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EXACT CONVERGENCE RATES IN SOME MARTINGALE CENTRAL LIMIT THEOREMS

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Convergence rates are derived in central limit theorems for martingale difference arrays. The rates depend heavily on the behavior of the conditional variances and on moment conditions. It is also shown that the rates which are obtained are the exact ones under the stated conditions.

1. Introduction and statements of results. We consider sequences $\mathbf{X} = (X_1, \cdots, X_n)$ of real valued random variables which are square integrable and satisfy

(1.1) $\quad E(X_i \mid \mathscr{F}_{i-1}) = 0$ a.s. *for* $1 \le i \le n$

where $\mathscr{F}_j$ is the σ-algebra generated by $X_1, \cdots, X_j$.

Let M_n denote the class of all such sequences of length n and $M = \cup_{n \in N} M_n$. Let us fix some notation: If $\mathbf{X} = (X_1, \cdots, X_n) \in M_n$ we write

$$\sigma_j^2 = E(X_j^2 \mid \mathscr{F}_{j-1}), \qquad \bar{\sigma}_j^2 = E(X_j^2)$$

$$s^2 = \textstyle\sum_{j=1}^n \bar{\sigma}_j^2$$

$$\| \mathbf{X} \|_p = \max_{1 \le j \le n} \| X_j \|_p \quad \text{for} \quad 1 \le p \le \infty$$

$$V^2 = \textstyle\sum_{j=1}^n \sigma_j^2 / s^2$$

$$S = \textstyle\sum_{j=1}^n X_j.$$

We sometimes write $\sigma_j^2(\mathbf{X})$, $\bar{\sigma}_j^2(\mathbf{X})$ etc. to indicate the dependence on $\mathbf{X}$. If $\mathbf{X} = (X_1, \cdots, X_n) \in M_n$ and $m \le n$ we call $(X_1, \cdots, X_m)$ a beginning of $\mathbf{X}$.

Quite good central limit theorems have been established for such sequences. The following statement is a special case of results obtained by Dvoretzky [4] and Brown [2]:

If $\mathbf{X}_m \in M_m$, $m \in N$, $V^2(\mathbf{X}_m) \to 1$ in probability, and some Lindeberg type condition is satisfied, then

$$\lim_{m \to \infty} P(S(\mathbf{X}_m)/s(\mathbf{X}_m) \le t) = \Phi(t)$$

for all $t \in R$, where Φ is the standard normal distribution function.

Of course it is desirable to have convergence rates in such limit theorems. Several results in this direction have been obtained by Ibragimov [7], Grams [5], Strobel [11], Hall and Heyde [6] Section 3.6., Chow and Teicher [3] Theorem 9.3.2, Kato [8], and others, but the results appear somewhat incomplete. With the one exception of Kato's, there seem to be no results which in the case $\bar{\sigma}_j^2 = 1$ give better rates than $n^{-1/4}$. Kato obtains for uniformly bounded variables with $\sigma_j^2 = \bar{\sigma}_j^2$ a.s. the rate $n^{-1/2}(\log n)^3$. We shall obtain the better rate $n^{-1/2} \log n$ under somewhat weakened conditions and shall show in Section 6 that this rate is exact.

For variables with bounded third moments, the rate $n^{-1/4}$ has never been surpassed, for very good reasons as will become clear soon. The following theorem is contained in the results of Grams [5]. As the proof is very easy and is the departing point of our further considerations, we shall include it in Section 3.

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THEOREM 1. *If* $0 < \alpha \le \beta < \infty$, $0 < \gamma < \infty$, *then there is a constant* $0 < L(\alpha, \beta, \gamma) < \infty$, *such that for all* $\mathbf{X} \in M_n$ *satisfying* $\sigma_j^2 = \bar{\sigma}_j^2$ a.s., $\alpha \le \bar{\sigma}_j^2 \le \beta$ *for* $1 \le j \le n$ *and* $\|\mathbf{X}\|_3 \le \gamma$, *one has*

$$\sup_t |P(S/s \le t) - \Phi(t)| \le Ln^{-1/4}.$$

This appears quite unsatisfactory for several reasons. The rate appears very poor when compared with the i.i.d. case. But it is sharp as will be shown in Section 6. (Example 1). Actually we give an example of a sequence $\mathbf{X}_m \in M_m$ satisfying the conditions in Theorem 1, where one can show that there exists a sequence $\mathbf{Y}_n \in M_n$ of beginnings of the $\mathbf{X}_m$, such that

$$\limsup_{n\to\infty} n^{1/4} \sup_t |P(S(\mathbf{Y}_n)/s(\mathbf{Y}_n) \le t) - \Phi(t)| > 0.$$

The condition $\sigma_j^2 = \bar{\sigma}_j^2$ a.s. is also very strong. This can be relaxed, and one can derive bounds depending on the behaviour of V^2. As the rates are never better than $n^{-1/4}$, we shall not pursue this here, but rather seek for conditions giving better rates. Indeed they are much better for bounded variables.

THEOREM 2. *Let* $0 < \gamma < \infty$. *There exists a constant* $0 < L(\gamma) < \infty$ *depending only on* γ, *such that for all* $\mathbf{X} \in M_n$, $n \ge 2$, *satisfying*

$$\|\mathbf{X}\|_\infty \le \gamma \quad and \quad V^2(\mathbf{X}) = 1 \text{ a.s.}$$

$\sup_t |P(S(\mathbf{X})/s(\mathbf{X}) \le t) - \phi(t)| \le L(\gamma) n \log n / s^3(\mathbf{X})$ *holds.*

If we specialize to $\bar{\sigma}_j^2 = 1$ one obtains a rate $n^{-1/2} \log n$, which improves on the rate of Kato. Again our rate is sharp. We shall show that there exists an example of a sequence $\mathbf{X}_n \in M_n$, $n \ge 1$, with $\sigma_j^2(\mathbf{X}_n) = \bar{\sigma}_j^2(\mathbf{X}_n) = 1$ a.s. $(1 \le j \le n)$ and $\sup_n \|\mathbf{X}_n\|_\infty < \infty$, but where

$$\limsup_{n\to\infty} n^{1/2} (\log n)^{-1} \sup_t |P(S(\mathbf{X}_n)/s(\mathbf{X}_n) \le t) - \Phi(t)| > 0.$$

(Example 2 in Section 6).

Of course, $V^2(\mathbf{X}) = 1$ a.s. in our theorem is very restrictive, but one can easily derive corollaries where this condition is relaxed. We shall just prove one:

COROLLARY. *Let* $0 < \gamma < \infty$. *There exists a constant* $0 < L(\gamma) < \infty$, *such that for* $\mathbf{X} \in M_n$, $n \ge 2$, *with* $\|\mathbf{X}\|_\infty \le \gamma$ *one has*

$$\sup_t |P(S/s \le t) - \Phi(t)| \le L(\gamma) \{n \log n / s^3 + \min(\|V^2 - 1\|_\infty^{1/2}, \|V^2 - 1\|_1^{1/3})\}.$$

The proof of Theorem 2 and its corollary will be given in Section 4. The corollary exactly tallies with Theorem 3.7 in Hall and Heyde [6]. However, their result gives only rates $n^{-1/4}(\log n)$. They give an example of a sequence $\mathbf{X}_n \in M_n$, where $\|V^2(\mathbf{X}_n) - 1\|_\infty$ is of order $n^{-1/2}(\log n)^2$ and the rate is $n^{-1/4}(\log n)$ is exact. In this example $\|V^2(\mathbf{X}_n) - 1\|_1$ is of the same order, so one could think that in our corollary $\|V^2 - 1\|_1^{1/3}$ could be replaced by $\|V^2 - 1\|_1^{1/2}$. But this is not so. In Section 6 we shall show that there exists an example of a sequence $\mathbf{X}_n \in M_n$, $n \in \mathbb{N}$, with $\bar{\sigma}_j^2(\mathbf{X}_n) = 1 (0 \le j \le n)$, $\sup_n \|\mathbf{X}_n\|_\infty < \infty$,

$$\sup_n \textstyle\sum_{j=1}^n \|\sigma_j^2(\mathbf{X}_n) - 1\|_1 < \infty,$$

but where

$$\limsup_{n\to\infty} n^{1/3} \sup_t |P(S(\mathbf{X}_n)/s(\mathbf{X}_n) \le t) - \Phi(t)| > 0.$$

(Example 3 in Section 6). Therefore, the rate given in the corollary appears to be best possible of this type. However, the second summand of the estimate makes things quite unsatisfactory.

Our Example 3 and the example in [6] suggest that things become bad if the randomness of σ_j^2 is large for large j. The rates can indeed be improved if the σ_j^2 behave better for large j.

To simplify somewhat the notation, we shall specialize to triangular arrays X_{ni}, $1 \le i \le n$, where $\mathbf{X}_n = (X_{n1}, \cdots, X_{nn}) \in M_n$ for all n and where

$$\lim_{n,j\to\infty} \bar{\sigma}_j^2(\mathbf{X}_n) = \sigma^2 \quad \text{for some constant} \quad \sigma^2 > 0. \tag{1.2}$$

We shall adapt the notation introduced above by writing

$$\sigma_{nj}^2 = \sigma_j^2(\mathbf{X}_n), \qquad S_n = S(\mathbf{X}_n) \text{ etc.}$$

Boundedness of the variables can also be somewhat relaxed.

THEOREM 3. *Let X_{ni}, $1 \le i \le n$, be an array as defined above which satisfies*

$$\sup_{j,n} \| E(|X_{nj}|^3 | \mathscr{F}_{n,j-1}) \|_\infty < \infty (\mathscr{F}_{nk} = \sigma(X_{n1}, \cdots, X_{nk})). \tag{1.3}$$

(a) *If for some $0 < \alpha \le 1/2$*

$$\sup_{n,j} j^\alpha \| \sigma_{nj}^2 - \bar{\sigma}_{nj}^2 \|_1 < \infty \quad \textit{then}$$

$$\sup_n n^\alpha (\log n)^{-1} \sup_t | P(S_n/s_n \le t) - \Phi(t) | < \infty.$$

(b) *If for some $1 < p \le \infty$, $0 < \alpha < 1/2$*

$$\sup_{n,j} j^\alpha \| \sigma_{nj}^2 - \bar{\sigma}_{nj}^2 \|_p < \infty \quad \textit{then} \quad \sup_n n^\alpha \sup_t | P(S_n/s_n \le t) - \Phi(t) | < \infty.$$

I do not know if the theorem is sharp in every respect, e.g. if $p > 1$ is necessary in (b). However, the rate $n^{-\alpha}$ is sharp in (b). In Section 6 we shall give an example with

$$\| \sigma_{nj}^2 - \bar{\sigma}_{nj}^2 \|_\infty = O(j^{-\alpha}) \textit{ where } \limsup_{n\to\infty} n^\alpha \sup_t | P(S_n/s_n \le t) - \Phi(t) | > 0.$$

(Section 6, Example 4).

We remark that in the classical i.i.d. case, (1.3) is satisfied for variables with finite third absolute moments, so one obtains a rate $n^{-1/2} \log n$ from (a).

As a second remark, we compare Theorem 3 with the corollary to Theorem 2. If $\| \sigma_{nj}^2 - \bar{\sigma}_{nj}^2 \|_1 = O(j^{-1/2})$, Theorem 3 yields a convergence rate $n^{-1/2} \log n$, where the corollary would only give $n^{-1/6}$. So the Theorem 3 gives much better bounds if σ_{nj}^2 becomes more and more nonrandom for large j. On the other hand, Theorem 2 and its corollaries have the advantage that they are based on assumptions on $V^2 - 1$. However, the rates then become very bad if $V^2 - 1$ is not very small.

To obtain bounds of order $n^{-1/2}$ one needs some assumptions on the conditional third moments.

THEOREM 4. *Let X_{ni}, $1 \le i \le n$, be an array as introduced above, which in addition to* (1.1) *and* (1.2) *satisfies*

$$\sup_{j,n} \| E(X_{nj}^4 | \mathscr{F}_{n,j-1}) \|_\infty < \infty. \tag{1.4}$$

If for some $1 < p \le \infty$ we have $\sup_{n,j} j^{1/2} \| \sigma_{nj}^2 - \bar{\sigma}_{nj}^2 \|_p < \infty$ and if either
(a) *for some $1 < p' < \infty$*

$$\sup_{j,n} j^{1/2p'} \| E(X_{nj}^3 | \mathscr{F}_{n,j-1}) - E(X_{nj}^3) \|_{p'} < \infty, \quad \textit{or}$$

(b)

$$\sup_{j,n} \log j \| E(X_{nj}^3 | \mathscr{F}_{n,j-1}) - E(X_{nj}^3) \|_\infty < \infty \quad \textit{then}$$

$$\sup_n n^{1/2} \sup_t | P(S_n/s_n \le t) - \Phi(t) | < \infty.$$

In some of the before-mentioned papers, to get bounds beyond $n^{-1/4}$, it is assumed that conditional higher moments become close to the normal moments. This is not assumed in Theorem 4. It is only assumed that the conditional third moments become more and more nonrandom. The theorem falls very short of including the classical i.i.d. Berry-Esseen theorem. Indeed, it gives the Berry-Esseen bound for i.i.d. variables with finite fourth moments. I conjecture that (1.4) can be replaced by (1.3). However, a proof eludes me.

2. Preliminary lemmas.

LEMMA 1. *Let* X, ξ *be real valued random variables and let*

$$\delta = \sup_t | P(X \le t) - \Phi(t) |, \quad \delta^* = \sup_t | P(X + \xi \le t) - \Phi(t) |$$

(Φ *the standard normal distribution function*). *Then*

$$\delta \le 2\delta^* + (5/(2\pi)^{1/2}) \| E(\xi^2 | X) \|_\infty^{1/2}$$

and

$$\delta^* \le 2\delta + (3/2\pi^{1/2}) \| E(\xi^2 | X) \|_\infty^{1/2}.$$

PROOF. There is nothing to prove if $\| E(\xi^2 | X) \|_\infty = \infty$. So we assume $\| E(\xi^2 | X) \|_\infty = \gamma < \infty$.

$$\begin{aligned} P(X + \xi \le t) &= E(P(\xi \le t - X | X)) \\ &\ge E(1_{X \le t-a} P(\xi \le t - X | X)) \quad \text{for any} \quad a > 0 \\ &= P(X \le t - a) - E(1_{X \le t-a} P(\xi > t - X | X)) \end{aligned}$$

$$\begin{aligned} E(1_{X \le t-a} P(\xi > t - X | X)) &\le \gamma\, E(1_{X \le t-a} (t - X)^{-2}) \\ &= \gamma \left\{ a^{-2} P(X \le t - a) - \int_{-\infty}^{t-a} 2(t - x)^{-3} P(X \le x)\, dx \right\} \\ &\le \gamma \left\{ a^{-2} \Phi(t - a) - \int_{-\infty}^{t-a} 2(t - x)^{-3} \Phi(x)\, dx + 2\delta a^{-2} \right\} \\ &= \gamma \left\{ \int_{-\infty}^{t-a} (t - x)^{-2} \varphi(x)\, dx + 2\delta a^{-2} \right\} \le \gamma (2\pi)^{-1/2} a^{-1} + 2\gamma\delta a^{-2}. \end{aligned}$$

And therefore

$$P(X + \xi \le t) \ge P(X \le t - a) - \gamma (2\pi)^{-1/2} a^{-1} - 2\gamma\delta a^{-2}. \tag{2.1}$$

On the other hand

$$\begin{aligned} P(X + \xi \le t) &= E(1_{X \le t+a} P(\xi \le t - X | X)) + E(1_{X > t+a} P(\xi \le t - X | X)) \\ &\le P(X \le t + a) + E(1_{X > t+a} P(\xi \le t - X | X)) \\ &\le P(X \le t + a) + \gamma E(1_{X > t+a} (t - X)^{-2}), \end{aligned}$$

and by a similar reasoning as above one obtains

$$P(X + \xi \le t) \le P(X \le t + a) + \gamma (2\pi)^{-1/2} a^{-1} + 2\gamma\delta a^{-2}. \tag{2.2}$$

(2.1) implies

$$\begin{aligned} \delta^* &\ge P(X + \xi \le t) - \Phi(t) \\ &\ge P(X \le t - a) - \Phi(t - a) - a(2\pi)^{-1/2} - \gamma (2\pi)^{-1/2} a^{-1} - 2\gamma\delta a^{-2} \end{aligned}$$

so $\sup_t (P(X \le t) - \Phi(t)) \le \delta^* + a(2\pi)^{-1/2} + \gamma (2\pi)^{-1/2} a^{-1} + 2\gamma\delta a^{-2}$, and similarly from (2.2)

$$\sup_t (\Phi(t) - P(X \le t)) \le \delta^* + a(2\pi)^{-1/2} + \gamma (2\pi)^{-1/2} a^{-1} + 2\gamma\delta a^{-2}$$

and taking $a = 2\gamma^{1/2}$ gives the first statement of the lemma. The second follows by an obvious modification of the last argument.

LEMMA 2. *Let $k \geq 0$ and f be a function $\mathbb{R} \to \mathbb{R}$, which has k derivatives $f^{(1)}, \cdots, f^{(k)}$ which together with f belong to $L_1(\mathbb{R})$. If $f^{(k)}$ is of bounded variation $\|f^{(k)}\|_V$, if X is a random variable and if $a \neq 0$, b are real numbers, then*

$$|Ef^{(k)}(aX + b)| \leq \|f^{(k)}\|_V \sup_t |P(X \leq t) - \Phi(t)| + |a|^{-(k+1)} \|f\|_1 \sup_x \left| \frac{d^k}{dx^k} \varphi(x) \right|$$

where $\varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

PROOF. If $\bar{X}$ is a standard normal variable, then clearly

$$|Ef^{(k)}(aX + b) - Ef^{(k)}(a\bar{X} + b)| \leq \|f^{(k)}\|_V \sup_t |P(X \leq t) - \Phi(t)|$$

and

$$|Ef^{(k)}(a\bar{X} + b)| \leq |a|^{-k-1} \sup_x \left| \frac{d^k}{dx^k} \varphi(x) \right| \|f\|_1$$

by partial integration.

3. Proof of Theorem 1. Let $\mathbf{X} = (X_1, \cdots, X_n)$ be as in the statement of the theorem and $Z_1, \cdots, Z_n, \xi$ be independent normally distributed random variables with mean 0 and $EZ_i^2 = \bar{\sigma}_i^2$, $E\xi^2 = n^{1/2}$. $\sum_{i=1}^n Z_i/s$ is standard normally distributed. Therefore, according to Lemma 1,

$$\begin{aligned} \sup_t |P(S/s \leq t) - \Phi(t)| \leq 2 \sup_t |P((S + \xi)/s \leq t) \\ - P((\textstyle\sum_{i=1}^n Z_i + \xi)/s \leq t) + cn^{-1/4}\alpha^{-1/2} \end{aligned} \tag{3.1}$$

(c an absolute constant). For $1 \leq m \leq n$, let $U_m = \sum_{j=1}^{m-1} X_j/s$, $W_m = (\sum_{j=m+1}^n Z_j + \xi)/s$.

According to an idea which goes back to Lindeberg [9], one writes

$$\begin{aligned} &P((S + \xi) \leq t) - P((\textstyle\sum_{j=1}^n Z_j + \xi)/s \leq t) \\ &\qquad = \textstyle\sum_{m=1}^n \{P(U_m + W_m + X_m/s \leq t) - P(U_m + W_m + Z_m/s \leq t)\}. \end{aligned}$$

W_m is normally distributed with mean 0 and variance $\lambda_m^2 = (\sum_{j=m+1}^n \bar{\sigma}_j^2 + n^{1/2})/s^2$. It is further independent of U_m, X_m and Z_m, so the above sum may be written as

$$\begin{aligned} &\textstyle\sum_{m=1}^n E\left\{\Phi\left(\frac{t - U_m}{\lambda_m} - \frac{X_m}{\lambda_m s}\right) - \Phi\left(\frac{t - U_m}{\lambda_m} - \frac{Z_m}{\lambda_m s}\right)\right\} \\ &\quad = \textstyle\sum_{m=1}^n E\left\{\left(-\frac{X_m}{\lambda_m s} + \frac{Z_m}{\lambda_m s}\right)\varphi\left(\frac{t - U_m}{\lambda_m}\right) + \left(\frac{X_m^2}{2\lambda_m s} - \frac{Z_m^2}{2\lambda_m s}\right)\varphi'\left(\frac{t - U_m}{\lambda_m}\right)\right. \\ &\qquad \left. - \frac{X_m^3}{6\lambda_m^3 s^3}\varphi''\left(\frac{t - U_m}{\lambda_m} - \theta_m \frac{X_m}{\lambda_m s}\right) + \frac{Z_m^3}{6\lambda_m^3 s^3}\varphi''\left(\frac{t - U_m}{\lambda_m} - \theta_m' \frac{Z_m}{\lambda_m s}\right)\right\} \end{aligned}$$

where $0 \leq \theta_m, \theta_m' \leq 1$.

As U_m is $\mathscr{F}_{m-1}$-measurable, it follows from (1.1) and the assumption $\sigma_j^2 = \bar{\sigma}_j^2$ a.s. that the first two types of summands in the above expression vanish. Combining this with (3.1), one obtains

$$\sup_t |P(S/s \leq t) - \Phi(t)| \leq c \textstyle\sum_{m=1}^n \lambda_m^{-3} s^{-3} + c' n^{-1/4}$$

for some constants c, c' which depend only on α, β, γ and this is

$$\leq c'' n^{-1/4}.$$

Therefore, the theorem is proved.,

REMARK. It is clear that such a simple type of argument cannot give anything better than $O(n^{-1/4})$ even in the i.i.d. case.

4. Proof of Theorem 2 and its Corollary. We make the following notational convention in this section (and only in this we shall change it in Section 5!) : c, c', c_1, c_2, etc. are absolute constants > 0. They may vary from formula to formula, but not in the same one.

For $n \in N$, $s, \gamma > 0$ let $\mathscr{G}_n(s, \gamma) = \{\mathbf{X} \in M_n: s(\mathbf{X}) = s, \|\mathbf{X}\|_\infty \le \gamma, V^2(\mathbf{X}) = 1 \text{ a.s.}\}$ and

$$\delta(n, s, \gamma) = \sup\{\sup_t |P(S(\mathbf{X})/s \le t) - \Phi(t)| : \mathbf{X} \in \mathscr{G}_n\}.$$

If $\mathbf{X} = (X_1, \cdots, X_n) \in \mathscr{G}_n(s, \gamma)$, then $\mathbf{X}' = (X_1, X_2, \cdots, X_{n-1} + X_n)$ obviously is in $\mathscr{G}_{n-1}(s, 2\gamma)$ and satisfies $S(\mathbf{X}') = S(\mathbf{X})$. So clearly

$$\delta(n, s, \gamma) \le \delta(n-1, s, 2\gamma). \tag{4.1}$$

I hope it will amuse the reader that this simple estimate will be the key fact for a recursion argument leading to our theorem.

We fix now an element $\mathbf{X} \in \mathscr{G}_n(s, \gamma)$, where we assume that $\gamma \ge 1$. Let $Z_1, \cdots, Z_n$ be i.i.d. standard normal variables and ξ be an extra centered normal with variance κ^2 which is independent of anything else. κ^2 will be specified later on, but in any case $\kappa^2 > 2\gamma^2$.

The first few steps now run parallel to the proof of Theorem 1. As $\sum_{i=1}^n \sigma_i Z_i/s$ is standard normal, one obtains from Lemma 1

$$\begin{aligned} \sup_t |P(S/s \le t) - \Phi(t)| \le 2 \sup_t |P((S + \xi)/s \le t) \\ - P((\textstyle\sum_{i=1}^n \sigma_i Z_i + \xi)/s \le t) + c\kappa/s. \end{aligned} \tag{4.2}$$

Let

$$U_m = \textstyle\sum_{j=1}^{m-1} X_j/s, \quad W_m = (\sum_{j=m+1}^n \sigma_j Z_j + \xi)/s$$
$$\lambda_m^2 = (\textstyle\sum_{j=m+1}^n \sigma_j^2 + \kappa^2)/s, \quad T_m = (t - U_m)/\lambda_m.$$

(Note that λ_m is random.)

Conditioned on $\sigma(X_1, \cdots, X_n, Z_m)$, W_m is centered normal with variance λ_m^2, so

$$\begin{aligned} P((S + \xi)/s \le t) - P((\textstyle\sum_{j=1}^n \sigma_j Z_j + \xi)/s \le t) \\ = \textstyle\sum_{m=1}^n E\{\Phi(T_m - X_m/\lambda_m s) - \Phi(T_m - \sigma_m Z_m/\lambda_m s)\}. \end{aligned} \tag{4.3}$$

As λ_m is $\bar{\mathscr{F}}_{m-1}$-measurable where $\bar{\mathscr{F}}_{m-1}$ is the completion of $\mathscr{F}_{m-1}$ ($\sum_{j=1}^n \sigma_j^2 = s^2$ a.s.!) one obtains, as in the proof of Theorem 1, that the above sum equals

$$\frac{1}{6} \textstyle\sum_{m=1}^n E\left(- \dfrac{X_m^3}{\lambda_m^3 s^3} \varphi''(T_m - \theta_m X_m/\lambda_m s) + \dfrac{Z_m^3}{\lambda_m^3 s^3} \varphi''(T_m - \theta'_m \sigma_m Z_m/\lambda_m s)\right)$$

where $0 \le \theta_m, \theta'_m \le 1$. So

$$\begin{aligned} &|P((S + \xi)/s \le t) - P((\textstyle\sum_{j=1}^n \sigma_j Z_j + \xi)/s \le t)| \\ &\quad\le \frac{1}{6s^3} \left\{ \textstyle\sum_{m=1}^n E\left(\dfrac{|X_m|^3}{\lambda_m^3} \left| \varphi''\left(T_m - \dfrac{\theta_m X_m}{\lambda_m s}\right) \right| \right) \right. \\ &\qquad \left. + \textstyle\sum_{m=1}^n E\left(\dfrac{|\sigma_m Z_m|^3}{\lambda_m^3} \left| \varphi''\left(T_m - \dfrac{\sigma_m \theta'_m Z_m}{\lambda_m s}\right) \right| \right) \right\}. \end{aligned} \tag{4.4}$$

We define a sequence of stopping times τ_j, $0 \le j \le n$,

$$\tau_0 = 0, \quad \tau_j = \inf\left\{ k: \textstyle\sum_{i=1}^k \sigma_i^2 \ge \dfrac{js^2}{n} \right\} \quad \text{for} \quad 1 \le j \le n-1, \quad \tau_n = n.$$

$$\begin{aligned} &\textstyle\sum_{m=1}^n E\left(\dfrac{|X_m|^3}{\lambda_m^3} \left| \varphi''\left(T_m - \dfrac{\theta_m X_m}{\lambda_m s}\right) \right| \right) \\ &\qquad = \textstyle\sum_{j=1}^n E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} \dfrac{|X_m|^3}{\lambda_m^3} \left| \varphi''\left(T_m - \dfrac{\theta_m X_m}{\lambda_m s}\right) \right| \right). \end{aligned} \tag{4.5}$$

If $\tau_{j-1} < m \leq \tau_j$

$$\lambda_m^2 \geq \left(\textstyle\sum_{i=\tau_j+1}^n \sigma_i^2 + \kappa^2\right)/s^2 \geq \left(s^2 - \frac{js^2}{n} - \gamma^2 + \kappa^2\right)/s^2 = \underline{\lambda}_j^2 \quad \text{say,}$$

$$\lambda_m^2 \leq \left(\textstyle\sum_{i=\tau_{j-1}+1}^n \sigma_i^2 + \kappa^2\right)/s^2 \leq (s^2 - (j-1)s^2/n + \kappa^2)/s^2 = \bar{\lambda}_j^2 \quad \text{say,}$$

and $U_m = U_{\tau_{j-1}+1} + \sum_{i=\tau_{j-1}+1}^{m-1} X_i/s$. We abbreviate $\sum_{i=\tau_{j-1}+1}^{m-1} X_i/s$ by R_m.

$$(4.6)\qquad \begin{aligned} &E\left(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} \frac{|X_m|^3}{\lambda_m^3} \left| \varphi''\left(T_m - \frac{\theta_m X_m}{\lambda_m s}\right)\right|\right) \\ &\leq \gamma\, \underline{\lambda}_j^{-3} E\left(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} X_m^2 \left|\varphi''\left(T_m - \frac{\theta_m X_m}{\lambda_m s}\right)\right| (1_{A_m} + 1_{A_m^c})\right) \end{aligned}$$

where $A_m = \left\{\frac{|R_m|}{\underline{\lambda}_j} \leq \frac{1}{2}\frac{|t - U_{\tau_{j-1}+1}|}{\bar{\lambda}_j}\right\}$.

As $\kappa^2 \geq 2\gamma^2$ we have $\theta_m |X_m|/\lambda_m s \leq 1$. If we put $\psi(x) = \sup\{|\varphi''(y)| : y \geq |x|/2 - 1\}$, then ψ is of bounded total variation and summable. On A_m

$$|\varphi''(T_m - \theta_m X_m/\lambda_m s)| \leq \psi((t - U_{\tau_{j-1}+1})/\bar{\lambda}_j).$$

As $U_{\tau_{j-1}+1}$ is $\mathscr{F}_{\tau_{j-1}}$ measurable one therefore obtains

$$(4.7)\qquad \begin{aligned} &E\left(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} X_m^2 |\varphi''(T_m - \theta_m X_m/\lambda_m s)| 1_{A_m}\right) \\ &\leq E(\psi((t - U_{\tau_{j-1}+1})/\bar{\lambda}_j) E(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} X_m^2 | \mathscr{F}_{\tau_{j-1}})) \\ &= E(\psi((t - U_{\tau_{j-1}+1})/\bar{\lambda}_j) E(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^2 | \mathscr{F}_{\tau_{j-1}})) \\ &\leq 2\gamma^2 E(\psi(t - U_{\tau_{j-1}+1})/\bar{\lambda}_j). \end{aligned}$$

Now

$$E((\textstyle\sum_{i=\tau_{j-1}+1}^n X_i)^2 | \mathscr{F}_{\tau_{j-1}}) = E(\textstyle\sum_{i=\tau_{j-1}+1}^n \sigma_i^2 | \mathscr{F}_{\tau_{j-1}}) \leq s^2\left(1 - \frac{j-1}{n}\right) \quad \text{a.s.,}$$

so, from Lemma 1, 2 and (4.1), one obtains

$$(4.8)\qquad E(\psi((t - U_{\tau_{j-1}+1})/\bar{\lambda}_j)) \leq c_1 \delta(n-1, s, 2\gamma) + c_2\sqrt{1 - \frac{j-1}{n}} + c_3 \bar{\lambda}_j.$$

We now look what happens on A_m^c. On $\tau_{j-1} < m \leq \tau_j$

$$A_m^c \subset B_j = \{\max_{\tau_{j-1} < i \leq \tau_j} |\textstyle\sum_{k=\tau_{j-1}+1}^i X_k|/s\underline{\lambda}_j > \tfrac{1}{2}|t - U_{\tau_{j-1}+1}|/\bar{\lambda}_j\}.$$

A_m is $\mathscr{F}_{m-1} \vee \mathscr{F}_{\tau_{j-1}}$ measurable, so

$$\begin{aligned} E(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} X_m^2 1_{A_m^c}) &= E(\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^2 1_{A_m^c}) \\ &\leq E(1_{B_j} \textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^2) \leq 2\gamma^2 P(B_j) \quad (\textstyle\sum_{m=\tau_{j-1}+1}^{\tau_j} \sigma_m^2 \leq 2\gamma^2 \quad \text{a.s.}) \\ &\leq 2\gamma^2 E\left(\min\left\{1, s^{-2}\underline{\lambda}_j^{-2} \left(\frac{|t - U_{\tau_{j-1}+1}|}{2\bar{\lambda}_j}\right)^{-2}\right.\right. \\ &\qquad \left.\left. \cdot E((\max_{\tau_{j-1} < i \leq \tau_j} |\textstyle\sum_{k=\tau_{j-1}+1}^i X_k|)^2 | \mathscr{F}_{\tau_{j-1}})\right\}\right) \\ &\leq 2\gamma^2 E\left(\min\left\{1, 2s^{-2}\underline{\lambda}_j^{-2} \left(\frac{|t - U_{\tau_{j-1}+1}|}{2\bar{\lambda}_j}\right)^{-2} E((\textstyle\sum_{i=\tau_{j-1}+1}^{\tau_j} X_i)^2 | \mathscr{F}_{\tau_{j-1}})\right\}\right) \end{aligned}$$

(by a well known martingale inequality, see e.g. [10] proposition IV 2.8. The extension of this proposition to our stopping time setting is straightforward)

$$\le 8\gamma^4 E\left(\min\left\{1, s^{-2}\underline{\lambda}_j\left(\frac{|t-U_{\tau_{j-1}+1}|}{2\bar{\lambda}_j}\right)^{-2}\right\}\right)$$

$$\le c\gamma^4\left\{\delta(n-1, s, 2\gamma) + \sqrt{1-\frac{j-1}{n}} + \bar{\lambda}_j\right\}$$

by applying again Lemma 1, 2 and (4.1).

Implementing this estimate with (4.8) in (4.6) and combining with (4.5), one obtains

$$\sum_{m=1}^n E\left(\frac{|X_m|^3}{\lambda_m^3}|\varphi''(T_m - \theta_m X_m/\lambda_m s)|\right)$$

$$\le c\gamma^5\left[\delta(n-1, s, 2\gamma)\sum_{j=1}^n \underline{\lambda}_j^{-3} + \sum_{j=1}^n\left(1-\frac{j-1}{n}\right)^{1/2}\underline{\lambda}_j^{-3} + \sum_{j=1}^n \underline{\lambda}_j\bar{\lambda}_j^{-3}\right] \tag{4.9}$$

$$\le c'\gamma^5[\delta(n-1, s, 2\gamma)sn(\kappa^2-2\gamma^2)^{-1/2} + n\log n]$$

if $\kappa^2 > 2\gamma^2$.

We need a similar estimate where X_m is replaced by $\sigma_m Z_m$. One cannot use exactly the same argument, because Z_m is not bounded.

We introduce

$$A'_m = \left\{|R_m| \le \frac{\underline{\lambda}_j}{4}\frac{|t-U_{\tau_{j-1}+1}|}{\bar{\lambda}_j}\right\}$$

$$B_m = \left\{|\sigma_m Z_m| \le \frac{s\underline{\lambda}_j}{4}\frac{|t-U_{\tau_{j-1}+1}|}{\bar{\lambda}_j}\right\}.$$

Then

$$E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j}\frac{\sigma_m^3|Z_m|^3}{\lambda_m^3}|\varphi''(T_m - \theta'_m\sigma_m Z_m/\lambda_m s)|\right)$$

$$\le E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} 1_{A'_m\cap B_m}\frac{\sigma_m^3|Z_m|^3}{\lambda_m^3}|\varphi''(T_m - \theta'_m\sigma_m Z_m/\lambda_m s)|\right)$$

$$+ \|\varphi''\|_\infty E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} 1A'^c_m\frac{\sigma_m^3|Z_m|^3}{\lambda_m^3}\right) + \|\varphi''\|_\infty E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} 1B^c_m\frac{\sigma_m^3|Z_m|^3}{\lambda_m^3}\right).$$

Using the independence of Z_m, the first and the second summand can be estimated as above. As for the third, we remark that

$$E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j} 1B^c_m\frac{\sigma_m^3|Z_m|^3}{\lambda_m^3}\right) \le \underline{\lambda}_j^{-3}\gamma E\left(\sum_{m=\tau_{j-1}+1}^{\tau_j}\sigma_m^2 Z_m^3 1_{\{|Z_m|>c|t-U_{\tau_{j-1}+1}|/\bar{\lambda}_j\}}\right)$$

$$\le c'\underline{\lambda}_j^{-3}\gamma^3 E(\psi((t-U_{\tau_{j-1}+1})/\bar{\lambda}_j))$$

where $\psi(x) = E(Z_m^3; |Z_m| \ge |x|)$.

Handling this expression as above, one obtains

$$\sum_{m=1}^n E\left(\frac{|\sigma_m Z_m|^3}{\lambda_m^3}|\varphi''(T_m - \theta'_m\sigma_m Z_m/\lambda_m s)|\right)$$

$$\le c\gamma^5[\delta(n-1, s, 2\gamma)sn(\kappa^2-2\gamma^2)^{-1/2} + n\log n].$$

Combining this with (4.9), (4.4) and (4.2) one obtains

$$\delta(n, s, \gamma) \le c_1\gamma^5\left[\delta(n-1, s, 2\gamma)\frac{n}{s^2}(\kappa^2 - 2\gamma^2)^{-1/2} + \frac{n}{s^3}\log n\right] + c_2\kappa/s. \tag{4.10}$$

Let now $K_n = \sup_{\gamma\ge 1, 0<s\le n\gamma^2}\delta(n, s, \gamma)/\left(\gamma^5\left(\frac{n}{s^3}\right)\log n\right)$. Clearly $K_n < \infty$ for all $n \ge 2$.

We take now $\kappa^2 = 2\gamma^2 + c_1^2 2^{12}\gamma^{10}\left(\frac{n}{s^2}\right)^2$ (c_1 from (4.10)). Then from (4.10)

$$K_n \le \tfrac{1}{2}K_{n-1} + c \quad \text{for} \quad n \ge 4, \tag{4.11}$$

and from this

$$\limsup_{n\to\infty} K_n \le 2c \ (c \text{ the same as in (4.11)}).$$

Therefore, the theorem is proved.

Proof of the Corollary. Let $\mathbf{X} \in M_n$ with $\|\mathbf{X}\|_\infty \le \gamma$, $a = \|s^2V^2 - s^2\|_\infty$. We define $X_{n+1}, \cdots, X_{n+[2a/\gamma^2]+1}$ as follows: Let $k = [(s^2 + a - s^2V^2)/\gamma^2]$. Conditioned on $\mathscr{F}_n$

$$X_{n+j} = \begin{cases} \pm\gamma \text{ w.p. } \tfrac{1}{2} & \text{for } j \le k \\ \pm(s^2 + a - s^2V^2 - k\gamma^2)^{1/2} & \text{for } j = k+1 \\ 0 & \text{else,} \end{cases}$$

and let $\hat{\mathbf{X}} = (X_1, \cdots, X_{n+[2a/\gamma^2]+1})$. Clearly $\hat{V}^2 = V^2(\hat{X}) = 1$ a.s. and

$$\begin{aligned}
&\sup_t |P(S/s \le t) - \Phi(t)| \\
&\quad\le \sup_t |P(S/\hat{s} \le t) - \Phi(t)| + \sup_t \left|\Phi\left(\frac{st}{\hat{s}}\right) - \Phi(t)\right| \\
&\quad\le 2\sup_t |P(\hat{S}/\hat{s} \le t) - \Phi(t)| + c_1\sqrt{a}/\hat{s} + c_2(\hat{s}/s - 1) \\
&\qquad\qquad \text{by Lemma 2 and an elementary calculation} \\
&\quad\le 2L(\gamma)\frac{\hat{n}}{\hat{s}^3}\log\hat{n} + c_1\sqrt{a}/\hat{s} - (\hat{s}/s - 1) \quad \text{by Theorem 2} \\
&\quad\le 6L(\gamma)\frac{n}{s^3}\log n + c\sqrt{a}/s \quad \text{if} \quad \sqrt{a}/s \le 1 \quad \text{and} \quad n \text{ is sufficiently large.}
\end{aligned}$$

Therefore, we obtain in this case

$$\sup_t |P(S/s \le t) - \Phi(t)| \le L'(\gamma)\frac{n}{s^3}\log n + c\sqrt{a}/s. \tag{4.12}$$

This is then also true for all n and $\sqrt{a}/s > 1$ if one chooses c and $L(\gamma)$ suitably.

The estimate with $\|V^2 - 1\|_1$ runs a bit differently: Again let $\mathbf{X} = (X_1, \cdots, X_n) \in M_n$ with $\|\mathbf{X}\|_\infty \le \gamma$, and let $\tau = \sup\{k : \sum_{i=1}^k \sigma_i^2 \le s^2\}$. We define $\hat{\mathbf{X}} = (\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_{2n}) \in M_{2n}$ as follows: $\hat{X}_i = X_i$ for $i \le \tau$. Let $r = [(s^2 - \sum_{j=1}^{\tau}\sigma_j^2)/\gamma^2]$. As $s^2 \le n\gamma^2$ we have $r \le n$. Conditioned on $\mathscr{F}_\tau$, we define $\hat{X}_{\tau+1}, \cdots, \hat{X}_{2n}$ as follows:

For $1 \le i \le r$, $\hat{X}_{\tau+1} = \pm\gamma$ w.p. ½. If $\tau + r < 2n$ we proceed by setting $\hat{X}_{\tau+r+1} = \pm(s^2 - \sum_{j=1}^{\tau}\sigma_\gamma^2 - r\gamma^2)^{1/2}$, $\hat{X}_{\tau+r+i} = 0$ for $i \ge 2$. Clearly $\sum_{i=1}^{2n}\hat{\sigma}_i^2 = \hat{s}^2 = s^2$ a.s. and $\|\hat{\mathbf{X}}\|_\infty \le \gamma$.

By Theorem 2: $\sup_t |P(\hat{S}/s \le t) - \Phi(t)| \le 4L(\gamma)\frac{n}{s^3}\log n$, and, further, it is easy to see that $E(\hat{S} - S)^2 \le c\|s^2V^2 - s^2\|_1$ for some c. If $x > 0$ then

$$\begin{aligned}
P(S/s \le t) &\le P(S/s \le t, |S - \hat{S}|/s \le x) + P(|S - \hat{S}|/s > x) \\
&\le P(\hat{S}/s \le t + x) + \frac{1}{x^2}E\left(\frac{(S - \hat{S})^2}{s^2}\right)
\end{aligned}$$

$$\leq \Phi(t+x) + 4L(\gamma)\frac{n}{s^3}\log n + \frac{c}{x^2}\| V^2 - 1 \|_1$$

$$\leq \Phi(t) + 4L(\gamma)\frac{n}{s^3}\log n + \frac{x}{\sqrt{2\pi}} + \frac{c}{x^2}\| V^2 - 1 \|_1$$

and now putting $x = \| V^2 - 1 \|_1^{1/3}$ one has

$$\sup_t(P(S/s \leq t) - \Phi(t)) \leq 4L(\gamma)\frac{n}{s^3}\log n + c\| V^2 - 1 \|_1^{1/3}$$

with a similar estimate for $\sup_t(\Phi(t) - P(S/s \leq t))$. Combining this with (4.12) yields the corollary.

5. Proof of Theorem 3 and Theorem 4. We slightly change the notational convention about the constants $c, c', c_1, \cdots$ etc. In this section they may depend on moment properties of the X_{nj} such as the numbers appearing in (1.3), (1.4), or the other moment conditions in the theorems. They do not depend on n, m, t, s and other such running variables. If (1.2) is assumed, the statement and conditions of the theorems remain unchanged, if we replace X_{ni} by $X_{ni}/\bar{\sigma}_{ni}$. So we assume in the future that $\bar{\sigma}_{ni} = 1$.

We shall consider subclasses $\mathscr{L}_n \subset M_n$, $n \geq 1$, consisting of elements with $\bar{\sigma}_i^2 = 1$ and which have the following property:

If $(X_1, \cdots, X_n) \in \mathscr{L}_n$, $m \leq n$, then $(X_1, \cdots, X_m) \in \mathscr{L}_m$. A sequence $\{\mathscr{L}_n\}$, having these properties, will be called of R-type. Then if $\mathbf{X} \in \mathscr{L}_n$ we write $\delta(\mathbf{X}) = \sup_t |P(S(\mathbf{X})/\sqrt{n} \leq t) - \Phi(t)|$ and $\delta(\mathscr{L}_n) = \sup\{\delta(\mathbf{X}) : \mathbf{X} \in \mathscr{L}_n\}$. If no danger of confusion can arise we shall write $\delta(n)$.

Let now Z_i, $i \in N$, be independent standard normally distributed random variables and ξ be an extra independent centered normal variable with variance κ^2 (to be specified later, but in any case $\kappa^2 > 1$).

As in the preceding proofs we have for $\mathbf{X} \in M_n$ with $\bar{\sigma}_i^2 = 1$, $1 \leq i \leq n$:

$$\delta(\mathbf{X}) \leq 2 \left| E \textstyle\sum_{m=1}^n (\Phi(T_m - X_m/\lambda_m\sqrt{n}) - \Phi(T_m - Z_m/\lambda_m\sqrt{n})) \right| + c\kappa/\sqrt{n}. \ (\lambda_m^2 = (n - m + \kappa^2)/n), \tag{5.1}$$

where, as in Section 4, $T_m = (t - U_m)/\lambda_m$, $U_m = \sum_{j=1}^{m-1} X_j/\sqrt{n}$. We shall now introduce some abbreviations: If $1 \leq p \leq \infty$ let

$$\beta_m^{(p)}(\mathbf{X}) = \| \sigma_i^2(\mathbf{X}) - \bar{\sigma}_i^2(\mathbf{X}) \|_p, \qquad \gamma_m^{(p)}(\mathbf{X}) = \| E(X_i^3 \mid \mathscr{F}_{i-1}) - E(X_i^3) \|_p.$$

Let

$$H_3(u, v) = \Phi(u - v) - \Phi(u) - v\varphi(u) + (v^2/2)\varphi'(u)$$

$$H_4(u, v) = \Phi(u - v) - \Phi(u) - v\varphi(u) + (v^2/2)\varphi'(u) - (v^3/6)\varphi''(u).$$

Clearly $H_3(u, v) = (v^3/6)\,\varphi''(u - \theta v)$ for some $0 \leq \theta \leq 1$, but we shall also use the fact that $|H_3(u, v)| \leq c_1 + c_2 |v|^2$. H_4 has similar properties.

$$\begin{aligned} &E(\Phi(T_m - X_m/\lambda_m\sqrt{n}) - \Phi(T_m - Z_m/\lambda_m\sqrt{n})) \\ &\quad = EH_3\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right) - EH_3\left(T_m, \frac{Z_m}{\lambda_m\sqrt{n}}\right) - \frac{1}{2\lambda_m^2 n} E((\sigma_m^2 - 1)\varphi'(T_m)) \end{aligned} \tag{5.2}$$

and if $\| \mathbf{X} \|_3 < \infty$ we have

$$E(\Phi(T_m - X_m/\lambda_m\sqrt{n}) - \Phi(T_m - Z_m/\lambda_m\sqrt{n}))$$

$$\begin{aligned} &= EH_4\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right) - EH_4\left(T_m, \frac{Z_m}{\lambda_m\sqrt{n}}\right) - \frac{1}{2\lambda_m^2 n} E((\sigma_m^2 - 1)\varphi'(T_m)) \\ &\quad + \frac{1}{6\lambda_m^3 n^{3/2}} E((E(X_m^3 \mid \mathscr{F}_{m-1}) - E(X_m^3))\varphi''(T_m)) + \frac{1}{6\lambda_m^3 n^{3/2}} E(X_m^3) E\varphi''(T_m). \end{aligned} \tag{5.2'}$$

PROOF OF THEOREM 3. If $\gamma_1, \gamma_2 > 0$, $0 < \alpha \le \frac{1}{2}$, $1 \le p \le \infty$ we set

$$\mathscr{L}_n = \{\mathbf{X} \in M_n : \bar{\sigma}_i^2 = 1, \sup_{1\le j\le n} \| E(|X_j|^3 \mid \mathscr{F}_{j-1})\|_\infty \le \gamma_1, \sup_{1\le j\le n} j^\alpha \|\sigma_j^2 - 1\|_p \le \gamma_2\}.$$

Clearly $\{\mathscr{L}_n\}$ is of R-type.

If $\mathbf{X} \in \mathscr{L}_n$, $m \le n$

$$|E(\sigma_m^2 - 1)\varphi'(T_m)| \le \|\sigma_m^2 - 1\|_p \|\varphi'(T_m)\|_q \tag{5.3}$$

where $1/p + 1/q = 1$, and from Lemma 2 and the fact that $\{\mathscr{L}_n\}$ is R-type one has for $q < \infty$ that this is

$$\le \|\sigma_m^2 - 1\|_p (c_1\delta(m-1)^{1/q} + c_2\lambda_m^{1/q}) \quad \text{for} \quad m \ge \frac{n}{2}.$$

We shall now estimate $E\left|H_3\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right)\right|$. Let

$$A_m = \left\{\frac{|X_m|}{\lambda_m\sqrt{n}} \le \frac{1}{2}|T_m|\right\}, \qquad B_m = \left\{\frac{1}{2}|T_m| < \frac{|X_m|}{\lambda_m\sqrt{n}} \le |T_m|^2\right\}$$

$$C_m = \left\{\frac{|X_m|}{\lambda_m\sqrt{n}} > |T_m|^2\right\}$$

$$\begin{aligned} E&\left|H_3\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right)\right| \\ &= E\left(\left|H_3\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right)\right|; |T_m| \le 1\right) + E\left(\left|H_3\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right)\right|; T_m > 1, A_m\right) \\ &\quad + E\left(\left|H_3\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right)\right|; T_m > 1, B_m\right) + E\left(\left|H_3\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right)\right|; T_m > 1, C_m\right) \\ &= I_1 + I_2 + I_3 + I_4 \quad \text{say.} \end{aligned}$$

We use here and elsewhere the notation $E(X; A) = E(1_A X)$. Clearly $E\left|H_3\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right)\right| \le c\lambda_m^{-3} n^{-3/2}$ which is good enough for small m, say for $m \le n/2$. For $m > n/2$ we need better estimates:

$$I_1 \le \frac{\|\varphi''\|_\infty}{6\lambda_m^3 n^{3/2}} E(|X_m|^3; T_m \in [-1, 1]) \le \frac{\gamma_1 \|\varphi''\|_\infty}{6\lambda_m^3 n^{3/2}} P(|T_m| \le 1)$$

$$\le c\lambda_m^{-3} n^{-3/2}(\delta(m-1) + \lambda_m) \quad \text{from Lemma 2}$$

(if $m > n/2$, which is assumed here. In general one has to replace λ_m by $\lambda_m\sqrt{n}/\sqrt{m-1}$).

$$I_2 \le \frac{1}{6\lambda_m^3 n^{3/2}} E(|X_m|^3 \Psi(T_m))$$

where $\Psi(u) = \sup_{|x|\ge|u|/2} |\varphi''(x)|$. So $I_2 \le c\lambda_m^{-3} n^{-3/2}(\delta(m-1) + \lambda_m)$.

As $\sup_{|v|\le|u|^2} |H(u, v)| < \infty$ one has

$$I_3 \le cP\left(\frac{|X_m|}{\lambda_m\sqrt{n}} > \frac{1}{2}|T_{m'}|, |T_m| > 1\right) \le c'\lambda_m^{-3}n^{-3/2}E(|T_m|^{-3}; |T_m| > 1)$$

$$\le c''\lambda_m^{-3}n^{-3/2}(\delta(m-1) + \lambda_m)$$

$$I_4 \le E\left(\left(c_1 + c_2\frac{|X_m|^2}{\lambda_m^2 n}\right); \frac{X_m}{\lambda_m\sqrt{n}} > |T_m|^2 > 1\right)$$

$$\le c\lambda_m^{-3}n^{-3/2}(E(|T_m|^{-6}; |T_m| > 1) + E(|T_m|^{-2}; |T_m| > 1))$$

$$\le c'\lambda_m^{-3}n^{-3/2}(\delta(m-1) + \lambda_m).$$

Combining these estimates yields

$$E\left|H_3\left(T_m, \frac{X_m}{\lambda_m\sqrt{n}}\right)\right| \le c\lambda_m^{-3}n^{-3/2}(\delta(m-1) + \lambda_m) \tag{5.4}$$

for $m \ge n/2$ and similarly

$$E\left|H_3\left(T_m, \frac{Z_m}{\lambda_m\sqrt{n}}\right)\right| \le c\lambda_m^{-3}n^{-3/2}(\delta(m-1) + \lambda_m) \tag{5.5}$$

for $m \ge n/2$.

REMARK. $|EH_3(T_m, Z_m/\lambda_m\sqrt{n})|$ behaves much better because Z_m is independent of T_m. It is $< c\lambda_m^{-3}n^{-3/2}(\delta(m-1) + \lambda_m^3)$ for $m \ge n/2$. If one had the same type of estimate for $|EH_3(T_m, X_m/\lambda_m\sqrt{n})|$ one could obtain the rate $n^{-1/2}$ (if $\sigma_i^2 = \bar{\sigma}_i^2$) (see Bergström [1]). Even $\lambda_m^{1+\delta}$ for some $\delta > 0$ in the above estimate would be sufficient. But our estimates of $|EH_3(T_m, X_m/\lambda_m\sqrt{n})|$ by $E|H_3(T_m, X_m/\lambda_m\sqrt{n})|$ and then (5.4) are sharp.

We shall now combine (5.1) through (5.5) to obtain a recursion relation for $\delta(n)$. We have the following elementary estimates of $\beta_m^{(p)} = O(m^{-\alpha})$:

$$\textstyle\sum_{m=1}^{n/2} \lambda_m^{-2}n^{-1}\delta(m-1)^{1/q}\beta_m^{(p)} \le cn^{-\alpha}$$

$$\textstyle\sum_{m=1}^{n/2} \lambda_m^{-2}n^{-1}\lambda_m^{1/q}\beta_m^{(p)} \le cn^{-\alpha}$$

$$\textstyle\sum_{m=n/2+1}^{n} \lambda_m^{-2}n^{-1}\lambda_m^{1/q}\beta_m^{(p)} \le cn^{-\alpha} \quad \text{if} \quad q < \infty$$

$$\le c'n^{-\alpha}\log n \quad \text{if} \quad q = \infty.$$

(Summation ends and starts at the integer part of a positive real number).

$$\textstyle\sum_{m=1}^{n/2} n^{-3/2}\lambda_m^{-3} \le cn^{-1/2}, \quad \sum_{m=n/2+1}^{n} n^{-3/2}\lambda_m^{-2} \le cn^{-1/2}\log n.$$

So together with (5.1) through (5.5) this gives

$$\begin{aligned}\delta(n) \le c_1 n^{-\alpha-1}\textstyle\sum_{m=n/2+1}^{n}\lambda_m^{-2}\delta(m-1)^{1/q} + c_2\sum_{m=n/2+1}^{n} n^{-3/2}\lambda_m^{-3}\delta(m-1)\\ + c_3 r_n + c_4\kappa/n^{1/2}\end{aligned} \tag{5.6}$$

where r_n for $\alpha < ½$ and $p > 1$ is $n^{-\alpha}$ and for $p = 1$ or $\alpha = ½$ is $n^{-\alpha}\log n$. Let now $K_n^{(0)} = \delta(n)/r_n$; $K_n = \max_{j\le n} K_j^{(0)}$ Then one obtains from (5.6)

$$K_n^{(0)} = \frac{\delta(n)}{r_n} \le c_1 n^{-\alpha}r_n^{1/q-1}K_{n-1}^{1/q}\textstyle\sum_{m=n/2+1}^{n} n^{-1}\lambda_m^{-2}$$

$$+ c_2 K_{n-1}\textstyle\sum_{m=n/2}^{n} n^{-3/2}\lambda_m^{-3} + c_3 + c_4\dfrac{\kappa}{\sqrt{nr_n}}.$$

Now $n^{-\alpha}r_n^{1/q-1}\sum_{m=n/2+1}^{n} n^{-1}\lambda_m$ is arbitrary small for large n and $\sum_{m=n/2}^{n} n^{-3/2}\lambda_m^{-3}$ is arbitrary small for large κ uniformly in n. So there exists a n_0 such that for $n \ge n_0$

$$K_n^{(0)} \le \frac{1}{2}K_{n-1} + c$$

and from this $\lim\sup_{n\ge\infty} K_n < \infty$ follows. The theorem is proved.

PROOF OF THEOREM 4. The proof is almost the same as that of Theorem 3. So we shall only point out the changes which are needed. The departing formula is (5.2′): If γ_1, γ_2, $\gamma_3 > 0$, $1 < p \le \infty$, $1 < p' < \infty$ we set

$$\mathscr{L}_n = \{\mathbf{X} \in M_n : \bar{\sigma}_i^2 = 1, \quad \sup_{1\le j\le n} \| E(X_j^4 \mid \mathscr{F}_{j-1}) \|_\infty \le \gamma_1, \quad \sup_{1\le j\le n} j^{1/2} \| \sigma_j^2 - 1 \|_p \le \gamma_2, \quad \sup_{1\le j\le n} j^{1/2p'} \| E(X_j^3 \mid \mathscr{F}_{j-1}) - EX_j^3 \|_p \le \gamma_3\}$$

and in the case $p' = \infty$ the expression with p' is replaced by $\log j \| E(X_j^3 \mid \mathscr{F}_{j-1}) - EX_j^3 \|_\infty$.

As above, let $\delta(n) = \delta(\mathscr{L}_n) \cdot \mathscr{L}_n$ is clearly of R-type. If $1 < p' \le \infty$, $1/q' = 1 - 1/p'$ one has for $m \ge n/2$

$$(5.7) \qquad | E((E(X_m^3 \mid \mathscr{F}_{m-1}) - EX_m^3)\varphi''(T_m)) | \le \gamma_m^{(p')}(c_1\delta(m-1)^{1/q'} + c_2\lambda_m^{1/q'}).$$

To estimate $| E\varphi''(T_m) |$ we use Lemma 2 with $k = 2$ and obtain for $m \ge n/2$

$$(5.8) \qquad | E(X_m^3)E\varphi''(T_m) | \le c_1\delta(m-1) + c_2\lambda_m^3 E \left| H_4\left(T_m, \frac{X_m}{\lambda_m \sqrt{n}}\right) \right|$$

and $E \left| H_4\left(T_m, \frac{Z_m}{\lambda_m \sqrt{n}}\right) \right|$ can be treated in the same way as $E \left| H_3\left(T_m, \frac{X_m}{\lambda_m \sqrt{n}}\right) \right|$, so one obtains

$$(5.9) \qquad E \left| H_4\left(T_m, \frac{X_m}{\lambda_m \sqrt{n}}\right) \right| \le c\lambda_m^{-4} n^{-2}(\delta(m-1) + \lambda_m) \quad \text{if} \quad m \ge n/2$$

and the same estimate for $E | H_4(T_m, Z_m/(\lambda_m \sqrt{n})) |$. If now $\beta_m^{(p)}$, $\gamma_m^{(p')}$ fulfill the conditions of Theorem 3, then the following expressions are bounded by some multiple of $n^{-1/2}$:

$$\sum_{m=1}^{n/2} n^{-1}\beta_m^{(p)}\lambda_m^{-2}, \quad \sum_{m=n/2+1}^{n} n^{-1}\beta_m^{(p)}\lambda_m^{-2+1/q}, \quad \sum_{m=1}^{n/2} n^{-3/2}\gamma_m^{(p')}\lambda_m^{-3}, \quad \sum_{m=n/2+1}^{n} n^{-3/2}\gamma_m^{(p')}\lambda_m^{-3+1/q'},$$
$$\sum_{m=1}^{n/2} n^{-3/2}\lambda_m^{-3}, \quad \sum_{m=n/2+1}^{n} n^{-3/2}, \quad n^{-2}\sum_{m=1}^{n/2} \lambda_m^{-4},$$
$$n^{-2}\sum_{m=n/2+1}^{n} \lambda_m^{-3} \quad (\text{whereas} \quad n^{-3/2}\sum_{m=n/2+1}^{n} \lambda_m^{-2} \quad \text{is not!}).$$

So one arrives at

$$(5.10) \qquad \begin{aligned}\delta(n) \le{}& c_1 n^{-2} \sum_{m=n/2+1}^{n} \delta(m-1)\lambda_m^{-4} + c_2 n^{-3/2} \sum_{m=n/2+1}^{n} \delta(m-1)\lambda_m^{-3/2} \\ &+ c_3 n^{-3/2} \sum_{m=n/2+1}^{n} \delta(m-1)^{1/q'}\gamma_m^{(p')}\lambda_m^{-3} + c_4 n^{-1/2} + c_5\kappa n^{-1/2}.\end{aligned}$$

By an identical argument as that above, this leads to $\delta(n) = O(n^{-1/2})$, proving the theorem.

6. Examples.

EXAMPLE 1. We define a triangular array X_{ni}, $i \le n$, such that for each $n (X_{n1}, \cdots, X_{nn}) \in M_n$. If $i \le \underline{k}_n = [n - 2\sqrt{n}]$, X_{ni} is i.i.d. standard normal. For $\underline{k}_n < i \le \bar{k}_n = [n - \sqrt{n}]$ X_{ni} is defined such that for fixed n, $S_{ni} = \sum_{j=1}^{i} X_{nj}$ is an inhomogenous Markov chain with transitions given by

$$P(X_{ni} \in du \mid S_{n,i-1} = x) = 1_{J_{n,i-1}^c}(x)\ (\tfrac{1}{2}(\delta_{-1} + \delta_1)) + 1_{J_{n,i-1}}(x)[16\lambda_i^2\delta_{\rho_i} + (1 - 16\lambda_i^2)\delta_{-1/\rho_i}]$$

where $\lambda_i = (1 - i/n)^{1/2}$, $\rho_i = ((1 - 16\lambda_i^2)/16\lambda_i^2)^{1/2}$, $J_{n,i} = [-\sqrt{n}\lambda_i/4, \sqrt{n}\lambda_i/4]$ (δ_x: the one point measure in x) for $i \ge \bar{k}_n + 1$ X_{ni} are again i.i.d. standard normal. Clearly $(X_{n1}, \cdots, X_{nn}) \in M_n$ with $\sigma_{ni}^2 = 1$ for $i \le n$.

Of course, the above definition makes sense only for large n where $1 - 16\lambda_i^2 \ge 0$ for $\underline{k}_n < i \le \bar{k}_n$. In the future we assume that n is large enough, such that $1 - 16\lambda_i^2 \ge \tfrac{1}{2}$ for $\underline{k}_n < i \le \bar{k}_n$.

PROPOSITION 1.

a) $\sup_{n,i} i^{1/4} \sup_t | P(S_{ni}/\sqrt{i} \le t) - \phi(t) | < \infty$

b) $\sup_{n,i} E | X_{ni} |^3 < \infty$

c) $\lim \sup_{j\to\infty} \sup_{n \ge i \ge j} i^{1/4} \sup_t | P(S_{ni}/\sqrt{i} \le t) - \phi(t) | > 0$.

Clearly a) follows from b) and Theorem 1. However, it seems difficult to prove b) directly. We shall first prove a) and with the help of a) prove b).

We shall consider the following subclasses of M_n: For given $\gamma_1, \gamma_2, \gamma_3 > 0$, $\mathscr{G}_n$ contains those elements $\mathbf{X}$ of M_n which satisfy $\sigma_i^2 = 1$ a.s., $1 \le i \le n$, and for each $i \le n$ there exists an interval $J \subset \mathbb{R}$ of length $\le \gamma_1 i^{-1/4}$ such that

$$E(| X_i |^3 | \mathscr{F}_{i-1}) \le \gamma_2 + \gamma_3 1_{\{S_{i-1}/\sqrt{i-1} \in J\}} i^{1/4}.$$

Clearly, $\{ \mathscr{G}_n \}$ is of R-type.

LEMMA 4. $\sup_n n^{1/4} \delta(\mathscr{G}_n) < \infty$.

PROOF. We copy the proof of Theorem 1 with only some changes. So let $(X_1, \cdots, X_n) \in \mathscr{G}_n$ and $Z_1, \cdots, Z_n$ be i.i.d. standard normal variables and ξ an independent centered normal variable with variance $\kappa n^{1/2}$ where κ is fixed and will be specified later. Then, as in the proof of Theorem 1,

$$\sup_t | P(S/s \le t) - \phi(t) | \le c \textstyle\sum_{m=1}^n n^{-3/2} \bar{\lambda}_m^{-3} (E | X_m |^3 + E | Z_m |^3) + c' \sqrt{\kappa n}^{-1/4} \tag{6.1}$$

where $\bar{\lambda}_m^2 = (n - m + \kappa \sqrt{n})/n$

$$E | X_m |^3 = E(E(| X_m |^3 | \mathscr{F}_{m-1})) \le \gamma_2 + \frac{1}{\sqrt{2\pi}} \gamma_1 \gamma_3 + \gamma_3 \delta(m-1) m^{1/4}.$$

So we arrive at

$$\delta(n) \le c(\kappa) n^{-1/4} + c' \textstyle\sum_{m=1}^n \dfrac{\delta(m-1) m^{1/4}}{(n - m + \kappa \sqrt{n})^{3/2}}$$

where c' does not depend on κ. By taking κ suitably large, one obtains $\sup_n n^{1/4} \delta(n) < \infty$ as in the proofs of Theorems 2 through 4.

We now come to the proof of Proposition 1.

Proof of a). For suitably large $\gamma_1, \gamma_2, \gamma_3$, $\mathscr{G}_n$ contains $(X_{N1}, \cdots, X_{Nn})$ for all n and $N \ge n$, so a) follows from Lemma 4.

Proof of b). b) follows from a) since

$$E | X_{ni} |^3 = E(E(| X_{ni} |^3 | \mathscr{F}_{n,i-1})) = E(E(| X_{ni} |^3 | S_{n,i-1})).$$

Proof of c). Let $\mathscr{G}_n^0$ consist of all sequences $(X_{N1}, \cdots, X_{Nn})$ $N \ge n$. Clearly $\{ \mathscr{G}_n^0 \}$ is of R-type. Let $\delta(n) = \delta(\mathscr{G}_n^0)$. We shall show that $\lim \sup_{n\to\infty} n^{1/4} \delta(n) > 0$. Assume to the contrary that $n^{1/4} \delta(n) = o(1)$. We shall show that this leads to a contradiction.

If $n \in \mathbb{N}$ let $(X_1, \cdots, X_n) = (X_{n1}, \cdots, X_{nn})$. We shall use the same abbreviations as in Sections 3 through 5 and take $Z_1, \cdots, Z_n$ to be i.i.d. standard normal.

$$P(S/\sqrt{n} \le 0) - \frac{1}{2} = \textstyle\sum_{m=k_n+1}^{\bar{k}_n} \left\{ EH_3\left(-\dfrac{U_m}{\lambda_m}, \dfrac{X_m}{\lambda_m \sqrt{n}} \right) - EH_3\left(-\dfrac{U_m}{\lambda_m}, \dfrac{Z_m}{\lambda_m \sqrt{n}} \right) \right\} \tag{6.2}$$

$$\left| EH_3\left(-\frac{U_m}{\lambda_m}, \frac{Z_m}{\lambda_m \sqrt{n}} \right) \right| \le c \lambda_m^{-3} n^{-3/2} \quad \text{for all } m \quad \text{and}$$

$$\le c' \lambda_m^{-3} n^{-3/2} (\delta(m-1) + \lambda_m^3) \quad \text{if } m \ge \frac{n}{2}.$$

Therefore if $\delta(k) = o(k^{-1/4})$, one has

$$\sum_{m=1}^{n} \left| EH_3\left(-\frac{U_m}{\lambda_m}, \frac{Z_m}{\lambda_m \sqrt{n}}\right) \right| = o(n^{-1/4}). \tag{6.3}$$

$$EH_3\left(-\frac{U_m}{\lambda_m}, \frac{X_m}{\lambda_m\sqrt{n}}\right) = E\left(-\frac{X_m^3}{6\lambda_m^3 n^{-3/2}} \varphi''\left(-\frac{U_m}{\lambda_m} - \frac{\theta_m X_m}{\lambda_m \sqrt{n}}\right); \frac{U_m}{\lambda_m} \in \left[-\frac{1}{4}, \frac{1}{4}\right]\right)$$
$$+ E\left(-\frac{X_m^3}{6\lambda_m^3 n^{-3/2}} \varphi''\left(-\frac{U_m}{\lambda_m} - \frac{\theta_m X_m}{\lambda_m \sqrt{n}}\right); \frac{U_m}{\lambda_m} \notin \left[-\frac{1}{4}, \frac{1}{4}\right]\right),$$

$$E\left(X_m^3 \varphi''\left(-\frac{U_m}{\lambda_m} - \theta_m \frac{X_m}{\lambda_m\sqrt{n}}\right); U_m/\lambda_m \in \left[-\frac{1}{4}, \frac{1}{4}\right]\right)$$
$$= \int_{[-1/4,1/4]} E\left(X_m^3\, \varphi''\left(-u - \theta_m \frac{X_m}{\lambda_m\sqrt{n}}\right) \,\middle|\, \frac{U_m}{\lambda_m} = u\right) P\left(\frac{U_m}{\lambda_m} \in du\right)$$
$$= \int_{[-1/4,1/4]} \left\{16\lambda_m^2 \frac{(1-16\lambda_m^2)^{3/2}}{64\lambda_m^3} \varphi''\left(-u - \theta_m \frac{(1-16\lambda_m^2)^{1/2}}{4\lambda_m\sqrt{n}}\right)\right.$$
$$\left. - (1-16\lambda_m^2)\frac{64\lambda_m^3}{(1-16\lambda_m^2)^{3/2}} \varphi''\left(u + \theta_m' \frac{4}{(1-16\lambda_m^2)^{1/2}\sqrt{n}}\right)\right\} \cdot P\left(\frac{U_m}{\lambda_m} \in du\right)$$

where $0 \le \theta_m, \theta_m' \le 1$ but depend on u.

For n large enough $4/(1-16\lambda_m^2)^{1/2}\sqrt{n}$ and $(1-16\lambda_m^2)^{1/2}/4\lambda_m\sqrt{n}$ are < ¼ if $K_n < m \le \bar{k}_n$. On the interval [−½, ½], φ'' is negative and bounded away from 0, so

$$E\left(X_m^3 \varphi''\left(-\frac{U_m}{\lambda_m} - \theta_m \frac{X_m}{\lambda_m\sqrt{n}}\right); \frac{U_m}{\lambda_m} \in \left[-\frac{1}{4}, \frac{1}{4}\right]\right)$$
$$\le \left(-c_1\frac{1}{\lambda_m} + c_2 \lambda_m^3\right) P\left(\frac{U_m}{\lambda_m} \in \left[-\frac{1}{4}, \frac{1}{4}\right]\right) \tag{6.4}$$
$$\le -c_1' + c_2' \frac{\delta(m-1)}{\lambda_m}$$

for $\underline{k}_n < m \le \bar{k}_n$ and n larger than a suitable n_0. If $\delta(k) = o(k^{-1/4})$ then $\delta(m-1)/\lambda_m = 0(1)$, so the above is $\le -\bar{c}$ for some $\bar{c} > 0$ and $k_n < m \le \bar{k}_n$, $n \ge \bar{n}_0$.

$$\left| E\left(X_m^3 \varphi''\left(-\frac{U_m}{\lambda_m} - \theta_m \frac{X_m}{\lambda_m\sqrt{n}}\right); \frac{U_m}{\lambda_m} \notin \left[-\frac{1}{4}, \frac{1}{4}\right]\right) \right|$$
$$\le E\left| \varphi''\left(-\frac{U_m}{\lambda_m} - \frac{\theta_m X_m}{\lambda_m\sqrt{n}}\right)\right| \le c_1\lambda_m + c_2\delta(m-1) \tag{6.5}$$
$$= o(1) \quad \text{if} \quad \underline{k}_n < m \le \bar{k}_n, \quad n \to \infty.$$

So there exists an N such that for $n \ge N$ and $\underline{k}_n < m \le \underline{k}_n$ one has

$$EH_3\left(-\frac{U_m}{\lambda_m}, \frac{X_m}{\lambda_m\sqrt{n}}\right) \ge cn^{-3/2}\lambda_m^{-3} \quad \text{and so} \quad \sum_{m=\underline{k}_n+1}^{\bar{k}_n} EH_3\left(-\frac{U_m}{\lambda_m}, \frac{X_m}{\lambda_m\sqrt{n}}\right) \ge cn^{-1/4}.$$

Combining this with (6.2) and (6.3), one obtains $P(S/\sqrt{n} \le 0) - ½ \ge cn^{-1/4}$ for large n, which clearly is a contradiction to $\delta(n) = o(n^{-1/4})$.

REMARK. We did not prove that for our example X_{ni}, $i \le n$ one has $\limsup_{n\to\infty} n^{1/4} \sup_t |P(S_{nn}/\sqrt{n} \le t) - \phi(t)| > 0$, but only showed that there exists a sequence $N_n \ge n$ such that $\limsup_{n\to\infty} n^{1/4} \sup_t |P(S_{N_n n}/\sqrt{n} \le t) - \phi(t)| > 0$.

EXAMPLE 2. The idea is similar to that of Example 1. Let X_{ni}, $i \le n$ be defined such that S_{ni} is for fixed n a Markov chain with

$$P(X_{ni} \in du \mid S_{n,i-1} = x) = 1_{J^c_{n,i-1}}(x)\left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\right) + 1_{J_{n,i-1}}(x)\left(\frac{1}{5}\delta_{-2} + \frac{4}{5}\delta_{1/2}\right)$$

where here $J_{n,k} = [-\sqrt{n}\lambda_k/2, \sqrt{n}\lambda_k/2]$ and $\lambda_k = \lambda_{n,k} = ((n-k+1)/n)^{1/2}$.

Let $\mathscr{G}^0_n$ here be the class of sequences $(X_{N1}, \cdots, X_{Nn})$, $N \ge n$ and let $\delta(n) = \delta(\mathscr{G}^0_n)$. With the notation as in Section 5 and Z_i, ξ standard normal and independent one has:

$$\begin{aligned}(6.6)\quad & P((\textstyle\sum_{i=1}^n X_{ni} + \xi)/\sqrt{n} \le 0) - P((\textstyle\sum_{i=1}^n Z_i + \xi)/\sqrt{n} \le 0) \\ &= \sum_{m=1}^n \left(EH_4\left(-\frac{U_m}{\lambda_m}, \frac{X_m}{\lambda_m\sqrt{n}}\right) - EH_4\left(-\frac{U_m}{\lambda_m}, \frac{Z_m}{\lambda_m\sqrt{n}}\right)\right) \\ &\quad - \sum_{m=1}^n \frac{1}{6\lambda_m^3 n^{3/2}} E\left(X_m^3\varphi''\left(-\frac{U_m}{\lambda_m}\right)\right).\end{aligned}$$

As in Section 5:

$$\begin{aligned}\left|EH_4\left(-\frac{U_m}{\lambda_m}, \frac{X_m}{\lambda_m\sqrt{n}}\right)\right| &\le c\lambda_m^{-4}n^{-2} \quad \text{for all} \quad m \\ &\le c\lambda_m^{-4}n^{-2}(\lambda_m + \delta(m-1)) \quad \text{for} \quad m \ge n/2\end{aligned}$$

and the same estimate for $\left|EH_4\left(-\frac{U_m}{\lambda_m}, \frac{Z_m}{\lambda_m\sqrt{n}}\right)\right|$. If now $\delta(k) = o(k^{-1/2}\log k)$, then

$$(6.7)\quad \left|\sum_{m=1}^n \left(EH_4\left(-\frac{U_m}{\lambda_m}, \frac{X_m}{\lambda_m\sqrt{n}}\right) - EH_4\left(-\frac{U_m}{\lambda_m}, \frac{Z_m}{\lambda_m\sqrt{n}}\right)\right)\right| = o(n^{-1/2}\log n).$$

But

$$EX_m^3\varphi''\left(-\frac{U_m}{\lambda_m}\right) = E\left(X_m^3\varphi''\left(-\frac{U_m}{\lambda_m}\right); \frac{U_m}{\lambda_m} \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \ge c\lambda_m - c'\delta(m-1).$$

If $\delta(k) = o(k^{-1/2}\log k)$, then $\sum_{k=1}^n \delta(k-1)/(\lambda_k^3 k^{3/2}) = o(n^{-1/2}\log n)$, but $\sum_{m=1}^n \lambda_m^{-2}n^{-3/2} \ge cn^{-1/2}\log n$.

Combining this with (6.6) and (6.7) yields

$$P((\textstyle\sum_{i=1}^n X_{ni} + \xi)/\sqrt{n} \le 0) - P((\textstyle\sum_{i=1}^n Z_i + \xi)/\sqrt{n} \le 0) \le -cn^{-1/2}\log n$$

for suitably large n. Using Lemma 1 one obtains a contradiction to $\delta(n) = o(n^{-1/2}\log n)$. Now we have proved that $\limsup_{n\to\infty}\delta(n)n^{1/2}(\log n)^{-1} > 0$.

EXAMPLE 3. The example is of the same type as those above, and we shall only sketch it.

Let X_{ni}, $i \le n$ be such that S_{ni}, $1 \le i \le n$ is a Markov chain. For $i \le n - n^{1/3}$ X_{ni} is i.i.d. ± 1 w.p. ½. For $i > n - n^{1/3}$, X_{ni} is uniformly bounded and constructed in such a way that

$$E(X_{ni}^2 \mid \mathscr{F}_{n,i-1}) - 1 = \frac{1}{2} \quad \text{for} \quad S_{i-1}/\sqrt{i-1} \in [0, \lambda_i]$$

$$E(X_{ni}^2 \mid \mathscr{F}_{n,i-1}) - 1 = -\frac{1}{2} \quad \text{for} \quad S_{i-1}/\sqrt{i-1} \in [-\lambda_i, 0]$$

and 0 otherwise, where $\lambda_i^2 = \lambda_{n,i}^2 = (n-i+1)/n$.

Let $\mathscr{G}^0_n$ and $\delta(n)$ be defined in the same way as in Examples 1 and 2. Using the same type of arguments, one obtains

$$\lim_{n,i\to\infty} E(X_{ni}^2) = 1; \ \limsup_{n\to\infty} \sum_{i=1}^n \| E(X_{ni}^2 \mid \mathscr{F}_{n,i-1}) - E(X_{ni}^2) \|_1 < \infty,$$

but $E((E(X_{ni}^2 | \mathscr{F}_{n,i-1}) - E(X_{ni}^2))\varphi'(- U_{ni}/\lambda_i))$ remains for $i \geq n - n^{1/3}$ of order λ_i. This then leads in the same way in Examples 1 and 2 to $\lim \sup_{n\to\infty} n^{1/3}\delta(n) > 0$.

EXAMPLE 4. As a change, we give an example where one can show that $\sup_t | P(S_{nn}/s_n \leq t) - \phi(t) |$ itself has bad properties. Actually we give an array $X_{n,k'}$, $k \leq n$ with $\sup_{k\leq n} k^\alpha \beta_{nk}^{(\infty)} < \infty$ $(0 < \alpha < ½)$ and $\lim \sup_{n\to\infty} n^\alpha | P(S_{nn} > 0) - ½ | > 0$. We set $P(X_{n1} \in dx) = \delta_0 (dx)$

$$P(X_{nk} \in dx | X_{n1}, \cdots, X_{n,k-1}) = \begin{cases} \frac{1}{2}(\delta_{-1} + \delta_1)(dx) \quad \text{if} \quad k \leq \left[\frac{n}{2}\right] \quad \text{or} \quad S_{n,k-1} > 0 \\ \left(\frac{1 + (k-1)^{-\alpha}}{2 + (k-1)^{-\alpha}}\delta_1 + \frac{1}{2 + (k-1)^{-\alpha}}\delta_{-1-(k-1)^{-\alpha}}\right)(dx) \\ \quad \text{if} \quad k > \left[\frac{n}{2}\right] \quad \text{and} \quad S_{n,k-1} = 0 \\ \frac{1}{2}(\delta_{-1-r^{-\alpha}} + \delta_{1+r^{-\alpha}})(dx) \quad \text{if} \\ \quad k > \left[\frac{n}{2}\right] \quad \text{and} \quad S_{n,k-1} < 0, \end{cases}$$

where $r = L_{n,k-1}$ if $L_{n,k-1} = \max\{i \leq k - 1 : S_{ni} = 0\} \geq \left[\frac{n}{2}\right]$ and $r = \infty$ else.

A somewhat lengthy but elementary calculation leads to the above stated assertions. Alternatively, one could also give an example which is similarly constructed as Examples 1 through 3.

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