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EXACT CONTROLLABILITY OF A NONLINEAR KORTEWEG–DE VRIES EQUATION ON A CRITICAL SPATIAL DOMAIN*

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Abstract. We consider the boundary controllability problem for a nonlinear Korteweg–de Vries equation with the Dirichlet boundary condition. We study this problem for a spatial domain with a critical length for which the linearized control system is not controllable. In order to deal with the nonlinearity, we use a power series expansion of second order. We prove that the nonlinear term gives the local exact controllability around the origin provided that the time of control is large enough.

Key words. controllability, Korteweg–de Vries equation, power series expansion

AMS subject classifications. 93B05, 35Q53

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1. Introduction. Let $L > 0$ be fixed. Let us consider the following Korteweg–de Vries (KdV) control system with the Dirichlet boundary condition

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \end{cases} \tag{1.1}$$

where the state is $y(t,\cdot) : [0,L] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. This is a well-known example of a nonlinear dispersive partial differential equation. This equation has been introduced by Korteweg and de Vries in [14] to describe approximately long waves in water of relatively shallow depth. A very good book to understand both physical motivation and deduction of the KdV equation is the book by Whitham [23].

We are concerned with the exact controllability properties of (1.1). In [17] Rosier has proved that this control system is locally exactly controllable around the origin provided that the length of the spatial domain is not critical. This was done using multiplier techniques and the Hilbert Uniqueness Method (HUM) method introduced by Lions (see [15]).

THEOREM 1.1 (see [17, Theorem 1.3]). *Let $T > 0$, and assume that*

$$L \notin N := \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}};\ k,l \in \mathbb{N}^* \right\}. \tag{1.2}$$

Then there exists $r > 0$ such that, for every $(y_0, y_T) \in L^2(0,L)^2$ with $\|y_0\|_{L^2(0,L)} < r$ and $\|y_T\|_{L^2(0,L)} < r$, there exist $u \in L^2(0,T)$ and

$$y \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$$

satisfying (1.1)*, $y(0,\cdot) = y_0$, and $y(T,\cdot) = y_T$.*

Moreover, Rosier proved that the linearized control system of (1.1) around the origin, which is given by

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \end{cases} \tag{1.3}$$

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is not controllable if $L \in N$. Indeed, there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by M, which is unreachable for the linear system. More precisely, for every nonzero state $\psi \in M$, for every $u \in L^2(0,T)$, and for every $y \in C([0,T],L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ satisfying (1.3) and $y(0,\cdot) = 0$, one has $y(T,\cdot) \neq \psi$.

Remark 1.2. If one is allowed to use more than one boundary control input, there is no critical spatial domain, and the exact controllability holds for any $L > 0$. More precisely, let us consider the nonlinear control system

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = u_1(t), \quad y(t,L) = u_2(t), \quad \partial_x y(t,L) = u_3(t), \end{cases} \tag{1.4}$$

where the controls are $u_1(t), u_2(t)$, and $u_3(t)$. As has been pointed out by Rosier in [17], for every $L > 0$ the system (1.4) with $u_1 \equiv 0$ is locally exactly controllable in $L^2(0,L)$ around the origin. Moreover, using all three control inputs, Zhang proved in [24] that for every $L > 0$ the system (1.4) is exactly controllable in the space $H^s(0,L)$ for any $s \geq 0$ in a neighborhood of a given smooth solution of the KdV equation.

Recently, Coron and Crépeau in [8] have proved Theorem 1.1 for the critical lengths $L = 2k\pi$, with $k \in \mathbb{N}^*$ satisfying

$$\nexists (m,n) \in \mathbb{N}^* \times \mathbb{N}^*, \text{ with } m^2 + mn + n^2 = 3k^2 \text{ and } m \neq n. \tag{1.5}$$

For these values of L, the subspace M of missed directions is one-dimensional and is generated by the function $f(x) = 1 - \cos(x)$. Their method consists, first, in moving along this direction by performing a power series expansion of the solution and then in using a fixed point theorem.

Remark 1.3. The condition (1.5) has been communicated to the author by Coron and Crépeau. They pointed out that if it is not satisfied, then the dimension of the missed directions subspace is higher than one, and the proof given in [8] does not work anymore.

In this paper, we follow the method of Coron and Crépeau to investigate the case of critical lengths for which the subspace M is two-dimensional. The set of lengths for which it holds is denoted by N'. We will see in section 2 that N' contains an infinite number of lengths.

This paper is organized as follows. First, in section 2, we study the linearized control system (1.3), and we provide a complete description of the space M in terms of the length L of the spatial domain $(0,L)$. Then, in section 3, we prove in the case $L \in N'$ that the nonlinear term $y\partial_x y$ allows us to reach all of the missed directions provided that the time of control is large enough. We give an explicit expression of the minimal time required by our method. Finally, in section 4, we get the local exact controllability by means of a fixed point theorem; i.e., we prove our main result.

THEOREM 1.4. *Let $L \in N'$. There exists $T_M > 0$ such that for any $T > T_M$ there exist $C > 0$ and $r > 0$ such that for every $(y_0, y_T) \in L^2(0,L)^2$ with $\|y_0\|_{L^2(0,L)} < r$ and $\|y_T\|_{L^2(0,L)} < r$ there exist $u \in L^2(0,T)$ with*

$$\|u\|_{L^2(0,T)} \leq C(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)})^{1/2} \tag{1.6}$$

and

$$y \in C([0,T],L^2(0,L)) \cap L^2(0,T,H^1(0,L))$$

satisfying (1.1), $y(0,\cdot) = y_0$, *and* $y(T,\cdot) = y_T$.

Remark 1.5. The power $1/2$ in the estimate (1.6) comes, as we will see, from performing a power series expansion of second order to deal with the nonlinearity. The same estimate holds with power $1/3$ for the critical lengths studied in [8] (third-order expansion) and with power 1 for the noncritical lengths studied in [17] (first-order expansion).

Remark 1.6. In order to complete the study of the exact controllability of system (1.1), it is necessary to investigate the case where the dimension of the space M is bigger than 2. An approach would be to use the exact controllability of the nonlinear equation around nontrivial stationary solutions proved by Crépeau (in [10] for the domains $(0, 2\pi k)$ and in [11] for any other domain $(0, L)$) and then to apply the method introduced in [5] (see also [1, 2]), that is, the return method (see [3, 4]), together with quasi-static deformations (see also [9]). With such a method, one should obtain the exact controllability of (1.1) for a large time. However, it seems that the minimal time required with this approach is far from being optimal.

Remark 1.7. In Theorem 1.4, we get the local controllability for (1.1) provided that the time of control is large enough. However, we may wonder if this condition on the time is really necessary. This is an interesting open problem since one knows that even if the speed of propagation of the KdV equation is infinite, it may exist a minimal time of control. This is, for example, the case of a nonlinear control system for the Schrödinger equation studied by Beauchard and Coron in [2]. They proved the local controllability of this system along the ground state trajectory for a large time. More recently, Coron proved in [6] and [7, Theorem 9.8] that this local controllability does not hold in small time, even if the Schrödinger equation has an infinite speed of propagation.

Remark 1.8. In [1, 2], there appear Schrödinger linear control systems which are not controllable. One could apply the method used in this paper to prove the local controllability of the corresponding nonlinear control systems. The main difficulty is that in those cases the subspace of missed directions for the linear system is not finite-dimensional.

Remark 1.9. Concerning the stabilization of the KdV equation, some results in the case of periodic boundary conditions can be found in [13] (damping distributed all along the domain), [20] (damping distributed with localized support), and [19] (boundary damping). In the case of the Dirichlet boundary condition, exponential decay of the solution has been obtained in [16] by adding a localized damping term (see also [18] for a generalization of this result). However, the decay rate is unknown. A natural open problem is to design for the control system (1.1) (or the linearized one (1.3)) stabilizing feedback laws which give us an explicit decay rate. This kind of result, even with a prescribed arbitrarily large decay rate, has been obtained in [12, 22] for a general class of second-order (in time) systems including the wave equation and platelike systems. It uses the fact that these systems are time-reversible. This is not the case of the control system (1.1).

2. Linearized control system.

We first recall some properties proved by Rosier in [17]. Let $L > 0$ and $T > 0$. In order to study the following linear KdV equation:

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = f, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \\ y(0,\cdot) = y_0, \end{cases} \tag{2.1}$$

we define the space $\mathcal{B} := C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ endowed with the norm

$$\|y\|_{\mathcal{B}} = \max_{t\in[0,T]} \|y(t)\|_{L^2(0,L)} + \left(\int_0^T \|y(t)\|^2_{H^1(0,L)} dt\right)^{1/2}.$$

Let A denote the operator $Aw = -w' - w'''$ on the domain $D(A) \subset L^2(0,L)$ defined by

$$D(A) := \left\{w \in H^3(0,L);\, w(0) = w(L) = w'(L) = 0\right\}.$$

One can see that both A and its adjoint A^* are closed and dissipative. Hence A generates a strongly continuous semigroup of contractions. Using this fact and the multiplier method, Rosier proved the following existence and uniqueness result.

Proposition 2.1 (see [17, Propositions 3.2 and 3.7]). *There exist unique continuous linear maps Ψ and δ*

$$\begin{array}{rcl} \Psi : L^2(0,L)\times L^2(0,T)\times L^1(0,T,L^2(0,L)) & \longrightarrow & \mathcal{B}, \\ (y_0,u,f) & \longmapsto & \Psi(y_0,u,f), \\ \delta : L^2(0,L)\times L^2(0,T)\times L^1(0,T,L^2(0,L)) & \longrightarrow & L^2(0,T), \\ (y_0,u,f) & \longmapsto & \delta(y_0,u,f), \end{array}$$

such that, for $y_0 \in D(A)$, $u \in C^2([0,T])$, with $u(0)=0$, and $f \in C^1([0,T],L^2(0,L))$, then $\Psi(y_0,u,f)$ is the unique classical solution of (2.1) *and*

$$\delta(y_0,u,f) = \partial_x \Psi(y_0,u,f)(\cdot,0).$$

The function $\Psi(y_0,u,f)$ is called the mild solution or simply the solution of (2.1) in the context of this paper.

Now we focus our attention on the domains of critical length. In particular, we describe the space M of unreachable states for the linear control system (1.3). Let $L \in N$. There exists a finite number of pairs $\{(k_j,l_j)\}_{j=1}^n \subset \mathbb{N}^*\times\mathbb{N}^*$, with $k_j \geq l_j$, such that

$$L = 2\pi\sqrt{\frac{k_j^2 + k_j l_j + l_j^2}{3}}. \tag{2.2}$$

From the work of Rosier in [17], we know that for each $j \in \{1,\dots,n\}$ there exist two nonzero real-valued functions $\varphi_1^j = \varphi_1^j(x)$ and $\varphi_2^j = \varphi_2^j(x)$ such that $\varphi^j := \varphi_1^j + i\varphi_2^j$ is a solution of

$$\begin{cases} -ip(k_j,l_j)\varphi^j + \varphi^{j\prime} + \varphi^{j\prime\prime\prime} = 0, \\ \varphi^j(0) = \varphi^j(L) = 0, \\ \varphi^{j\prime}(0) = \varphi^{j\prime}(L) = 0, \end{cases} \tag{2.3}$$

where, for $(k,l) \in \mathbb{N}^*\times\mathbb{N}^*$, $p(k,l)$ is defined by

$$p(k,l) := \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}}.$$

Easy computations lead to

$$\begin{aligned} \varphi_1^j &= C\left(\cos(\gamma_1^j x) - \frac{\gamma_1^j-\gamma_3^j}{\gamma_2^j-\gamma_3^j}\cos(\gamma_2^j x) + \frac{\gamma_1^j-\gamma_2^j}{\gamma_2^j-\gamma_3^j}\cos(\gamma_3^j x)\right), \\ \varphi_2^j &= C\left(\sin(\gamma_1^j x) - \frac{\gamma_1^j-\gamma_3^j}{\gamma_2^j-\gamma_3^j}\sin(\gamma_2^j x) + \frac{\gamma_1^j-\gamma_2^j}{\gamma_2^j-\gamma_3^j}\sin(\gamma_3^j x)\right), \end{aligned} \tag{2.4}$$

where C is a constant and the numbers γ_m^j, with $m = 1, 2, 3$, are the three roots of $x^3 - x + p(k_j, l_j) = 0$. One can easily verify that these roots are given by

$$\gamma_1^j = -\frac{2\pi}{L}\left(\frac{2k_j + l_j}{3}\right), \quad \gamma_2^j = \gamma_1^j + \frac{2\pi k_j}{L}, \quad \gamma_3^j = \gamma_2^j + \frac{2\pi l_j}{L}. \tag{2.5}$$

Moreover, by choosing the constant C, we can assume that

$$\|\varphi_1^j\|_{L^2(0,L)} = \|\varphi_2^j\|_{L^2(0,L)} = 1.$$

Roughly speaking, the functions φ_1^j and φ_2^j for $j = 1, \ldots, n$ are unreachable states for the linear KdV control system (1.3) since the following functions:

$$y_1(t, x) = \mathrm{Re}(e^{-ip(k_j,l_j)t}\varphi^j(x)) \quad \text{and} \quad y_2(t, x) = \mathrm{Im}(e^{-ip(k_j,l_j)t}\varphi^j(x))$$

are solutions of (1.3) with $u(t) \equiv 0$, but they do not satisfy the next observability inequality leading to the exact controllability

$$\|y(0, x)\|_{L^2(0,L)} \leq C\|\partial_x y(t, 0)\|_{L^2(0,T)}.$$

Let us define the following subspaces of $L^2(0, L)$:

$$M := \langle\{\varphi_1^1, \varphi_2^1, \ldots, \varphi_1^n, \varphi_2^n\}\rangle \quad \text{and} \quad H := M^\perp.$$

Remark 2.2. If $p(k_j, l_j) = 0$ for some $j \in \{1, \ldots, n\}$, then $\varphi_1^j = \varphi_2^j = 1 - \cos(x)$. It occurs when $k_j = l_j$, i.e., if $L = 2\pi k_j$. If k_j satisfies the condition (1.5), then the space M is one-dimensional. This is the case treated in [8]. It corresponds, for example, to the length $L = 2\pi$.

Remark 2.3. If $p(k_j, l_j) \neq 0$, it is easy to see that $\varphi_1^j \perp \varphi_2^j$. Moreover, for distinct $j_1, j_2 \in \{1, \ldots, n\}$, $\varphi_m^{j1} \perp \varphi_s^{j2}$ for $m, s = 1, 2$. Let us give some examples. The pair $(2, 1)$ defines a critical length for which the space M is two-dimensional. The pair $(11, 8)$ defines a critical length for which the space M is four-dimensional since the pairs $(11, 8)$ and $(16, 1)$ define the same critical length.

At this point, we can state the following controllability result which follows directly from the work of Rosier in [17, Propositions 3.3 and 3.9].

THEOREM 2.4. *Let $T > 0$. For every $(y_0, y_T) \in H \times H$, there exist $u \in L^2(0, T)$, and $y \in \mathcal{B}$ satisfying* (1.3), *$y(0, \cdot) = y_0$, and $y(T, \cdot) = y_T$.*

Now let us define the set N' by

$$N' := \left\{2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}; (k, l) \in \mathbb{N}^* \times \mathbb{N}^* \text{ satisfying } k > l \text{ and } (2.7)\right\} \tag{2.6}$$

$$\forall m, n \in \mathbb{N}^*\backslash\{k\}, \; k^2 + kl + l^2 \neq m^2 + mn + n^2. \tag{2.7}$$

It is easy to see that N' is the set of critical lengths for which the space of unreachable states is two-dimensional. Indeed, let $L \in N'$; from (2.7) there exists a unique pair $(k_1, l_1) := (k, l)$ satisfying (2.2), and since $k_1 > l_1$, $p(k_1, l_1) > 0$, and therefore the functions φ_1^1, φ_2^1 are orthogonal.

Let us follow the proof of Proposition 8.3 in [7] in order to see that N' contains an infinite number of elements. Let $q \geq 1$ be an integer satisfying

$$\forall m, n \in \mathbb{N}^*\backslash\{q\}, \quad m^2 + mn + n^2 \neq 7q^2. \tag{2.8}$$

Let us consider the critical length L_q defined by the pair $(2q, q)$, that is,

$$L_q := 2\pi\sqrt{\frac{(2q)^2 + 2q^2 + q^2}{3}} = 2\pi q\sqrt{\frac{7}{3}}.$$

From (2.8), it is easy to see that $L_q \in N'$. One can verify that (2.8) holds for $q = 1, 2, 3$, and therefore $L_1, L_2, L_3 \in N'$. Moreover, the following lemma says that the set N' contains an infinite number of lengths L_q.

LEMMA 2.5. *There are infinitely many positive integers q satisfying* (2.8).

Proof. Let $q > 3$ be a prime integer which does not satisfy (2.8), that is, such that

$$\exists m, n \in \mathbb{N}^* \backslash \{q\}, \quad m^2 + mn + n^2 = 7q^2. \tag{2.9}$$

From (2.9) one gets

$$-3mn = (m - n)^2 \ (\mathrm{mod}\, q), \quad mn = (m + n)^2 \ (\mathrm{mod}\, q). \tag{2.10}$$

It is easy to see that $m + n \neq 0\,(\mathrm{mod}\, q)$, and consequently from (2.10) we have

$$-3 = \big((m + n)^{-1}(m - n)\big)^2 \quad (\mathrm{mod}\, q); \tag{2.11}$$

that is, -3 is a square modulo q. Let us introduce the Legendre symbol, where s is a prime and $x \in \mathbb{Z}$ is an integer not divisible by s:

$$\left(\frac{x}{s}\right) := \begin{cases} 1 & \text{if } x \text{ is a square modulo}\, s, \\ -1 & \text{if } x \text{ is not a square modulo}\, s. \end{cases}$$

We have the quadratic reciprocity law due to Gauss for every prime integer $z > 2$, $s > 2$ (see [21, Chapter 3])

$$\left(\frac{s}{z}\right) = \left(\frac{z}{s}\right)(-1)^{\epsilon(z)\epsilon(s)}, \tag{2.12}$$

where

$$\epsilon(z) = \begin{cases} 0 & \text{if } \ z = 1 \ (\mathrm{mod}\, 4), \\ 1 & \text{if } \ z = -1 \ (\mathrm{mod}\, 4). \end{cases}$$

From [21, Chapter 3], we also have that for every x, y coprime to s

$$\left(\frac{xy}{s}\right) = \left(\frac{x}{s}\right)\left(\frac{y}{s}\right) \tag{2.13}$$

and for every $s > 2$ prime integer

$$(-1)^{\epsilon(s)} = \left(\frac{-1}{s}\right). \tag{2.14}$$

Using (2.12), (2.14), (2.13), and (2.11) with $s = q$, $z = 3$, and since $\epsilon(3) = 1$, one obtains

$$\left(\frac{q}{3}\right) = \left(\frac{3}{q}\right)(-1)^{\epsilon(q)} = \left(\frac{3}{q}\right)\left(\frac{-1}{q}\right) = \left(\frac{-3}{q}\right) = 1;$$

that is, $q = 1$ (mod 3).

Hence, if $q > 3$ is a prime integer such that $q = 2 \pmod 3$, then q satisfies (2.8). As there are two possible nonzero congruences modulo 3, the Dirichlet density theorem (see [21, Chapter 4]) says that (2.8) holds on a set of prime integers of density 1/2. In particular, there are infinitely many positive integers q satisfying (2.8). $\square$

From now on and until the end of this paper, we consider $L \in N'$. From (2.7), for each $L \in N'$ we can define a unique

$$p := \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}},$$

and the space M is then defined by

$$M := \langle \varphi_1, \varphi_2 \rangle = \{\alpha\varphi_1 + \beta\varphi_2\,;\, \alpha, \beta \in \mathbb{R}\},$$

where φ_1 and φ_2 are given by (2.4) with γ_m^j replaced by γ_m, where γ_1, γ_2, and γ_3 are the three roots of $x^3 - x + p = 0$. From (2.3) we also have that φ_1 and φ_2 satisfy

$$\left\{\begin{array}{l} \varphi_1' + \varphi_1''' = -p\,\varphi_2, \\ \varphi_1(0) = \varphi_1(L) = 0, \\ \varphi_1'(0) = \varphi_1'(L) = 0, \end{array}\right. \tag{2.15}$$

and

$$\left\{\begin{array}{l} \varphi_2' + \varphi_2''' = p\,\varphi_1, \\ \varphi_2(0) = \varphi_2(L) = 0, \\ \varphi_2'(0) = \varphi_2'(L) = 0. \end{array}\right. \tag{2.16}$$

Now we investigate the evolution of the projection on the subspace M of a solution of (1.3). Let us consider $(y, u) \in \mathcal{B} \times L^2(0, T)$ satisfying (1.3). Let us multiply (2.15) by y and integrate on $[0, L]$. Using integrations by parts we get

$$\frac{d}{dt}\left(\int_0^L y(t,x)\varphi_1(x)dx\right) = -p\int_0^L y(t,x)\varphi_2(x)dx. \tag{2.17}$$

Similarly, multiplying (2.16) by y, we get

$$\frac{d}{dt}\left(\int_0^L y(t,x)\varphi_2(x)dx\right) = p\int_0^L y(t,x)\varphi_1(x)dx. \tag{2.18}$$

Hence, from (2.17) and (2.18), we obtain

$$\int_0^L y(t,x)\varphi_1(x)\,dx = \int_0^L y(0,x)(\cos(p\,t)\varphi_1(x) - \sin(p\,t)\varphi_2(x))\,dx, \tag{2.19}$$

$$\int_0^L y(t,x)\varphi_2(x)\,dx = \int_0^L y(0,x)(\sin(p\,t)\varphi_1(x) + \cos(p\,t)\varphi_2(x))\,dx. \tag{2.20}$$

From (2.19) and (2.20), we see that the projection on M of $y(t,\cdot)$, denoted by $P_M(y(t,\cdot))$, only turns in this two-dimensional subspace and therefore conserves its $L^2(0, L)$-norm. The period of this rotation is $2\pi/p$. Furthermore, we see that if the initial condition $y(0,\cdot)$ lies in H, the solution does too for every time t. Combining this rotation with Theorem 2.4, we obtain the following proposition.

PROPOSITION 2.6. *Let $y_0, y_1 \in L^2(0,L)$ be such that*

$$\|P_M(y_0)\|_{L^2(0,L)} = \|P_M(y_1)\|_{L^2(0,L)}.$$

Then there exists $t^* \leq \frac{2\pi}{p}$ and $u \in L^2(0,t^*)$ such that the solution $y = y(t,x)$ of (1.3), *with $y(0,\cdot) = y_0$, satisfies $y(t^*,\cdot) = y_1$.*

Proof. Let $y_M = y_M(t,x)$ be the solution of (1.3), with $y_M(0,\cdot) = P_M(y_0)$ and without control ($u \equiv 0$). We know that there exists a time $0 < t^* \leq \frac{2\pi}{p}$ such that $y_M(t^*,\cdot) = P_M(y_1)$. On the other hand, from Theorem 2.4 there exists a control $u_H \in L^2(0,t^*)$ such that the corresponding solution $y_H = y_H(t,x)$ of (1.3) satisfies

$$y_H(0,\cdot) = P_H(y_0) \in H \quad \text{and} \quad y_H(t^*,\cdot) = P_H(y_1).$$

Then $y(t,x) := y_H(t,x) + y_M(t,x)$ satisfies (1.3), with $u = u_H$, $y(0,\cdot) = y_0$, and $y(t^*,\cdot) = y_1$, which ends the proof of this proposition. ◻

3. Motion in the missed directions. Let us first explain the general idea of the method. Let $y = y(t,x)$ be a solution of (1.1) with control $u = u(t)$. We consider a power series expansion of (y,u) with the same scaling on the state and on the control

$$\begin{aligned} y &= \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 \ldots, \\ u &= \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 \ldots. \end{aligned}$$

In this way, we see that the nonlinear term is given by

$$y\partial_x y = \epsilon^2 y_1 \partial_x y_1 + \epsilon^3 y_1 \partial_x y_2 + \epsilon^3 y_2 \partial_x y_1 + (\text{higher terms}),$$

and therefore, for a small ϵ, we have the expansion of second order $y \approx \epsilon y_1 + \epsilon^2 y_2$, where y_1 and y_2 are given by

$$\begin{cases} \partial_t y_1 + \partial_x y_1 + \partial_x^3 y_1 = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ \partial_x y_1(t,L) = u_1(t), \end{cases}$$

and

$$\begin{cases} \partial_t y_2 + \partial_x y_2 + \partial_x^3 y_2 = -y_1 \partial_x y_1, \\ y_2(t,0) = y_2(t,L) = 0, \\ \partial_x y_2(t,L) = u_2(t), \end{cases}$$

respectively. The strategy consists first in proving that the expansion to the second order of $y = y(t,x)$, i.e., $\epsilon y_1 + \epsilon^2 y_2$, can reach all of the missed directions and then in using a fixed point argument to prove that it is sufficient to get Theorem 1.4. This is a classical approach to study the local controllability of a finite-dimensional control system, and it has been applied in [8] to prove the local exact controllability around the origin of the control system (1.1) for some critical domains.

Now we see that we can "enter" into the subspace M. More precisely, the result we prove is the following one.

PROPOSITION 3.1. *Let $T > 0$. There exists $(u,v) \in L^2(0,T)^2$ such that if $\alpha = \alpha(t,x)$ and $\beta = \beta(t,x)$ are the solutions of*

$$\begin{cases} \partial_t \alpha + \partial_x \alpha + \partial_x^3 \alpha = 0, \\ \alpha(t,0) = \alpha(t,L) = 0, \\ \partial_x \alpha(t,L) = u(t), \\ \alpha(0,\cdot) = 0, \end{cases} \tag{3.1}$$

and

$$
\left\{
\begin{array}{l}
\partial_t \beta + \partial_x \beta + \partial_x^3 \beta = -\alpha \partial_x \alpha, \\
\beta(t,0) = \beta(t,L) = 0, \\
\partial_x \beta(t,L) = v(t), \\
\beta(0,\cdot) = 0,
\end{array}
\right.
\tag{3.2}
$$

then

$$
\alpha(T,\cdot) = 0 \quad \textit{and} \quad \beta(T,\cdot) \in M\backslash\{0\}.
$$

Proof. In order to study the trajectory $\beta = \beta(t,x)$, we set $\beta = \beta^u + \beta^v$, where $\beta^u = \beta^u(t,x)$ and $\beta^v = \beta^v(t,x)$ are the solutions of

$$
\left\{
\begin{array}{l}
\partial_t \beta^u + \partial_x \beta^u + \partial_x^3 \beta^u = -\alpha \partial_x \alpha, \\
\beta^u(t,0) = \beta^u(t,L) = 0, \\
\partial_x \beta^u(t,L) = 0, \\
\beta^u(0,\cdot) = 0,
\end{array}
\right.
\tag{3.3}
$$

and

$$
\left\{
\begin{array}{l}
\partial_t \beta^v + \partial_x \beta^v + \partial_x^3 \beta^v = 0, \\
\beta^v(t,0) = \beta^v(t,L) = 0, \\
\partial_x \beta^v(t,L) = v(t), \\
\beta^v(0,\cdot) = 0,
\end{array}
\right.
\tag{3.4}
$$

respectively. If $u \in L^2(0,T)$ is given, by Theorem 2.4 one can find $v \in L^2(0,T)$ such that

$$
\beta^v(T,\cdot) = -P_H(\beta^u(T,\cdot))
$$

and thus $\beta(T,\cdot) = P_M(\beta^u(T,\cdot))$. From this fact, one sees that the proof of Proposition 3.1 can be reduced to prove

$$
\exists u \in L^2(0,T) \quad \text{such that} \quad \alpha(T,\cdot) = 0 \quad \text{and} \quad P_M(\beta^u(T,\cdot)) \neq 0. \tag{3.5}
$$

Let $u \in L^2(0,T)$. Let us multiply (3.3) by φ_1 and integrate the resulting equality on $[0,L]$. Then, using integration by parts, (2.15), and the boundary and initial conditions in (3.3), one gets

$$
\frac{d}{dt}\left(\int_0^L \beta^u(t,x)\varphi_1(x)dx\right) = -p\int_0^L \beta^u(t,x)\varphi_2(x)dx + \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_1'(x)dx.
$$

In a similar way, if we now multiply (3.3) by φ_2, we get

$$
\frac{d}{dt}\left(\int_0^L \beta^u(t,x)\varphi_2(x)dx\right) = p\int_0^L \beta^u(t,x)\varphi_1(x)dx + \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_2'(x)dx.
$$

If we call

$$
\eta_k(t) := \int_0^L \beta^u(t,x)\varphi_k(x)dx \quad \text{for } k = 1, 2,
$$

we can write the system

$$
\left\{
\begin{array}{l}
\begin{pmatrix} \dot\eta_1(t) \\ \dot\eta_2(t) \end{pmatrix} = \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_1'(x)dx \\ \frac{1}{2}\int_0^L \alpha^2(t,x)\varphi_2'(x)dx \end{pmatrix}, \\
\eta_1(0) = 0, \quad \eta_2(0) = 0.
\end{array}
\right.
\tag{3.6}
$$

The solution of (3.6) is given by

$$
\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = \begin{pmatrix} \cos(p\,t) & -\sin(p\,t) \\ \sin(p\,t) & \cos(p\,t) \end{pmatrix} \begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix},
$$

where

$$
\begin{aligned}
I_1(t) &:= \frac{1}{2}\int_0^t \int_0^L \alpha^2(s,x)(\cos(ps)\varphi_1'(x) + \sin(ps)\varphi_2'(x))dx\,ds, \\
I_2(t) &:= \frac{1}{2}\int_0^t \int_0^L \alpha^2(s,x)(-\sin(ps)\varphi_1'(x) + \cos(ps)\varphi_2'(x))dx\,ds.
\end{aligned}
$$

If we work with complex numbers calling $\varphi := \varphi_1 + i\varphi_2$, we get

$$
\eta_1(t) + i\eta_2(t) = \frac{1}{2}e^{ip\,t}\int_0^t \int_0^L e^{-ips}\alpha^2(s,x)\varphi'(x)dx\,ds.
$$

Now let us assume that (3.5) fails to be true; i.e., let us suppose that

$$
\forall\, u \in L^2(0,T), \quad \eta_1(T) = \eta_2(T) = 0 \quad \text{or} \quad \alpha(T,\cdot) \neq 0. \tag{3.7}
$$

If we define

$$
U_{ad} := \left\{u \in L^2(0,T);\ \text{the solution } \alpha \text{ of (3.1) satisfies } \alpha(T,\cdot) = 0\right\},
$$

then condition (3.7) implies that

$$
\forall\, u \in U_{ad}, \quad \int_0^T \int_0^L e^{-ips}\alpha^2(s,x)\varphi'(x)dx\,ds = 0. \tag{3.8}
$$

Let $\alpha_1 = \alpha_1(t,x)$ and $\alpha_2 = \alpha_2(t,x)$ be two solutions of (3.1) such that

$$
\alpha_1(T,\cdot) = \alpha_2(T,\cdot) = 0.
$$

Now, for $(\rho_1,\rho_2) \in \mathbb{R}^2$, let $\alpha := \rho_1\alpha_1 + \rho_2\alpha_2$ and $u := \alpha_x(\cdot,L)$. By linearity, we see that $\alpha = \alpha(t,x)$ is a solution of (3.1) and $u \in U_{ad}$. Consequently, (3.8) implies that, for every $(\rho_1,\rho_2) \in \mathbb{R}^2$,

$$
\begin{aligned}
\rho_1^2 \int_0^T \int_0^L e^{-ips}\alpha_1^2(s,x)\varphi'(x)dx\,ds + 2\rho_1\rho_2 \int_0^T \int_0^L e^{-ips}\alpha_1(s,x)\alpha_2(s)\varphi'(x)dx\,ds& \\
+ \rho_2^2 \int_0^T \int_0^L e^{-ips}\alpha_2^2(s,x)\varphi'(x)dx\,ds = 0.&
\end{aligned}
$$

Looking at the coefficient of $\rho_1\rho_2$, we get

$$
\int_0^T \int_0^L e^{-ips}\alpha_1(s,x)\alpha_2(s,x)\varphi'(x)dx\,ds = 0. \tag{3.9}
$$

Let t_1, t_2 be such that $0 < t_1 < t_2 < T$. We choose the trajectories $\alpha_1 = \alpha_1(t,x)$ and $\alpha_2 = \alpha_2(t,x)$ such that

$$\alpha_2 \text{ is not identically equal to } 0, \tag{3.10}$$

$$\alpha_2(t,x)|_{([0,t_1]\cup[t_2,T])\times[0,L]} = 0 \quad \text{and} \quad \alpha_1(t,x)|_{[t_1,t_2]\times[0,L]} = \mathrm{Re}(e^{\lambda t} y_\lambda(x)), \tag{3.11}$$

where $\lambda \in \mathbb{C}\backslash\{\pm ip\}$ and $y_\lambda = y_\lambda(x)$ is a complex-valued function which satisfies

$$\begin{cases} \lambda y_\lambda + y_\lambda' + y_\lambda''' = 0, \\ y_\lambda(0) = y_\lambda(L) = 0. \end{cases} \tag{3.12}$$

If $\lambda \neq \pm ip$, one can see that $\mathrm{Re}(y_\lambda), \mathrm{Im}(y_\lambda) \in H$, and then by Theorem 2.4 there exists such a trajectory $\alpha_1 = \alpha_1(t,x)$.

Let us introduce the operator $\tilde{A}w = -w' - w'''$ on the domain $D(\tilde{A}) \subset L^2(0,L)$ defined by

$$D(\tilde{A}) := \left\{ w \in H^3(0,L);\, w(0) = w(L) = 0, w'(0) = w'(L) \right\}.$$

It is not difficult to see that $i\tilde{A}$ is a self-adjoint operator on $L^2(0,L)$ with compact resolvent. Hence, the spectrum $\sigma(\tilde{A})$ of $\tilde{A}$ consists only of eigenvalues. Furthermore, the spectrum is a discrete subset of $i\mathbb{R}$.

If we take λ such that $(-ip+\lambda) \notin \sigma(\tilde{A})$, the operator $(\tilde{A}-(-ip+\lambda)I)$ is invertible, and thus, there exists a unique complex-valued function $\phi_\lambda = \phi_\lambda(x)$ solution of

$$\begin{cases} (-ip+\lambda)\phi_\lambda + \phi_\lambda' + \phi_\lambda''' = y_\lambda \varphi', \\ \phi_\lambda(0) = \phi_\lambda(L) = 0, \\ \phi_\lambda'(0) = \phi_\lambda'(L). \end{cases} \tag{3.13}$$

We multiply (3.13) by $\alpha_2(t,x)e^{(-ip+\lambda)t}$, integrate on $[0,L]$, and use integrations by parts together with (3.1), and the boundary and initial conditions in (3.13) to get

$$e^{-ip\,t}\int_0^L e^{\lambda t} y_\lambda \alpha_2(t,x)\varphi'(x)dx = \frac{d}{dt}\left(\int_0^L e^{(-ip+\lambda)t}\phi_\lambda(x)\alpha_2(t,x)dx\right) - e^{(-ip+\lambda)t}\phi_\lambda'(L)\partial_x\alpha_2(t,x)\Big|_{x=0}^{L}.$$

Then let us integrate this equality on $[0,T]$ and use the fact that $\alpha_2(0,\cdot) = 0$ and $\alpha_2(T,\cdot) = 0$. We obtain

$$\int_0^T\int_0^L e^{-ip\,t}e^{\lambda t} y_\lambda \alpha_2(t,x)\varphi'(x)dx\,dt = -\phi_\lambda'(L)\int_0^T e^{(-ip+\lambda)t}\left(\partial_x\alpha_2(t,L) - \partial_x\alpha_2(t,0)\right)dt. \tag{3.14}$$

On the other hand, by (3.9) and (3.11), it follows that

$$\int_0^T\int_0^L e^{-ip\,t}\mathrm{Re}(e^{\lambda t}y_\lambda)\alpha_2(t,x)\varphi'(x)dx\,dt = 0, \tag{3.15}$$

and, since one can also take a trajectory $\tilde{\alpha}_1 = \tilde{\alpha}_1(t,x)$ such that

$$\tilde{\alpha}_1(t,x)|_{[t_1,t_2]\times[0,L]} = \mathrm{Im}(e^{\lambda t}y_\lambda(x)),$$

one deduces from (3.9) that

$$\int_0^T \int_0^L e^{-ip\,t}\mathrm{Im}(e^{\lambda t}y_\lambda)\alpha_2(t,x)\varphi'(x)dx\,dt = 0. \tag{3.16}$$

Therefore, from (3.15) and (3.16), one gets

$$\int_0^T \int_0^L e^{-ip\,t}e^{\lambda t}y_\lambda\alpha_2(t,x)\varphi'(x)dx\,dt = 0,$$

and consequently from (3.14), for every $\lambda \neq \pm ip$ such that $(-ip+\lambda) \notin \sigma(\tilde{A})$, one has

$$\phi'_\lambda(L)\int_0^T e^{(-ip+\lambda)t}\left(\partial_x\alpha_2(t,L) - \partial_x\alpha_2(t,0)\right)dt = 0. \tag{3.17}$$

Let $a \in \mathbb{R}\backslash[-1/\sqrt{3}, 1/\sqrt{3}]$. We take $\lambda = 2ai(4a^2-1)$. Let

$$y_\lambda(x) = Ce^{(-\sqrt{3a^2-1}-ai)x} + (1-C)e^{(\sqrt{3a^2-1}-ai)x} - e^{2aix}, \tag{3.18}$$

where

$$C = \frac{e^{2aiL} - e^{(\sqrt{3a^2-1}-ai)L}}{e^{(-\sqrt{3a^2-1}-ai)L} - e^{(\sqrt{3a^2-1}-ai)L}}.$$

One easily checks that such a $y_\lambda = y_\lambda(x)$ satisfies (3.12) and $y_\lambda \neq 0$. Let us define

$$\Sigma := \left\{a \in \mathbb{R}\backslash[-1/\sqrt{3}, 1/\sqrt{3}]\,;\, \lambda \notin \sigma(\tilde{A}), (\lambda - ip) \notin \sigma(\tilde{A})\right\},$$

where $\lambda = 2ai(4a^2-1)$. Then the function $S : \Sigma \to \mathbb{C}$, $S(a) = \phi'_\lambda(L)$ is continuous. Now we use the fact that S is not identically equal to the function 0 (the proof of this statement will be given in Lemma 3.6 at the end of this section). Then there exist $\widehat{a} \in \Sigma$ and $\epsilon > 0$ such that, for every $a \in \Sigma$ with $|a - \widehat{a}| < \epsilon$, $S(a) \neq 0$. From (3.17) one gets

$$\forall\, a \in \Sigma, \quad |a-\widehat{a}| < \epsilon, \quad \int_0^T e^{(-p+2a(4a^2-1))i\,t}\left(\partial_x\alpha_2(t,L) - \partial_x\alpha_2(t,0)\right)dt = 0,$$

and since the function $\beta \in \mathbb{C} \mapsto \int_0^T e^{\beta t}\left(\partial_x\alpha_2(t,L) - \partial_x\alpha_2(t,0)\right)dt \in \mathbb{C}$ is holomorphic, it follows that

$$\forall\, \beta \in \mathbb{C}, \quad \int_0^T e^{\beta t}\big(\partial_x\alpha_2(t,L) - \partial_x\alpha_2(t,0)\big)dt = 0,$$

which implies that $\partial_x\alpha_2(t,0) - \partial_x\alpha_2(t,L) = 0$ for every t. In summary, one has that $\alpha_2 = \alpha_2(t,x)$ satisfies

$$\begin{cases} \partial_t\alpha_2 + \partial_x\alpha_2 + \partial_x^3\alpha_2 = 0, \\ \alpha_2(t,0) = \alpha_2(t,L) = 0, \\ \partial_x\alpha_2(t,0) = \partial_x\alpha_2(t,L), \\ \alpha_2(0,\cdot) = 0, \\ \alpha_2(T,\cdot) = 0. \end{cases} \tag{3.19}$$

If we multiply (3.19) by α_2, integrate on $[0,L]$, and use integration by parts together with the boundary conditions, we obtain that

$$\frac{d}{dt}\int_0^L |\alpha_2(t,x)|^2 dx = 0,$$

which, together with $\alpha_2(0,\cdot) = 0$, implies that

$$\alpha_2(t,x) = 0 \quad \forall\, x \in [0,L],\, \forall\, t \in [0,T]. \tag{3.20}$$

But this is in contradiction with (3.10). Thus, we have proved (3.5) and therefore Proposition 3.1. ☐

From now on, for each $T_c > 0$, we denote by $(u_c, v_c) \in L^2(0,T)^2$ the controls given by Proposition 3.1 and by (α_c, β_c) the corresponding trajectories. Let us define $\tilde{\varphi}_1 := \beta_c(T_c,\cdot)$. Let us notice that, by scaling the controls, we can assume that $\|\tilde{\varphi}_1\|_{L^2(0,L)} = 1$. We will prove now that in any time $T > \pi/p$, we can reach all of the states lying in M.

PROPOSITION 3.2. *Let $T > \pi/p$. Let $\psi \in M$. There exists $(u,v) \in L^2(0,T)^2$ such that if $\alpha = \alpha(t,x)$ and $\beta = \beta(t,x)$ are the solutions of* (3.1) *and* (3.2), *then*

$$\alpha(T,\cdot) = 0 \quad \textit{and} \quad \beta(T,\cdot) = \psi.$$

Proof. Let $\hat{T} > 0$ be such that $T = (\pi/p) + \hat{T}$. Let T_c be such that $0 < T_c < \hat{T}$. Let $T_a := T - T_c$. If we take in (3.1) and (3.2) the controls

$$(u^1, v^1)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a), \\ (u_c(t - T_a), v_c(t - T_a)) & \text{if } t \in (T_a, T), \end{cases}$$

we obtain that $\beta^1(T,\cdot) = \tilde{\varphi}_1$, where $\beta^1 = \beta^1(t,x)$ is the corresponding solution of (3.2). Now we use the rotation showed in section 2 (see, in particular, (2.19) and (2.20)) in order to reach other states lying in M. Let us define $\tilde{\varphi}_2 := \beta^2(T,\cdot)$, where $\beta^2 = \beta^2(t,x)$ is defined by the controls

$$(u^2, v^2)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a - \frac{\pi}{2p}), \\ (u_c(t - T_a + \frac{\pi}{2p}), v_c(t - T_a + \frac{\pi}{2p})) & \text{if } t \in (T_a - \frac{\pi}{2p}, T - \frac{\pi}{2p}), \\ (0,0) & \text{if } t \in (T - \frac{\pi}{2p}, T). \end{cases}$$

In a similar way, the controls

$$(u^3, v^3)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a - \frac{\pi}{p}), \\ (u_c(t - T_a + \frac{\pi}{p}), v_c(t - T_a + \frac{\pi}{p})) & \text{if } t \in (T_a - \frac{\pi}{p}, T - \frac{\pi}{p}), \\ (0,0) & \text{if } t \in (T - \frac{\pi}{p}, T) \end{cases}$$

allow us to define $\tilde{\varphi}_3 := \beta^3(T,\cdot)$. Notice that $\tilde{\varphi}_3 = -\tilde{\varphi}_1$.

Let T_θ be such that $0 < T_\theta < \min\{\pi/(2p), \hat{T} - T_c\}$, and let $T_b := (\pi/p) + T_\theta$. Let us define $\tilde{\varphi}_4 := \beta^4(T,\cdot)$, where $\beta^4 = \beta^4(t,x)$ is the solution of (3.2), with

$$(u^4, v^4)(t) = \begin{cases} (0,0) & \text{if } t \in (0, T_a - T_b), \\ (u_c(t - T_a + T_b), v_c(t - T_a + T_b)) & \text{if } t \in (T_a - T_b, T - T_b), \\ (0,0) & \text{if } t \in (T - T_b, T). \end{cases}$$

We have thus proved that we can reach the missed directions $\{\tilde{\varphi}_k\}_{k=1}^4$. Let us now define the cones

$$\begin{aligned}
M_1 &:= \{d_1\tilde{\varphi}_1 + d_2\tilde{\varphi}_2; d_1 > 0, d_2 \geq 0\},\\
M_2 &:= \{d_1\tilde{\varphi}_2 + d_2\tilde{\varphi}_3; d_1 > 0, d_2 \geq 0\},\\
M_3 &:= \{d_1\tilde{\varphi}_3 + d_2\tilde{\varphi}_4; d_1 > 0, d_2 \geq 0\},\\
M_4 &:= \{d_1\tilde{\varphi}_4 + d_2\tilde{\varphi}_1; d_1 > 0, d_2 \geq 0\}.
\end{aligned}$$

By construction of these cones, one has that $M = \cup_{k=1}^4 M_k$.

Remark 3.3. It is easy to see that if one chooses T_c, T_θ such that $T_c < T_\theta$, then the supports of the trajectories $\alpha^k = \alpha^k(t,x)$ for $k = 1, \dots, 4$ are disjoint.

For each $w = (w_1, w_2) \in \mathbb{R}^2$, let us define

$$\rho_w := \sqrt{w_1^2 + w_2^2} \quad \text{and} \quad z_w := (w_1\varphi_1 + w_2\varphi_2)/\rho_w \in M.$$

We have that $z_w \in M_i$ for some $i \in \{1, \dots, 4\}$, and hence there exist $d_{1w} > 0$ and $d_{2w} \geq 0$ such that $z_w = d_{1w}\tilde{\varphi}_i + d_{2w}\tilde{\varphi}_{i+1}$. If we take the control

$$(u_w, v_w) = (d_{1w}^{1/2}u^i + d_{2w}^{1/2}u^{i+1}, d_{1w}v^i + d_{2w}v^{i+1})$$

and use the fact that the trajectories α^k for $k = 1, \dots, 4$ are disjoints, then we see that the corresponding solution $\beta_w = \beta_w(t,x)$ of (3.2) satisfies $\beta_w(T, \cdot) = z_w$.

Finally, let $\psi \in M$. With $R := \|\psi\|_{L^2(0,L)}$ we can write $\psi = Rz_w$ for a $(w_1, w_2) \in \mathbb{R}^2$ such that $w_1^2 + w_2^2 = 1$. It is easy to see that the control $(u, v) = (R^{1/2}u_w, Rv_w)$ allows us to reach the state ψ, and so the proof of this proposition is ended. □

Remark 3.4. The proof of Proposition 3.2 is the only part which needs a time large enough. Hence, Theorem 1.4 holds for $T_M := \pi/p$.

Remark 3.5. In [8] an expansion to the second order is not sufficient, and the authors must go to the third order to enter into the subspace of missed directions. Since in their case this subspace is one-dimensional and since they use an odd order expansion, one can reach all of the missed states with a scaling argument. Our case is different. We can also enter into the subspace of missed directions in any time, but, in order to reach all of these states, our method needs a time large enough.

It remains to prove the following lemma to complete the proof of Proposition 3.1.

LEMMA 3.6. *The function S is not identically equal to* 0.

Proof. Let $a \in \Sigma$ and $\lambda = 2ai(4a^2 - 1)$. Let $\mu \in \mathbb{C}$, and let $y_\mu = y_\mu(x)$ be a solution of

$$\begin{cases} \mu y_\mu + y_\mu' + y_\mu''' = 0,\\ y_\mu(0) = y_\mu(L) = 0.\end{cases}$$

We multiply (3.13) by y_μ and integrate by parts on $[0, L]$. Thus, we get

$$(3.21) \qquad (\lambda - ip + \mu)\int_0^L \phi_\lambda y_\mu dx - \phi_\lambda'(L)\big(y_\mu'(L) - y_\mu'(0)\big) = \int_0^L y_\lambda \varphi' y_\mu dx.$$

From now on, we set $\mu = \mu(a) := -\lambda + ip$. With this choice we obtain from (3.21)

$$-S(a)\big(y_\mu'(L) - y_\mu'(0)\big) = \int_0^L y_\lambda \varphi' y_\mu dx.$$

Therefore, if we prove that the function

$$a \in \Sigma \longrightarrow J(a) := \int_0^L y_\lambda \varphi' y_\mu dx \in \mathbb{C}$$

is not identically equal to 0, the proof of this lemma is ended. Let $b \in \mathbb{R}$ be such that $\mu = 2bi(4b^2 - 1)$. We take the function y_μ given by

$$y_\mu(x) = De^{(-\sqrt{3b^2-1}-bi)x} + (1-D)e^{(\sqrt{3b^2-1}-bi)x} - e^{2bix}, \tag{3.22}$$

where

$$D = \frac{e^{2biL} - e^{(\sqrt{3b^2-1}-bi)L}}{e^{(-\sqrt{3b^2-1}-bi)L} - e^{(\sqrt{3b^2-1}-bi)L}}.$$

In the next computations, we use the fact that $e^{i\gamma_1 L} = e^{i\gamma_2 L} = e^{i\gamma_3 L}$ (see (2.5)) and the following formula:

$$\int_0^L e^{(v+iw)x}\varphi' = \frac{\left(1+\gamma_1^2 - 2p/\gamma_1\right)\left(1 - e^{(v+iw+i\gamma_1)L}\right)(vi-w)}{(vi-w)^3 - (vi-w) + p}, \tag{3.23}$$

which holds if $v + iw \neq -i\gamma_m$ for $m = 1, 2, 3$.

We want to show that as $a \to \infty$, the following expression diverges, which is in contradiction with the fact that $J(a) \equiv 0$:

$$R(a) := \frac{(e^{(-\sqrt{3a^2-1}-ai)L} - e^{(\sqrt{3a^2-1}-ai)L})(e^{(-\sqrt{3b^2-1}-bi)L} - e^{(\sqrt{3b^2-1}-bi)L})}{1+\gamma_1^2 - 2p/\gamma_1} J(a).$$

In fact, by using (3.23), one computes explicitly $J(a)$, and thus one sees that, as a tends to infinity, the dominant term of $R(a)$ is given by

$$\begin{aligned}
Z(a) := e^{(\sqrt{3a^2-1}+\sqrt{3b^2-1})L}\Big\{ &\frac{(e^{(-ai-bi)L} - e^{(ai+bi+\gamma_1 i)L})(-2a-2b)}{(-2a-2b)^3 - (-2a-2b) + p} \\
&+ \frac{e^{(-ai-bi)L}(-i\sqrt{3a^2-1} - i\sqrt{3b^2-1} + a + b)}{(-i\sqrt{3a^2-1} - i\sqrt{3b^2-1} + a + b)^3 - (-i\sqrt{3a^2-1} - i\sqrt{3b^2-1} + a + b) + p} \\
&- \frac{e^{(ai+bi+\gamma_1 i)L}(i\sqrt{3a^2-1} + i\sqrt{3b^2-1} + a + b)}{(i\sqrt{3a^2-1} + i\sqrt{3b^2-1} + a + b)^3 - (i\sqrt{3a^2-1} + i\sqrt{3b^2-1} + a + b) + p} \\
&+ \frac{e^{(ai+bi+\gamma_1 i)L}(i\sqrt{3a^2-1} + a - 2b)}{(i\sqrt{3a^2-1} + a - 2b)^3 - (i\sqrt{3a^2-1} + a - 2b) + p} \\
&- \frac{e^{(-ai-bi)L}(-i\sqrt{3b^2-1} - 2a + b)}{(-i\sqrt{3b^2-1} - 2a + b)^3 - (-i\sqrt{3b^2-1} - 2a + b) + p} \\
&+ \frac{e^{(ai+bi+\gamma_1 i)L}(i\sqrt{3b^2-1} - 2a + b)}{(i\sqrt{3b^2-1} - 2a + b)^3 - (i\sqrt{3b^2-1} - 2a + b) + p} \\
&- \frac{e^{(-ai-bi)L}(-i\sqrt{3a^2-1} + a - 2b)}{(-i\sqrt{3a^2-1} + a - 2b)^3 - (-i\sqrt{3a^2-1} + a - 2b) + p}\Big\}.
\end{aligned}$$

Using that as $a \to \infty$, $b \to -\infty$ and $a + b \sim -p/(24a^2)$, we obtain the following asymptotical expression for the right-hand factor of $Z(a)$:

$$\frac{-(e^{\frac{p}{24a^2}iL} - e^{-\frac{p}{24a^2}iL+i\gamma_1 L})}{12a^2} \sim \begin{cases} -\frac{(1-e^{i\gamma_1 L})}{12a^2} & \text{if } e^{i\gamma_1 L} \neq 1, \\ -\frac{ipL}{144a^4} & \text{if } e^{i\gamma_1 L} = 1. \end{cases}$$

One can see that in both cases $Z(a)$ diverges as $a \to \infty$, and therefore $R(a)$ does, which implies that $J(a)$ is not identically equal to 0. It ends the proof of this lemma. □

4. Proof of Theorem 1.4.

4.1. Existence and uniqueness results. Let us recall the existence property proved by Coron and Crépeau in [8] for the following nonlinear KdV equation:

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = f, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \\ y(0,\cdot) = y_0. \end{cases} \tag{4.1}$$

Proposition 4.1 (see [8, Proposition 14]). *Let $L > 0$ and $T > 0$. There exist $\epsilon > 0$ and $C > 0$ such that, for every $f \in L^1(0,T,L^2(0,L))$, $u \in L^2(0,T)$, and $y_0 \in L^2(0,L)$ such that*

$$\|f\|_{L^1(0,T,L^2(0,L))} + \|u\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)} \leq \epsilon,$$

there exists at least one solution of (4.1) *which satisfies*

$$\|y\|_{\mathcal{B}} \leq C(\|f\|_{L^1(0,T,L^2(0,L))} + \|u\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}). \tag{4.2}$$

For the uniqueness, one has the following.

Proposition 4.2 (see [8, Proposition 15]). *Let $T > 0$, and let $L > 0$. There exists $C > 0$ such that for every $(y_{01}, y_{02}) \in L^2(0,L)^2$, $(u_1,u_2) \in L^2(0,T)^2$, and $(f_1,f_2) \in L^1(0,T,L^2(0,L))^2$ for which there exist solutions $y_1 = y_1(t,x)$ and $y_2 = y_2(t,x)$ of* (4.1)*, one has the following estimates:*

$$\int_0^T\int_0^L |\partial_x y_1(t,x) - \partial_x y_2(t,x)|^2 dxdt \leq e^{C(1+\|y_1\|^2_{L^2(0,T,H^1(0,L))}+\|y_2\|^2_{L^2(0,T,H^1(0,L))})} \cdot \left(\|u_1-u_2\|^2_{L^2(0,T)} + \|f_1-f_2\|^2_{L^1(0,T,L^2(0,L))} + \|y_{01}-y_{02}\|^2_{L^2(0,L)}\right),$$

$$\int_0^L |y_1(t,x) - y_2(t,x)|^2 dx \leq e^{C(1+\|y_1\|^2_{L^2(0,T,H^1(0,L))}+\|y_2\|^2_{L^2(0,T,H^1(0,L))})} \cdot \left(\|u_1-u_2\|^2_{L^2(0,T)} + \|f_1-f_2\|^2_{L^1(0,T,L^2(0,L))} + \|y_{01}-y_{02}\|^2_{L^2(0,L)}\right)$$

for every $t \in [0,T]$.

4.2. Settings and a technical lemma. Until the end of this paper, we adopt some important notations. Let us denote, for $D > 0$ and $R > 0$,

$$B_R^D := \left\{\xi \in L^2(0,D)\,;\, \|\xi\|_{L^2(0,D)} \leq R\right\}$$

and recall that for each $w = (w_1, w_2) \in \mathbb{R}^2$, we write $\rho_w := \sqrt{w_1^2 + w_2^2}$ and $z_w := (w_1\varphi_1 + w_2\varphi_2)/\rho_w$. We also write $(u_w, v_w) \in L^2(0,T)$ the controls defined in section 3 in order to drive the solutions $\beta_w = \beta_w(t,x)$ from zero at $t = 0$ to z_w at $t = T$.

By the work of Rosier in [17], we know that for each $y_0 \in L^2(0,L)$ there exists a continuous linear affine map (it is a consequence of applying the HUM method to prove Theorem 2.4)

$$\Gamma_0 : h \in H \subset L^2(0,L) \longmapsto \Gamma_0(h) \in L^2(0,T)$$

such that the solution of

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = \Gamma_0(h), \\ y(0,\cdot) = P_H(y_0) \end{cases}$$

satisfies $y(T,\cdot) = h$. Moreover, there exist constants $D_1, D_2 > 0$ such that

$$\forall y_0 \in L^2(0,L),\ \forall h \in H, \quad \|\Gamma_0(h)\|_{L^2(0,T)} \le D_1(\|h\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)}), \tag{4.3}$$

$$\forall y_0 \in L^2(0,L),\ \forall h, g \in H, \quad \|\Gamma_0(h) - \Gamma_0(g)\|_{L^2(0,T)} \le D_2\|h - g\|_{L^2(0,L)}. \tag{4.4}$$

Let $y_0 \in L^2(0,L)$ be such that $\|y_0\|_{L^2(0,L)} < r$, $r > 0$ to be chosen later. Let us define the functions G and F

$$\begin{aligned} G : L^2(0,L) &\longrightarrow \quad L^2(0,T), \\ z = P_H(z) + w_1\varphi_1 + w_2\varphi_2 &\mapsto \quad G(z) = \Gamma_0(P_H(z)) + \rho_w^{1/2} u_w + \rho_w v_w, \end{aligned}$$

$$\begin{aligned} F : B_{\epsilon_1}^T \cap L^2(0,T) &\longrightarrow \quad L^2(0,L), \\ u &\longmapsto \quad F(u) = y(T,\cdot), \end{aligned}$$

where $y = y(t,x)$ is the solution of

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = u(t), \\ y(0,\cdot) = y_0, \end{cases} \tag{4.5}$$

and ϵ_1 is small enough so that the function F is well defined. It holds if $\epsilon_1 + r < \epsilon$, where ϵ is given by Proposition 4.1. The map F is even continuous according to Proposition 4.2. Let $y_T \in L^2(0,L)$ be such that $\|y_T\| < r$. Let Λ_{y_0,y_T} denote the map

$$\begin{aligned} \Lambda_{y_0,y_T} : B_{\epsilon_2}^L \cap L^2(0,L) &\longrightarrow \quad L^2(0,L), \\ z &\longmapsto \quad \Lambda_{y_0,y_T}(z) = z + y_T - F \circ G(z), \end{aligned}$$

where ϵ_2 is small enough so that Λ_{y_0,y_T} is well defined (ϵ_2 exists according to Proposition 4.1 and to the continuity of G).

Let us notice that if we find a fixed point $\tilde{z} \in L^2(0,L)$ of the map Λ_{y_0,y_T}, then we will have $F \circ G(\tilde{z}) = y_T$, and this means that the control $u := G(\tilde{z}) \in L^2(0,T)$ drives the solution of (4.5) from y_0 at $t = 0$ to y_T at $t = T$.

Let us assert the following technical result which will be needed to study the map Λ_{y_0,y_T}.

Lemma 4.3. *There exist $\epsilon_3 > 0$ and $C_3 > 0$ such that for every $z, y_0 \in B^L_{\epsilon_3}$ the following estimate holds:*

$$\|z - F(G(z))\|_{L^2(0,L)} \le C_3(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}).$$

Proof. Let $z, y_0 \in L^2(0,L)$. Let $w = (w_1, w_2) \in \mathbb{R}^2$ be such that $z = P_H(z) + \rho_w z_w$. Let $y = y(t,x)$ be a solution of

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = G(z), \\ y(0,\cdot) = y_0. \end{cases} \tag{4.6}$$

From (4.3) and since $\rho_w \le \|z\|_{L^2(0,L)}$, one deduces that if $\|z\|_{L^2(0,L)}$ is smaller than 1 (and therefore $\|z\|_{L^2(0,L)} \le \|z\|^{1/2}_{L^2(0,L)}$), then there exists a constant D_3 such that

$$\|G(z)\|_{L^2(0,T)} \le D_3(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}). \tag{4.7}$$

Remark 4.4. Let us notice that the controls u_w, v_w in the definition of the map G drive the solution β_w from the origin at $t = 0$ to the state z_w at $t = T$, with $\|z_w\|_{L^2(0,L)} = 1$, and therefore they are uniformly bounded.

By using (4.2) and (4.7), one can find $\epsilon_2, C_2 > 0$ such that for every $z, y_0 \in B^L_{\epsilon_2}$ the unique solution of (4.6) satisfies

$$\|y\|_{\mathcal{B}} \le C_2(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}). \tag{4.8}$$

Let $\tilde{y} = \tilde{y}(t,x)$, $\alpha_w = \alpha_w(t,x)$, $\beta_w = \beta_w(t,x)$, and $\beta^0 = \beta^0(t,x)$ be the solutions of

$$\begin{cases} \partial_t \tilde{y} + \partial_x \tilde{y} + \partial_x^3 \tilde{y} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \partial_x \tilde{y}(t,L) = \Gamma_0(P_H(z)), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases}$$

$$\begin{cases} \partial_t \alpha_w + \partial_x \alpha_w + \partial_x^3 \alpha_w = 0, \\ \alpha_w(t,0) = \alpha_w(t,L) = 0, \\ \partial_x \alpha_w(t,L) = u_w(t), \\ \alpha_w(0,\cdot) = 0, \end{cases}$$

$$\begin{cases} \partial_t \beta_w + \partial_x \beta_w + \partial_x^3 \beta_w = -\alpha_w \partial_x \alpha_w, \\ \beta_w(t,0) = \beta_w(t,L) = 0, \\ \partial_x \beta_w(t,L) = v_w(t), \\ \beta_w(0,\cdot) = 0, \end{cases}$$

$$\begin{cases} \partial_t \beta^0 + \partial_x \beta^0 + \partial_x^3 \beta^0 = 0, \\ \beta^0(t,0) = \beta^0(t,L) = 0, \\ \partial_x \beta^0(t,L) = 0, \\ \beta^0(0,\cdot) = P_M(y_0), \end{cases}$$

respectively. Let us define

$$\phi := y - \tilde{y} - \rho_w^{1/2}\alpha_w - \rho_w \beta_w - \beta^0.$$

We have that $\phi = \phi(t, x)$ satisfies

$$\begin{cases} \partial_t\phi + \partial_x\phi + \partial_x^3\phi + \phi\partial_x\phi = -\partial_x(\phi a) - b, \\ \phi(t, 0) = \phi(t, L) = 0, \\ \partial_x\phi(t, L) = 0, \\ \phi(0) = 0, \end{cases}$$

where

$$\begin{aligned} a &:= \tilde{y} + \rho_w^{1/2}\alpha_w + \rho_w\beta_w + \beta^0, \\ b &:= \tilde{y}\partial_x\tilde{y} + \partial_x(\tilde{y}(\rho_w^{1/2}\alpha_w + \rho_w\beta_w + \beta^0)) + \rho_w^{3/2}\partial_x(\alpha_w\beta_w) \\ &\quad + \rho_w^2\beta_w\partial_x(\beta_w) + \rho_w^{1/2}\partial_x(\alpha_w\beta^0) + \rho_w\partial_x(\beta_w\beta^0) + \beta^0\partial_x\beta^0. \end{aligned}$$

It is easy to see that there exists $C_4 > 0$ such that for every $z, y_0 \in B^L_{\epsilon_2}$

$$\|\phi\|_{\mathcal{B}} \le C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}), \tag{4.9}$$

$$\|a\|_{\mathcal{B}} \le C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{1/2}_{L^2(0,L)}), \tag{4.10}$$

$$\|b\|_{L^1(0,T,L^2(0,L))} \le C_4(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}). \tag{4.11}$$

One can also prove that there exists $C_5 > 0$ such that for every $f, g \in \mathcal{B}$

$$\|\partial_x(fg)\|_{L^1(0,T,L^2(0,L))} \le C_5\|f\|_{\mathcal{B}}\|g\|_{\mathcal{B}}. \tag{4.12}$$

By (4.2), (4.11), and (4.12), there exists $C_6 > 0$ such that

$$\|\phi\|^2_{\mathcal{B}} \le C_6(\|\phi\|^2_{\mathcal{B}}\|a\|^2_{\mathcal{B}} + \|y_0\|^2_{L^2(0,L)} + \|z\|^3_{L^2(0,L)});$$

i.e., one has

$$\|\phi\|^2_{\mathcal{B}}(1 - C_6\|a\|^2_{\mathcal{B}}) \le C_6(\|y_0\|^2_{L^2(0,L)} + \|z\|^3_{L^2(0,L)}),$$

which, together with (4.10), implies the existence of ϵ_3 and C_7 such that for every $z, y_0 \in B^L_{\epsilon_3}$

$$\|\phi\|_{\mathcal{B}} \le C_7(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}). \tag{4.13}$$

Finally, from (4.13) and using that $\|\beta^0(0)\|_{L^2(0,L)} = \|\beta^0(T)\|_{L^2(0,L)}$ (β^0 turns only in the subspace M), one obtains with $C_3 := C_7 + 1$

$$\begin{aligned} \|z - F\circ G(z)\|_{L^2(0,L)} &\le \|z - F\circ G(z) + \beta^0(T)\|_{L^2(0,L)} + \|\beta^0(T)\|_{L^2(0,L)} \\ &= \|\phi(T)\|_{L^2(0,L)} + \|\beta^0(0)\|_{L^2(0,L)} \\ &\le \|\phi\|_{\mathcal{B}} + \|y_0\|_{L^2(0,L)} \\ &\le C_7(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}) + \|y_0\|_{L^2(0,L)} \\ &\le C_3(\|y_0\|_{L^2(0,L)} + \|z\|^{3/2}_{L^2(0,L)}), \end{aligned}$$

which ends the proof of Lemma 4.3. $\square$

4.3. Fixed point in the subspace $\boldsymbol{H}$. For $w = (w_1, w_2) \in \mathbb{R}^2$ fixed, let us study the map $\Pi := P_H \circ \Lambda_{y_0,y_T}(\cdot + \rho_w z_w)$ on the subspace H (recall that $\rho_w z_w = w_1\varphi_1 + w_2\varphi_2$):

$$\begin{array}{rcl} \Pi : H & \longrightarrow & H, \\ h & \longmapsto & \Pi(h) = h + P_H(y_T) - P_H(F \circ G(h + \rho_w z_w)). \end{array}$$

In order to prove the existence of a fixed point of the map Π, we will apply the Banach fixed point theorem to the restriction of Π to the closed ball $B_R^L \cap H$, with $R > 0$ small enough. By using Lemma 4.3 we see that

$$\begin{array}{rcl} \|\Pi(h)\|_{L^2(0,L)} & \leq & \|y_T\|_{L^2(0,L)} + \|h + \rho_w z_w - F \circ G(h + \rho_w z_w)\|_{L^2(0,L)} \\ & \leq & \|y_T\|_{L^2(0,L)} + C_3(\|y_0\|_{L^2(0,L)} + \|h + \rho_w z_w\|^{3/2}_{L^2(0,L)}) \\ & \leq & C_3'(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} + \rho_w^{3/2}) + C_3\|h\|^{3/2}_{L^2(0,L)} \\ & \leq & C_3'(2r + \rho_w^{3/2}) + C_3\|h\|^{3/2}_{L^2(0,L)}, \end{array}$$

where $C_3' := C_3 + 1$. Hence, if we choose R such that $R^{3/2} \leq \frac{R}{2C_3}$ and r, ρ_w such that

$$C_3'(2r + \rho_w^{3/2}) \leq \frac{R}{2},$$

then it follows that

$$\|\Pi(h)\|_{L^2(0,L)} \leq R \quad \text{and so} \quad \Pi(B_R^L \cap H) \subset (B_R^L \cap H).$$

It remains to prove that the map Π is a contraction. Let $g, h \in B_R^L \cap H$. Let $y = y(t,x)$, $q = q(t,x)$, $\tilde{y} = \tilde{y}(t,x)$, and $\tilde{q} = \tilde{q}(t,x)$ be the solutions of the following problems:

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + y\partial_x y = 0, \\ y(t,0) = y(t,L) = 0, \\ \partial_x y(t,L) = G(g + \rho_w z_w), \\ y(0,\cdot) = y_0, \end{cases}$$

$$\begin{cases} \partial_t q + \partial_x q + \partial_x^3 q + q\partial_x q = 0, \\ q(t,0) = q(t,L) = 0, \\ \partial_x q(t,L) = G(h + \rho_w z_w), \\ q(0,\cdot) = y_0, \end{cases}$$

$$\begin{cases} \partial_t \tilde{y} + \partial_x \tilde{y} + \partial_x^3 \tilde{y} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \partial_x \tilde{y}(t,L) = \Gamma_0(g), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases}$$

$$\begin{cases} \partial_t \tilde{q} + \partial_x \tilde{q} + \partial_x^3 \tilde{q} = 0, \\ \tilde{q}(t,0) = \tilde{q}(t,L) = 0, \\ \partial_x \tilde{q}(t,L) = \Gamma_0(h), \\ \tilde{q}(0,\cdot) = P_H(y_0), \end{cases}$$

repsectively. Let us define $\phi := y - \tilde{y}$, $\psi := q - \tilde{q}$, and $\gamma := \phi - \psi$. One sees that γ satisfies

$$\begin{cases} \partial_t \gamma + \partial_x \gamma + \partial_x^3 \gamma + \gamma\partial_x \gamma = -\partial_x(\gamma a) - b, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \partial_x \gamma(t,L) = 0, \\ \gamma(0) = 0, \end{cases} \tag{4.14}$$

where

$$a := \tilde{y} + \psi \quad \text{and} \quad b := \partial_x \left(q(\tilde{y} - \tilde{q}) \right) + (\tilde{y} - \tilde{q})\partial_x(\tilde{y} - \tilde{q}).$$

It is easy to see that there exists a constant $C_8 > 0$ such that

$$\|b\|_{L^1(0,T,L^2(0,L))} \leq C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}} \right) \|\tilde{y} - \tilde{q}\|_{\mathcal{B}}, \tag{4.15}$$

$$\|\partial_x(a\gamma)\|_{L^1(0,T,L^2(0,L))} \leq C_8 \left(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}} \right) \|\gamma\|_{\mathcal{B}}. \tag{4.16}$$

By using Proposition 4.2, (4.15), and (4.16) we get the existence of $C_9 > 0$ such that

$$\|\gamma\|_{\mathcal{B}}^2 \leq C_9(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}})^2(\|\tilde{y} - \tilde{q}\|_{\mathcal{B}}^2 + \|\gamma\|_{\mathcal{B}}^2). \tag{4.17}$$

In addition, since $w := \tilde{y} - \tilde{q}$ satisfies the following linear equation:

$$\begin{cases} \partial_t w + \partial_x w + \partial_x^3 w = 0, \\ w(t,0) = w(t,L) = 0, \\ \partial_x w(t,L) = \Gamma_0(g) - \Gamma_0(h), \\ w(0,\cdot) = 0, \end{cases}$$

there exists $C_{10} > 0$ such that

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leq C_{10}\|\Gamma_0(g) - \Gamma_0(h)\|_{L^2(0,T)},$$

and so, from (4.4), one gets

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leq C_{10}D_2\|g - h\|_{L^2(0,L)}. \tag{4.18}$$

Moreover, it is easy to see that there exists a constant $C_{11} > 0$ such that

$$\|q\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} \leq C_{11}(\|y_0\|_{L^2(0,L)} + \|h\|_{L^2(0,L)} + \|g\|_{L^2(0,L)} + \rho_w^{1/2}). \tag{4.19}$$

Thus, using (4.17)–(4.19) we see that if R, ρ_w, r are small enough, it follows that

$$\|\gamma\|_{\mathcal{B}} \leq \tfrac{1}{2}\|g - h\|_{L^2(0,L)}.$$

Therefore, we have

$$\begin{array}{rcl} \|\Pi(g) - \Pi(h)\|_{L^2(0,L)} & \leq & \|g - F \circ G(g + \rho_w z_w) - h + F \circ G(h + \rho_w z_w)\|_{L^2(0,L)} \\ & = & \|\gamma(T)\|_{L^2(0,L)} \leq \|\gamma\|_{\mathcal{B}} \\ & \leq & \tfrac{1}{2}\|g - h\|_{L^2(0,L)}, \end{array}$$

which implies the existence of a unique fixed point $h(y_0, y_T, w_1, w_2) \in B_R^L \cap H$ of the map $\Pi|_{B_R^L \cap H}$. Moreover, the more precise proposition follows.

PROPOSITION 4.5. *There exist $R_0 > 0$, $D > 1$ such that for every $0 < R < R_0$, for every $y_0, y_T \in B_{R/D}^L$, $(w_1, w_2) \in \mathbb{R}^2$, with $\rho_w < R/D$, there exists a unique $h(y_0, y_T, w_1, w_2) \in B_R^L \cap H$ fixed point of the map $\Pi|_{B_R^L \cap H}$.*

4.4. Fixed point in the subspace M. We now apply the Brouwer fixed point theorem to the restriction of the map

$$\begin{array}{rcl} \tau : M & \longrightarrow & M, \\ w_1\varphi_1 + w_2\varphi_2 & \rightarrow & P_M(\rho_w z_w + y_T - F \circ G(\rho_w z_w + h(y_0, y_T, w_1, w_2))) \end{array}$$

to the closed ball $B^L_{\hat{R}} \cap M$, with $\hat{R}$ small enough. Using Lemma 4.3, the continuity (in a neighborhood of $0 \in (L^2(0,L))^2 \times \mathbb{R}^2$) of the map $(y_0, y_T, w_1, w_2) \longmapsto h(y_0, y_T, w_1, w_2)$ and choosing r small enough, we get the existence of a radius $\hat{R} > 0$ such that $\tau(B^L_{\hat{R}} \cap M) \subset B^L_{\hat{R}} \cap M$. This inclusion and the continuity of the map τ allow us to apply the Brouwer fixed point theorem. Therefore, there exists $(\tilde{w}_1, \tilde{w}_2) \in \mathbb{R}^2$, with $\tilde{w}_1^2 + \tilde{w}_2^2 \leq \hat{R}^2$, such that $\tilde{h} := h(y_0, y_T, \tilde{w}_1, \tilde{w}_2)$ satisfies

$$P_M(y_T - F \circ G(\tilde{h} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2)) = 0. \tag{4.20}$$

Using the fact that

$$\Pi(\tilde{h}) = P_H(\tilde{h} + y_T - F \circ G(\tilde{h} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2)) = \tilde{h},$$

we obtain

$$P_H(y_T - F \circ G(\tilde{h} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2)) = 0,$$

which together with (4.20) implies that

$$y_T = F \circ G(\tilde{h} + \tilde{w}_1\varphi_1 + \tilde{w}_2\varphi_2),$$

which ends the proof of Theorem 1.4. Let us remark that from our proof it follows that if r is chosen small enough, one can take $\hat{R} := rD$, where $D > 0$ is given by Proposition 4.5. By using this proposition one obtains the estimate (1.6).

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