that

$$|x^0 - x^1| = \max_{\Gamma} |x^0 - x|.$$

It is clear that $m(x^1) \cdot \nu(x^1) > 0$. Since Ω is of class C^0 (in fact of class C^2), there exists an open ball B of center x^1 such that $\Gamma \cap B$ is connected. If we choose the radious of B sufficiently small, then we also have $m \cdot \nu > 0$ on $\Gamma \cap B$ whence $\Gamma \cap B \subset \Gamma_+$. $\square$

6.2. The wave equation. Dirichlet condition

Let us return to the probem studied in section 3.1. First we shall apply theorem 5.2 in order to obtain a variant of the uniqueness theorem 3.1 concerning the problem

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{4}$$

$$u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}, \tag{5}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{6}$$

Fix $x^0 \in \mathbb{R}^n$ arbitrarily.

Theorem 6.4. — *Let I be an interval of length $|I| > 2R$. There exists a constant $c' > 0$ such that the solution of (4)–(6) satisfies*

$$\int_I \int_{\Gamma_+} |\partial_\nu u|^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt} \geq c'E, \ \forall (u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega). \tag{7}$$

If $\Gamma_+ \neq \Gamma$, then the estimate (7) is slightly weaker than the inequality

$$\int_I \int_{\Gamma} |\partial_\nu u|^2 \mathrm{d}\Gamma_m \, \mathrm{dt} \geq c'E$$

of theorem 3.1, because $\mathrm{d}\Gamma_m \leq 0$ on Γ_- ; nevertheless, this weaker estimate is still sufficient to prove the exact controllability of the corresponding problem : see theorem 6.5 below. On the other hand, and this is the key point, the condition $|I| > 2R$ is weaker than the condition $(1 - Q_1)|I| > 2R$ of theorem 3.1.

Proof. — We are going to apply theorem 5.2 with

$$H = L^2(\Omega), \quad V = H_0^1(\Omega) \text{ and } p(v^0, v^1) := \Big(\int_{\Gamma_+} |\partial_\nu v^0|^2 \, \mathrm{d}\Gamma_m \Big)^{1/2};$$

cf. example 5.1. Condition (5.18) of theorem 5.2 is fulfilled by theorem 2.2, regardless of the choice of k and for every interval I_2, even if U^0 does not satisfy any orthogonality condition of the type (5.19).

Now fix $l > 1$ such that

$$Q_l < 1 \quad \text{and} \quad |I| > 2R/(1 - Q_l) \tag{8}$$

and set $k = 2l - 1$; then we have

$$\{\omega_1, \ldots, \omega_{k-1}\} = \{\pm\sqrt{\lambda_1}, \ldots, \pm\sqrt{\lambda_{l-1}}\}.$$

We deduce from theorem 3.1 and remark 3.3 that condition (5.17) is fulfilled for every interval I_1 of length

$$|I_1| > 2R/(1 - Q_l).$$

By (8) we may choose I_1 such that $|I_1| < |I|$.

If $Q_1 < 1$, then we conclude from theorem 3.1 that condition (5.28) (for $q = p$) is fulfilled for every interval I' of length

$$|I'| > 2R/(1 - Q_1).$$

This implies (5.16). If $Q_1 \geq 1$, then we verify (5.16) directly. Let $U = (v, \pm i\sqrt{\lambda}v)$ be an eigenvector of $\mathcal{A}$ satisfying $p(U) = 0$. Then we deduce from the definition of $\mathcal{A}$ that

$$-\Delta v + qv = \lambda v \quad \text{in} \quad \Omega,$$
$$v = 0 \quad \text{on} \quad \Gamma$$

and

$$\partial_\nu v = 0 \quad \text{on} \quad \Gamma_+.$$

Applying corollary 6.3 we conclude that $v \equiv 0$ and therefore $U \equiv 0$.

Since the estimate (7) is equivalent to the first inequality in (5.20), we conclude by applying theorem 5.2. $\square$

Theorem 6.4 leads to a strong improvement of theorem 4.1 concerning the exact controllability of the problem

$$y'' - \Delta y + qy = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{9}$$
$$y = v \quad \text{on} \quad \Gamma \times (0, T), \tag{10}$$
$$y(0) = y^0 \quad \text{and} \quad y'(0) = y^1 \tag{11}$$

by eliminating the condition (4.10) on q and by weakening the condition (4.11) on T :

Theorem 6.5. — *Let $T > 2R$. Given (y^0, y^1), $(y_T^0, y_T^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ arbitrarily, there exists $v \in L^2(0,T; L^2(\Gamma))$ such that*

$$v = 0 \quad \textit{a.e. on} \quad \Gamma_- \times (0,T)$$

and the solution of (9)–(11) *satisfies*

$$y(T) = y_T^0 \quad \textit{and} \quad y'(T) = y_T^1.$$

Proof. — We may repeat the proof of theorem 4.1 by applying theorem 6.4 instead of theorem 3.1. □

Remark 6.6. — It is easy to give a formal recipe leading to the condition $T > 2R$ in theorem 6.5. Assume for simplicity that $n \geq 2$. Then the corresponding condition (4.11) of theorem 4.1 may be written explicitly as

$$T > \frac{2R}{1 - 2RQ\lambda_1^{-1/2}}.$$

Letting $\lambda_1 \to +\infty$ we obtain the weaker condition $T > 2R$.

6.3. The first Petrovsky system

Now consider the problem

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{12}$$

$$u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}, \tag{13}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{14}$$

We begin by establishing a variant of the uniqueness theorem 3.7 by eliminating hypothesis (3.28) on the length of the interval I.

Theorem 6.7. — *For every interval I there exists a constant $c' > 0$ such that*

$$\int_I \int_{\Gamma_+} |\Delta u|^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt} \geq c'E, \ \forall (u^0, u^1) \in H_0^2(\Omega) \times L^2(\Omega). \tag{15}$$

Proof. — Fix an interval I arbitrarily. We are going to apply theorem 5.2 with

$$H = L^2(\Omega), \quad V = H_0^2(\Omega) \text{ and } p(v^0, v^1) := \Big(\int_{\Gamma_+} |\Delta v^0|^2 \, \mathrm{d}\Gamma_m \Big)^{1/2}$$

(see example 5.1). Since (15) is equivalent to the first inequality in (5.20), it is sufficient to verify the properties (5.17), (5.18) and (5.28) (for $q = p$) with a suitable integer $k > 1$ and three intervals I_1, I_2, I' such that $|I_1| < |I|$.

By theorem 2.6 condition (5.18) is satisfied with any integer $k \ge 1$ and with any interval I_2, even without the orthogonality assumption (5.19).

To prove (5.17) and (5.28) choose $l > 1$ such that

$$|I| > 2R\lambda_l^{-1/4}$$

and put $k = 2l - 1$; then

$$\{\omega_1, \ldots, \omega_{k-1}\} = \{\pm\sqrt{\lambda_1}, \ldots, \pm\sqrt{\lambda_{l-1}}\}.$$

It follows from theorem 3.7 and remark 3.13 that (5.17) and (5.28) are satisfied if

$$|I_1| > 2R\lambda_l^{-1/4} \quad \text{and} \quad |I'| > 2R\mu_1^{-1/2}.$$

Thus we may choose $|I_1|$ such that $|I_1| < |I|$ and we may conclude by applying theorem 5.2. $\square$

Let us apply this result to the exact controllability problem

$$y'' + \Delta^2 y = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{16}$$

$$y = 0 \quad \text{and} \quad \partial_\nu y = v \quad \text{on} \quad \Gamma \times (0, T), \tag{17}$$

$$y(0) = y^0 \quad \text{and} \quad y'(0) = y^1. \tag{18}$$

Repeating the proof of theorem 4.5 by using theorem 6.7 instead of theorem 3.7 at the end, we obtain the

Theorem 6.8. — *Fix $T > 0$ arbitrarily (arbitrarily small). Given (y^0, y^1), $(y_T^0, y_T^1) \in L^2(\Omega) \times H^{-2}(\Omega)$ arbitrarily, there exists $v \in L^2(0, T; L^2(\Gamma))$ such that*

$$v = 0 \quad \text{a.e. on} \quad \Gamma_- \times (0, T)$$

and the solution of (16)–(18) satisfies

$$y(T) = y_T^0 \quad \text{and} \quad y'(T) = y_T^1.$$

Theorems 6.7 and 6.8 are due to Zuazua [1]. He proved them by using an indirect compactness–uniqueness argument.

Remark 6.9. — We may formally obtain the condition $T > 0$ from the stronger condition $T > 2R\lambda_1^{-1/4}$ of theorem 4.5 and remark 3.13 by letting $\lambda_1 \to +\infty$.

6.4. The second Petrovsky system. Uniqueness theorems

We consider here the problem (cf. section 3.3)

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{19}$$

$$u = \Delta u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}, \tag{20}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{21}$$

In the following variant of the uniqueness theorem 3.14 we do not need any hypothesis on the length of the interval I. If $\Gamma_+ = \Gamma$, then the left-hand sides of (22) and (23) below are majorized by the left-hand side of (3.42).

Theorem 6.10. — *For every interval I there exists a constant $c' > 0$ such that*

$$\int_I \int_{\Gamma_+} |\partial_\nu u'|^2 d\Gamma \ dt \geq c' E_{1/4}, \ \forall (u^0, u^1) \in D_{3/4} \times D_{1/4} \tag{22}$$

and

$$\int_I \int_{\Gamma_+} |\partial_\nu \Delta u|^2 d\Gamma \ dt \geq c' E_{1/4}, \ \forall (u^0, u^1) \in D_{3/4} \times D_{1/4}. \tag{23}$$

Proof. — It is sufficient to prove the inequalities (22), (23) for $u^0, u^1 \in Z$; the general case then follows by density, using theorem 2.13.

We are going to apply theorem 5.5 with

$$H = D_{1/4} (= H_0^1(\Omega)),$$
$$V = D_{3/4} (= \{v \in H^3(\Omega) : v = \Delta v = 0 \quad \text{on} \quad \Gamma\}),$$
$$p(v^0, v^1) = \|\partial_\nu v^1\|_{L^2(\Gamma_+)} \quad \text{for (22)},$$
$$p(v^0, v^1) = \|\partial_\nu \Delta v^0\|_{L^2(\Gamma_+)} \quad \text{for (23)}$$

and with $q = p$.

It is clear that (22), (23) are equivalent to (5.36). Therefore it suffices to verify (5.28), (5.29), (5.30), (5.32) and (5.33) with a suitable integer $l > 1$ and with suitable intervals I', I_1', I_2 such that $|I_1'| < |I|$. (Recall that (5.28) implies (5.21) and note that (5.22) is obvious here.)

It follows from the standard trace theorems that condition (5.29) is fulfilled with any $\alpha > 1$.

Condition (5.30) means that in some sense the eigenvalues λ_k tend to $+\infty$ sufficiently quickly. y a well-known theorem of H. Weyl, see e.g. Agmon [1], in the present case we the estimate

$$\lambda_k = (c + o(1)) k^{4/n}, \quad k \to +\infty$$

holds, which implies (5.30) for every $d > n/2$.

It follows from theorem 2.13 that inequality (5.33) is fulfilled for all $U^0 \in \mathcal{Z}$ (even in the absence of conditions (5.34) and (5.35)), for every interval I_2.

It remains to prove (5.28) and (5.32). These will be deduced from (the apparently weaker) theorem 3.14 and remark 3.17 by using a small "trick".

Observe that (by the special boundary conditions (20)) for $(u^0, u^1) \in \mathcal{Z}_\pm$ the solution of (19)–(21) is also solution of the problem

$$u' \pm i\Delta u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{24}$$

$$u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}, \tag{25}$$

$$u(0) = u^0. \tag{26}$$

Since the solutions of (24)–(26) clearly satisfy

$$|\partial_\nu u'| = |\partial_\nu \Delta u| \quad \text{on} \quad \Gamma \times \mathbb{R}, \tag{27}$$

(3.42) reduces to

$$\int_J \int_{\Gamma_+} |\partial_\nu u'|^2 \, \mathrm{d}\Gamma \, \mathrm{dt} = \int_J \int_{\Gamma_+} |\partial_\nu \Delta u|^2 \, \mathrm{d}\Gamma \, \mathrm{dt} \geq \frac{c'}{2} E_{1/4}$$

i.e. to

$$\int_J p(U(t))^2 \, \mathrm{dt} \geq \frac{c'}{2} \|U^0\|^2_{\mathcal{H}}$$

if $|J| > R/\sqrt{\lambda_1}$. Hence (5.28) is satisfied if we choose the interval I' such that $|I'| > R/\sqrt{\lambda_1}$.

The proof of (5.32) is similar. First we choose $k > 1$ such that $|I| > R/\sqrt{\lambda_k}$ and then we choose an interval I'_1 satisfying $|I| > |I'_1| > R/\sqrt{\lambda_k}$. Using again (27), (5.32) follows from (3.42) and from remark 3.17 if we choose $l = 2k - 1$.

We may apply theorem 5.5 and the proof is completed. $\square$

Next we deduce from theorem 6.10 the inverse inequality of the direct inequality obtained in theorem 2.10.

Theorem 6.11. — *For every interval I there exists a constant $c' > 0$ such that the solution of* (19)–(21) *satisfies*

$$\int_I \int_{\Gamma_+} |\partial_\nu u|^2 \mathrm{d}\Gamma \, \mathrm{dt} \geq c' (\|u^0\|^2_{H^1_0(\Omega)} + \|u^1\|^2_{H^{-1}(\Omega)}) \tag{28}$$

for every $(u^0, u^1) \in H^1_0(\Omega) \times H^{-1}(\Omega)$.

Proof. — (Compare with that of theorem 2.10.) Using a density argument based on theorem 2.10, it is sufficient to prove (28) for $u^0, u^1 \in Z$.

Applying the inequality (23) for $(A^{-1/2}u^0, A^{-1/2}u^1)$ instead of (u^0, u^1) we obtain

$$\int_I \int_{\Gamma_+} |\partial_\nu \Delta A^{-1/2} u|^2 \ \mathrm{d}\Gamma \ \mathrm{d}t \geq \frac{c'}{2} (\|A^{-1/2}u^0\|^2_{3/4} + \|A^{-1/2}u^1\|^2_{1/4})$$
$$= \frac{c'}{2} (\|u^0\|^2_{1/4} + \|u^1\|^2_{-1/4}) = \frac{c'}{2} (\|u^0\|^2_{H^1_0(\Omega)} + \|u^1\|^2_{H^{-1}(\Omega)}).$$

Since $\Delta A^{-1/2} u \equiv -u$, (28) hence follows (with $c'/2$ instead of c'). $\square$

6.5. The second Petrovsky system. Exact controllability

Theorems 2.10 and 6.11 permit us to study the exact controllability of the problem

$$y'' + \Delta^2 y = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{29}$$
$$y = 0 \quad \text{and} \quad \Delta y = v \quad \text{on} \quad \Gamma \times (0, T), \tag{30}$$
$$y(0) = y^0, \quad y'(0) = y^1. \tag{31}$$

It follows from theorem 2.14 that for any given $(y^0, y^1) \in H^1_0(\Omega) \times H^{-1}(\Omega)$ and $v \in L^2(0, T; L^2(\Gamma))$ this problem has a unique solution satisfying $(y(T), y'(T)) \in H^1_0(\Omega) \times H^{-1}(\Omega)$.

Definition. — We say that the problem (29)–(31) is exactly controllable if for any given $(y^0, y^1), (y^0_T, y^1_T) \in H^1_0(\Omega) \times H^{-1}(\Omega)$ there exists $v \in L^2(0, T; L^2(\Gamma))$ such that the solution of (29)–(31) satisfies $y(T) = y^0_T$ and $y'(T) = y^1_T$.

Theorem 6.12. — *The problem* (29)–(31) *is exactly controllable for every* $T > 0$ (*arbitrarily small*). *Moreover, there exist controls* v *satisfying*

$$v = 0 \quad a.e. \ on \quad \Gamma_- \times (0, T). \tag{32}$$

Proof. — We apply HUM as in chapter 4. Fixing $u^0 \in H^1_0(\Omega)$ and $u^1 \in H^{-1}(\Omega$ arbitrarily, we solve (29)–(31), then we solve the problem

$$y'' + \Delta^2 y = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{33}$$
$$y = 0 \quad \text{and} \quad \Delta y = -\partial_\nu u \quad \text{on} \quad \Gamma_+ \times (0, T), \tag{34}$$
$$y = \Delta y = 0 \quad \text{on} \quad \Gamma_- \times (0, T), \tag{32}$$
$$y(T) = y'(T) = 0, \tag{36}$$

and we set

$$\Lambda(u^0, u^1) := (y'(0), -y(0)).$$

It follows from theorems 2.10 and 2.14 that Λ is a bounded linear map from $F := H_0^1(\Omega) \times H^{-1}(\Omega)$ into $F' = H^{-1}(\Omega) \times H_0^1(\Omega)$; it suffices to show that it is surjective. Using the Lax-Milgram theorem it is sufficient to prove the inequality

$$\langle \Lambda(u^0, u^1), (u^0, u^1) \rangle_{F',F} \geq c \|(u^0, u^1)\|_F^2$$

with a constant $c > 0$, independent of $(u^0, u^1) \in H_0^1(\Omega) \times H^{-1}(\Omega)$. By theorem 6.11 it is sufficient to show that

$$\langle \Lambda(u^0, u^1), (u^0, u^1) \rangle_{F',F} \geq c \int_I \int_{\Gamma_+} |\partial_\nu u|^2 \, \mathrm{d}\Gamma \, \mathrm{dt}. \tag{37}$$

As usual, we may restrict ourselves to the real case and we may asume that $(u^0, u^1) \in Z \times Z$. Multiplying the equation (33) by u and integrating by parts we obtain that

$$\begin{aligned} 0 = \int_0^T \int_\Omega u(y'' + \Delta^2 y) \, \mathrm{dx} \, \mathrm{dt} = \\ \int_0^T \int_\Omega (u'' + \Delta^2 u) y \, \mathrm{dx} \, \mathrm{dt} + \Big[\int_\Omega u y' - u' y \, \mathrm{dx} \Big]_0^T \\ + \int_0^T \int_\Gamma u(\partial_\nu \Delta y) - (\partial_\nu u)(\Delta y) + (\Delta u)(\partial_\nu y) - (\partial_\nu \Delta u) y \, \mathrm{d}\Gamma \, \mathrm{dt} \\ = \int_\Omega u^1 y(0) - u^0 y'(0) \, \mathrm{dx} + \int_0^T \int_{\Gamma_+} (\partial_\nu u)^2 \, \mathrm{d}\Gamma \, \mathrm{dt}. \end{aligned}$$

Hence (37) follows. □

Remark 6.13. — Improving a former result of Lions [5; p. 310], Zuazua [1] earlier obtained (with an indirect method) a theorem analogous to theorem 6.12, by using two controls. See also Lasiecka [1] for another result analogous to theorem 6.12.

Now it is easy to study the exact controllability of the problem by acting in the other boundary condition :

$$z'' + \Delta^2 z = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{38}$$

$$z = v \quad \text{et} \quad \Delta z = 0 \quad \text{on} \quad \Gamma \times (0, T), \tag{39}$$

$$z(0) = z^0 \quad \text{and} \quad z'(0) = z^1. \tag{40}$$

We have the following result (we recall from lemma 1.7 that $D_{-1/4} = H^{-1}(\Omega) \subset D_{-3/4}$).

Theorem 6.14. — *Fix $T > 0$ arbitrarily. Given*

$$(z^0, z^1), (z_T^0, z_T^1) \in D_{-1/4} \times D_{-3/4}$$

arbitrarily, there exists $w \in L^2(0,T; D_{-1/4})$ such that

$$w(t) = 0 \quad \text{on} \quad \Gamma_- \quad \text{for almost every} \quad t \in (0,T)$$

and the solution of (38)–(40) satisfies $z(T) = z_T^0$, $z'(T) = z_T^1$.

Remark 6.15. — We *define* the solution of (38)–(40) in the following way : we solve (29)–(32) with $(y^0, y^1) := (A^{-1/2}z^0, A^{-1/2}z^1) \in D_{1/4} \times D_{-1/4} = H_0^1(\Omega) \times H^{-1}(\Omega)$ and $w := -v$, and then we set $z := A^{1/2}y$. To justify this definition we observe that if y is a sufficiently smooth solution of (29)–(32), then $z := A^{1/2}y$ is given in fact by $z = -\Delta y$ and therefore (38)–(40) is satisfied in the usual sense :

$$\begin{gathered}
z'' + \Delta^2 z = -\Delta(y'' + \Delta y) = 0 \quad \text{in} \quad \Omega \times (0,T), \\
z = -\Delta y = -v = w \quad \text{and} \quad \Delta z = -\Delta^2 y = y'' = 0 \quad \text{on} \quad \Gamma \times (0,T), \\
z(0) = -\Delta y^0 = -\Delta A^{1/2} z^0 = z^0, \\
z'(0) = -\Delta y'(0) = -\Delta A^{1/2} z^1 = z^1
\end{gathered}$$

Proof of theorem 6.14. — The theorem follows at once from theorem 6.13 and from remark 6.15. □

6.6. The wave equation. Neumann or Robin condition

Let us return to the problem (cf. §§3.4 and 4.3)

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{41}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}, \tag{42}$$

$$\partial_\nu u + au = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}, \tag{43}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1. \tag{44}$$

Assume that

$$m \cdot \nu \le 0 \quad \text{on} \quad \Gamma_0, \quad m \cdot \nu \ge 0 \quad \text{on} \quad \Gamma_1, \tag{45}$$

and that a has the form

$$a = (m \cdot \nu)b, \quad b \in C^1(\Gamma_1), \quad b \ge 0 \text{ on } \Gamma_1. \tag{46}$$

First we improve proposition 3.21 by eliminating hypothesis (3.60) on q and by weakening hypothesis (3.61) on the length of I :

Proposition 6.16. — *Assume* (45), (46) *and let* I *be an interval of length* $> 2R$. *Then there exists a constant* $c' > 0$ *such that*

$$\int_I \int_{\Gamma_1} |u|^2 + |u'|^2 \, \mathrm{d}\Gamma_m \, \mathrm{d}t \ge c'E, \ \forall (u^0, u^1) \in D_1 \times D_{1/2}. \tag{47}$$

Proof. — It suffices to show (47) for $(u^0, u^1) \in Z \times Z$; the general case then follows by density. We are going to apply theorem 5.3 with V and H defined as in section 1.4 (cf. example 5.1) and with

$$p(v^0, v^1) := \Big(\int_{\Gamma_1} |v^0|^2 + |v^1|^2 \, \mathrm{d}\Gamma_m \Big)^{1/2}, \quad q(v^0, v^1) := \Big(\int_{\Gamma_1} |v^0|^2 \mathrm{d}\Gamma_m \Big)^{1/2}.$$

Then (47) is equivalent to (5.26), and (5.22), (5.24) are obviously satisfied.

To prove (5.23) fix $l > 1$ such that

$$Q_l < 1,$$

and

$$|I| > 2R_l/(1 - Q_l)$$

(this is possible because the right-hand side tends to $2R$ as $l \to +\infty$, see remark 3.24), choose an interval I_1 such that

$$|I| > |I_1| > 2R_l/(1 - Q_l).$$

Then putting $k = 2l - 1$ property (5.23) follows from proposition 3.21 and remark 3.24.

It remains to verify (5.21). If $Q_1 < 1$, then we can show easily that the stronger condition (5.28) is also satisfied for every interval I' of length

$$|I'| > 2R_1/(1 - Q_1).$$

Indeed, given $j \ge 1$ and $U^0 \in \mathcal{Z}_j$ arbitrarily, the solution of (5.12) satisfies

$$p(U(t))^2 = (1 + \omega_j^2) q(U(t))^2;$$

using proposition 3.21 hence (5.28) follows.

If $Q_1 \ge 1$, then we verify (5.21) directly. It is sufficient to show that if an eigenfunction $v \in Z$ satisfies

$$-\Delta v + qv = \lambda v \quad \text{in} \quad \Omega,$$

$$v = 0 \quad \text{on} \quad \Gamma_0,$$

and

$$v = \partial_\nu v + av = 0 \quad \text{on} \quad \Gamma_1$$

with some real number λ, then in fact $v \equiv 0$. And this follows from corollary 6.3. $\square$

We are going to improve proposition 6.16 by eliminating the term $|u|^2$ from (47). We need a lemma. Fix an interval I of length $> 2R$ and denote by $X = X(I)$ the completion of the vector space of the solutions of (41)–(44) corresponding to the initial data $(u^0, u^1) \in D_1 \times D_{1/2}$, with respect to the *norm*

$$u \mapsto \Big(\int_I \int_{\Gamma_1} |u|^2 + |u'|^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt} \Big)^{1/2}.$$

By the preceding proposition X is a Hilbert space and we have

$$\int_I \int_{\Gamma_1} |u|^2 + |u'|^2 \, \mathrm{d}\Gamma_m \, \mathrm{dt} \geq c'E, \quad \forall u \in X \tag{48}$$

by density.

Lemma 6.17. — *Let $|I|$ be an interval of length $> 2R$ and let $u \in X = X(I)$ be such that*

$$u' = 0 \quad \textit{on} \quad \Gamma_1 \times I. \tag{49}$$

Then

$$u \equiv 0. \tag{50}$$

Proof. — Let us introduce the function

$$U(t) := \int_0^t u'(s) \, \mathrm{ds}, \quad t \in \mathbb{R}.$$

Then $U \in X$, and applying (48) for U we obtain that $U \equiv 0$. Hence $u' \equiv 0$ and therefore u does not depend on $t \in \mathbb{R} : u(t) = u^0$ for all $t \in \mathbb{R}$. It remains to show that $u^0 \equiv 0$. Since $u' \equiv 0$, we deduce from (41)–(43) that u^0 belongs to the kernel of the operator A. Since A is injective, the lemma follows. $\square$.

Proposition 6.18. — *Assume* (45), (46) *and let I be an interval of length $> 2R$. Then there exists a constant $c' > 0$ such that*

$$\int_I \int_{\Gamma_1} |u'|^2 \mathrm{d}\Gamma_m \, \mathrm{dt} \geq c'E, \ \forall (u^0, u^1) \in D_1 \times D_{1/2}. \tag{52}$$

Proof. — By proposition 6.16 it is sufficient to show the existence of a constant c such that

$$\int_I \int_{\Gamma_1} |u|^2 \mathrm{d}\Gamma_m \ \mathrm{dt} \le c \int_I \int_{\Gamma_1} |u'|^2 \mathrm{d}\Gamma_m \ \mathrm{dt}, \ \forall (u^0, u^1) \in D_1 \times D_{1/2}.$$

Assume on the contrary that there is a sequence (u_n^0, u_n^1) in $D_1 \times D_{1/2}$ such that the corresponding solutions satisfy

$$\int_I \int_{\Gamma_1} |u_n|^2 \mathrm{d}\Gamma_m \ \mathrm{dt} = 1, \ \forall n \tag{53}$$

and

$$\int_I \int_{\Gamma_1} |u_n'|^2 \mathrm{d}\Gamma_m \ \mathrm{dt} \to 0. \tag{54}$$

Then the sequence (u_n) is bounded in X. Extracting a subsequence if needed, we may assume that

$$u_n \rightharpoonup u \quad \text{(weakly) in} \quad X$$

for some $u \in X$. By (54) we have

$$u' \equiv 0 \quad \text{on} \quad \Gamma_1 \times I,$$

therefore $u \equiv 0$ by lemma 6.17.

On the other hand, since (u_n) is bounded in X, we deduce from (48) that it is bounded in

$$L^\infty(\mathbb{R}; H^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}; L^2(\Omega))$$

and hence also in

$$L^2(I; H^1(\Omega)) \cap H^1(I; L^2(\Omega)).$$

Consequently, using an Ascoli type result (see Lions [1; theorem 1.5.1]) the sequence is precompact in $L^2(I; H^{1-\varepsilon}(\Omega))$ for all $\varepsilon > 0$ and therefore its trace is precompact in $L^2(\Gamma \times I)$. Thus we deduce from (53) that

$$\int_I \int_{\Gamma_1} |u|^2 \mathrm{d}\Gamma_m \ \mathrm{dt} = 1$$

contradicting $u \equiv 0$. □

Now we may improve theorem 3.26 by eliminating the hypotheses on q, b, and by weakening the hypothesis on the length of I :

Theorem 6.19. — *Assume* (45), (46) *and let I be an interval of length $> 2R$. Then there exists a constant $c' > 0$ such that*

$$\int_I \int_{\Gamma_1} |u|^2 \, \mathrm{d}\Gamma_m \, \mathrm{d}t \geq c' E_{-1/2}, \ \forall (u^0, u^1) \in V \times H.$$

Proof. — We repeat the proof of theorem 3.26, using proposition 6.18 instead of the estimate (3.68) in remark 3.23. ∎

Finally, we can strongly improve theorem 4.8 on the exact controllability of the problem

$$y'' - \Delta y + qy = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{55}$$

$$y = 0 \quad \text{on} \quad \Gamma_0 \times (0,T), \tag{56}$$

$$\partial_\nu y + ay = v \quad \text{on} \quad \Gamma_1 \times (0,T), \tag{57}$$

$$y(0) = y^0 \quad \text{and} \quad y'(0) = y^1 \quad \textit{on} \quad \Omega. \tag{58}$$

Theorem 6.20. — *Assume* (45), (46) *and let $T > 2R$. Then, given*

$$(y^0, y^1), \ (y_T^0, y_T^1) \in V \times H$$

arbitrarily, there exists a control $v \in L^2(0,T; L^2(\Gamma_1))$ such that the solution of (55)–(58) *satisfies*

$$y(T) = y_T^0 \quad \textit{and} \quad y'(T) = y_T^1 \quad \textit{on} \quad \Omega.$$

For $q \equiv 0$ this theorem is due to Lions [5; p. 203]. He used unique continuation theorems for the wave equation. Let us note that the above proof does not use any unique continuation theorem if $q \equiv 0$.

Proof. — We repeat the proof of theorem 4.8, using the preceding theorem instead of theorem 3.26. ∎

7. Dissipative evolutionary problems

In this chapter we introduce some dissipative systems governed by the wave equation or by a plate equation; their stabilization properties will be studied in the following chapters. In order to simplify the notation we limit ourselves to the real case.

7.1. Maximal monotone operators

In this section we recall some existence and perturbation results concerning the evolutionary problem

$$U' + \mathcal{A}U = 0 \quad \text{in} \quad \mathbb{R}_+ := [0, +\infty), \quad U(0) = U^0 \tag{1}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is an operator, non necessarily linear, in a real Hilbert space $\mathcal{H}$. We refer e.g. to Barbu [1] or Brézis [1] for proof.

We say that $\mathcal{A}$ is *maximal monotone* if the following two properties are fulfilled :

$$\mathcal{A} \quad \text{is } \textit{monotone}: \quad (\mathcal{A}U - \mathcal{A}V, U - V)_{\mathcal{H}} \ge 0, \quad \forall U, V \in D(\mathcal{A}), \tag{2}$$

$$I + \mathcal{A} \quad \text{is } \textit{surjective}: \quad R(I + \mathcal{A}) = \mathcal{H}. \tag{3}$$

Theorem 7.1. — *Let $\mathcal{A}$ be a maximal monotone operator in a Hilbert space $\mathcal{H}$. Then for every $U^0 \in \overline{D(\mathcal{A})}$ the problem* (1) *has a unique solution*

$$U \in C(\mathbb{R}_+; \mathcal{H}) \tag{4}$$

(*defined in some suitable sense*)*. If $V^0 \in \overline{D(\mathcal{A})}$ and if V is the corresponding solution of* (1)*, then the function*

$$t \mapsto \|U(t) - V(t)\|_{\mathcal{H}} \quad \textit{is non-increasing in} \quad \mathbb{R}_+. \tag{5}$$

If $U^0 \in D(\mathcal{A})$, then the solution is more regular :

$$U \in W^{1,\infty}(\mathbb{R}_+; \mathcal{H}) \tag{6}$$

and the function

$$t \mapsto \|\mathcal{A}U(t)\|_{\mathcal{H}} \quad \textit{is defined everywhere and is non-increasing in} \quad \mathbb{R}_+. \tag{7}$$

Remark 7.2. — One can show that $\overline{D(\mathcal{A})}$ is always convex. Moreover, if $\mathcal{A}$ is *linear*, then $\overline{D(\mathcal{A})} = \mathcal{H}$. In the linear case the conclusion of the theorem is also stronger : for $U^0 \in D(\mathcal{A})$ the solution satisfies

$$U \in C^1(\mathbb{R}_+; \mathcal{H})$$

instead of (6). We note that theorem 7.1 also remains valid for *multivalued* maximal monotone operators.

Let us also recall the following perturbation theorem (see e.g. Barbu [1; proposition 4.2.1 and theorem 4.2.1]).

Theorem 7.3. — *Let $\mathcal{A}$ and $\mathcal{A}_k$ $(k = 1, 2, , \ldots)$ be maximal monotone operators in a Hilbert space $\mathcal{H}$ and assume that*

$$(I + \mathcal{A}_k)^{-1} W \to (I + \mathcal{A})^{-1} W \quad in \quad \mathcal{H}$$

for every $W \in \mathcal{H}$, as $k \to +\infty$. Choose U^0, $U_k^0 \in \mathcal{H}$ such that

$$U_k^0 \to U^0 \quad in \quad \mathcal{H}.$$

Then the corresponding solutions of (1) *and*

$$U_k' + \mathcal{A}_k U_k = 0 \quad in \quad \mathbb{R}_+, \quad U_k(0) = U_k^0$$

satisfy

$$U_k(t) \to U(t) \quad in \quad \mathcal{H}$$

for every $t \in \mathbb{R}_+$.

7.2. The wave equation

Let Ω be a bounded domain of class C^2 in $\mathbb{R}^n$ and let $\{\Gamma_0, \Gamma_1\}$ be a partition of its boundary Γ. Fix three *nonnegative* functions $q : \Omega \to \mathbb{R}$, $a, l : \Gamma_1 \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, non-decreasing function such that $g(0) = 0$. Consider the problem

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{8}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+, \tag{9}$$

$$\partial_\nu u + au + lg(u') = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{10}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{on} \quad \Omega. \tag{11}$$

The particular case $l \equiv 0$ corresponds to the conservative system of section 1.3.

Assume that

$$q \in L^\infty(\Omega), \quad a, l \in C^1(\Gamma_1). \tag{12}$$

Furthermore, in order to avoid some extra difficulties we shall always assume that

$$\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset, \tag{13}$$

$$\Gamma_0 \neq \emptyset \text{ or } q \not\equiv 0 \text{ or } a \not\equiv 0, \tag{14}$$

$$n \geq 3, \tag{15}$$

and that there exists a constant $c > 0$ such that

$$|g(x)| \leq 1 + c|x|^{n/(n-2)}, \quad \forall x \in \mathbb{R}. \tag{16}$$

(See Komornik and Zuazua [1], Barucq and Hanouzet [1], Komornik [10] and Tcheugoue [1] for the study of the problem if one of conditions (13)–(16) is not satisfied.)

Set

$$V = H^1_{\Gamma_0}(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\},$$

$$\|v\|_V^2 := \int_\Omega |\nabla v|^2 + qv^2 \, \mathrm{dx} + \int_{\Gamma_1} av^2 \, \mathrm{d}\Gamma.$$

By hypothesis (14) the last expression defines a norm on V, which is equivalent to the norm induced by $H^1(\Omega)$; consequently, $V = H^1_{\Gamma_0}(\Omega)$ is a Hilbert space.

We shall prove the

Theorem 7.4. — *Assume* (12)–(16). *Given* $(u^0, u^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *arbitrarily, the problem* (8)–(11) *has a unique solution satisfying*

$$u \in C(\mathbb{R}_+; H^1_{\Gamma_0}(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega)). \tag{17}$$

The energy $E : \mathbb{R}_+ \to \mathbb{R}_+$ *of the solution, defined by*

$$E = E(u) := \tfrac{1}{2} \int_\Omega |u'|^2 + |\nabla u|^2 + qu^2 \, \mathrm{dx} + \tfrac{1}{2} \int_{\Gamma_1} au^2 \, \mathrm{d}\Gamma \tag{18}$$

is non-increasing. Moreover, if u *and* v *are two solutions* (*corresponding to different initial data*), *then the function* $E(u-v)$ *is non-increasing*; *in particular,*

$$\|E(u-v)\|_{L^\infty(\mathbb{R}_+)} \leq E(u-v)(0). \tag{19}$$

Under a stronger growth assumption on g we have a simple characterization of $D(\mathcal{A})$:

Theorem 7.5. — *Assume* (12)–(15) *and assume that* g *is globally Lipschitz continuous : there exists a constant* $c' > 0$ *such that*

$$|g(x_1) - g(x_2)| \le c'|x_1 - x_2, \quad \forall x_1,\ x_2 \in \mathbb{R}. \tag{20}$$

Then $D(\mathcal{A})$ *consists of the couples*

$$(u^0, u^1) \in \big(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\big) \times H^1_{\Gamma_0}(\Omega) \tag{21}$$

which satisfy

$$\partial_\nu u^0 + au^0 + lg(u^1) = 0 \quad on \quad \Gamma_1. \tag{22}$$

Hence for $(u^0, u^1) \in D(\mathcal{A})$ *the solution of* (8)–(11) *satisfies*

$$u \in L^\infty(\mathbb{R}_+; H^2(\Omega)), \quad u' \in L^\infty(\mathbb{R}_+; H^1_{\Gamma_0}(\Omega)) \tag{23}$$

and

$$u'' \in L^\infty(\mathbb{R}_+; L^2(\Omega)). \tag{24}$$

The rest of this section is devoted to the proof of these theorems. Set

$$H := L^2(\Omega), \quad \|v\|_H^2 := \int_\Omega v^2 \ \mathrm{dx},$$

then V and H are separable Hilbert spaces with a dense and compact inclusion $V \subset H$. As usual, we introduce the duality mapping $A : V \to V'$ and we identify H with H'.

By hypothesis (16) the formula

$$\langle Bz, v\rangle_{V',V} := \int_{\Gamma_1} lg(z)v \ \mathrm{d}\Gamma, \quad z, v \in V$$

defines a map $B : V \to V'$ (not linear in general). Indeed, using (16) and the trace and imbedding theorems

$$V \subset H^1(\Omega) \hookrightarrow H^{1/2}(\Gamma_1) \subset L^{(2n-2)/(n-2)}(\Gamma_1) \tag{25}$$

we have, using for brevity the notation $\|\cdot\|_s := \|\cdot\|_{L^s(\Gamma_1)}$,

$$\Big|\int_{\Gamma_1} lg(z)v \ \mathrm{d}\Gamma\Big| \le c\|g(z)\|_{(2n-2)/n}\|v\|_{(2n-2)/(n-2)}$$

$$\le c\big(1 + \|z\|^{n/(n-2)}_{(2n-2)/(n-2)}\big)\|v\|_{(2n-2)/(n-2)} \le c\big(1 + \|z\|_V^{n/(n-2)}\big)\|v\|_V < +\infty.$$

Hence $Bz \in V'$ for every $z \in V$.

In order to find a reasonable definition of the (weak) solution of the problem (8)–(11) we multiply the equation (8) by $v \in V$ and we integrate by parts. Using the boundary conditions (9), (10) we obtain that

$$\begin{aligned}0 &= \int_\Omega (u'' - \Delta u + qu)v \ \mathrm{dx} \\ &= \int_\Omega u''v + \nabla u \cdot \nabla v + quv \ \mathrm{dx} - \int_\Gamma (\partial_\nu u)v \ \mathrm{d}\Gamma \\ &= \int_\Omega u''v + \nabla u \cdot \nabla v + quv \ \mathrm{dx} + \int_{\Gamma_1} auv + lg(u')v \ \mathrm{d}\Gamma \\ &= \langle u'' + Au + Bu', v\rangle_{V',V},\end{aligned}$$

whence

$$u'' + Au + Bu' = 0 \quad \text{on} \quad \mathbb{R}_+. \tag{26}$$

Putting

$$U = (U_1, U_2) := (u, u') \quad \text{and} \quad \mathcal{A}U := (-U_2, AU_1 + BU_2)$$

we may write (26), (11) in the form

$$U' + \mathcal{A}U = 0 \quad \text{in} \quad \mathbb{R}_+, \quad U(0) = (u^0, u^1). \tag{27}$$

If u is a (sufficiently smooth) solution of (8)–(11), then

$$U(t), \quad \mathcal{A}U(t) \in V \times H, \quad \forall t \in \mathbb{R}_+.$$

This leads us to set

$$\mathcal{H} := V \times H, \quad D(\mathcal{A}) := \{U \in V \times V : AU_1 + BU_2 \in H\}, \tag{28}$$

and to *define* the solution of (8)–(11) as that of (27). The definition is justified by

Proposition 7.6. — *$\mathcal{A}$ is a maximal monotone operator in $\mathcal{H}$.*

Proof. — The monotonicity of $\mathcal{A}$ follows from the nonnegativity of l and from the non-decreasingness of g. Indeed, given $U, V \in D(\mathcal{A})$ arbitrarily, we have

$$\begin{aligned}&(\mathcal{A}U - \mathcal{A}V, U - V)_{\mathcal{H}} \\ &= (V_2 - U_2, U_1 - V_1)_V + (AU_1 - AV_1 + BU_2 - BV_2, U_2 - V_2)_H \\ &= (V_2 - U_2, U_1 - V_1)_V + \langle AU_1 - AV_1 + BU_2 - BV_2, U_2 - V_2\rangle_{V',V} \\ &= \langle BU_2 - BV_2, U_2 - V_2\rangle_{V',V} = \int_{\Gamma_1} l\big(g(U_2) - g(V_2)\big)(U_2 - V_2) \ \mathrm{d}\Gamma \ge 0.\end{aligned}$$

It remains to show that for $W = (W_1, W_2) \in \mathcal{H} = V \times H$ given arbitrarily, there exists $U = (U_1, U_2) \in D(\mathcal{A})$ such that $(I + \mathcal{A})U = W$. It is sufficient to show that the map $I + A + B : V \to V'$ is onto. Indeed, then there exists $U_2 \in V$ satisfying

$$(I + A + B)U_2 = W_2 - AW_1.$$

Choosing U_2 in this way and setting $U_1 = U_2 + W_1$ we clearly have $U \in V \times V$, $AU_1 + BU_2 = W_2 - U_2 \in H$ (hence $U \in D(\mathcal{A})$) and $(I + \mathcal{A})U = W$.

To prove the surjectivity of the map $I + A + B : V \to V'$ fix $f \in V'$ arbitrarily, set

$$G(t) = \int_0^t g(s)\ \mathrm{ds}, \quad t \in \mathbb{R},$$

and consider the map $F : V \to \mathbb{R}$ defined by

$$F(u) = \tfrac{1}{2}\|u\|_H^2 + \tfrac{1}{2}\|u\|_V^2 + \int_\Gamma G(u)\ \mathrm{d}\Gamma - \langle f, u\rangle_{V',V}.$$

Using the growth assumption (16) one may readily verify that the map F is well-defined, continuously differentiable and that

$$F'(u)v = \langle (I + A + B)u - f, v\rangle_{V',V}, \quad \forall u,\ v \in V.$$

Furthermore, the monotonicity of g implies the convexity of F. Finally, F is coercive : $F(v) \to +\infty$ if $\|v\|_V \to +\infty$. This follows at once from the obvious inequality

$$F(v) \geq \left(\tfrac{1}{2}\|v\|_V - \|f\|_{V'}\right)\|v\|_V.$$

It follows that the infimum of F is attained at some point $u \in V$. Then $F'(u) = 0$ i.e. $(I + A + B)u = f$. $\square$

Now set

$$D_0 := \{U \in V \times V : U_1 \in H^2(\Omega), \quad \partial_\nu U_1 + aU_1 + lg(U_2) = 0 \quad \text{on} \quad \Gamma_1\} \tag{29}$$

and

$$D := \{U \in V \times H_0^1(\Omega) : U_1 \in H^2(\Omega) \text{ and } \partial_\nu U_1 + aU_1 = 0 \quad \text{on} \quad \Gamma_1\}. \tag{30}$$

Lemma 7.7. — *We have $D \subset D_0 \subset D(\mathcal{A})$, and D is dense in $\mathcal{H}$. Consequently, $D(\mathcal{A})$ is dense in $\mathcal{H}$.*

Proof. — It is clear that $D \subset D_0$ and that D is dense in $\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$. It remains to show that $D_0 \subset D(\mathcal{A})$.

Fix $(U_1, U_2) \in D_0$ arbitrarily ; it suffices to prove the estimate

$$\big|\langle AU_1 + BU_2, v\rangle_{V',V}\big| \le c\|v\|_H, \quad \forall v \in V \tag{31}$$

with a suitable constant c. Using the definition of A and B we have

$$\langle AU_1+BU_2, v\rangle_{V',V} = \int_\Omega \nabla U_1 \cdot \nabla v + qU_1 v \text{ dx} + \int_{\Gamma_1} aU_1 v + lg(U_2)v \text{ d}\Gamma. \tag{32}$$

Since $(U_1, U_2) \in D_0$ implies $U_1 \in H^2(\Omega)$, we may apply Green's formula to the right-hand side. We obtain that

$$\begin{aligned} &\langle AU_1 + BU_2, v\rangle_{V',V} \\ &= \int_\Omega (-\Delta U_1 + qU_1)v \text{ dx} + \int_\Gamma (\partial_\nu U_1)v \text{ d}\Gamma + \int_{\Gamma_1} aU_1 v + lg(U_2)v \text{ d}\Gamma \end{aligned}$$

It follows from the definition of D_0 that the boundary integrals vanish. Hence

$$\langle AU_1 + BU_2, v\rangle_{V',V} = \int_\Omega (-\Delta U_1 + qU_1)v \text{ dx}.$$

Since $-\Delta U_1 + qU_1 \in L^2(\Omega) = H$, hence (31) follows. □

Proof of theorem 7.4. — Theorem 7.4 is an immediate consequence of theorem 7.1, proposition 7.6 and lemma 7.7 : observe that

$$E(u) = \tfrac{1}{2}\|U(t)\|^2_{\mathcal{H}} \text{ and } E(u-v) = \tfrac{1}{2}\|U(t) - V(t)\|^2_{\mathcal{H}}. \quad \square$$

For the proof of theorem 7.5 we need another lemma.

Lemma 7.8. — *Let g be globally Lipschitz continuous. Then*

$$D(\mathcal{A}) \subset D_0. \tag{33}$$

Proof. — Fix $(U_1, U_2) \in D(\mathcal{A})$ arbitrarily and set $f := AU_1 + BU_2$, $h := -aU_1 - lg(U_2)$. We will show that U_1 is the weak solution of the problem

$$-\Delta U_1 + qU_1 = f \quad \text{in} \quad \Omega, \tag{34}$$

$$U_1 = 0 \quad \text{on} \quad \Gamma_0, \tag{35}$$

$$\partial_\nu U_1 = h \quad \text{on} \quad \Gamma_1. \tag{36}$$

This will imply at once the boundary condition in the definition of D_0.

Let us recall that by definition $U_1 \in V$ is the weak solution of (34)–(36) if

$$\int_\Omega \nabla U_1 \cdot \nabla v + qU_1 v \text{ dx} = \langle f, v\rangle_{V',V} + \int_{\Gamma_1} hv \text{ d}\Gamma$$

for all $v \in V$. We may rewrite this equation in the form

$$\langle AU_1 + BU_2, v\rangle_{V',V} = \int_\Omega \nabla U_1 \cdot \nabla v + qU_1 v \ \mathrm{dx} + \int_{\Gamma_1} aU_1 v + lg(U_2)v \ \mathrm{d}\Gamma,$$

and (as we have seen in the proof of the preceding lemma, cf. (32)) this is a direct consequence of the definitions of A and B.

It remains to prove that $U_1 \in H^2(\Omega)$; applying the elliptic regularity theory to the problem (34)–(36), it is sufficient to establish that

$$f \in L^2(\Omega) \quad \text{and} \quad h \in H^1(\Omega). \tag{37}$$

The first relation in (37) follows from the definition of $D(\mathcal{A})$.

For the second relation it suffices to show that

$$g(U_2) \in L^2(\Omega) \quad \text{and} \quad \nabla g(U_2) \in L^2(\Omega). \tag{38}$$

Both properties are obvious because the global Lipschitz continuity implies the inequality

$$|g(x)| \le 1 + c|x|, \quad \forall x \in \mathbb{R}: \tag{39}$$

we have

$$\int_\Omega |g(U_2)|^2 \ \mathrm{dx} \le c \int_\Omega 1 + |U_2|^2 \ \mathrm{dx} \le c\big(1 + \|U_2\|_V^2\big) < +\infty$$

and

$$\begin{aligned}\int_\Omega |\nabla g(U_2)|^2 \ \mathrm{dx} &= \int_\Omega |g'(U_2)\nabla U_2|^2 \ \mathrm{dx} \\ &\le c \int_\Omega |\nabla U_2|^2 \ \mathrm{dx} \le \|U_2\|_V^2 < +\infty. \quad \square\end{aligned}$$

Proof of theorem 7.5. — We apply the second half of theorem 7.1 and we use lemmas 7.7 and 7.8. $\square$

7.3. Kirchhoff plates

Let Ω be a bounded domain of class C^4 in $\mathbb{R}^2$ and let $\{\Gamma_0, \Gamma_1\}$ be a partition of its boundary Γ such that

$$\Gamma_0 \ne \emptyset \quad \text{and} \quad \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset. \tag{40}$$

Fix a number $\mu \in (0,1)$ and a nonnegative function $l \in C^1(\Gamma_1)$. Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, continuous function such that $g(0) = 0$ and consider the problem

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{41}$$

$$u = u_\nu = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+, \tag{42}$$

$$u_{\nu\nu} + \mu u_{\tau\tau} = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{43}$$

$$u_{\nu\nu\nu} + (2-\mu) u_{\tau\tau\nu} = lg(u') \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{44}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{on} \quad \Omega. \tag{45}$$

This models the small transversal vibrations of a thin plate whose Poisson coefficient is equal to μ; see e.g. Lagnese and Lions [1] or Lagnese [2]. Here and in the sequel the subscripts ν and τ stand for the normal and tangential derivatives; the unit normal and tangential vectors are given by $\nu = (\nu_1, \nu_2)$ and $\tau := (-\nu_2, \nu_1)$, respectively. We shall use the notation $\nabla u = (u_x, u_y)$ for the gradient of u, and we introduce the quadratic form

$$Q(u) = u_{xx}^2 + u_{yy}^2 + 2\mu u_{xx} u_{yy} + 2(1-\mu) u_{xy}^2. \tag{46}$$

We shall write $\mathrm{dX} := \mathrm{dx}\ \mathrm{dy}$.

Setting

$$H^2_{\Gamma_0}(\Omega) := \{v \in H^2(\Omega) : v = v_\nu = 0 \quad \text{on} \quad \Gamma_0\}, \tag{47}$$

we have the following result :

Theorem 7.9. — *Given $(u^0, u^1) \in H^2_{\Gamma_0}(\Omega) \times L^2(\Omega)$ arbitrarily, the problem (41)–(45) has a unique solution satisfying*

$$u \in C(\mathbb{R}_+; H^2_{\Gamma_0}(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega)). \tag{48}$$

The energy $E : \mathbb{R}_+ \to \mathbb{R}_+$ of the solution, defined by

$$E = E(u) := \tfrac{1}{2} \int_\Omega (u')^2 + Q(u)\ \mathrm{dX}, \tag{49}$$

is non-increasing. Moreover, if u and v are two solutions (corresponding to different initial data), then the function $E(u-v)$ is non-increasing; in particular,

$$\|E(u-v)\|_{L^\infty(\mathbb{R}_+)} \le E(u-v)(0). \tag{50}$$

If g is globally Lipschitz and if

$$(u^0, u^1) \in \left(H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)\right) \times H^2_{\Gamma_0}(\Omega) \tag{51}$$

and

$$u^0_{\nu\nu} + \mu u^0_{\tau\tau} = 0 \text{ and } u^0_{\nu\nu\nu} + (2-\mu)u^0_{\tau\tau\nu} = lg(u^1) \text{ sur } \Gamma_1 \times \mathbb{R}_+, \tag{52}$$

then the solution is more regular :

$$u \in L^\infty(\mathbb{R}_+; H^4(\Omega)), \quad u' \in L^\infty(\mathbb{R}_+; H^2_{\Gamma_0}(\Omega)) \tag{53}$$

and

$$u'' \in L^\infty(\mathbb{R}_+; L^2(\Omega)). \tag{54}$$

We begin by *defining* the solution of (41)–(45). Set

$$V := H^2_{\Gamma_0}(\Omega) \quad \text{and} \quad H := L^2(\Omega)$$

with

$$\|v\|_V := \Big(\int_\Omega Q(v)\ \mathrm{dX}\Big)^{1/2} \quad \text{and} \quad \|v\|_H := \Big(\int_\Omega v^2\ \mathrm{dX}\Big)^{1/2};$$

by hypothesis $\Gamma_0 \neq \emptyset$ they are separable Hilbert spaces with a dense and compact imbedding $V \subset H$. We introduce the duality mapping $A : V \to V'$ and we identify H with H'.

By the continuity of g and to the Sobolev imbedding $H^2(\Omega) \subset C(\overline{\Omega})$ the formula

$$\langle Bz, v\rangle_{V',V} := \int_{\Gamma_1} lg(z)v\ \mathrm{d}\Gamma, \quad z, v \in V$$

defines a (non-linear) map $B : V \to V'$. As in the preceding section for the wave equation, one can readily verify that B is monotone and hemicontinuous.

Next we need two lemmas.

Lemma 7.10. — *Given $v \in H^3(\Omega)$ and $w \in H^2(\Omega)$ arbitrarily, the following identity holds true :*

$$\int_\Omega v_{xx}w_{yy} + v_{yy}w_{xx} - 2v_{xy}w_{xy}\ \mathrm{dX} = \int_\Gamma v_{\tau\tau}w_\nu - v_{\nu\tau}w_\tau\ \mathrm{d}\Gamma. \tag{55}$$

If $v \in H^4(\Omega)$, then we also have

$$\int_\Omega v_{xx}w_{yy} + v_{yy}w_{xx} - 2v_{xy}w_{xy}\ \mathrm{dX} = \int_\Gamma v_{\tau\tau}w_\nu + v_{\nu\tau\tau}w\ \mathrm{d}\Gamma. \tag{56}$$

Proof. — We apply Green's formula as follows :

$$\begin{aligned}\int_\Omega v_{xx}w_{yy}\ \mathrm{dX} &= -\int_\Omega v_{xxy}w_y\ \mathrm{dX} + \int_\Gamma v_{xx}\nu_2 w_y\ \mathrm{d}\Gamma\\ &= \int_\Omega v_{xy}w_{xy}\ \mathrm{dX} + \int_\Gamma w_y(\nu_2 v_{xx} - \nu_1 v_{xy})\ \mathrm{d}\Gamma\\ &= \int_\Omega v_{xy}w_{xy}\ \mathrm{dX} - \int_\Gamma v_{x\tau}w_y\ \mathrm{d}\Gamma.\end{aligned}$$

Using the obvious differential relations

$$u_x = \nu_1 u_\nu - \nu_2 u_\tau, \qquad u_y = \nu_2 u_\nu + \nu_1 u_\tau \tag{57}$$

hence we deduce that

$$\int_\Omega v_{xx}w_{yy} - v_{xy}w_{xy}\ \mathrm{dX} = -\int_\Gamma (\nu_1 v_{\nu\tau} - \nu_2 v_{\tau\tau})(\nu_2 w_\nu + \nu_1 w_\tau)\ \mathrm{d}\Gamma. \tag{58}$$

Analogously, we have

$$\int_\Omega v_{yy}w_{xx} - v_{xy}w_{xy}\ \mathrm{dX} = \int_\Gamma (\nu_2 v_{\nu\tau} + \nu_1 v_{\tau\tau})(\nu_1 w_\nu - \nu_2 w_\tau)\ \mathrm{d}\Gamma. \tag{59}$$

Adding (58) to (59) we find (55).

If $v \in H^4(\Omega)$, then (56) follows from (55) by integration by parts. ∎

Let us introduce the notation (compare with (46))

$$Q(u,v) = u_{xx}v_{xx} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1-\mu)u_{xy}v_{xy}. \tag{60}$$

Lemma 7.11. — *Given $v \in H^4(\Omega)$ and $w \in H^2(\Omega)$ arbitrarily, the following identity holds true* :

$$\int_\Omega (\Delta^2 v)w - Q(v,w)\ \mathrm{dX} = \int_\Gamma (v_{\nu\nu} + (2-\mu)v_{\tau\tau})_\nu w - (v_{\nu\nu} + \mu v_{\tau\tau})w_\nu\ \mathrm{d}\Gamma. \tag{61}$$

Proof. — We use (56) and (60) in the following way :

$$\begin{aligned}\int_\Omega (\Delta^2 v)w - Q(v,w)\ \mathrm{dX} &= \int_\Omega \Delta v\Delta w - Q(v,w)\ \mathrm{dX} + \int_\Gamma (\Delta v)_\nu w - (\Delta v)w_\nu\ \mathrm{d}\Gamma\\ &= (1-\mu)\int_\Omega v_{xx}w_{yy} + v_{yy}w_{xx} - 2v_{xy}w_{xy}\ \mathrm{dX}\\ &\quad + \int_\Gamma (v_{\nu\nu\nu} + v_{\tau\tau\nu})w - (v_{\nu\nu} + v_{\tau\tau})w_\nu\ \mathrm{d}\Gamma\\ &= (1-\mu)\int_\Gamma v_{\tau\tau}w_\nu + v_{\nu\tau\tau}w\ \mathrm{d}\Gamma + \int_\Gamma (v_{\nu\nu\nu} + v_{\tau\tau\nu})w - (v_{\nu\nu} + v_{\tau\tau})w_\nu\ \mathrm{d}\Gamma\\ &= \int_\Gamma (v_{\nu\nu\nu} + (2-\mu)v_{\tau\tau\nu})w - (v_{\nu\nu} + \mu v_{\tau\tau})w_\nu\ \mathrm{d}\Gamma.\end{aligned}$$

∎

Let $v \in V$ and multiply the equation (41) by v. Integrating on Ω, applying the preceding lemma and using the boundary conditions (42)–(44) we obtain by a formal computation that

$$\begin{aligned}
0 = \int_\Omega (u'' + \Delta^2 u)v \,\mathrm{dX} &= \int_\Omega u''v + Q(u,v)\,\mathrm{dX} \\
+ \int_\Gamma \big(u_{\nu\nu} + (2-\mu)u_{\tau\tau}\big)_\nu v &- \big(u_{\nu\nu} + \mu u_{\tau\tau}\big)v_\nu \,\mathrm{d}\Gamma \\
&= \langle u'' + Au, v\rangle_{V',V} + \int_\Gamma lg(u')v\,\mathrm{d}\Gamma \\
&= \langle u'' + Au + Bu', v\rangle_{V',V}
\end{aligned}$$

whence

$$u'' + Au + Bu' = 0 \quad \text{on} \quad \mathbb{R}_+.$$

Putting

$$U = (U_1, U_2) := (u, u') \quad \text{and} \quad \mathcal{A}U := (-U_2, AU_1 + BU_2) \tag{62}$$

we may rewrite this equation as

$$U' + \mathcal{A}U = 0 \quad \text{on} \quad \mathbb{R}_+.$$

Taking into account the boundary and initial conditions it is natural to define $\mathcal{H}$ and $D(\mathcal{A})$ by

$$\mathcal{H} := V \times H\big(= H^2_{\Gamma_0}(\Omega) \times L^2(\Omega)\big), \tag{63}$$

$$D(\mathcal{A}) := \{U \in V \times V : AU_1 + BU_2 \in H\}, \tag{64}$$

and to define the solution of (41)–(45) as that of

$$U' + \mathcal{A}U = 0 \quad \text{on} \quad \mathbb{R}_+, \quad U(0) = (u^0, u^1). \tag{65}$$

Proposition 7.12. — *$\mathcal{A}$ is a maximal monotone operator in $\mathcal{H}$.*

Proof. — We may repeat word by word the proof of proposition 7.6 in the preceding section. ∎

Set

$$\begin{aligned}
D_0 := \{U \in V \times V : U_1 \in H^4(\Omega), \quad & U_{1,\nu\nu} + \mu U_{1,\tau\tau} = 0 \\
\text{and} \quad U_{1,\nu\nu\nu} + (2-\mu)U_{1,\tau\tau\nu} &= lg(U_2) \text{ on } \Gamma_1\}
\end{aligned} \tag{66}$$

and

$$D := \{U \in V \times H_0^2(\Omega) : U_1 \in H^4(\Omega) \text{ and}$$
$$U_{1,\nu\nu} + \mu U_{1,\tau\tau} = U_{1,\nu\nu\nu} + (2-\mu)U_{1,\tau\tau\nu} = 0 \text{ on } \Gamma_1\}. \tag{67}$$

Lemma 7.13. — *We have $D \subset D_0 = D(\mathcal{A})$ and D is dense in $\mathcal{H}$. Consequently, $D(\mathcal{A})$ is dense in $\mathcal{H}$.*

Proof. — Observe that for $U \in V \times V$ the equality $AU_1 + BU_2 = f$ means that U_1 is the weak solution of the problem

$$\Delta^2 U_1 = f \quad \text{in} \quad \Omega, \tag{68}$$
$$U_1 = U_{1,\nu} = 0 \quad \text{on} \quad \Gamma_0, \tag{69}$$
$$U_{1,\nu\nu} + \mu U_{1,\tau\tau} = 0 \quad \text{on} \quad \Gamma_1, \tag{70}$$
$$U_{1,\nu\nu\nu} + (2-\mu)U_{1,\tau\tau\nu} = lg(U_2) \quad \text{on} \quad \Gamma_1. \tag{71}$$

Indeed, if $AU_1 + BU_2 = f$, then for every fixed $v \in V$ we have, using lemma 7.11,

$$\langle f, v\rangle_{V',V} = \int_\Omega Q(U_1, v)\, \mathrm{dX} + \int_{\Gamma_1} lg(U_2)v\, \mathrm{d}\Gamma, \quad \forall v \in V$$

and this is the usual *definition* of the weak solution $U_1 \in V$ of (68)–(71). (It is easy to show, using lemma 7.11, that every regular solution is also a weak solution : adapt the formal computation leading to (65).)

Using this observation one can readily verify that $D \subset D_0 \subset D(\mathcal{A})$ and that D is dense in $\mathcal{H}$. To show the inclusion $D(\mathcal{A}) \subset D_0$ we verify that $U \in D(\mathcal{A})$ implies $g(U_2) \in H^1(\Omega)$; indeed, then $U_1 \in H^4(\Omega)$ by the elliptic regularity theory applied to the problem (68)–(71).

Since $U_2 \in V \subset H^2(\Omega) \subset L^\infty(\Omega)$ and g, g' are locally bounded, we have $g(U_2) \in L^\infty(\Omega)$ and $g'(U_2) \in L^\infty(\Omega)$; in particular, $g(U_2) \in L^2(\Omega)$. It remains to show that $\nabla g(U_2) \in L^2(\Omega)$. This follows from the equality $\nabla g(U_2) = g'(U_2)\nabla U_2$ because $g'(U_2) \in L^\infty(\Omega)$ and $\nabla U_2 \in H^1(\Omega) \subset L^2(\Omega)$. ∎

Proof of theorem 7.9. — We apply theorem 7.1 and we use proposition 7.12 and lemma 7.13. ∎

8. Linear stabilization

The aim of this chapter is to prove the *exponential* energy decay of the solutions of the wave equation under suitable linear boundary feedbacks (i.e. $g(x) \equiv x$). We shall also apply a principle of RUSSELL [2] relating the problem of stabilization to that of exact controllability. In the last section we shall study Maxwell's equations, which are closely related to the wave equation.

As usual, for a given point $x^0 \in \mathbb{R}^n$ we shall use the notation

$$\begin{gathered} m(x) = x - x^0, \quad x \in \mathbb{R}^n, \\ R = R(x^0) = \sup\{|x - x^0| : x \in \Omega\}, \\ \Gamma_+ = \{x \in \Gamma : m(x) \cdot \nu(x) > 0\}, \\ \mathrm{d}\Gamma_m = (m \cdot \nu)\, \mathrm{d}\Gamma. \end{gathered}$$

As in §7.2, we restrict ourselves for brevity to the case of dimension $n \geq 3$. We shall consider the real case only; the generalization to the complex case is obvious.

8.1. An integral inequality

In this section we recall the following simple result (used e.g. in HARAUX [2] and LAGNESE [2]); it will play an important role in this chapter.

THEOREM 8.1. — *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ := [0, +\infty)$) be a non-increasing function and assume that there exists a constant $T > 0$ such that*

$$\int_t^\infty E(s)\, \mathrm{d}s \leq T E(t), \quad \forall t \in \mathbb{R}_+. \tag{1}$$

Then

$$E(t) \leq E(0) e^{1 - t/T}, \quad \forall t \geq T. \tag{2}$$

Observe that the inequality (2) is also satisfied for $0 \leq t < T$: indeed, then it is weaker than the trivial inequality $E(t) \leq E(0)$.

PROOF OF THE THEOREM. — Define

$$f(x) := e^{x/T} \int_x^\infty E(s)\, \mathrm{d}s, \quad x \in \mathbb{R}_+;$$

then f is locally absolutely continuous and it is also non-increasing by (1) :

$$f'(x) = T^{-1}e^{x/T}\Big(\int_x^\infty E(s)\ \mathrm{ds} - TE(x)\Big) \le 0$$

almost everywhere in $\mathbb{R}_+$. Hence, using (1) again,

$$f(x) \le f(0) = \int_0^\infty E(s)\ \mathrm{ds} \le TE(0), \quad \forall x \in \mathbb{R}_+,$$

i.e.

$$\int_x^\infty E(s)\ \mathrm{ds} \le TE(0)e^{-x/T}, \quad \forall x \in \mathbb{R}_+. \tag{3}$$

Since E is nonnegative and non-increasing, we have

$$\int_x^\infty E(s)\ \mathrm{ds} \ge \int_x^{x+T} E(s)\ \mathrm{ds} \ge TE(x+T).$$

Substituting into (3) we obtain that

$$E(x+T) \le E(0)e^{-x/T}, \quad \forall x \in \mathbb{R}_+;$$

setting $t := x + T$ hence we conclude (2). □

Remark 8.2. — The theorem is optimal in the following sense : given $T > 0$ and $t' \ge T$ arbitrarily, there exists a non-increasing function $E : \mathbb{R}_+ \to \mathbb{R}_+$, non identically zero, satisfying (1) and such that

$$E(t') = E(0)e^{1-t'/T};$$

see the limit $\alpha \to 0$ of the example given by formula (9.5) in the next chapter.

Remark 8.3. — If the function E is also continuous, then the inequalities (2) are strict ; in particular, $E(T) < E(0)$. This result is also optimal, see Komornik [12].

8.2. Uniform stabilization of the wave equation I

Consider the linear case of the problem introduced in section 7.2 :

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{4}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+, \tag{5}$$

$$\partial_\nu u + au + lu' = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{6}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{sur} \quad \Omega \tag{7}$$

where we assume (7.12)–(7.15).

Recall the definition of the energy :

$$E(t) := \tfrac{1}{2}\int_\Omega (u'(t))^2 + |\nabla u(t)|^2 + q(u(t))^2 \,\mathrm{dx} + \tfrac{1}{2}\int_{\Gamma_1} a(u(t))^2 \,\mathrm{d}\Gamma. \tag{8}$$

The following result shows in particular that the energy is non-increasing.

Lemma 8.4. — *Given $(u^0, u^1) \in D(\mathcal{A})$ arbitrarily, the solution of* (4)–(7) *satisfies the energy equalities*

$$E(S) - E(T) = \int_S^T \int_{\Gamma_1} l(u'(t))^2 \mathrm{d}\Gamma, \quad 0 \le S < T < \infty. \tag{9}$$

Indeed, by the nonnegativity of l the right-hand side of (9) is nonnegative whence $E(S) \ge E(T)$.

Proof. — We multiply equation (4) by u' and we integrate by parts in $\Omega \times (S,T)$. Using (5) and (6) we obtain that

$$\begin{aligned} 0 &= \int_S^T \int_\Omega u'(u'' - \Delta u + qu) \,\mathrm{dx}\,\mathrm{dt} \\ &= \Big[\tfrac{1}{2}\int_\Omega (u')^2 + |\nabla u|^2 + qu^2 \,\mathrm{dx}\Big]_S^T - \int_S^T \int_\Gamma u'\partial_\nu u \,\mathrm{d}\Gamma\,\mathrm{dt} \\ &= \Big[\tfrac{1}{2}\int_\Omega (u')^2 + |\nabla u|^2 + qu^2 \,\mathrm{dx} + \tfrac{1}{2}\int_{\Gamma_1} au^2 \,\mathrm{d}\Gamma\,\mathrm{dt}\Big]_S^T + \int_S^T \int_{\Gamma_1} l(u')^2 \,\mathrm{d}\Gamma\,\mathrm{dt} \end{aligned}$$

and (9) follows from the definition of the energy. □

Remark 8.5. — Lemma 8.4 permits us to define $\sqrt{l}u'$ by density as an element of $L^2(\mathbb{R}_+; L^2(\Gamma_1))$, for every $(u^0, u^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$. Then we have in particular

$$lu' \in L^2(\mathbb{R}_+; L^2(\Gamma_1)). \tag{10}$$

The purpose of this section is to show that a particular choice of the feedback (i.e. of l and a) leads to fast energy decay.

Assume that there is a point $x^0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \le 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad m \cdot \nu \ge 0 \quad \text{on} \quad \Gamma_1. \tag{11}$$

We define $Q_1 \ge 0$ as in section 3.4 :

$$Q_1 := 2R\lambda_1^{-1/2} \sup_\Omega q \tag{12}$$

where λ_1 is the biggest constant such that

$$\int_\Omega |\nabla v|^2 + qv^2 \,\mathrm{dx} + \int_{\Gamma_1} av^2 \,\mathrm{d}\Gamma \ge \lambda_1 \int_\Omega v^2 \,\mathrm{dx}$$

for every $v \in H^1_{\Gamma_0}(\Omega)$.

Theorem 8.6. — *Assume* (7.12)–(7.15), (11),

$$Q_1 < 1, \tag{13}$$

and choose

$$l := (m\cdot\nu)/R \quad \textit{and} \quad a := (n-1)(m\cdot\nu)/(2R^2). \tag{14}$$

Then for every given $(u^0, u^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *the solution of* (4)–(7) *satisfies the following estimate* :

$$E(t) \le E(0)\exp\bigl(1-(1-Q_1)t/2R\bigr), \quad \forall t \in \mathbb{R}_+. \tag{15}$$

Theorem 8.6 generalizes a former result in Komornik [4] which improved earlier theorems of Slemrod [1], Quinn and Russell [1], Russell [1], Chen [1], [2], Lagnese [1], Triggiani [1], Komornik and Zuazua [1]. As for the case $n = 2$ we refer to Tcheugoue [1].

Remark 8.7. — More general feedbacks will be considered later in section 8.4 (but the decay estimates will be weaker).

The proof of the theorem will be based on the following identity where we set

$$k := 1/R, \quad b := (n-1)/(2R^2) \quad \text{and} \quad Mu := 2m\cdot\nabla u + (n-1)u$$

for brevity.

Lemma 8.8. — *Given* $(u^0, u^1) \in D(\mathcal{A})$ *and* $0 \le S < T < \infty$ *arbitrarily, the solution of* (4)–(7) *satisfies the following identity* :

$$\begin{aligned} 2\int_S^T E\ dt &= \Bigl[\int_\Omega u' Mu\ \mathrm{dx}\Bigr]_T^S \\ &- \int_S^T \int_\Omega (n-2)qu^2 + 2qum\cdot\nabla u\ \mathrm{dx\ dt} + \int_S^T \int_{\Gamma_0} |\partial_\nu u|^2\ \mathrm{d}\Gamma_m\ \mathrm{dt} \\ &+ \int_S^T \int_{\Gamma_1} (u')^2 - |\nabla u|^2 + bu^2 - (ku' + bu)Mu\ \mathrm{d}\Gamma_m\ \mathrm{dt}. \end{aligned} \tag{16}$$

Proof. — Recall the identity (3.18) which was proved for every sufficiently smooth function u satisfying (4) :

$$\begin{aligned} &\int_S^T \int_\Gamma (\partial_\nu u)Mu + (m\cdot\nu)((u')^2 - |\nabla u|^2)\mathrm{d}\Gamma\ \mathrm{dt} \\ &= \Bigl[\int_\Omega u' Mu\ \mathrm{dx}\Bigr]_S^T + \int_S^T \int_\Omega (u')^2 + |\nabla u|^2 + (n-1)qu^2 + 2qum\cdot\nabla u\ \mathrm{dx\ dt}. \end{aligned}$$

Using (8) hence we deduce that

$$\begin{aligned}\int_S^T \int_\Gamma (\partial_\nu u)Mu + (m\cdot\nu)((u')^2 - |\nabla u|^2)\ \mathrm{d}\Gamma\ \mathrm{dt}\\ = \Big[\int_\Omega u'Mu\ \mathrm{dx}\Big]_S^T + 2\int_S^T E\ \mathrm{dt} \qquad (17)\\ + \int_S^T \int_\Omega (n-2)qu^2 + 2qum\cdot\nabla u\ \mathrm{dx}\ \mathrm{dt} - \int_S^T \int_{\Gamma_1} bu^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}.\end{aligned}$$

Using the boundary conditions (5) et (6) we may replace the left-hand side of (17) by

$$\int_S^T \int_{\Gamma_0} |\partial_\nu u|^2\ \mathrm{d}\Gamma_m\ \mathrm{dt} + \int_S^T \int_{\Gamma_1} (u')^2 - |\nabla u|^2 - (ku'+bu)Mu\ \mathrm{d}\Gamma_m\ \mathrm{dt},$$

and (16) follows. ∎

Proof of theorem 8.6. — Introduce R_1 as in section 3.4, then $R_1 = R$ by the particular choice of b and by remark 3.20.

Let us first fix $(u^0, u^1) \in D(\mathcal{A})$ arbitrarily. Using lemma 3.22 (it remains valid for every $t \in \mathbb{R}_+$) the first term on the right-hand side of (16) is majorized by $2RE(S) + 2RE(T)$. Using the definition of Q_1 and E, the second term is majorized by $2Q_1 \int_S^T E\ \mathrm{dt}$. By (11) the third term is ≤ 0. Thus we deduce from (16) the following inequality :

$$\begin{aligned}2(1-Q_1)\int_S^T E\ \mathrm{dt} \le 2RE(S) + 2RE(T)\\ \qquad (18)\\ + \int_S^T \int_{\Gamma_1} (u')^2 - |\nabla u|^2 + bu^2 - (ku'+bu)Mu\ \mathrm{d}\Gamma_m\ \mathrm{dt}.\end{aligned}$$

In order to majorize the last term of this inequality we note that, by the particular choice (cf. (14)) of the coefficients k and b we have

$$\begin{aligned}&(u')^2 - |\nabla u|^2 + bu^2 - (ku'+bu)Mu\\ &\le (u')^2 - |\nabla u|^2 + bu^2 - 2(ku'+bu)m\cdot\nabla u + (1-n)u(ku'+bu)\\ &\le (u')^2 + bu^2 + R^2(ku'+bu)^2 + (1-n)u(ku'+bu)\\ &= (u')^2 + b(2-n+R^2b)u^2 + R^2k^2(u')^2 + (1-n+2R^2b)kuu'\\ &= 2(u')^2 + ((3-n)/2)bu^2.\end{aligned}$$

In view of condition (8) we conclude from (18) that

$$2(1-Q_1)\int_S^T E\ \mathrm{dt} \le 2RE(S) + 2RE(T)$$
$$+2\int_S^T\int_{\Gamma_1}(u')^2\ \mathrm{d}\Gamma_m\ \mathrm{dt} + \frac{3-n}{2}\int_S^T\int_{\Gamma_1} bu^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}.$$

Applying lemma 8.4 and using the definition of k we conclude that

$$2(1-Q_1)\int_S^T E\ \mathrm{dt} \le 4RE(S) + \frac{3-n}{2}\int_S^T\int_{\Gamma_1} bu^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}. \tag{19}$$

Since $n \ge 3$, the last term is ≤ 0. Therefore we deduce from (19) that

$$2(1-Q_1)\int_S^T E\ \mathrm{dt} \le 4RE(S).$$

Letting $T \to +\infty$ we obtain for every fixed $S \in \mathbb{R}_+$ the estimate

$$2(1-Q_1)\int_S^\infty E\ \mathrm{dt} \le 4RE(S).$$

Using (13) and applying theorem 8.1 hence (15) follows for every $(u^0, u^1) \in D(\mathcal{A})$.

Now fix $(u^0, u^1) \in \mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ arbitrarily. Choose (using lemma 7.7) a sequence of initial data $(u^0_j, u^1_j) \in D(\mathcal{A})$ converging to (u^0, u^1) in $\mathcal{H}$ and apply the estimate (15) for each (u^0_j, u^1_j) :

$$E_j(t) \le E_j(0)\exp\Big(1-(1-Q_1)t/(2R)\Big), \quad \forall t \in \mathbb{R}_+.$$

It follows from property (7.19) in theorem 7.4 we have $E_j(t) \to E(t)$ as $j \to \infty$, for each fixed $t \in \mathbb{R}_+$. Passing to the limit in the above estimates hence (15) follows. $\square$

Remark 8.9. — We observed in remarks 3.19 and 3.25 that hypotheses (7.13) and (11) *together* are very restrictive. It was shown in Komornik and Zuazua [1] that in dimension $n \le 3$ hypothesis (8) is not necessary. For $n > 3$ its necessity remains an interesting open problem.

8.3. Application to the exact controllability. Russell's principle

By a general principle of Russell [2] the stabilizability of a linear reversible system implies its exact controllability. We apply here the corresponding construction to deduce from theorem 8.6 the following generalization (for $n \geq 3$) of theorem 4.8 on the exact controllability of the problem

$$y'' - \Delta y + qy = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{20}$$
$$y = 0 \quad \text{on} \quad \Gamma_0 \times (0,T), \tag{21}$$
$$\partial_\nu y + ay = v \quad \text{on} \quad \Gamma_1 \times (0,T), \tag{22}$$
$$y(0) = y^0 \quad \text{and} \quad y'(0) = y^1 \quad \text{on} \quad \Omega \tag{23}$$

where we eliminate hypotheses (4.39) and (4.40) sur a.

Theorem 8.10. — *Assume* (11), (13) *and let*

$$T > 2R/(1 - Q_1). \tag{24}$$

Then for every given

$$(y^0, y^1),\ (y_T^0, y_T^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$$

there exists a control $v \in L^2(0,T; L^2(\Gamma_1))$ *such that the solution of* (20)–(23) *satisfies*

$$y(T) = y_T^0 \quad \textit{and} \quad y'(T) = y_T^1 \quad \textit{on} \quad \Omega. \tag{25}$$

(Compare with theorem 6.20.)

Proof. — As in chapter 4, we may assume that $y_T^0 = y_T^1 = 0$. First consider the case

$$a \equiv (n-1)(m \cdot \nu)/(2R^2)$$

and set $l := (m \cdot \nu)/R$ for brevity.

Given $(u^0, u^1) \in \mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ arbitrarily, first solve the problem

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+,$$
$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+,$$
$$\partial_\nu u + au + lu' = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+,$$
$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{on} \quad \Omega,$$

then the problem

$$z'' - \Delta z + qz = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+,$$
$$z = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+,$$
$$\partial_\nu z + az + lz' = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+,$$
$$z(0) = -u(T) \quad \text{and} \quad z'(0) = u'(T) \quad \text{on} \quad \Omega,$$

and define

$$y(t) := u(t) + z(T-t), \quad t \in [0,T],$$

$$v(t) := -l\big((u'(t) + z'(T-t)\big), \quad t \in [0,T].$$

Then $v \in L^2(0,T;L^2(\Gamma_1))$ by remark 8.5 and y satisfies (20), (21), (22), (25) and

$$y(0) = u^0 + z(T) \quad \text{and} \quad y'(0) = u^1 - z'(T) \quad \text{on} \quad \Omega.$$

Therefore it is sufficient to show that for any given $(y^0, y^1) \in \mathcal{H}$ there exists $(u^0, u^1) \in \mathcal{H}$ such that

$$y^0 = u^0 + z(T) \quad \text{and} \quad y^1 = u^1 - z'(T) \quad \text{on} \quad \Omega.$$

In other words, it suffices to show that the linear map

$$L : \mathcal{H} \to \mathcal{H}$$

defined by

$$L(u^0, u^1) := (u^0 + z(T), u^1 - z'(T))$$

is onto. Since $L = I - K$ where the linear map

$$K : \mathcal{H} \to \mathcal{H}$$

is defined by

$$K(u^0, u^1) := (-z(T), z'(T)),$$

it is sufficient to verify that $\|K\| < 1$. Indeed, then L is invertible with

$$L^{-1} = I + K + K^2 + K^3 + \cdots$$

Now a twofold application of theorem 8.6 gives

$$\|K(u^0, u^1)\| \le \exp\big(1 - (1 - Q_1)T/(2R)\big)^{1/2} \|(-u(T), u'(T))\|$$

$$\le \exp\big(1 - (1 - Q_1)T/(2R)\big) \|(u^0, u^1)\| =: \gamma \|(u^0, u^1)\|$$

and the result follows because $\gamma < 1$ by hypothesis (24).

Now consider the general case. Using the above special case, for arbitrarily given

$$(y^0, y^1),\ (y_T^0, y_T^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$$

there exists $v \in L^2(0,T; L^2(\Gamma_1))$ such that the solution of problem (20)–(23), with a replaced by $a' := (n-1)(m \cdot \nu)/(2R^2)$, satisfies (25). It follows by an obvious algebraic manipulation that the solution of the original problem also has this property if we apply the control $v + (a - a')y$ instead of v. Since this new control also belongs to $L^2(0,T; L^2(\Gamma_1))$ by the regularity of the solutions of (20)–(23) (cf. theorem 7.4), the proof is completed. $\square$

8.4. Uniform stabilization of the wave equation II

Let us return to the dissipative problem of sections 7.2 and 8.2 :

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{26}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+, \tag{27}$$

$$\partial_\nu u + au + lu' = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{28}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{on} \quad \Omega. \tag{29}$$

Assume that there is a point $x^0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \le 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad \min_{\Gamma_1} m \cdot \nu > 0 \tag{30}$$

and assume that

$$\min_{\Gamma_1} a > 0 \tag{31}$$

and

$$\min_{\Gamma_1} l > 0. \tag{32}$$

Furthermore, we continue to assume that

$$Q_1 < 1. \tag{33}$$

Theorem 8.11. — *Assume* (30)–(33)*. Then there exist two positive constants C, ω such that for any given $(u^0, u^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ the solution of* (26)–(29) *satisfies the estimate*

$$E(t) \le CE(0)e^{-\omega t}, \quad \forall t \in \mathbb{R}_+. \tag{34}$$

Proof. — Repeating the density argument of the proof of theorem 8.6 we may assume that $(u^0, u^1) \in \mathcal{D}(\mathcal{A})$.

By (30)–(32) we may write $a = (m \cdot \nu)b$ and $l = (m \cdot \nu)k$ with suitable *positive* functions $b, k \in C^\infty(\Gamma_1)$. Then lemmas 8.4 and 8.8 remain valid. Applying lemma 3.22 we deduce from identity (16) the inequality

$$
\begin{aligned}
2(1-Q_1)\int_S^T E \ \mathrm{d}t &\le 2R_1E(S) + 2R_1E(T) \\
&+ \int_S^T \int_{\Gamma_1} (u')^2 - |\nabla u|^2 + bu^2 - (ku' + bu)Mu \ \mathrm{d}\Gamma_m \ \mathrm{d}t
\end{aligned}
\tag{35}
$$

(Compare with (18).) Since

$$
2(ku' + bu)(m \cdot \nabla u) \le |\nabla u|^2 + 2R^2k^2(u')^2 + R^2b^2u^2
$$

and

$$
2(ku' + bu)u \le k^2|u'|^2 + (b+1)|u|^2,
$$

hence we deduce that

$$
\begin{aligned}
2(1-Q_1)\int_S^T E \ \mathrm{d}t &\le cE(S) + cE(T) \\
&+c_1 \int_S^T \int_{\Gamma_1} (u')^2 + u^2 \ \mathrm{d}\Gamma \ \mathrm{d}t;
\end{aligned}
\tag{36}
$$

here and in the sequel we shall denote by c diverse constants, independent of the initial data and of S, T.

By (9) and (32) we have $E(T) \le E(S)$ and

$$
\int_S^T \int_{\Gamma_1} (u')^2 \ \mathrm{d}\Gamma \ \mathrm{d}t \le cE(S).
$$

Therefore we deduce from (36) that

$$
2(1-Q_1)\int_S^T E \ \mathrm{d}t \le cE(S) + c_1 \int_S^T \int_{\Gamma_1} u^2 \ \mathrm{d}\Gamma \ \mathrm{d}t. \tag{37}
$$

We shall eliminate the term u^2 on the right-hand side by using a method of Conrad and Rao [1] :

Lemma 8.12. — *There exists a constant $c > 0$ such that for every $\varepsilon \in (0,1)$ the solutions of* (26)–(29) *satisfy the inequality*

$$\int_S^T \int_{\Gamma_1} u^2 \, \mathrm{d}\Gamma \, \mathrm{d}t \le \frac{c}{\varepsilon} E(S) + \varepsilon \int_S^T E \, \mathrm{d}t. \tag{38}$$

Proof. — We define (for each fixed $t \in \mathbb{R}_+$) $z = z(t)$ by

$$\Delta z = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad z = u(t) \quad \text{on} \quad \Gamma; \tag{39}$$

then we have the estimates

$$\int_\Omega z^2 \, \mathrm{dx} \le c \int_\Gamma u^2 \, \mathrm{d}\Gamma \le cE \tag{40}$$

by the elliptic regularity theory. Since

$$\int_\Omega \nabla z \cdot \nabla(u - z) \, \mathrm{dx} = -\int_\Omega (\Delta z)(u - z) \, \mathrm{dx} + \int_\Gamma (\partial_\nu z)(u - z) \, \mathrm{d}\Gamma = 0$$

by (39), we have

$$\int_\Omega \nabla z \cdot \nabla u \, \mathrm{dx} = \int_\Omega |\nabla z|^2 \, \mathrm{dx} \ge 0. \tag{41}$$

Since z' satisfies (46) with u replaced by u', we also have

$$\int_\Omega |z'|^2 \, \mathrm{dx} \le \int_\Gamma |u'|^2 \, \mathrm{d}\Gamma \le -cE'. \tag{42}$$

(Here we use (30), (32) and lemma 8.4.)

Now we have

$$\begin{aligned}
0 &= \int_S^T \int_\Omega z(u'' - \Delta u) \, \mathrm{dx} \, \mathrm{dt} \\
&= \Big[\int_\Omega zu' \, \mathrm{dx}\Big]_S^T + \int_S^T \int_\Omega -z'u' + \nabla z \cdot \nabla u \, \mathrm{dx} \, \mathrm{dt} - \int_S^T \int_\Gamma z\partial_\nu u \mathrm{d}\Gamma \, \mathrm{dt} \\
&= \Big[\int_\Omega zu' \, \mathrm{dx}\Big]_S^T + \int_S^T \int_\Omega -z'u' + \nabla z \cdot \nabla u \, \mathrm{dx} \, \mathrm{dt} + \int_S^T \int_{\Gamma_1} u(au + lu') \mathrm{d}\Gamma \, \mathrm{dt}.
\end{aligned}$$

Using (40)–(42) hence we deduce that

$$\begin{aligned}
\int_S^T \int_{\Gamma_1} au^2 \, \mathrm{d}\Gamma \, \mathrm{dt} &\le \Big[\int_\Omega zu' \, \mathrm{dx}\Big]_T^S + \int_S^T \int_\Omega z'u' \, \mathrm{dx} \, \mathrm{dt} - \int_S^T \int_{\Gamma_1} luu' \, \mathrm{d}\Gamma \, \mathrm{dt} \\
&\le \|z(S)\|_H \|u'(S)\|_H + \|z(T)\|_H \|u'(T)\|_H \\
&\quad + \int_S^T \|z'(t)\|_H \|u'(t)\|_H \, \mathrm{dt} + c\int_S^T \int_{\Gamma_1} |uu'| \, \mathrm{d}\Gamma \, \mathrm{dt} \\
&\le cE(S) + cE(T) + c\int_S^T (-E')^{1/2} E^{1/2} \, \mathrm{dt} + c\int_S^T \int_{\Gamma_1} |uu'| \, \mathrm{d}\Gamma \, \mathrm{dt}.
\end{aligned}$$

Fix $\alpha > 0$ such that $a \ge 2\alpha$ on Γ_1 (cf. (31)). Since the energy is non-increasing, using (9), (32) we obtain that

$$\begin{aligned}2\alpha \int_S^T \int_{\Gamma_1} u^2 \,\mathrm{d}\Gamma \,\mathrm{dt} &\le \int_S^T \int_{\Gamma_1} au^2 \,\mathrm{d}\Gamma \,\mathrm{dt}\\ \le cE(S) + \int_S^T \alpha\varepsilon E - \frac{c}{\alpha\varepsilon} E' \,\mathrm{dt} &+ \int_S^T \int_{\Gamma_1} \alpha u^2 + \frac{c}{\alpha}(u')^2 \,\mathrm{d}\Gamma \,\mathrm{dt}\\ \le \frac{c}{\varepsilon} E(S) + \alpha\varepsilon \int_S^T E \,\mathrm{dt} &+ \alpha \int_S^T \int_{\Gamma_1} u^2 \,\mathrm{d}\Gamma \,\mathrm{dt}\end{aligned}$$

which implies (38). $\square$

We conclude from (37) and (38) that

$$2(1 - Q_1 - c_1\varepsilon) \int_S^T E \,\mathrm{dt} \le \frac{c}{\varepsilon} E(S), \quad \forall 0 \le S < T < +\infty$$

for every $0 < \varepsilon < 1$. By hypothesis (33) we may choose ε such that $1 - Q_1 - \varepsilon > 0$. Then the estimate (34) follows by applying theorem 8.1. $\square$

8.5. Strong stabilization. LaSalle's principle

We consider in this section the problem

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \tag{43}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}, \tag{44}$$

$$\partial_\nu u + au + lu' = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}, \tag{45}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{on} \quad \Omega \tag{46}$$

under weaker hypotheses as before. Assume that there is a point $x^0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \le 0 \quad \text{on} \quad \Gamma_0, \quad m \cdot \nu \ge 0 \quad \text{on} \quad \Gamma_1, \tag{47}$$

and that

$$l > 0 \quad \text{on} \quad \Gamma_+. \tag{48}$$

Theorem 8.13. — *Assume* (47) *and* (48). *Then the solution of* (43)–(46) *satisfies*

$$E(t) \to 0, \quad \forall (u^0, u^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega). \tag{49}$$

The hypotheses of this theorem are much weaker than those of theorem 8.11, but the conclusion is also weaker : in general there is no exponential energy decay.

Note that there is no hypothesis involving Q_1.

The theorem remains valid (with a simple modification of the proof) for the *nonlinear* problem (7.8)–(7.11) if the function g is locally Lipschitz and if it satisfies the following two conditions :

$$g(x) \neq 0 \text{ if } x \neq 0;$$

$$\exists c' > 0 : \quad |g'(x)| \leq 1 + c'|x|^{n/(n-2)}, \quad \forall x \in \mathbb{R}.$$

(For this we need a nonlinear generalization of lemma 8.4; cf. lemma 9.7 in the following chapter.)

More general results are obtained in Chen and Wang [1], Lasiecka [1], [3], Zuazua [6].

Proof. — Using the density of $D(\mathcal{A})$ in $\mathcal{H}$ and the inequality (7.19) we may assume that

$$(u^0, u^1) \in D(\mathcal{A}).$$

Fix $(u^0, u^1) \in D(\mathcal{A})$ arbitrarily. Then the set

$$\{(u(t), u'(t)) : t \in \mathbb{R}_+\}$$

is bounded in $H^2(\Omega) \times H^1(\Omega)$ by theorem 7.5 and therefore it is precompact in $\mathcal{H}$ by Rellich's theorem. In order to prove the relation $E(t) \to 0$ it suffices to show that if for some increasing sequence $t_n \to \infty$ the sequence $U(t_n) = (u(t_n), u'(t_n))$ converges in $\mathcal{H}$ to some point (z^0, z^1), then $z^0 = z^1 \equiv 0$.

Set

$$z_n(t) := u(t_n + t), \quad t \in \mathbb{R}_+, \quad n = 1, 2, \ldots,$$

then z_n is the solution of (43)–(46) with (u^0, u^1) replaced by $(u(t_n), u'(t_n))$. It follows from inequality (7.19) of theorem 7.4 that the sequence (z_n, z_n') is precompact in $L^\infty(\mathbb{R}_+; \mathcal{H})$. Extracting a subsequence if needed, we may assume that

$$(z_n, z_n') \to (z, z') \quad \text{in} \quad L^\infty(\mathbb{R}_+; \mathcal{H}).$$

Clearly, z satisfies (43)–(46) with (u^0, u^1) replaced by (z^0, z^1). We are going to show that $z \equiv 0$.

Observe that $E(z)$ is constant. Indeed, we have

$$E(z;t) = \lim_{n\to\infty} E(z_n;t) = \lim_{n\to\infty} E(u;t_n+t) = \lim_{s\to\infty} E(u;s), \quad \forall t \in \mathbb{R}_+;$$

the last limit exists by the non-increasingness of the energy and it is independent of t. Using (9) and (48) hence we deduce that

$$z' \equiv 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+. \tag{50}$$

Now we may apply lemma 6.16 from chapter 6. Indeed, it follows from (50) that z satisfies (6.41)–(6.44) with (u^0, u^1) replaced by (z^0, z^1), and condition (6.49) is satisfied with any interval J. Applying the lemma we obtain $z \equiv 0$ and hence $z^0 = z^1 \equiv 0$. ☐

The method of the above proof is due to LaSalle [1]; also see Haraux [5] for various applications of this method to nonlinear partial differential equations.

Remark 8.14. — Alternatively, theorem 8.12 could have been obtained by applying a general theorem of Benchimol [1], based on the decomposition theory of semigroups developed by Sz.-Nagy and Foias [1] and Foguel [1]. This theorem permits one to reduce the problem, in the *linear* case, to the study of the eigenvalues of $\mathcal{A}$. Generalizing some standard finite-dimensional results, it is sufficient to prove that $\mathcal{A}$ does not have purely imaginary eigenvalues. (Since $\mathcal{A}$ is monotone, it is already known that no eigenvalue may have negative real part.) See e.g. Lagnese [2] for the application of Benchimol's theorem to the stabilization of some plate models.

8.6. Uniform stabilization of the wave equation III

We shall improve here theorem 8.10 of section 8.4 concerning the problem

$$u'' - \Delta u + qu = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{51}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+, \tag{52}$$

$$\partial_\nu u + au + lu' = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{53}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{on} \quad \Omega, \tag{54}$$

by applying the strong stabilization result obtained in the preceding section; we shall eliminate hypothesis (33) on Q_1. The method of proof, introduced in Komornik and Rao [1], plays a similar role in stabilization problems to the method of chapter 5 in exact controllability problems.

Assume that there is a point $x^0 \in \mathbb{R}^n$ such that

$$m \cdot \nu \le 0 \quad \text{on} \quad \Gamma_0, \quad \min_{\Gamma_1} m \cdot \nu > 0, \tag{55}$$

and assume that

$$\min_{\Gamma_1} a > 0, \quad \min_{\Gamma_1} l > 0. \tag{56}$$

Theorem 8.15. — *Assume* (55), (56). *There exist two positive constants* C, ω *such that for every given* $(u^0, u^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *the solution of* (51)–(54) *satisfies the estimate*

$$E(t) \le CE(0)e^{-\omega t}, \quad \forall t \in \mathbb{R}_+. \tag{57}$$

The proof, given in Komornik and Rao [1], is based on the following useful general result of Gibson [1] which is admitted here without proof.

Theorem 8.16. — *Let* $\mathcal{A}$ *be a maximal monotone* linear *operator in a Hilbert space* $\mathcal{H}$ *and assume that the solutions of the problem*

$$U' + \mathcal{A}U = 0 \quad in \quad \mathbb{R}_+, \quad U(0) = U^0 \tag{58}$$

are strongly stable :

$$\|U(t)\|_{\mathcal{H}} \to 0 \text{ as } t \to +\infty, \quad \forall U^0 \in \mathcal{H}. \tag{59}$$

Assume that there exists a compact linear *operator* $\mathcal{B}$ *in* $\mathcal{H}$ *such that the solutions of the problem*

$$V' + \mathcal{A}V + \mathcal{B}V = 0 \quad in \quad \mathbb{R}_+, \quad V(0) = V^0 \tag{60}$$

are uniformly exponentially stable : *there exist two positive constants* C_1, ω_1 *such that*

$$\|V(t)\|_{\mathcal{H}} \le C_1 \|V^0\|_{\mathcal{H}} e^{-\omega_1 t}, \quad \forall t \in \mathbb{R}_+, \quad \forall V^0 \in \mathcal{H}. \tag{61}$$

Then the solutions of (58) *are also uniformly exponentially stable* : *there exist two positive constants* C, ω *such that*

$$\|U(t)\|_{\mathcal{H}} \le C \|U^0\|_{\mathcal{H}} e^{-\omega t}, \quad \forall t \in \mathbb{R}_+, \quad \forall U^0 \in \mathcal{H}. \tag{62}$$

Gibson's theorem generalized an earlier theorem of Russell [1].

Note that the operator $\mathcal{A} + \mathcal{B}$ is not necessarily maximal monotone; nevertheless, the problem (60) has a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ for every

$V^0 \in \mathcal{H}$. Indeed, the following more general holds true (see e.g. Brézis [1]) : if $\mathcal{A}$ is a (not necessarily linear) maximal monotone operator in a Hilbert space $\mathcal{H}$ and if $\mathcal{B}$ is a (non necessarily linear) Lipschitz continuous operator in $\mathcal{H}$, then problem (60) has a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ for every $V^0 \in \overline{D(\mathcal{A})}$.

Proof of theorem 8.15. — By hypotheses (55), (56) the conditions (47), (48) of theorem 8.13 are satisfied. Consequently, the solutions of problem (51)–(54) (i.e. of (58)) are strongly stable.

Furthermore, the formula

$$\mathcal{B}V := (0, -qV_1), \quad V = (V_1, V_2) \in \mathcal{H}$$

defines a compact linear operator in $\mathcal{H}$ (because of the compactness of the imbedding $H^1(\Omega) \subset L^2(\Omega)$). The solutions of the corresponding problem (60) are uniformly exponentially stable. Indeed, (60) coincides with the problem (26)–(29) where we replace q by zero and therefore we may apply theorem 8.11 : conditions (30)–(32) follow from hypotheses (55), (56), and (33) is satisfied because $Q_1 = 0$.

We conclude by applying theorem 8.16. $\square$

8.7. Uniform stabilization of Maxwell's equations

We consider here the problem

$$E' - \operatorname{curl} H = H' + \operatorname{curl} E \quad \text{in} \quad \Omega \times \mathbb{R}_+ \tag{63}$$

$$\operatorname{div} E = \operatorname{div} H = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+ \tag{64}$$

$$\nu \times (E \times \nu + H) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+ \tag{65}$$

$$E(0) = E^0 \quad \text{and} \quad H(0) = H^0 \quad \text{in} \quad \Omega \tag{66}$$

where Ω is a bounded open domain in $\mathbb{R}^3$ having a boundary Γ of class C^1 and ν denotes the outward unit normal vector to Γ.

Let us introduce the Hilbert space

$$\mathcal{H} := \{(E, H) \in L^2(\Omega)^6 \mid \operatorname{div} E = \operatorname{div} H = 0 \text{ in } \Omega\},$$

$$\|(E, H)\|_{\mathcal{H}} := \Big(\tfrac{1}{2} \int_\Omega |E|^2 + |H|^2 \, \mathrm{dx}\Big)^{1/2},$$

and let us define in $\mathcal{H}$ a linear operator by setting

$$D(\mathcal{A}) := \{(E, H) \in H^1(\Omega)^6 \cap \mathcal{H} \mid \nu \times (E \times \nu + H) = 0 \text{ on } \Gamma\}$$

and

$$\mathcal{A}(E,H) := (-\text{curl } H, \text{curl } E).$$

Then we can rewrite the problem (63)–(66) in the following form :

$$(E,H)' + \mathcal{A}(E,H) = 0 \quad \text{in} \quad \mathbb{R}_+, \quad (E,H)(0) = (E^0, H^0).$$

It was proved by Barucq and Hanouzet [2] that the operator $\mathcal{A}$ is maximal monotone in $\mathcal{H}$. We admit this result here. Applying the Hille-Yosida theorem it follows that for every given $(E^0, H^0) \in \mathcal{H}$ the problem (63)–(66) has a unique (mild) solution

$$(E,H) \in C(\mathbb{R}_+; \mathcal{H});$$

moreover, for $(E^0, H^0) \in D(\mathcal{A})$ we have

$$(E,H) \in C(\mathbb{R}_+; D(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H}),$$

whence in particular

$$(E,H) \in C(\mathbb{R}_+; H^1(\Omega)^6) \cap C^1(\mathbb{R}_+; L^2(\Omega)^6). \tag{67}$$

We define the energy of the solutions by

$$\mathcal{E}(t) := \tfrac{1}{2} \int_\Omega |E(t)|^2 + |H(t)|^2 \text{ dx}, \quad t \in \mathbb{R}_+. \tag{68}$$

Now assume that Ω is *strictly star-shaped* with respect to the origin :

$$x \cdot \nu(x) > 0 \text{ for all } x \in \Gamma \tag{69}$$

and set

$$R := \sup_{x \in \Omega} |x|, \tag{70}$$

$$R_1 := \max_{x \in \Gamma} \frac{(x \cdot \nu(x))^2 + |x|^2}{2x \cdot \nu(x)}. \tag{71}$$

Considering a point $x \in \Gamma$ with $|x| = R$ one can readily verifiy that $R_1 \geq R$. If Ω is a ball centered at the origin, then R is equal to its radious and $R_1 = R$.

We shall prove the

Theorem 8.17. — *Assume* (69). *Then for any given* $(E^0, H^0) \in \mathcal{H}$ *the solution of the problem* (63)-(66) *satisfies the energy estimates*

$$\mathcal{E}(t) \leq \mathcal{E}(0) e^{1-(t/(R+R_1))}, \quad \forall t \geq 0. \tag{72}$$

Applying Russell's principle one can deduce from theorem 8.17 an exact controllability result which improves some earlier theorems of Russell [3] and Lagnese [3]. Moreover, one can show that in the special case of the ball $\Omega = \{x \in \mathbb{R}^3 \big| |x| < R\}$ (then $T_0 = 2R$) these results are optimal. We refer to Komornik [13] for the proof of these results.

Turning to the proof of the theorem first we note that it suffices to prove the estimates (72) for the case of smooth initial data $(E^0, H^0) \in D(\mathcal{A})$: the general case then follows easily by density. Henceforth we only consider such solutions; in this case the regularity (67) of the solution is sufficient to justify the computations of this section.

We need some lemmas. The first shows in particular that the energy is non-increasing.

Lemma 8.18. — *The solution of the problem* (63)–(66) *satisfies the energy equalities*

$$\mathcal{E}(S) - \mathcal{E}(T) = \int_S^T \int_\Gamma |E_\tau|^2 \mathrm{d}\Gamma\ \mathrm{dt} = \int_S^T \int_\Gamma |H_\tau|^2 \mathrm{d}\Gamma\ \mathrm{dt} \tag{73}$$

for all $0 \le S < T < +\infty$, *where* E_τ, H_τ *denote the tangential components of* E, Γ.

Proof. — Applying Green's formula we deduce easily from (63) and (68) that

$$\mathcal{E}'(t) \equiv -\int_\Gamma (E(t) \times H(t)) \cdot \nu \ \mathrm{d}\Gamma, \quad t \ge 0. \tag{74}$$

It remains to show that

$$(E \times H) \cdot \nu = |E_\tau|^2 = |H_\tau|^2 \tag{75}$$

at every fixed point $x \in \Gamma$. Let us choose the system of coordinates such that $\nu = (0, 0, 1)$ at this point. Then

$$\nu \times (E \times \nu + H) = (E_1 - H_2, E_2 + H_1, 0). \tag{76}$$

Using (65) hence we conclude that $E_1 = H_2$, $E_2 = -H_1$ and therefore

$$(E \times H) \cdot \nu = E_1 H_2 - E_2 H_1 = E_1^2 + E_2^2 = H_1^2 + H_2^2,$$

which is just another form of (73). $\square$

Remark 8.19. — The formula (76) expresses the geometric meaning of the boundary condition (65) : H_τ is obtained from E_τ by a rotation of angle $\pi/2$ in the positive direction in the tangent plane.

The main tool in our proof of theorem 8.17 is the following identity. In what follows we shall denote by m the identity mapping $m(x) \equiv x$, $x \in \mathbb{R}^3$.

Lemma 8.20. — *The solution of* (63)–(66) *satisfies the identity*

$$\int_S^T \int_\Omega |E|^2 + |H|^2 \,\mathrm{dx}\,\mathrm{dt} = \Big[2\int_\Omega (E \times H)\cdot m \,\mathrm{dx}\Big]_S^T \tag{77}$$

$$+\int_S^T \int_\Gamma (m\cdot\nu)\big(|E|^2+|H|^2\big) - 2(m\cdot E)(\nu\cdot E) - 2(m\cdot H)(\nu\cdot H)\,\mathrm{d\Gamma}\,\mathrm{dt}$$

for all $0 \le S < T < +\infty$.

Proof. — The identity (77) will be obtained by the multiplier method. Let us write the equations (63) explicitly. Putting $E = (E_1, E_2, E_3)$, $H = (H_1, H_2, H_3)$ and writing for brevity $f_{,j} = \partial_j f$ we have

$$E_1' = H_{3,2} - H_{2,3} \quad \text{in} \quad \Omega \times (0, +\infty), \tag{78}$$

$$E_2' = H_{1,3} - H_{3,1} \quad \text{in} \quad \Omega \times (0, +\infty), \tag{79}$$

$$E_3' = H_{2,1} - H_{1,2} \quad \text{in} \quad \Omega \times (0, +\infty), \tag{80}$$

$$H_1' = E_{2,3} - E_{3,2} \quad \text{in} \quad \Omega \times (0, +\infty), \tag{81}$$

$$H_2' = E_{3,1} - E_{1,3} \quad \text{in} \quad \Omega \times (0, +\infty), \tag{82}$$

$$H_3' = E_{1,2} - E_{2,1} \quad \text{in} \quad \Omega \times (0, +\infty). \tag{83}$$

Using (78) and (82) we have

$$\begin{aligned} 2(E_1H_2m_3)' &= 2(H_{3,2}-H_{2,3})H_2m_3 + 2E_1(E_{3,1}-E_{1,3})m_3 \\ &= 2H_{3,2}H_2m_3 - m_3(H_2^2)_3 + 2E_1E_{3,1}m_3 - m_3(E_1^2)_3. \end{aligned}$$

Analogously, using (79) and (81) we have

$$\begin{aligned} 2(E_2H_1m_3)' &= 2(H_{1,3}-H_{3,1})H_1m_3 + 2E_2(E_{2,3}-E_{3,2})m_3 \\ &= m_3(H_1^2)_3 - 2H_{3,1}H_1m_3 + m_3(E_2^2)_3 - 2E_2E_{3,2}m_3. \end{aligned}$$

Integrating by parts their difference in $\Omega \times (S, T)$ and writing $\nu = (\nu_1, \nu_2, \nu_3)$ we obtain

$$\begin{aligned} &\Big[2\int_\Omega E_1H_2m_3 - E_2H_1m_3 \,\mathrm{dx}\Big]_S^T \\ &= \int_S^T \int_\Omega \big(H_1^2+H_2^2+E_1^2+E_2^2\big) - 2m_3H_3(H_{1,1}+H_{2,2}) \\ &\qquad -2m_3E_3(E_{1,1}+E_{2,2})\,\mathrm{dx}\,\mathrm{dt} \\ &\int_S^T \int_\Gamma -m_3\nu_3\big(H_1^2+H_2^2+E_1^2+E_2^2\big) + 2m_3\nu_2(H_2H_3+E_2E_3) \\ &\qquad +2m_3\nu_1(H_1H_3+E_1E_3)\,\mathrm{d\Gamma}\,\mathrm{dt}. \end{aligned}$$

Next we observe that using (64) we have

$$\begin{aligned}\int_\Omega -2m_3H_3(H_{1,1}+H_{2,2})\ \mathrm{dx} &= \int_\Omega 2m_3H_3H_{3,3}\ \mathrm{dx}\\ &= \int_\Omega m_3\big(H_3^2\big)_3\ \mathrm{dx} = \int_\Gamma m_3\nu_3H_3^2\ \mathrm{d\Gamma} - \int_\Omega H_3^2\ \mathrm{dx},\end{aligned}$$

and analogously

$$\int_\Omega -2m_3E_3(E_{1,1}+E_{2,2})\ \mathrm{dx} = \int_\Gamma m_3\nu_3E_3^2\ \mathrm{d\Gamma} - \int_\Omega E_3^2\ \mathrm{dx}.$$

Using these equalities the above identity may be rewritten as

$$\begin{aligned}\Big[2\int_\Omega E_1H_2m_3-E_2H_1m_3\ \mathrm{dx}\Big]_S^T &= \int_S^T\int_\Omega H_1^2+H_2^2-H_3^2+E_1^2+E_2^2-E_3^2\ \mathrm{dx\ dt}\\ &+\int_S^T\int_\Gamma -m_3\nu_3\big(|E|^2+|H|^2\big)+2m_3H_3(\nu\cdot H)+2m_3E_3(\nu\cdot E)\ \mathrm{d\Gamma\ dt}.\end{aligned}$$

Two analogous identities may be obtained by cyclical permutation of the indices $1,2,3$. Summing the three identities we obtain that

$$\begin{aligned}\Big[2\int_\Omega (E\times H)\cdot m\ \mathrm{dx}\Big]_S^T &= \int_S^T\int_\Omega |E|^2+|H|^2\ \mathrm{dx\ dt}\\ &+\int_S^T\int_\Gamma -(m\cdot\nu)\big(|E|^2+|H|^2\big)+2(m\cdot H)(\nu\cdot H)+2(m\cdot E)(\nu\cdot E)\ \mathrm{d\Gamma\ dt}\end{aligned}$$

and this is equivalent to the identity (77). $\square$

Observe that we did not use the boundary condition (65) in the proof of the preceding lemma. Thus the identity (77) remains valid for every function (E,H) satisfying (63), (64) and (67). Now we shall use the boundary condition (65) in order to majorize the boundary integral in (77).

Lemma 8.21. — *We have*

$$\begin{aligned}(m\cdot\nu)\big(|E|^2+|H|^2\big)-2(m\cdot E)(\nu\cdot E)-2(m\cdot H)(\nu\cdot H)&\\ \le R_1\big(|E_\tau|^2+|H_\tau|^2\big)&\end{aligned}\tag{84}$$

on Γ.

Proof. — Putting for brevity $E_\nu := E\cdot\nu$ and $H_\nu := H\cdot\nu$ we have

$$E=E_\tau+E_\nu\nu,\quad H=H_\tau+H_\nu\nu$$

and the left-hand side of (2.15) may be written as

$$(m \cdot \nu)\big(|E_\tau|^2 + |H_\tau|^2 - E_\nu^2 - H_\nu^2\big) - 2(m_\tau \cdot E_\tau)E_\nu - 2(m_\tau \cdot H_\tau)H_\nu.$$

Since we have obviously

$$-2(m_\tau \cdot E_\tau)E_\nu \le (m \cdot \nu)E_\nu^2 + \frac{(m_\tau \cdot E_\tau)^2}{m \cdot \nu}$$

and

$$-2(m_\tau \cdot H_\tau)H_\nu \le (m \cdot \nu)H_\nu^2 + \frac{(m_\tau \cdot H_\tau)^2}{m \cdot \nu},$$

the left-hand side of (84) is less than or equal to

$$(m \cdot \nu)\big(|E_\tau|^2 + |H_\tau|^2\big) + \frac{(m_\tau \cdot E_\tau)^2 + (m_\tau \cdot H_\tau)^2}{m \cdot \nu}. \tag{85}$$

It follows from remark 8.19 that E_τ, H_τ are orthogonal and they have the same length. Therefore the expression (85) is equal to

$$\Big((m \cdot \nu) + \frac{|m_\tau|^2}{2m \cdot \nu}\Big)\big(|E_\tau|^2 + |H_\tau|^2\big) = \frac{(m \cdot \nu)^2 + |m|^2}{2m \cdot \nu}\big(|E_\tau|^2 + |H_\tau|^2\big).$$

Using (71) hence (84) follows. □

Now we are ready to complete the proof of the theorem. Using (73) and (84) the last integral in the identity (77) is less than or equal to

$$2R_1(\mathcal{E}(S) - \mathcal{E}(T)).$$

Furthermore, we have obviously

$$\Big|2\int_\Omega E \cdot (m \times H)\ \mathrm{dx}\Big| \le 2\int_\Omega |m||E||H|\ \mathrm{dx} \le 2R\mathcal{E}.$$

Therefore we deduce from (77) the following inequality :

$$\int_S^T \mathcal{E}(t)\ \mathrm{dt} \le R(\mathcal{E}(S) + \mathcal{E}(T)) + R_1(\mathcal{E}(S) - \mathcal{E}(T)).$$

Since $R_1 \ge R$, the right-hand side is less than or equal to $(R + R_1)\mathcal{E}(S)$. Letting $T \to +\infty$ we obtain that

$$\int_S^{+\infty} \mathcal{E}(t)\ \mathrm{dt} \le (R + R_1)\mathcal{E}(S), \quad \forall S \ge 0. \tag{86}$$

Applying theorem 8.1 hence the estimate (72) follows. □

9. Nonlinear stabilization

The aim of this chapter is to obtain *polynomial* energy decay estimates for the solutions of the wave equation and of a plate model by the application of suitable boundary feedbacks.

As before, we shall use for any given $x^0 \in \mathbb{R}^n$ the notation

$$m(x) = x - x^0, \quad x \in \mathbb{R}^n,$$
$$R = R(x^0) = \sup\{|x - x^0| : x \in \Omega\},$$
$$\mathrm{d}\Gamma_m = (m \cdot \nu)\mathrm{d}\Gamma.$$

9.1. A nonlinear integral inequality

In this section we give a nonlinear generalization of theorem 8.1, which improves some earlier results of Haraux [2] and Lagnese [1].

Theorem 9.1. — *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ := [0, +\infty)$) be a non-increasing function and assume that there are two constants $\alpha > 0$ and $T > 0$ such that*

$$\int_t^\infty E^{\alpha+1}(s)\ \mathrm{ds} \le TE(0)^\alpha E(t), \quad \forall t \in \mathbb{R}_+. \tag{1}$$

Then we have

$$E(t) \le E(0)\Big(\frac{T+\alpha t}{T+\alpha T}\Big)^{-1/\alpha}, \quad \forall t \ge T. \tag{2}$$

Note that inequality (2) is also satisfied for $0 \le t < T$: then it follows from the obvious inequality $E(t) \le E(0)$.

Observe that letting $\alpha \to 0$ in this theorem we obtain theorem 8.1.

Proof. — If $E(0) = 0$, then $E \equiv 0$ and there is nothing to prove. Otherwise, replacing the function E by the function $E/E(0)$ we may assume that $E(0) = 1$ and we have to prove the following estimation holds :

$$E(t) \le \Big(\frac{T+\alpha t}{T+\alpha T}\Big)^{-1/\alpha}, \quad \forall t \ge T. \tag{3}$$

Introduce the function

$$F : \mathbb{R}_+ \to \mathbb{R}_+, \qquad F(t) = \int_t^\infty E^{\alpha+1} ds.$$

It is non-increasing and locally absolutely continuous. Differentiating and using (1) we find that

$$-F' \geq T^{-\alpha-1}F^{\alpha+1} \quad \text{a.e. in} \quad (0,+\infty),$$

whence

$$(F^{-\alpha})' \geq \alpha T^{-\alpha-1} \quad \text{a.e. in} \quad (0,B), \quad B := \sup\{t:\ E(t)>0\}.$$

(Observe that $F^{-\alpha}(t)$ is defined for $t<B$.) Integrating in $[0,s]$ we obtain that

$$(F(s)^{-\alpha} - F(0)^{-\alpha}) \geq \alpha T^{-\alpha-1}s \quad \text{for every} \quad s\in[0,B),$$

whence

$$F(s) \leq (F(0)^{-\alpha} + \alpha T^{-\alpha-1}s)^{-1/\alpha} \quad \text{for every} \quad s\in[0,B). \tag{4}$$

Since $F(s)=0$ if $s\geq B$, this inequality holds in fact for every $s\in\mathbb{R}_+$. Since $F(0)\leq TE(0)^{\alpha+1}=T$ by (1), the right-hand side of (4) is less than equal to

$$(T^{-\alpha}+\alpha T^{-\alpha-1}s)^{-1/\alpha} = T^{(\alpha+1)/\alpha}(T+\alpha s)^{-1/\alpha}.$$

On the other hand, E being nonnegative and non-increasing, the left-hand side of (4) may be estimated as follows :

$$F(s)=\int_s^{+\infty} E^{\alpha+1}\ \mathrm{dt} \geq \int_s^{T+(\alpha+1)s} E^{\alpha+1}\ \mathrm{dt} \geq (T+\alpha s)E(T+(\alpha+1)s)^{\alpha+1}.$$

Therefore we deduce from (4) the estimate

$$(T+\alpha s)E(T+(\alpha+1)s)^{\alpha+1} \leq T^{(\alpha+1)/\alpha}(T+\alpha s)^{-1/\alpha},$$

whence

$$E(T+(\alpha+1)s) \leq \Big(1+\frac{\alpha s}{T}\Big)^{-1/\alpha}, \quad \forall s\geq 0.$$

Choosing $t := T+(\alpha+1)s$ hence the inequality (3) follows. $\square$

Remark 9.2. — The theorem is optimal in the following sense : given $\alpha>0$, $T>0$, $C>0$ and $t'\geq T$ arbitrarily, there exists a non-increasing function $E:\mathbb{R}_+\to\mathbb{R}_+$ satisfying (1) and such that

$$E(0)=C \quad \text{and} \quad E(t')=E(0)\Big(\frac{T+\alpha t'}{T+\alpha T}\Big)^{-1/\alpha}.$$

We leave to the reader to verify that the following example has these properties :

$$E(t) := \begin{cases} C\bigl(1+\alpha C^{-\alpha}t/T\bigr)^{-1/\alpha}, & \text{if } 0 \le t \le t'' \\ C(1+\alpha)^{1/\alpha}\bigl(1+\alpha C^{-\alpha}t'/T\bigr)^{-1/\alpha}, & \text{if } t'' \le t \le t' \\ 0, & \text{if } t > t' \end{cases} \tag{5}$$

where $t'' = (t' - TC^{\alpha})/(\alpha+1)$. Let us also note that for $t < T$ we cannot state more than the trivial estimate $E(t) \le E(0)$. Indeed, for any given $\alpha > 0$, $T > 0$, $C > 0$ and $t' < T$ the function

$$E(t) := \begin{cases} C, & \text{if } 0 \le t \le T \\ 0, & \text{if } t > T \end{cases} \tag{6}$$

satisfies (1) and $E(t') = E(0) = C$.

Remark 9.3. — Assume that E is also continuous. Then the inequalities (3) are strict; in particular, $E(T) < E(0)$. See Komornik [9], [12] for this result, for a detailed study of integral inequalities of type (1) (also for $\alpha < 0$) and for the study of closely related differential inequalities.

9.2. Uniform stabilization of the wave equation I

Fix a point $x^0 \in \mathbb{R}^n$ and consider the nonlinear problem

$$u'' - \Delta u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{7}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+, \tag{8}$$

$$\partial_\nu u + (m \cdot \nu) g(u') = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{9}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{on} \quad \Omega, \tag{10}$$

a particular case of the problem in section 7.2.

We assume that $g : \mathbb{R} \to \mathbb{R}$ is a non-decreasing, continuous function satisfying $g(0) = 0$ and we assume that

$$n \ge 3, \tag{11}$$

$$\Gamma_0 \ne \emptyset \quad \text{and} \quad \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset, \tag{12}$$

$$m \cdot \nu \le 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad m \cdot \nu \ge 0 \quad \text{on} \quad \Gamma_1. \tag{13}$$

(More general situations are considered in Zuazua [5], Conrad and Rao [1] and in Komornik [10].)

We recall that the energy is defined by

$$E := \tfrac{1}{2}\int_\Omega |u'|^2 + |\nabla u|^2 \, \mathrm{dx}.$$

In this section we prove the

Theorem 9.4. — *Assume* (11)–(13). *Assume that there exist $p > 1$ and four constants $c_1, c_2, c_3, c_4 > 0$ such that*

$$c_1|x|^p \le |g(x)| \le c_2|x|^{1/p} \quad \textit{if} \quad |x| \le 1 \tag{14}$$

and

$$c_3|x| \le |g(x)| \le c_4|x| \quad \textit{if} \quad |x| > 1. \tag{15}$$

Then for any given $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ the solution of (7)–(10) *satisfies the estimates*

$$E(t) \le Ct^{2/(1-p)}, \ \forall t > 0 \tag{16}$$

with a constant C only depending on the initial energy $E(0)$ (and in a continuous way).

Theorem 9.4 improves some earlier results of Zuazua [5] and Conrad and Rao [1]. Let us note that the dependence of the constant C on $E(0)$ was studied in detail by Carpio [1] and later by Souplet [1] and Kouémou [1].

Remark 9.5. — A similar result holds for $p = 1$: then (16) is replaced by

$$E(t) \le CE(0)e^{-\omega t}, \ \forall t > 0$$

with two positive constants C, ω, independent of the initial data. Thus we may recover certain results of the preceding chapter.

First we shall prove the theorem under the additional hypothesis that

$$g \quad \text{is globally Lipschitz continuous.}$$

This assumption will be removed at the end of this section.

By lemma 7.7 and by the inequality (7.19) in theorem 7.4 it is sufficient to prove the estimate (16) for smooth initial data $(u_0, u_1) \in D(\mathcal{A})$: the general case then follows by an obvious density argument.

In this case by theorem 7.5 the solution of (7)–(10) is sufficiently smooth to justify all computations that follow.

We begin by generalizing lemma 8.4 :

Lemma 9.6. — *The function $E:\mathbb{R}_+\to\mathbb{R}_+$ is non-increasing, locally absolutely continuous and*

$$E' = -\int_{\Gamma_1}(m\cdot\nu)u'g(u')\,\mathrm{d}\Gamma \quad \textit{a.e. in} \quad \mathbb{R}_+. \tag{17}$$

Proof. — Given $0\le S<T<\infty$ arbitrarily, we have

$$\begin{aligned} 0 &= \int_S^T\int_\Omega u'(u''-\Delta u)\,\mathrm{dx}\,\mathrm{dt}\\ &= \int_S^T\int_\Omega u'u''+\nabla u\cdot\nabla u'\,\mathrm{dx}\,\mathrm{dt} - \int_S^T\int_\Gamma u'\partial_\nu u\,\mathrm{d}\Gamma\,\mathrm{dt}\\ &= \int_S^T\int_\Omega u'u''+\nabla u\cdot\nabla u'\,\mathrm{dx}\,\mathrm{dt} + \int_S^T\int_{\Gamma_1}(m\cdot\nu)u'g(u')\,\mathrm{d}\Gamma\,\mathrm{dt}\end{aligned}$$

whence

$$E(S)-E(T) = \int_S^T\int_{\Gamma_1}(m\cdot\nu)u'g(u')\,\mathrm{d}\Gamma\,\mathrm{dt}. \tag{18}$$

Since $m\cdot\nu\ge 0$ on Γ_1 and $xg(x)\ge 0,\quad \forall x\in\mathbb{R}$, the right-hand side of (18) is nonnegative; hence E is non-increasing. Furthermore, (18) implies that E is locally absolutely continuous and that (17) is satisfied. ∎

Next we generalize the identity of lemma 8.8 :

Lemma 9.7. — *Putting for brevity*

$$Mu := 2m\cdot\nabla u + (n-1)u, \tag{19}$$

for any fixed $0\le S<T<\infty$ *we have*

$$\begin{aligned} &2\int_S^T E^{(p+1)/2}\mathrm{dt} - \int_S^T E^{(p-1)/2}\int_{\Gamma_0}(\partial_\nu u)^2\mathrm{d}\Gamma_m\,\mathrm{dt}\\ &= \Big[E^{(p-1)/2}\int_\Omega u'Mu\mathrm{dx}\Big]_T^S + \frac{p-1}{2}\int_S^T E^{(p-3)/2}E'\int_\Omega u'Mu\mathrm{dx}\,\mathrm{dt}\\ &\quad + \int_S^T E^{(p-1)/2}\int_{\Gamma_1}(u')^2-|\nabla u|^2-g(u')Mu\mathrm{d}\Gamma_m\,\mathrm{dt}.\end{aligned} \tag{20}$$

Proof. — We have

$$\begin{aligned} 0 &= \int_S^T E^{(p-1)/2}\int_\Omega (Mu)(u''-\Delta u)\,\mathrm{dx}\,\mathrm{dt}\\ &= \Big[E^{(p-1)/2}\int_\Omega u'Mu\,\mathrm{dx}\Big]_S^T - \frac{p-1}{2}\int_S^T E^{(p-3)/2}E'\int_\Omega u'Mu\,\mathrm{dx}\,\mathrm{dt}\\ &\quad -\int_S^T E^{(p-1)/2}\int_\Omega u'Mu'+(Mu)(\Delta u)\,\mathrm{dx}\,\mathrm{dt}.\end{aligned} \tag{21}$$

Integrating by parts and using the relation div $m = n$ we transform the interior integral in the last term as follows :

$$\begin{aligned}
&\int_\Omega u'Mu' + (Mu)(\Delta u)\ \mathrm{dx}\\
&= \int_\Omega m\cdot\nabla(u')^2 + (n-1)(u')^2 - \nabla u\cdot\nabla(Mu)\ \mathrm{dx} + \int_\Gamma (Mu)\partial_\nu u\ \mathrm{d}\Gamma\\
&= \int_\Omega m\cdot\nabla(u')^2 + (n-1)(u')^2 - 2|\nabla u|^2 - m\cdot\nabla|\nabla u|^2\\
&\qquad -(n-1)|\nabla u|^2\ \mathrm{dx} + \int_\Gamma (Mu)\partial_\nu u\ \mathrm{d}\Gamma\\
&= -\int_\Omega (u')^2 + |\nabla u|^2\ \mathrm{dx} + \int_\Gamma (m\cdot\nu)((u')^2 - |\nabla u|^2) + (Mu)\partial_\nu u\ \mathrm{d}\Gamma\\
&= -\int_\Omega (u')^2 + |\nabla u|^2\ \mathrm{dx} + \int_{\Gamma_0} -(m\cdot\nu)|\nabla u|^2 + (2m\cdot\nabla u)\partial_\nu u\ \mathrm{d}\Gamma\\
&\qquad + \int_{\Gamma_1} (u')^2 - |\nabla u|^2 - (Mu)g(u')\ \mathrm{d}\Gamma_m.
\end{aligned}$$

Since (8) implies that $\nabla u = \nu\partial_\nu u$ on Γ_0, hence we conclude that

$$\begin{aligned}
\int_\Omega u'Mu' + (Mu)(\Delta u)\ \mathrm{dx} &= -2E + \int_{\Gamma_0} (\partial_\nu u)^2\ \mathrm{d}\Gamma_m\\
&+ \int_{\Gamma_1} (u')^2 - |\nabla u|^2 - (Mu)g(u')\ \mathrm{d}\Gamma_m.
\end{aligned}$$

Substituting into (21) we obtain (20). □

The following lemma is an immediate consequence of the definition of the energy and of the hypothesis $\Gamma_0 \neq \emptyset$. Here and in the sequel c will denote diverse positive constants only depending on $E(0)$.

Lemma 9.8. — *We have*

$$\Big|\int_\Omega u'Mu\ dx\Big| \le cE. \quad \square \tag{22}$$

We deduce from lemma 9.8 and from the non-increasingness of the energy that

$$\Big|E^{(p-1)/2}\int_\Omega u'Mu\ \mathrm{dx}\Big| \le cE^{(p+1)/2} \le cE$$

and

$$\Big|E^{(p-3)/2}E'\int_\Omega u'Mu\ \mathrm{dx}\Big| \le -cE^{(p-1)/2}E' \le -c\big(E^{(p+1)/2}\big)';$$

hence the first and second terms on the right-hand side of the identity (20) are majorized by $cE(S)$. Since the second integral on the left-hand side of (20) is $\le$ by (12), we deduce from (20) that

$$2\int_S^T E^{(p+1)/2}\ \mathrm{dt} \le cE(S) + \int_S^T E^{(p-1)/2} \int_{\Gamma_1} (u')^2 - |\nabla u|^2 - g(u')Mu\ \mathrm{d}\Gamma_m\ \mathrm{dt};$$

Using the definition of Mu and the relation $|m| \le R$, we obtain for any fixed $\varepsilon > 0$ the inequality

$$2\int_S^T E^{(p+1)/2}\ \mathrm{dt} \le cE(S)$$
$$+\int_S^T E^{(p-1)/2} \int_{\Gamma_1} (u')^2 + R^2 g(u')^2 + \varepsilon u^2 + \frac{(n-1)^2}{4\varepsilon} g(u')^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}.$$

Choosing $\varepsilon = \varepsilon(\Omega)$ such that

$$\varepsilon \int_{\Gamma_1} u^2\ \mathrm{d}\Gamma_m \le \tfrac{1}{2} \int_\Omega |\nabla u|^2\ \mathrm{dx}\ (\le E),$$

hence we conclude that

$$\int_S^T E^{(p+1)/2} dt \le cE(S) + c\int_S^T E^{(p-1)/2} \int_{\Gamma_1} (u')^2 + g(u')^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}. \quad (23).$$

We are going to majorize the last integral in (23). Set

$$\Gamma_2 = \{x \in \Gamma_1 : |u'(x)| \le 1\} \quad \text{and} \quad \Gamma_3 = \{x \in \Gamma_1 : |u'(x)| > 1\}. \tag{24}$$

(Note that Γ_2 and Γ_3 depend on $t \in \mathbb{R}_+$.) Using (15) and (17) we obtain that

$$\begin{aligned}&\int_S^T E^{(p-1)/2} \int_{\Gamma_3} (u')^2 + g(u')^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}\\ &\le -c\int_S^T E^{(p-1)/2} E'\ \mathrm{dt} \le cE^{(p+1)/2}(S) \le cE(S).\end{aligned} \tag{25}$$

Furthermore, using (14) we also have

$$\begin{aligned}\int_{\Gamma_2} (u')^2 + g(u')^2\ \mathrm{d}\Gamma_m &\le c\int_{\Gamma_2} (u'g(u'))^{2/(p+1)}\ \mathrm{d}\Gamma_m\\ &\le c\Big(\int_{\Gamma_2} u'g(u')\ \mathrm{d}\Gamma_m\Big)^{2/(p+1)} \le c(-E')^{2/(p+1)};\end{aligned}$$

hence, using the Young inequality we obtain for every $\varepsilon > 0$ the estimate

$$\begin{aligned}\int_S^T E^{(p-1)/2} \int_{\Gamma_2} (u')^2 + g(u')^2 \ \mathrm{d}\Gamma_m \ \mathrm{dt} \\ \leq c \int_S^T E^{(p-1)/2} (-E')^{2/(p+1)} \ \mathrm{dt} \qquad (26) \\ \leq \int_S^T \varepsilon E^{(p+1)/2} - c(\varepsilon) E' \ \mathrm{dt} \leq \varepsilon \int_S^T E^{(p+1)/2} \ \mathrm{dt} + c(\varepsilon) E(S).\end{aligned}$$

Combining (23), (25) and (26) we find that

$$\int_S^T E^{(p+1)/2} \ \mathrm{dt} \leq c(\varepsilon) E(S) + \varepsilon c \int_S^T E^{(p+1)/2} \ \mathrm{dt}.$$

Choosing ε such that $\varepsilon c < 1$, hence we conclude that

$$\int_S^T E^{(p+1)/2} \ \mathrm{dt} \leq c E(S). \qquad (27)$$

Letting $T \to +\infty$ we obtain that

$$\int_S^\infty E^{(p+1)/2} \ \mathrm{dt} \leq c E(S), \quad \forall S \geq 0,$$

and we conclude by applying theorem 9.1 with $\alpha = (p-1)/2$ and $T = cE(0)^{-\alpha}$.

Thus the proof of the theorem is completed in the case of globally Lipschitz continuous functions g. Turning to the general case, let us admit for a moment the following

Lemma 9.9. — *Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, continuous function satisfying the inequalities* (14) *and* (15) *for some $p \geq 1$ and for some positive constants c_i. Then there exists a sequence of non-decreasing, globally Lipschitz continuous functions $g_k : \mathbb{R} \to \mathbb{R}$ satisfying the inequalities*

$$c_1' |x|^p \leq |g_k(x)| \leq c_2' |x|^{1/p} \quad if \quad |x| \leq 1 \qquad (28)$$

and

$$c_3' |x| \leq |g_k(x)| \leq c_4' |x| \quad if \quad |x| > 1. \qquad (29)$$

with suitable positive constants c_i', independent of k, and such that

$$|g_k(x)| \leq |g(x)|, \quad \forall x \in \mathbb{R}, \quad k = 1, 2, \ldots, \qquad (30)$$

and

$$g_k(x) \to g(x), \quad \forall x \in \mathbb{R}, \tag{31}$$

If we replace the function g in the equation (9) by one of the functions g_k, the already proved part of the theorem applies. Using theorem 7.3 the general case of the theorem will follow if we prove that

$$(I + \mathcal{A}_k)^{-1} W \to (I + \mathcal{A})^{-1} W \quad in \quad \mathcal{H}, \quad \forall W \in \mathcal{H} \tag{32}$$

where $\mathcal{A}_k$ denotes the generator of the semigroup associated with the problem (7)–(10) where g is replaced by g_k. Putting $f = W_2 - AW_1$ and denoting by u, u_k the solutions of

$$(I + A + B)u = f \quad \text{and} \quad (I + A + B_k)u_k = f$$

where we use the same notations as in the proof of proposition 7.6 (and B_k denotes the operator analogous to B but corresponding to the function g_k), it is sufficient to prove that $u_k \to u$ in H as $k \to +\infty$.

We have

$$\langle (I + A + B_k)u_k - (I + A + B_k)u, u_k - u \rangle_{V',V} = \langle (B - B_k)u, u_k - u \rangle_{V',V}$$

whence

$$\|u_k - u\|_V^2 \leq \|(B - B_k)u\|_{V'} \|u_k - u\|_V$$

and therefore

$$\|u_k - u\|_V \leq \|g(u) - g_k(u)\|_{L^{(2n-2)/n}(\Gamma_1)}.$$

By (28) and (29) the expression on the right-hand side tends to zero by Lebesgue's dominated convergence theorem. Hence $u_k \to u$ in V and therefore also in H. Since

$$(I + \mathcal{A}_k)^{-1} W = (u_k + W_1, u_k) \quad \text{and} \quad (I + \mathcal{A})^{-1} W = (u + W_1, u),$$

hence (32) follows.

Proof of lemma 9.9. — Set

$$g_k(x) := g\big((\mathrm{id}_{\mathbb{R}} + k^{-1} g)^{-1}(x)\big), \quad x \in \mathbb{R}, \ k = 1, 2, \ldots.$$

One may readily verify that the functions g_k are well-defined, continuous, non-decreasing and that properties (30), (31) are satisfied. It follows from

(14), (15) and (30) that the right-hand side inequalities in (28), (29) are satisfied with $c_2' = c_2$ and $c_4' = c_4$.

To prove their Lipschitz continuity choose $x_1,\ x_2 \in \mathbb{R}$ arbitrarily and set

$$y_i := (\mathrm{id}_{\mathbb{R}} + k^{-1}g)^{-1}(x_i), \quad i = 1, 2.$$

Then we have $g_k(x_i) = g(y_i)$ and therefore

$$x_1 - x_2 = (y_1 - y_2) + k^{-1}\big(g_k(x_1) - g_k(x_2)\big).$$

Using the monotonicity of g hence we conclude that

$$|g_k(x_1) - g_k(x_2)| \le k|x_1 - x_2|.$$

To prove the first inequality in (28) fix $0 \le x \le 1$ arbitrarily (the case $-1 \le x \le 0$ is similar) and set $y := (\mathrm{id}_{\mathbb{R}} + k^{-1}g)^{-1}(x)$. Then $y + k^{-1}g(y) = x$ and $0 \le y \le x$. If $y \ge x/2$, then

$$g_k(x) = g(y) \ge g(x/2) \ge c_1(x/2)^p.$$

If $y < x/2$, then $k^{-1}g(y) \ge x/2$ and therefore

$$g_k(x) = g(y) \ge x/2 \ge (x/2)^p.$$

Hence the first inequality in (28) is satisfied with $c_1' := 2^{-p}\min\{c_1, 1\}$.

To prove the first inequality in (29) let us choose a number $0 < \varepsilon \le 1/2$ such that $\varepsilon + g(\varepsilon) < 1$ and $-\varepsilon + g(-\varepsilon) > -1$. Observe that (14) and (15) imply the existence of a positive constant c_3'' such that

$$|g(y)| \ge c_3''|y| \quad if \quad |y| \ge \varepsilon.$$

Now fix $x \ge 1$ arbitrarily (the case $x \le -1$ is analogous) and set $y := (\mathrm{id}_{\mathbb{R}} + k^{-1}g)^{-1}(x)$ as above. We have $y + k^{-1}g(y) = x$ and $\varepsilon \le y \le x$. If $y \ge x/2$, then

$$g_k(x) = g(y) \ge g(x/2) \ge (c_3''/2)x$$

because $x/2 \ge 1/2 \ge \varepsilon$. If $y < x/2$, then $k^{-1}g(y) \ge x/2$ and therefore

$$g_k(x) = g(y) \ge x/2.$$

Hence the first inequality in (29) is satisfied with $c_3' := 2^{-1}\min\{c_3'', 1\}$. $\square$

9.3. Uniform stabilization of the wave equation II

The theorem 9.4 proved in the preceding section has a serious drawback : it never applies for bounded functions g (because of $c_3 > 0$ in (15)). The purpose of this section is to obtain a variant of this theorem for bounded feedback functions.

Theorem 9.10. — *Assume* (11)–(13) *and assume that the function g is bounded, globally Lipschitz continuous and that the inequalities* (14) *are satisfied with some positive constants c_1, c_2 and with a number p satisfying*

$$p \geq n-1. \tag{30}$$

Then for every

$$(u^0, u^1) \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \tag{31}$$

satisfying

$$\frac{\partial u^0}{\partial \nu} + (m \cdot \nu) g(u^1) = 0 \quad on \quad \Gamma_1, \tag{32}$$

the solution of (7)-(10) *satisfies the estimates*

$$E(t) \leq C t^{2/(1-p)}, \ \forall t > 0 \tag{33}$$

with a constant C depending on the initial data.

Remark 9.11. — Observe that if the condition (30) is not satisfied initially, it will be satisfied if we replace p by $n-1$; at the same time (14) continues to hold with the same constants c_1, c_2.

Proof. — Repeating the proof of the preceding theorem, except that part where the first inequality of (15) involving c_3 is applied, now we obtain the following inequality :

$$\int_S^T E^{(p+1)/2} \,\mathrm{dt} \leq cE(S) + c \int_S^T E^{(p-1)/2} \int_{\Gamma_3} (u')^2 \,\mathrm{d}\Gamma_m \,\mathrm{dt}. \tag{34}$$

It remains to establish the estimate

$$E^{(p-1)/2} \int_{\Gamma_3} (u')^2 \,\mathrm{d}\Gamma_m \leq \varepsilon E^{(p+1)/2} - c(\varepsilon) E' \tag{35}$$

for every $\varepsilon > 0$. Then the theorem will follow. Indeed, choosing a sufficiently small ε, (34)–(35) imply (27) and the proof may be completed by applying theorem 9.1 as in the preceding section.

For brevity we shall denote the norm of $L^\beta(\Gamma)$ by $\| \ \|_\beta$. Set

$$s := 2/(p+1) \quad \text{and} \quad \alpha := (2-s)/(1-s);$$

we have $0 < s < 1$ and $\alpha = 2p/(p-1) > 2$. We establish for every $\varepsilon > 0$ the inequality

$$E^{(p-1)/2} \int_{\Gamma_3} (u')^2 \, d\Gamma_m \le \varepsilon E^{(p+1)/2} \|u'\|_\alpha^\alpha - c(\varepsilon) E'. \tag{36}$$

Indeed, we have

$$\begin{aligned} E^{(p-1)/2} \int_{\Gamma_3} (u')^2 \, d\Gamma_m &\le c E^{(p-1)/2} \int_{\Gamma_3} |u'|^{2-s} (u' g(u'))^s \, d\Gamma_m \\ &\le c E^{(p-1)/2} \| |u'|^{2-s} \|_{1/(1-s)} \| (u' g(u'))^s \|_{1/s} \\ &= c E^{(p-1)/2} \|u'\|_\alpha^{(1-s)\alpha} \|u' g(u')\|_1^s \\ &= c E^{(p-1)/2} \|u'\|_\alpha^{(1-s)\alpha} (-E')^s \le \varepsilon E^{(p-1)/(2(1-s))} \|u'\|_\alpha^\alpha - c(\varepsilon) E' \\ &= \varepsilon E^{(p+1)/2} \|u'\|_\alpha^\alpha - c(\varepsilon) E'. \end{aligned}$$

Using the trace theorem

$$H^1(\Omega) \hookrightarrow L^{2p/(p-1)}(\Gamma) = L^\alpha(\Gamma)$$

(following from (30)) and the regularity property (7.24) we deduce from (36) that

$$E^{(p-1)/2} \int_{\Gamma_3} (u')^2 \, d\Gamma_m \le c\varepsilon E^{(p+1)/2} - c(\varepsilon) E';$$

hence (35) follows (with another ε). $\square$

9.4. Uniform stabilization of Kirchhoff plates

Consider the problem of section 7.3 :

$$u'' + \Delta^2 u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{37}$$

$$u = u_\nu = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+, \tag{38}$$

$$u_{\nu\nu} + \mu u_{\tau\tau} = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{39}$$

$$u_{\nu\nu\nu} + (2-\mu) u_{\tau\tau\nu} = (m \cdot \nu) g(u') \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{40}$$

$$u(0) = u^0 \quad \text{and} \quad u'(0) = u^1 \quad \text{on} \quad \Omega. \tag{41}$$

Assume that there is a point $X^0 \in \mathbb{R}^2$ such that

$$m \cdot \nu \geq 0 \text{ on } \Gamma_1 \quad \text{and} \quad m \cdot \nu \leq 0 \text{ on } \quad \Gamma_0. \tag{42}$$

We shall prove the

Theorem 9.12. — *Assume* (42) *and assume that there exist two numbers* $p, q > 1$ *and two positive constants* c_1, c_2 *such that*

$$c_1|x|^p \leq |g(x)| \leq c_2|x|^{1/p} \quad if \quad |x| \leq 1 \tag{43}$$

and

$$c_1|x| \leq |g(x)| \leq c_2|x|^q \quad if \quad |x| > 1. \tag{44}$$

Then for any given $(u^0, u^1) \in H^2_{\Gamma_0} \times L^2(\Omega)$ *the solution of* (37)–(40) *satisfies the estimates*

$$E(t) \leq Ct^{2/(1-p)}, \quad \forall t > 0 \tag{45}$$

with a constant C *only depending on the initial energy* $E(0)$ (*and in a continuous way*). $\square$

Remark 9.13. — A similar result holds for $p = 1$: then the estimate (45) is replaced by

$$E(t) \leq Ce^{-\omega t}, \quad \forall t > 0$$

with two positive constants C, ω which do not depend on the initial data.

Remark 9.14. — Theorem 9.12 improves some earlier results of Lagnese [2] (whose method is followed here) and of Rao [1] by weakening their growth assumptions and by using one feedback only. Also see Komornik [11] for a more general result.

There are many other theorems concerning the strong or uniform stabilization of different plate and beam models; see e.g. Bartolomeo and Triggiani [1], Horn [1], Horn and I. Lasiecka [1], Lagnese [2], Lagnese and Leugering [1], I. Lasiecka [1], [2], Lasiecka and Tataru [1], Lasiecka and Triggiani [3], Lebeau [1], W. Littmann and L. Markus [1], Littmann and Taylor [1], Puel and Tucsnak [1], Tataru [1].

The rest of this section is devoted to the proof of theorem 9.12. Using lemma 9.9 and the argument following it in section 9.2 we may assume without loss of generality that g is globally Lipschitz. Furthermore, using theorem 7.9 and a density argument we may assume that the solutions verify the regularity conditions (7.53) and (7.54); then all computations that follow are justified.

Lemma 9.15. — *The function $E : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing, locally absolutely continuous and*

$$E' = -\int_{\Gamma_1} u' g(u') \, \mathrm{d}\Gamma_m \quad \textit{a.e. on} \quad \mathbb{R}_+. \tag{46}$$

Proof. — Given $0 \le S < T < +\infty$ arbitrarily, we apply lemma 7.12; we obtain that

$$\begin{aligned}
0 &= \int_S^T \int_\Omega u'(u'' + \Delta^2 u) \, \mathrm{dX} \, \mathrm{dt} = \int_S^T \int_\Omega u'u'' + Q(u, u') \, \mathrm{dX} \, \mathrm{dt} \\
&\quad + \int_S^T \int_\Gamma (u_{\nu\nu} + (2-\mu)u_{\tau\tau})_\nu u' - (u_{\nu\nu} + \mu u_{\tau\tau})u'_\nu \, \mathrm{d}\Gamma \, \mathrm{dt} \\
&= \Big[\frac{1}{2} \int_\Omega (u')^2 + Q(u) \, \mathrm{dX} \Big]_S^T + \int_S^T \int_{\Gamma_1} (m \cdot \nu) u' g(u') \, \mathrm{d}\Gamma \, \mathrm{dt} \\
&= \Big[\frac{1}{2} \int_\Omega (u')^2 + Q(u) \, \mathrm{dX} \Big]_S^T + \int_S^T \int_{\Gamma_1} (m \cdot \nu) u' g(u') \, \mathrm{d}\Gamma \, \mathrm{dt},
\end{aligned}$$

i.e.

$$E(S) - E(T) = \int_S^T \int_{\Gamma_1} u' g(u') \, \mathrm{d}\Gamma_m \, \mathrm{dt}. \tag{47}$$

This implies that E is locally absolutely continuous; differentiating (47) we find (46). Finally, E is non-increasing because the right-hand side of (47) is nonnegative by the increasingness of g and by hypothesis (42). □

Lemma 9.16. — *Given $v \in H^3(\Omega)$ arbitrarily, the following identity holds true:*

$$\int_\Omega Q(v, m \cdot \nabla v) \, \mathrm{dX} = \int_\Omega Q(v) \, dX + \tfrac{1}{2} \int_\Gamma Q(v) \, \mathrm{d}\Gamma_m. \tag{48}$$

Proof. — Integrating by parts we obtain the following five simple identities:

$$\begin{aligned}
\int_\Omega v_{xx}(m_1 v_x + m_2 v_y)_{xx} \, \mathrm{dX} &= \int_\Omega 2v_{xx}^2 + m_1 v_{xx} v_{xxx} + m_2 v_{xx} v_{xxy} \, \mathrm{dX} \\
&= \int_\Omega 2v_{xx}^2 + \tfrac{1}{2} m \cdot \nabla(v_{xx}^2) \, \mathrm{dX},
\end{aligned}$$

$$\begin{aligned}
\int_\Omega v_{yy}(m_1 v_x + m_2 v_y)_{yy} \, \mathrm{dX} &= \int_\Omega 2v_{yy}^2 + m_1 v_{yy} v_{yyx} + m_2 v_{yy} v_{yyy} \, \mathrm{dX} \\
&= \int_\Omega 2v_{yy}^2 + \tfrac{1}{2} m \cdot \nabla(v_{yy}^2) \, \mathrm{dX},
\end{aligned}$$

$$\int_\Omega v_{xx}(m_1v_x+m_2v_y)_{yy}\ \mathrm{dX}=\int_\Omega 2v_{xx}v_{yy}+m_1v_{xx}v_{xyy}+m_2v_{xx}v_{yyy}\ \mathrm{dX},$$

$$\int_\Omega v_{yy}(m_1v_x+m_2v_y)_{xx}\ \mathrm{dX}=\int_\Omega 2v_{xx}v_{yy}+m_1v_{yy}v_{xxx}+m_2v_{yy}v_{yxx}\ \mathrm{dX},$$

$$\begin{aligned}\int_\Omega v_{xy}(m_1v_x+m_2v_y)_{xy}\ \mathrm{dX}&=\int_\Omega 2v_{xy}^2+m_1v_{xy}v_{xxy}+m_2v_{xy}v_{xyy}\ \mathrm{dX}\\&=\int_\Omega 2v_{xy}^2+\tfrac12 m\cdot\nabla(v_{xy}^2)\ \mathrm{dX}.\end{aligned}$$

Combining these identities, using the definitions of $Q(v)$, $Q(u,v)$ and finally applying the divergence theorem we obtain that

$$\begin{aligned}\int_\Omega Q(v,m\cdot\nabla v)\ \mathrm{dX}&=\int_\Omega 2Q(v)+\tfrac12 m\cdot\nabla Q(v)\ \mathrm{dX}=\int_\Omega \big(2-\tfrac12\mathrm{div}m\big)Q(v)\ \mathrm{dX}\\&+\tfrac12\int_\Gamma(m\cdot\nu)Q(v)\ \mathrm{d}\Gamma=\int_\Omega Q(v)\ \mathrm{dX}+\tfrac12\int_\Gamma Q(v)\ \mathrm{d}\Gamma_m.\quad\square\end{aligned}$$

Now we prove our basic identity :

Lemma 9.17. — *Given $0\le S<T<+\infty$ arbitrarily, the following identity holds true* :

$$\begin{aligned}4\int_S^T E^{(p+1)/2}\ \mathrm{dt}-\int_S^T E^{(p-1)/2}\int_{\Gamma_0}(\Delta u)^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}\\=\Big[E^{(p-1)/2}\int_\Omega 2u'm\cdot\nabla u\ \mathrm{dX}\Big]_T^S\\+(p-1)\int_S^T E^{(p-3)/2}E'\int_\Omega u'm\cdot\nabla u\ \mathrm{dX}\ \mathrm{dt}\\+\int_S^T E^{(p-1)/2}\int_{\Gamma_1}(u')^2-Q(u)-2(m\cdot\nabla u)g(u')\ \mathrm{d}\Gamma_m\ \mathrm{dt}.\end{aligned}\tag{49}$$

Proof. — We multiply the equation (39) by $2E^{(p-1)/2}m\cdot\nabla u$ and we integrate by parts in $\Omega\times(S,T)$. Using lemmas 7.12 and 9.16 we obtain that

$$\begin{aligned}0&=\int_S^T E^{(p-1)/2}\int_\Omega 2m\cdot\nabla u(u''+\Delta^2u)\ \mathrm{dX}\ \mathrm{dt}\\&=\Big[E^{(p-1)/2}\int_\Omega 2u'm\cdot\nabla u\ \mathrm{dX}\Big]_S^T-(p-1)\int_S^T E^{(p-3)/2}E'\int_\Omega u'm\cdot\nabla u dXdt\end{aligned}$$

$$+\int_S^T E^{(p-1)/2}\int_\Omega -2u'm\cdot\nabla u' + 2Q(u, m\cdot\nabla u)\ \mathrm{dX\ dt}$$

$$+2\int_S^T E^{(p-1)/2}\int_\Gamma (u_{\nu\nu}+(2-\mu)u_{\tau\tau})_\nu m\cdot\nabla u - (u_{\nu\nu}+\mu u_{\tau\tau})(m\cdot\nabla u)_\nu\ \mathrm{d\Gamma\ dt}$$

$$= \Big[E^{(p-1)/2}\int_\Omega 2u'm\cdot\nabla u\ \mathrm{dX}\Big]_S^T - (p-1)\int_S^T E^{(p-3)/2}E'\int_\Omega u'm\cdot\nabla u\ \mathrm{dX\ dt}$$

$$+\int_S^T E^{(p-1)/2}\int_\Omega -m\cdot\nabla(u')^2 + 2Q(u)\ \mathrm{dX\ dt}$$

$$+\int_S^T E^{(p-1)/2}\int_\Gamma (m\cdot\nu)Q(u) + 2(u_{\nu\nu}+(2-\mu)u_{\tau\tau})_\nu m\cdot\nabla u$$
$$-2(u_{\nu\nu}+\mu u_{\tau\tau})(m\cdot\nabla u)_\nu\ \mathrm{d\Gamma\ dt}$$

$$= \Big[E^{(p-1)/2}\int_\Omega 2u'm\cdot\nabla u\ \mathrm{dX}\Big]_S^T - (p-1)\int_S^T E^{(p-3)/2}E'\int_\Omega u'm\cdot\nabla u\ \mathrm{dX\ dt}$$

$$+2\int_S^T E^{(p-1)/2}\int_\Omega (u')^2 + Q(u)\ \mathrm{dX\ dt}$$

$$+\int_S^T E^{(p-1)/2}\int_\Gamma (m\cdot\nu)(Q(u)-(u')^2) + 2(u_{\nu\nu}+(2-\mu)u_{\tau\tau})_\nu m\cdot\nabla u$$
$$-2(u_{\nu\nu}+\mu u_{\tau\tau})(m\cdot\nabla u)_\nu\ \mathrm{d\Gamma\ dt}.$$

Recalling the definition of the energy we conclude that

$$4\int_S^T E^{(p+1)/2}\ \mathrm{dt}$$

$$= \Big[E^{(p-1)/2}\int_\Omega 2u'm\cdot\nabla u\ \mathrm{dX}\Big]_T^S + (p-1)\int_S^T E^{(p-3)/2}E'\int_\Omega u'm\cdot\nabla u\ \mathrm{dX\ dt}$$

$$+\int_S^T E^{(p-1)/2}\int_\Gamma (m\cdot\nu)((u')^2-Q(u)) - 2(u_{\nu\nu}+(2-\mu)u_{\tau\tau})_\nu m\cdot\nabla u$$
$$+2(u_{\nu\nu}+\mu u_{\tau\tau})(m\cdot\nabla u)_\nu\ \mathrm{d\Gamma\ dt}.$$

It remains to show that

$$(m\cdot\nu)((u')^2-Q(u)) - 2(u_{\nu\nu}+(2-\mu)u_{\tau\tau})_\nu m\cdot\nabla u$$
$$+2(u_{\nu\nu}+\mu u_{\tau\tau})(m\cdot\nabla u)_\nu = (m\cdot\nu)\{(u')^2 - Q(u) - 2(m\cdot\nabla u)(\alpha u + g(u'))\}$$

on Γ_1 and

$$(m\cdot\nu)((u')^2-Q(u)) - 2(u_{\nu\nu}+(2-\mu)u_{\tau\tau})_\nu m\cdot\nabla u$$
$$+2(u_{\nu\nu}+\mu u_{\tau\tau})(m\cdot\nabla u)_\nu = (\Delta u)^2 \qquad (50)$$

on Γ_0.

The first equality follows at once from the boundary conditions (39) and (40).

For the proof of (50) first observe that the boundary conditions (38) imply $u' = u_x = u_y = 0$ on Γ_0. Consequently, the vectors $\nabla u_x = (u_{xx}, u_{xy})$ and $\nabla u_y = (u_{xy}, u_{yy})$ are orthogonal to Γ_0 and therefore they are parallel. Hence $u_{xx}u_{yy} = u_{xy}^2$ and $Q(u) = (\Delta u)^2$ on Γ_0. Furthermore, $m \cdot \nabla u = (m \cdot \nu)u_\nu$ and

$$(m \cdot \nabla u)_\nu = (m_1 u_x + m_2 u_y)_\nu = (m \cdot \nu)u_{\nu\nu} + (m \cdot \tau)u_{\nu\tau} = (m \cdot \nu)u_{\nu\nu}$$

on Γ_0 because $u_\nu = 0$ on Γ_0.

Combining these relations we conclude that the left-hand side of (50) is equal to

$$-(m \cdot \nu)(\Delta u)^2 + 2(m \cdot \nu)u_{\nu\nu}^2.$$

Since the boundary conditions (38) imply that $u_{\tau\tau} = 0$ on Γ_0, we have

$$-(m \cdot \nu)(\Delta u)^2 + 2(m \cdot \nu)u_{\nu\nu}^2 = (m \cdot \nu)(\Delta u)^2.$$

This completes the proof of (50) and that of the lemma. $\square$

In order to majorize the right-hand side of (49) we need the

Lemma 9.18. — *The semi-norm $p : H^2(\Omega) \to \mathbb{R}_+$ defined by*

$$p(v) := \Big(\int_\Omega v_{xx}^2 + v_{xy}^2 + v_{yy}^2 \, \mathrm{dX}\Big)^{1/2} \tag{51}$$

is equivalent to the norm $|| \ ||_{H^2(\Omega)}$ on the subspace $H^2_{\Gamma_0}(\Omega)$.

Proof. — First we show that p is a *norm* on $H^2_{\Gamma_0}(\Omega)$. Indeed, let $v \in H^2_{\Gamma_0}(\Omega)$ such that $p(v) = 0$. Then the second derivative of v vanishes. Since Ω is connexe, hence we conclude that v is affine. Since $\Gamma_0 \neq \emptyset$ and $v = v_x = v_y = 0$ on Γ_0 by hypotheses $\Gamma_0 \neq \emptyset$ and (40), it follows that $v = 0$.

The estimate

$$p \le c|| \ ||_{H^2(\Omega)} \tag{52}$$

is obvious. Assume that the inverse inequality is false. Then there exists a sequence v^k in $H^2_{\Gamma_0}(\Omega)$ such that

$$||v^k||_{H^2(\Omega)} = 1, \quad k = 1, 2, \dots \tag{53}$$

and

$$p(v^k) \to 0. \tag{54}$$

By the compactness of the imbedding $H^2(\Omega) \subset H^1(\Omega)$ we may assume, by extracting a subsequence if needed, that

$$v^k \to v \quad \text{in} \quad H^1(\Omega)$$

for some $v \in H^1(\Omega)$. Using (51) and (54) we conclude that

$$v^k \to v \quad \text{in} \quad H^2(\Omega);$$

consequently, using (52) and (54) we obtain that $v = 0$ and hence

$$v^k \to 0 \quad \text{in} \quad H^2(\Omega).$$

But this contradicts (53). ∎

In what follows c will denote diverse positive constants depending on $E(0)$ only.

Corollary 9.19. — *We have*

$$\Big| \int_\Omega u' m \cdot \nabla u \ \mathrm{dX} \Big| \leq cE. \tag{55}$$

Proof. — Using the trivial inequality

$$\Big| \int_\Omega u' m \cdot \nabla u \ \mathrm{dX} \Big| \leq c\big(||u'||^2_{L^2(\Omega)} + ||u||^2_{H^1(\Omega)} \big)$$

and the definition of the energy, (55) follows from the preceding lemma and from the hypothesis $0 < \mu < 1$. ∎

Lemma 9.20. — *Given* $0 \leq S < T < +\infty$ *arbitrarily, we have*

$$\int_S^T E^{(p+1)/2} \ \mathrm{dt} \leq cE^{(p+1)/2}(S)$$
$$+c \int_S^T E^{(p-1)/2} \int_{\Gamma_1} (u')^2 + |\nabla u||g(u')| \ \mathrm{d}\Gamma_m \ \mathrm{dt}. \tag{56}$$

Proof. — Since $m \cdot \nu \leq 0$ on Γ_0, the left-hand side of (49) is minorized by its first term. Applying corollary 9.19 and using the non-increasingness of E we may estimate the first and second term on the right-hand side of (49) as follows :

$$\Big[E^{(p-1)/2} \int_\Omega 2u' m \cdot \nabla u \ \mathrm{dX} \Big]_T^S + (p-1) \int_S^T E^{(p-3)/2} E' \int_\Omega u' m \cdot \nabla u \ \mathrm{dX} \ \mathrm{dt}$$
$$\leq cE^{(p+1)/2}(S) + cE^{(p+1)/2}(T) - c \int_S^T E^{(p-1)/2} E' \ \mathrm{dt} \leq cE^{(p+1)/2}(S).$$

Furthermore, using hypothesis $0 < \mu < 1$ we have

$$Q(u) = \mu(u_{xx} + u_{yy})^2 + (1-\mu)(u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2) \geq 0$$

and therefore we deduce from (49) that

$$4\int_S^T E^{(p+1)/2}\ \mathrm{dt} \leq cE^{(p+1)/2}(S) + \int_S^T E^{(p-1)/2}\int_{\Gamma_1}(u')^2 - 2(m\cdot\nabla u)g(u')\ \mathrm{d}\Gamma_m\ \mathrm{dt}$$

which implies (56). ∎

Now we introduce the notation

$$\Gamma_2 = \{x \in \Gamma_1 : |u'(x)| \leq 1\} \quad \text{and} \quad \Gamma_3 = \{x \in \Gamma_1 : |u'(x)| > 1\}.$$

Lemma 9.21. — *Given* $0 \leq S < T < +\infty$ *arbitrarily, we have*

$$\begin{aligned}\int_S^T E^{(p+1)/2}\ \mathrm{dt} &\leq cE^{(p+1)/2}(S)\\ &+c\int_S^T E^{(p-1)/2}\int_{\Gamma_2}(u')^2 + |g(u')|^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}\\ &+c\int_S^T E^{(p-1)/2}\int_{\Gamma_3}(u')^2 + |g(u')|^{1+q^{-1}}\ \mathrm{d}\Gamma_m\ \mathrm{dt}.\end{aligned} \tag{57}$$

Proof. — For any fixed $\varepsilon > 0$ we deduce from (56) that

$$\begin{aligned}\int_S^T E^{(p+1)/2}\ \mathrm{dt} &\leq cE^{(p+1)/2}(S)\\ &+c\int_S^T E^{(p-1)/2}\int_{\Gamma_1}(u')^2 + \varepsilon|\nabla u|^2 + c(\varepsilon)|g(u')|^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}\\ &+c\int_S^T E^{(p-1)/2}\int_{\Gamma_2}(u')^2 + \varepsilon|\nabla u|^2 + \varepsilon|\nabla u|^{1+q} + c(\varepsilon)|g(u')|^{1+q^{-1}}\ \mathrm{d}\Gamma_m\ \mathrm{dt}.\end{aligned} \tag{58}$$

Using the Sobolev inequality

$$||u||_{W^{1,\gamma}(\Gamma)} \leq cE^{1/2}, \quad \forall\gamma \geq 1$$

with $\gamma = 2$ and $\gamma = 1 + q$, we conclude from (58) that

$$\begin{aligned}\int_S^T E^{(p+1)/2}\ \mathrm{dt} &\leq cE^{(p+1)/2}(S) + c\varepsilon\int_S^T E^{(p+1)/2}\ \mathrm{dt}\\ &+c\int_S^T E^{(p-1)/2}\int_{\Gamma_2}(u')^2 + c(\varepsilon)|g(u')|^2\ \mathrm{d}\Gamma_m\ \mathrm{dt}\\ &+c\int_S^T E^{(p-1)/2}\int_{\Gamma_3}(u')^2 + c(\varepsilon)|g(u')|^{1+q^{-1}}\ \mathrm{d}\Gamma_m\ \mathrm{dt}.\end{aligned}$$

The lemma follows if we choose $\varepsilon > 0$ such that $c\varepsilon < 1$. ∎

Now we are ready to complete the proof of the theorem. Applying (43) and (44) we have

$$\begin{aligned}\int_{\Gamma_2} |g(u')|^2 + (u')^2 \, \mathrm{d}\Gamma &\le c \int_{\Gamma_2} (u'g(u'))^{2/(p+1)} \, \mathrm{d}\Gamma \\ &\le c\Big(\int_{\Gamma_2} u'g(u') \, \mathrm{d}\Gamma\Big)^{2/(p+1)} \le c(-E')^{2/(p+1)}\end{aligned}$$

and

$$\int_{\Gamma_3} |g(u')|^{1+q^{-1}} + (u')^2 \, \mathrm{d}\Gamma \le c \int_{\Gamma_3} u'g(u') \, \mathrm{d}\Gamma \le -cE'.$$

Substituting into (57) we obtain for every $\varepsilon > 0$ the estimate

$$\begin{aligned}\int_S^T E^{(p+1)/2} \, \mathrm{dt} &\le cE^{(p+1)/2}(S) \\ +c\int_S^T E^{(p-1)/2}(-E')^{2/(p+1)} \, \mathrm{dt} &- c\int_S^T E^{(p-1)/2}E' \, \mathrm{dt} \\ &\le cE^{(p+1)/2}(S) + \int_S^T \varepsilon E^{(p+1)/2} - c(\varepsilon)E' \, \mathrm{dt} \\ &\le c(\varepsilon)E(S) + \varepsilon\int_S^T E^{(p+1)/2} \, \mathrm{dt}.\end{aligned}$$

Choosing $\varepsilon < 1$ we conclude that E satisfies an inequality of the form

$$\int_S^T E^{(p+1)/2} \, \mathrm{dt} \le AE(S) \quad \text{for every} \quad 0 \le S < T < +\infty$$

with a suitable positive constant A. We may conclude by applying theorem 9.1. ∎

10. Stabilization of the Korteweg–de Vries equation

In this chapter we shall study, following Komornik, Russell and Zhang [1], [2] the stabilizability of the non-linear Korteweg–de Vries equation by linear distributed feedbacks. The proof is based on a remarkable property of this equation : the existence of an infinite sequence of conservation laws corresponding to an infinite sequence of useful multipliers.

10.1. Formulation of the results

Let $\Omega = (0,1)$, $k > 0$ and consider the problem

$$u' + uu_x + u_{xxx} = -k(u - [u]) \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{1}$$
$$u(0,t) = u(1,t), \quad \forall t \in \mathbb{R}_+, \tag{2}$$
$$u_x(0,t) = u_x(1,t), \quad \forall t \in \mathbb{R}_+, \tag{3}$$
$$u_{xx}(0,t) = u_{xx}(1,t), \quad \forall t \in \mathbb{R}_+, \tag{4}$$
$$u(0) = u^0 \quad \text{on} \quad \Omega \tag{5}$$

where $[u]$ denotes the mean-value of u defined by

$$[u] := \int_\Omega u \ \mathrm{dx}. \tag{6}$$

For $k = 0$ the equation (1) is a good model of shallow water : $u(x,t)$ denotes the depth of water at a point x at time t; see Miura [1], Temam [1]. The periodic boundary conditions correspond to a circular movement. In this model $[u]$ denotes the total volume of water.

For $k > 0$ the action of the "feedback" $-k(u - [u])$ consists in balancing the level of water, conserving at the same time its total volume. Indeed, the latter property follows, at least formally, from (1), (2) and (5) :

$$\begin{aligned}
[u]' = \int_\Omega u' \ \mathrm{dx} &= -\int_\Omega uu_x + u_{xxx} + k(u - [u]) \ \mathrm{dx} \\
&= -\int_\Omega (u^2/2 + u_{xx})_x \ \mathrm{dx} + k[u] - k\int_\Omega u \ \mathrm{dx} \\
&= -\int_\Omega (u^2/2 + u_{xx})_x \ \mathrm{dx} = [u^2/2 + u_{xx}]_0^1 = 0
\end{aligned}$$

whence

$$[u(t)] = [u^0], \quad \forall t \in \mathbb{R}_+. \tag{7}$$

The following formal computation shows that $u(t)$ converges exponentially to the constant $M := [u^0] = [u]$ in $L^2(\Omega)$ as $t \to +\infty$:

$$\begin{aligned}\Big(\int (u - [u])^2 \,\mathrm{dx}\Big)' &= \int 2(u - M)u' \,\mathrm{dx}\\ &= \int -2(u - M)(uu_x + u_{xxx} + k(u - M)) \,\mathrm{dx}\\ &= \int -2u^2u_x + 2Muu_x - 2uu_{xxx} - 2k(u - M)^2 \,\mathrm{dx}\\ &= \big[-(2/3)u^3 + Mu^2 - 2uu_{xx} + u_x^2\big]_0^1 - 2k \int (u - M)^2 \,\mathrm{dx}\\ &= -2k \int (u - M)^2 \,\mathrm{dx}\end{aligned}$$

whence

$$\|u(t) - [u^0]\|_{L^2(\Omega)} = \|u^0 - [u^0]\|_{L^2(\Omega)} e^{-kt}, \quad \forall t \geq 0. \tag{8}$$

Let us introduce the Hilbert spaces

$$H_p^m := \{w \in H^m(\Omega) : w^j(0) = w^j(1),\ j = 0, \ldots, m - 1\},\ m = 1, 2, \ldots$$

$$H_p^0 := L^2(\Omega) \quad \text{and} \quad H_p^{-1} := (H_p^1)'.$$

Identifying $(L^2(\Omega))'$ with $L^2(\Omega)$ we obtain the algebraical and topological inclusions

$$\cdots \subset H_p^2 \subset H_p^1 \subset H_p^0 \subset H_p^{-1}.$$

The problem (1)–(5) is well-posed in the following sense :

Theorem 10.1. — *Let $m \geq 2$ and $u^0 \in H_p^m$. Then the problem* (1)–(5) *has a unique solution*

$$u \in C(\mathbb{R}_+; H_p^m) \cap C^1(\mathbb{R}_+; H_p^{m-3}). \tag{9}$$

Furthermore, the mapping $u^0 \mapsto (u, u')$ is continuous from H_p^m into $H^m \times H^{m-3}$.

The solution $u(t)$ converges quickly to the constant $[u^0]$ as $t \to +\infty$:

Theorem 10.2. — *Let $m \geq 2$ and $u^0 \in H_p^m$. Then for every fixed $k' \in]0,k[$ there exists a constant $C = C(u^0, k')$ such that the solution of* (1)–(5) *satisfies the estimate*

$$\|(u(t) - [u^0], u'(t))\|_{H_p^m \times H_p^{m-3}} \leq Ce^{-k't}, \quad \forall t \in \mathbb{R}_+. \tag{10}$$

Theorems 10.1 et 10.2 were proved in Komornik, Russell and Zhang [2]. Here we admit theorem 10.1 without proof and we only prove theorem 10.2 in the particular case $m = 2$.

10.2. Uniform stabilization by linear feedbacks

We shall often use the equality (7) and therefore we shall write $[u]$ instead of $[u^0]$. For brevity we shall write $\int$ instead of $\int_\Omega$.

Applying a usual density argument it is sufficient to prove the estimates (10) for $u^0 \in H_p^5$. According to theorem 10.1 thus we may assume that

$$u \in C(\mathbb{R}_+; H_p^5) \cap C^1(\mathbb{R}_+; H_p^2). \tag{11}$$

This regularity property is sufficient to justify all computations which follow.

It is convenient to introduce the notations

$$M := [u^0], \quad v := u - M, \quad v^0 := u^0 - M; \tag{12}$$

then we deduce from (1), (5), (7) and (11) that

$$v \in C(\mathbb{R}_+; H_p^5), \tag{13}$$
$$v \in C^1(\mathbb{R}_+; H_p^2), \tag{14}$$
$$[v(t)] = 0, \ \forall t \in \mathbb{R}_+, \tag{15}$$
$$v' + vv_x + Mv_x + v_{xxx} + kv = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{16}$$
$$v(0) = v^0 \quad \text{on} \quad \Omega, \tag{17}$$

and the estimates (10) take the following form :

$$\|(v(t), v'(t))\|_{H_p^2 \times H_p^{-1}} \leq Ce^{-k't}, \quad \forall t \in \mathbb{R}_+. \tag{18}$$

Lemma 10.3. — *The function*

$$t \mapsto \int v(t)^2 \, \mathrm{dx}, \quad t \in \mathbb{R}_+ \tag{19}$$

is continuously differentiable and

$$\Big(\int v^2 \,\mathrm{dx}\Big)' \equiv -2k \int v^2 \,\mathrm{dx}. \tag{20}$$

Proof. — Since H_p^2 is a Banach algebra, it follows from (14) that the function v^2 is continuously differentiable. Hence the function (19), being the composition of two function of class C^2, is also continuously differentiable.

Using (16) and the periodicity of v (cf. (13)) we easily obtain the identity (20) :

$$\begin{aligned}\Big(\int v^2 \,\mathrm{dx}\Big)' &= \int 2vv' \,\mathrm{dx} = \int -2v(vv_x + Mv_x + v_{xxx} + kv) \,\mathrm{dx}\\ &= \int -2v^2v_x - 2Mvv_x - 2vv_{xxx} - 2kv^2 \,\mathrm{dx}\\ &= \big[-(2/3)v^3 - Mv^2 - 2vv_{xx} + v_x^2\big]_0^1 - 2k\int v^2 \,\mathrm{dx} = -2k\int v^2 \,\mathrm{dx}. \quad \square\end{aligned}$$

Lemma 10.4. — *The function*

$$t \mapsto \int v_x(t)^2 - (1/3)v^3(t)^2 \,\mathrm{dx}, \quad t \in \mathbb{R}_+ \tag{21}$$

is continuously differentiable and

$$\Big(\int v_x(t)^2 - (1/3)v^3(t)^2 \,\mathrm{dx}\Big)' \equiv -2k \int v_x(t)^2 - (1/2)v^3(t)^2 \,\mathrm{dx}. \tag{22}$$

Proof. — It follows easily from (14) that the function (21) is continuously differentiable. Using (16) and the periodicity of v hence the identity (22) follows :

$$\begin{aligned}\Big(\int v_x^2 - (1/3)v^3 \,\mathrm{dx}\Big)' &= \int 2v_xv_x' - v^2v' \,\mathrm{dx}\\ &= \big[2v_xv'\big]_0^1 + \int -v'(2v_{xx}+v^2) \,\mathrm{dx} = \int (vv_x+Mv_x+v_{xxx}+kv)(2v_{xx}+v^2) \,\mathrm{dx}\\ &= \int (v^2)_xv_{xx} + \big(Mv_x^2 + v_{xx}^2 + (1/4)v^4 + (M/3)v^3\big)_x + v^2v_{xxx} - 2kv_x^2 + kv^3 \,\mathrm{dx}\\ &= \big[v^2v_{xx} + Mv_x^2 + v_{xx}^2 + (1/4)v^4 + (M/3)v^3\big]_0^1 + \int -2kv_x^2 + kv^3 \,\mathrm{dx}\\ &= -2k\int -(1/2)v^3 \,\mathrm{dx}. \quad \square\end{aligned}$$

Lemma 10.5. — *The function*

$$t \mapsto \int v_{xx}(t)^2 - (5/3)v_x^2 v(t)^2 + (5/36)v(t)^4 \, \mathrm{dx}, \quad t \in \mathbb{R}_+ \tag{23}$$

is continuously differentiable and

$$\begin{aligned} &\Big(\int v_{xx}^2 - (5/3)v_x^2 v + (5/36)v^4 \, \mathrm{dx}\Big)' \\ &\equiv -2k \int v_{xx}^2 - (5/2)v_x^2 v + (5/18)v^4 \, \mathrm{dx}. \end{aligned} \tag{24}$$

Proof. — By (14) the function (23) is continuously differentiable. To show the identity (24) first we deduce from (16), using the periodicity of v, the following identity :

$$\begin{aligned} &\Big(\int v_{xx}^2 - (5/3)v_x^2 v + (5/36)v^4 \, \mathrm{dx}\Big)' \\ &= \int 2v_{xx}v_{xx}' - (10/3)v_x v_x' v - (5/3)v_x^2 v' + (5/9)v^3 v' \, \mathrm{dx} \\ &= \big[2v_{xx}v_x' - 2v_{xxx}v' - (10/3)v v_x v'\big]_0^1 \\ &+ \int v'\big(2v_{xxxx} + (5/3)v_x^2 + (10/3)v v_{xx} + (5/9)v^3\big) \, \mathrm{dx} \\ &= \int v'\big(2v_{xxxx} + (5/3)v_x^2 + (10/3)v v_{xx} + (5/9)v^3\big) \, \mathrm{dx} \\ &= -\int (v v_x + M v_x + v_{xxx} + kv)\big(2v_{xxxx} + (5/3)v_x^2 + (10/3)v v_{xx} + (5/9)v^3\big) \, \mathrm{dx} \\ &= -k \int 2v v_{xxxx} + (5/3)v v_x^2 + (10/3)v^2 v_{xx} + (5/9)v^4 \, \mathrm{dx} \\ &\quad -M \int 2v_x v_{xxxx} + (5/3)v_x^3 + (10/3)v v_x v_{xx} + (5/9)v^3 v_x \, \mathrm{dx} \\ &\quad -\int 2v v_x v_{xxxx} + (5/3)v v_x^3 + (10/3)v^2 v_x v_{xx} + (5/9)v^4 v_x + 2v_{xxx}v_{xxxx} \\ &\quad +(5/3)v_x^2 v_{xxx} + (10/3)v v_{xx} v_{xxx} + (5/9)v^3 v_{xxx} \, \mathrm{dx} \\ &=: -kI_1 - MI_2 - I_3. \end{aligned}$$

It suffices to show that

$$I_1 = \int 2v_{xx}^2 - 5v v_x^2 + (5/9)v^4 \, \mathrm{dx} \quad \text{and} \quad I_2 = I_3 = 0.$$

We have

$$I_1 = \Big[2vv_{xxx} - 2v_x v_{xx} + (10/3)v^2 v_x\Big]_0^1$$

$$+ \int 2v_{xx}^2 + (5/3)vv_x^2 - (20/3)vv_x^2 + (5/9)v^4 \text{ dx} = \int 2v_{xx}^2 - 5vv_x^2 + (5/9)v^4 \text{ dx}$$

and

$$I_2 = \Big[2v_x v_{xxx} - v_{xx}^2 + (5/3)vv_x^2 + (5/36)v^4\Big]_0^1 + \int (5/3)v_x^3 - (5/3)v_x^3 \text{ dx} = 0.$$

Finally, we have

$$\begin{aligned}
I_3 = \int & 2vv_x v_{xxxx} + (5/3)vv_x^3 + (5/3)v_x^2 v_{xxx} + (10/3)v^2 v_x v_{xx} + (10/3)vv_{xx}v_{xxx} \\
& + (5/9)v^4 v_x + (5/9)v^3 v_{xxx} + 2v_{xxx}v_{xxxx} \text{ dx} \\
= & \Big[2vv_x v_{xxx} + (1/9)v^5\Big]_0^1 + \int -2v_x^2 v_{xxx} - 2vv_{xx}v_{xxx} + (5/3)vv_x^3 \\
& + (5/3)v_x^2 v_{xxx} + (10/3)v^2 v_x v_{xx} + (10/3)vv_{xx}v_{xxx} + (5/9)v^3 v_{xxx} \text{ dx} \\
= \int & -2v_x^2 v_{xxx} - 2vv_{xx}v_{xxx} + (5/3)vv_x^3 + (5/3)v_x^2 v_{xxx} + (10/3)v^2 v_x v_{xx} \\
& + (10/3)vv_{xx}v_{xxx} + (5/9)v^3 v_{xxx} + (5/9)v^3 v_{xx} \text{ dx} \\
= & \Big[-2v_x^2 v_{xx} - vv_{xx}^2 + (5/3)vv_x^2 + (5/3)vv_{xx}^2\Big]_0^1 + \int 4v_x v_{xx}^2 + v_x v_{xx}^2 \\
& + (5/3)vv_x^3 - (10/3)v_x v_{xx}^2 + (10/3)v^2 v_x v_{xx} - (5/3)v_x v_{xx}^2 - (5/3)v^2 v_x v_{xx} \text{ dx} \\
= & \int (5/3)vv_x^3 + (5/3)v^2 v_x v_{xx} \text{ dx} = 0. \quad \Box
\end{aligned}$$

In order to simplify the notation we shall write $\|\cdot\|_p$ for the norm of $L^p(\Omega)$, $1 \le p \le \infty$. Since Ω is the unit interval, the Hölder inequality is particularly simple :

$$\|v\|_p \le \|v\|_q, \quad \forall v \in L^q(\Omega), \quad 1 \le p \le q \le \infty. \tag{25}$$

We shall also use the Poincaré–Wirtinger inequality :

$$\|v\|_\infty \le \|v_x\|_1, \quad \text{if} \quad v \in H^1(\Omega) \quad \text{and} \quad [v] = 0. \tag{26}$$

The proof is simple : since v is continuous, there exists $a \in \Omega$ such that $v(a) = 0$. Then for any $y \in \Omega$ we have

$$|v(y)| = |v(y) - v(a)| = \Big|\int_a^y v_x \text{ dx}\Big| \le \int_\Omega |v_x| \text{ dx} = \|v_x\|_1.$$

Noe that lemma 10.3 implies that

$$\|v(t)\|_2 = \|v^0\|_2 e^{-kt}, \quad \forall t \in \mathbb{R}_+. \tag{27}$$

Now let us show that for each fixed $k' \in (0,k)$ there exists a positive constant C' such that

$$\|v_x(t)\|_2 = C' e^{-k't}, \quad \forall t \geq 0. \tag{28}$$

Using (25)–(27) we have

$$\|v^3(t)\|_1 \leq \|v(t)\|_\infty^2 \|v(t)\|_1 \leq \|v_x(t)\|_1^2 \|v(t)\|_2 \leq \|v_x(t)\|_2^2 \|v^0\|_2 e^{-kt};$$

consequently, for any fixed $\varepsilon > 0$ (to be chosen later) there exists $T' > 0$ such that

$$\int v^3(t) \text{ dx} \leq \varepsilon \int v_x^2 \text{ dx}, \quad \forall t > T'. \tag{29}$$

If $\varepsilon \leq 2$, then we deduce from (29) the inequalities

$$\int \left(v_x^2 - (1/3)v^3\right)(t) \text{ dx} \geq (1/3) \int v_x^2(t) \text{ dx} \geq 0, \quad \forall t > T'. \tag{30}$$

If ε is sufficiently small, then we also deduce from (29) that

$$-2k \int \left(v_x^2 - (1/2)v^3\right)(t) \text{ dx} \leq -2k' \int \left(v_x^2 - (1/3)v^3\right)(t) \text{ dx}, \quad \forall t > T'. \tag{31}$$

(It suffices to choose $\varepsilon \leq (6k - 6k')/(3k - 2k')$.)

Thus, choosing a sufficiently small ε we deduce from (21), (30) and (31) that

$$\begin{aligned} (1/3) \int v_x^2(t) \text{ dx} &\leq \int \left(v_x^2 - (1/3)v^3\right)(t) \text{ dx} \\ &\leq \int \left(v_x^2 - (1/3)v^3\right)(0) \text{ dx} e^{-k'(t-T')} =: C' e^{-k't}, \quad \forall t > T' \end{aligned}$$

which implies (28) for all $t > T'$. The left-hand side of (28) being continuous, the estimate (28) remains valid for all $t \geq 0$ with some bigger constant C'.

Next we show similarly that for any fixed $k' < k$ there exists a positive constant C'' such that

$$\|v_{xx}\|_2 = C'' e^{-k't}, \quad \forall t \geq 0. \tag{32}$$

Using (25)–(28) we have

$$\|v^4(t)\|_1 \le \|v(t)\|_\infty^2 \|v(t)\|_2^2 \le \|v_x(t)\|_2^2 \|v(t)\|_2^2 \le \|v_{xx}(t)\|_2^2 \|v^0\|_2^2 e^{-2kt}$$

and

$$\Big|\int (v_x^2 v)(t)\ \mathrm{dx}\Big| \le \|v_x(t)\|_2^2 \|v(t)\|_\infty \le \|v_x(t)\|_2^3 \le \|v_{xx}(t)\|_2^2 C' e^{-k't}.$$

It follows that for any fixed $\varepsilon > 0$ (to be chosen later) there exists $T'' > 0$ such that

$$\int \big|(v_x^2 v)(t)\big| + v(t)^4\ \mathrm{dx} \le \varepsilon \int v_{xx}(t)^2\ \mathrm{dx}, \quad \forall t > T''. \tag{32}$$

Choosing $\varepsilon > 0$ sufficiently small we conclude from (32) that

$$\int \big(v_{xx}^2 - (5/3)v_x^2 v + (5/36)v^4\big)(t)\ \mathrm{dx} \ge (1/3) \int v_{xx}(t)^2\ \mathrm{dx} \ge 0 \tag{33}$$

and

$$\begin{aligned} &-2k \int \big(v_{xx}^2 - (5/2)v_x^2 v + (5/18)v^4\big)(t)\ \mathrm{dx} \\ &\le -2k'' \int \big(v_{xx}^2 - (5/3)v_x^2 v + (5/36)v^4\big)(t)\ \mathrm{dx} \end{aligned} \tag{34}$$

for all $t > T''$. We deduce from (23), (33) and (34) that

$$\begin{aligned} (1/3) \int v_{xx}(t)^2\ \mathrm{dx} &\le \int \big(v_{xx}^2 - (5/3)v_x^2 v + (5/36)v^4\big)(t)\ \mathrm{dx} \\ \le \int \big(v_{xx}^2 - (5/3)v_x^2 v &+ (5/36)v^4\big)(0)\ \mathrm{dx} e^{-2k'(t-T')} =: (C')^2 e^{-2k't}, \end{aligned}$$

proving (32) for all $t > T'$. The left-hand side of (32) being continuous, the estimate (32) remains valid for every $t \ge 0$ if we choose some larger constant C''.

Now we may easily complete the proof of the theorem. By (27), (28) and (32) for every fixed $k' < k$ there exists a positive constant $C_1 > 0$ such that

$$\|v(t)\|_{H_p^2} \le C_1 e^{-k't}, \quad \forall t \ge 0. \tag{35}$$

Using the equation (16) hence we conclude easily that

$$\|u'(t)\|_{H_p^{-1}} \le C_2 e^{-k't}, \quad \forall t \ge 0 \tag{36}$$

with some constant $C_2 > 0$. The estimate (18) follows from (35) and (36). □

References

S. Agmon [1], Elliptic Boundary Value Problems, Van Nostrand, Princeton, NJ, 1965.

S. A. Avdonin and S. A. Ivanov [1], Controllability of distributed systems and families of exponents, Polytechnical Institute, Kiev, 1989 (in Russian).

J. M. Ball and M. Slemrod [1], Nonharmonic Fourier series and the stabilization of distributed semi-linear control systems, Comm. Pure Appl. Math. 37(1979), 555-587.

V. Barbu [1], Analysis and control of nonlinear infinite dimensional systems, Academic Press, New York, 1993.

C. Bardos, G. Lebeau and J. Rauch [1] Contrôle et stabilisation dans les problèmes hyperboliques, in Lions [5], 492-537.

C. Bardos, G. Lebeau and J. Rauch [2] Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Control Opt. 30(1992), 1024-1065.

H. Barucq and B. Hanouzet [1], **, C. R. Acad. Sci. Paris Sér I Math. 316(1993), 547-552.

H. Barucq and B. Hanouzet [2], Étude asymptotique du système de Maxwell avec la condition aux limites absorbante de Silver-Müller II, C. R. Acad. Sci. Paris Sér I Math. 316(1993), 1019-1024.

C. D. Benchimol [1], A note on weak stabilizability of contraction semigroups, SIAM J. Control Opt. 16(1978), 373-379.

A. Bensoussan [1], On the general theory of exact controllability for skew-symmetric operators, Acta Appl. Math 20(1990), 197-229.

H. Brézis [1], Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North Holland, Amsterdam, 1973.

H. Brézis [2], Analyse fonctionnelle : théorie et applications, Masson, Paris, 1983.

F. E. Browder [1], Problèmes non-linéaires, Les Presses de l'Université de Montréal, 1966.

T. Carleman [1], Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes, Ark. Mat. 26B(1939), 1-9.

A. Carpio [1], Sharp estimates of the energy for the solutions of some dissipative second order evolution equations, Potential Analysis 1(1992), 265-289.

T. Cazenave [1], On the propagation of confined waves along the geodesics, J. Math. Anal. Appl. 146(1990), 591-603.

G. Chen [1], Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain, J. Math. Pures Appl. 58(1979), 249-274.

G. Chen [2], A note on the boundary stabilization of the wave equation, SIAM J. Control Opt. 19(1981), 106-113.

G. Chen and H. K. Wang [1], Asymptotic behavior of solutions on the one-dimensional wave equation with a nonlinear boundary stabilizer, SIAM J. Control Opt. 27(1989), 758-775.

P.-G. Ciarlet [1], Mathematical elasticity, Vol. 2, North Holland, Amsterdam, 1988.

F. Conrad, J. Leblond and J. P. Marmorat [1], Stabilization of second order evolution equations by unbounded nonlinear feedback, to appear.

F. Conrad and M. Pierre [1], Stabilization of second-order evolution equations by unbounded nonlinear feedback, preprint No. 14 de l'Institut Élie Cartan, Département de Mathématiques, Université de Nancy I, 1992.

F. Conrad and B. Rao [1], Decay of solutions of wave equations in a star-shaped domain with nonlinear boundary feedback, Asymptotic Anal. 7(1993), 159-177.

S. Dolecki and D. L. Russell [1], A general theory of observation and control, SIAM J. Control Opt. 15(1977), 185-220.

G. Duvaut and J.-L. Lions [1], Les inéquations en mécanique et en physique, Dunod, Paris, 1972.

H. O. Fattorini and D. L. Russell [1], Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations, Quart. Appl. Math. 32(1974), 45-69.

S. R. Foguel [1], Powers of contractions in Hilbert space, Pacific J. Math. 13(1963), 551-562.

N. Garofalo and F. Lin [1], Monotonicity properties of variational integrals, A_p weights and unique continuation, Indiana Univ. Math. J. 35(1986), 245-268.

N. Garofalo and F. Lin [2], Unique continuation for elliptic operators : a geometric–variational approach, Comm. Pure Appl. Math. 40(1987), 347-366.

J. S. Gibson [1], A note on stabilization of infinite dimensional linear oscillators by compact linear feedback, SIAM J. Control Opt. 18(1980), 311-316.

K. D. Graham and D. L. Russell [1], Boundary value control of the wave equation in a spherical region, SIAM J. Control 13(1975), 174-196.

P. Grisvard [1], Contrôlabilité exacte des solutions de l'équation des ondes en présence de singularités, J. Math. Pures Appl. 68(1989), 215-259.

S. Hansen [1], Bounds on functions biorthogonal to sets of complex potentials; control of damped elastic systems, J. Math. Anal. Appl. 158(1991), 487-508.

S. Hansen [2], Boundary control of a one-dimensional, linear, thermoelastic rod, to appear.

A. Haraux [1], Oscillations forcées pour certains systèmes dissipatifs non linéaires, Publication du Laboratoire d'Analyse Numérique No. 78010, Université Pierre et Marie Curie, Paris, 1978.

A. Haraux [2], Semi-groupes linéaires et équations d'évolution linéaires périodiques, Publication du Laboratoire d'Analyse Numérique No. 78011, Université Pierre et Marie Curie, Paris, 1978.

A. Haraux [3], Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire, J. Math. Pures Appl. 68(1989), 457-465.

A. Haraux [4], Semi-linear wave equations in bounded domains, Mathematical Reports, Vol. 3, Part 1, J. Dieudonné Editor, Harwood Academic Publishers, Gordon and Breach, 1987.

A. Haraux [5], Systèmes dynamiques dissipatifs et applications, Masson, Paris, 1991.

A. Haraux and E. Zuazua [1], Decay estimates for some semilinear damped hyperbolic problems, Arch. Rat. Mech. Anal. (1988), 191-206.

L. F. Ho [1], Observabilité frontière de l'équation des ondes, C. R. Acad. Sci. Paris Sér I Math. 302(1986), 443-446.

M.-A. Horn [1], Exact controllability of the Euler–Bernoulli plate via bending moments only on the space of optimal regularity, J. Math. Anal. Appl., to appear.

M. Horváth [1], Vibrating strings with free ends, Acta Math. Acad. Sci. Hungar. 51(1988), 171-198.

S. Jaffard [1], Contrôle interne exact des vibrations d'une plaque carrée, C. R. Acad. Sci. Paris Sér I Math. 307(1988), 759-762.

S. Jaffard [2], Contrôle interne exact des vibrations d'une plaque rectangulaire, Portugal.-Math. 47(1990), 423-429.

I. Joó [1], Contrôlabilité exacte et propriétés d'oscillation de l'équation des ondes par analyse non harmonique, C. R. Acad. Sci. Paris Sér I Math. 312(1991), 119-122.

I. Joó [2], On the control of a circular membrane and related problems, Annales Univ. Sci. Budapest. Sect. Math. 34(1991), 231-266.

V. Komornik [1], Contrôlabilité exacte en un temps minimal, C. R. Acad. Sci. Paris Sér I Math. 304(1987), 223-225.

V. Komornik [2], Exact controllability in short time for the wave equation, Ann. de l'I.H.P., Anal. non linéaire 6(1989), 153-164.

V. Komornik [3], A new method of exact controllability in short time and applications, Ann. Fac. Sci. Toulouse 10(1989), 415-464.

V. Komornik [4], Rapid boundary stabilization of the wave equation, SIAM J. Control Opt. 29(1991), 197-208.

V. Komornik [5], On the exact controllability of Kirchoff plates, Differential Integral Equations 4(1991), 1121-1132.

V. Komornik [6], On the boundary stabilization of plates, Metz Days, ed. M. Chipot and J. Saint Jean Paulin, Pitman Research Notes in Math., vol. 249, 105-111, Essex, UK, 1991.

V. Komornik [7], On the exact internal controllability of a Petrowsky system, J. Math. Pures Appl. 71(1992), 331-342.

V. Komornik [8], On the zeros of Bessel type functions and applications to exact controllability problems, Asymptotic Analysis 5(1991), 115-128.

V. Komornik [9], Well-posedness and decay estimates for the wave equation with internal damping, Proceedings of the Conference on control theory, Vorau, Austria, 1993, Birkhäuser, International Series Num. Analysis, to appear.

V. Komornik [9], Decay estimates for the wave equation with internal damping, Proceedings of the Conference on control theory, Vorau, Austria, 1993, Birkhäuser, International Series Num. Analysis, to appear.

V. Komornik [10], On the nonlinear boundary stabilization of the wave equation, Chin. Ann. of Math. 14B :2(1993), 153-164.

V. Komornik [11], On the nonlinear boundary stabilization of Kirchoff plates, Nonlinear Diff. Equations and Appl., to appear.

V. Komornik [12], On some integral and differential inequalities, preprint No. 490/P-283 du Département de Mathématique de l'Université Louis Pasteur, Strasbourg, 1992.

V. Komornik [13], Boundary stabilization and control of Maxwell's equations, to appear.

V. Komornik and B. Rao [1], On the stabilization of compactly perturbed distributed systems, to appear.

V. Komornik, D. L. Russell and B.-Y. Zhang [1], Stabilisation de l'équation de Korteweg–de Vries, C. R. Acad. Sci. Paris Sér I Math. 312(1991), 841-843.

V. Komornik, D. L. Russel and B.-Y. Zhang [2], Control and stabilization of the Korteweg–de Vries equation on a periodic domain, to appear.

V. Komornik and E. Zuazua [1], A direct method for the boundary stabilization of the wave equation, J. Math. Pures Appl. 69(1990), 33-54.

S. Kouémou-patcheu [1], to appear.

W. Krabs [1], On Moment Theory and Controllability of One-Dimensional Vibrating Systems and Heating Processes, Lect. Notes in Control and Inf. Sciences, vol. 173, Springer-Verlag, Berlin, 1992.

J. Lagnese [1], Decay of solutions of wave equations in a bounded region with boundary dissipation, J. Diff. Equations 50(1983), 163-182.

J. Lagnese [2], Boundary Stabilization of Thin Plates, SIAM Studies in Appl. Math., Philadelphia, 1989.

J. Lagnese [3], Exact boundary controllability of Maxwell's equations in a general region, SIAM J. Control Opt. 27(1989), 374-388.

J. Lagnese and G. Leugering [1], Uniform stabilization of a nonlinear beam by nonlinear boundary feedback, J. Diff. Equations 91(1991), 355-388.

J. Lagnese and J.-L. Lions [1], Modelling, Analysis and Control of Thin Plates, Masson, Paris, 1988.

J. P. LaSalle [1], Some extensions of Liapunov's second method, IEEE Trans. Circuit Theory CT-7 (1960), 520-527.

I. Lasiecka [1], Stabilization of wave and plate-like equations with nonlinear dissipative damping on the boundary, J. Diff. Equations 79(1989), 340-381.

I. Lasiecka [2], Exponential decay rates for the solutions of Euler–Bernoulli equations with boundary dissipation occurring in the moments only, J. Diff. Equations 95(1992), 169-182.

I. Lasiecka and R. Tataru [1], Uniform boundary stabilization of semilinear wave equation with nonlinear boundary conditions, Differential and integral equations, to appear.

I. Lasiecka and R. Triggiani [1], Regularity of hyperbolic equations under $L_2(0,T;L_2(\Gamma))$ boundary terms, Appl. Math. and Optimiz. 10(1983), 275-286.

P. D. Lax, C. S. Morawetz and R. S. Phillips [1], Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle, Comm. Pure Appl. Math. 16(1963), 477-486.

G. Lebeau [1], Contrôle de l'équation de Schrödinger, J. Math. Pures Appl. 71(1992), 267-291.

J.-L. Lions [1], Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod–Gauthiers-Villars, Paris, 1983.

J.-L. Lions [2], Contrôle des systèmes distribués singuliers, Gauthiers-Villars, Paris, 1968.

J.-L. Lions [3], Contrôlabilité exacte des systèmes distribués, C. R. Acad. Sci. Paris Sér I Math. 302(1986), 471-475.

J.-L. Lions [4], Exact controllability, stabilizability, and perturbations for distributed systems, Siam Rev. 30(1988), 1-68.

J.-L. Lions [5], Contrôlabilité exacte et stabilisation de systèmes distribués, Vol. 1, Masson, Paris, 1988.

J.-L. Lions and E. Magenes [1], Problèmes aux limites non homogènes et applications I-III, Dunod, Paris, 1968-1970.

W. Littman and S. Taylor [1], Smoothing evolution equations and boundary control theory, Journal d'Analyse, to appear.

E. Machtyngier [1], Contrôlabilité et stabilisation frontière de l'équation de Schrödinger, C. R. Acad. Sci. Paris Sér I Math. 310(1990), 801-806.

L. Markus [1], Introduction to the theory of distributed control systems, University of Minnesota Mathematics Reports No. 89-107.

R. M. Miura [1], The Korteweg–de Vries equation : a survey of results, SIAM Review (18)1976 , 412-459.

C. S. Morawetz [1], Notes on time decay and scattering for some hyperbolic poblems, Regional conference series in applied mathematics, vol. 19, SIAM, Philadelphia, 1975.

M. Nakao [1], On the decay of solutions of some nonlinear dissipative wave equations in higher dimensions, Math. Z. 193(1986), 227-234.

J. Nečas [1], Les méthodes directes en théorie des équations elliptiques, Masson, Paris, 1967.

S. Pohožaev [1], Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6(1965), 1408-1411 (traduit de Dokl. Akad. Nauk SSSR 165(1965), 36-39).

J.-P. Puel and M. Tucsnak [1], Stabilisation frontière pour les équations de von Karman, C. R. Acad. Sci. Paris Sér I Math. 314(1992), 609-612.

J. P. Quinn and D. L. Russell [1], Asymptotic stability and energy decay rates for solutions of hyperbolic equations with boundary damping, Proc. Roy. Soc. Edinburgh Sect. A, 77(1977), 97-127.

P.-A. Raviart and J.-M. Thomas [1], Introduction à l'analyse numérique des équations aux dérivées partielles, Masson, Paris, 1988.

F. Rellich [1], Darstellung der Eigenwerte von $\Delta u + \lambda u$ durch ein Randintegral, Math. Z. 18(1940), 635-636.

D. L. Russell [1], Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods, J. Diff. Equations 19(1975), 344-370.

D. L. Russell [2], Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions, SIAM Rev. 20(1978), 639-739.

D. L. Russell [3], The Dirichlet-Neumann boundary control problem associated with Maxwell's equations in a cylindrical region, SIAM J. Control Opt. 24(1986), 199-229.

M. Slemrod [1], Stabilization of boundary control systems, J. Diff. Equations 22(1976), 402-415.

Ph. Souplet [1], thèse, Paris 6, 1994.

B. Sz.-Nagy and C. Foias [1], Analyse harmonique des opérateurs de l'espace de Hilbert, Masson, Paris, 1967.

D. Tataru [1], Boundary controllability for conservative P.D.E., to appear.

D. Tataru [2], Decay rates and attractors for semilinear P.D.E. with boundary dissipation, to appear.

T. L. R. Tcheugoue [1], Deux remarques sur la stabilisation de l'équation des ondes en dimension 2, to appear.

R. Temam [1], Sur un problème non linéaire, J. Math. Pures et Appl. 48(1969), 157-172.

R. Triggiani [1], Wave equation on a bounded domain with boundary dissipation : an operator approach, J. Math. Anal. Appl. 137(1989), 438-461.

E. Zuazua [1], Contrôlabilité exacte en un arbitrairement petit de quelques modèles de plaques, dans Lions [4], 465-491.

E. Zuazua [2], Some remarks on the boundary stabilizability of the wave equation, Control of Boundaries and Stabilization (ed. J. Simon), Lecture Notes in Control and Information Sciences No. 125, Springer-Verlag, Berlin, 1989, 251-266.

E. Zuazua [3], Robustesse du feedback de stabilisation par contrôle frontière, C. R. Acad. Sci. Paris Sér I Math. 307(1988),587-591.

E. Zuazua [4], Stability and decay for a class of nonlinear hyperbolic problems, Asymptotic. Anal. 1,2 (1988), 161-185.

E. Zuazua [5], Uniform stabilization of the wave equation by nonlinear boundary feedback, SIAM J. Control Opt. 28(1989), 265-268.

E. Zuazua [6], Controlabilidad exacta y estabilizacion de la ecuacion de ondas, Textos de Métodos Matemáticos 23, Instituto de Matemática, Universidad Federal do Rio de Janeiro, 1990.

J. M. Ball and M. Slemrod [1], Nonharmonic Fourier series and the stabilization of distributed semi-linear control systems, Comm. Pure Appl. Math. 37(1979), 555-587.

J. Bartolomeo and R. Triggiani [1], Uniform energy decay rates for Euler–Bernoulli equations with feedback operators in the Dirichlet/Neumann boundary conditions, SIAM J. Math. Anal., to appear.

D. Cioranescu and P. Donato [1], Some remarks on the exact controllability in a neighbourhood of the boundary of a perforated domain, Control of Boundaries and Stabilization (ed. J. Simon), Lecture Notes in Control and Information Sciences No. 125, Springer-Verlag, Berlin, 1989, 75-94.

I. Lasiecka and R. Triggiani [2], Uniform exponential decay in a bounded region with $L_2(0,T;L_2(\Sigma))$- feedback control in the Dirichlet boundary conditions, J. Diff. Equations 66(1987), 340-390.

I. Lasiecka and R. Triggiani [3], Exact controllability of the Euler–Bernoulli equation with boundary controls for displacement and moment, J. Math. Anal. Appl. 145(1990), 1-33.

W. Littman and L. Markus [1], Stabilization of a hybrid system of elasticity by feedback boundary damping, Annali di mat. pura ed appl., 1988, 281-330.

W. Littman and L. Markus [2], Exact boundary controllability of a hybrid system of elasticity, Arch. Rat. Mech. Anal., 1988, 193-236.

R. M. Young [1], An introduction to nonharmonic Fourier series, Academic Press, 1980.