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Exact Coupling Method for Stratonovich Stochastic Differential Equation Using Non-Degeneracy for the Diffusion

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ABSTRACT The method of Davie's [5] describes an easily generated scheme based on the standard order-one Milstein scheme, which is order-one in the Vaserstein metric, provided that the stochastic differential equation has invertible diffusion term. We apply the exact coupling method from Davie's paper to Stratonovich stochastic differential equation and the convergence of this method is proved by MATLAB implementation. We examine the strong convergence for the Stratonovich SDE using a particular invertible SDE.

INDEX TERMS Stochastic differential equation, coupling, Stratonovich SDE, Euler scheme, Milstein scheme.

I. INTRODUCTION

A new method developed by Davie [5] is investigated that uses coupling and has order-one strong convergence for stochastic differential equations (SDEs). There are several numerical methods for solving SDEs. Kloeden and Platen [6] described a method based on the stochastic Taylor series expansion; however, the major difficulty with this approach is that the double stochastic integrals cannot be easily expressed in terms of simpler stochastic integrals when the Wiener process is multi-dimensional. In the multi-dimensional case, the Fourier series expansion of Wiener process has been used to represent the double integrals in [6], [10], and [11]. However, several random variables should be generated each time, and therefore the computation requires a large amount of time; moreover, this method is difficult to extend to higher order. In this study a modified interpretation for the normal random variables generated in the Taylor expansion will be considered for the Stratonovich SDE. This method has order-one convergence under a non-degeneracy condition for the diffusion term. In standard methods such as Milstein, the approximations for the Taylor expansion terms are separately generated. In the coupling method, the approximation for the Taylor expansion is generated as a combination of random variables. The modification consists in replacing the iterated integrals by different random variables with a good approximation in distribution. Then, a random vector will be obtained from the linear term that is a good approximation in distribution to the original Taylor expansion. There are several studies that used coupling for the numerical solution of SDEs. Kanagawa [12] investigated the rate of convergence in terms of two probability metrics between approximate solutions with i.i.d. random variables. Rachev and Ruschendorff [8] developed Kanagawa's method by using the Komlós *et al.* theorem in [7]. Fournier [13] used the quadratic Vaserstein distance for the approximation of the Euler scheme and the results of Rio [14], which give a very precise rate of convergence for the central limit theorem in the Vaserstein distance. Moreover, Rio [19] provided precise bound estimates. Under uniform ellipticity, Alfonsi *et al.* [3], [4] studied the Vaserstein bound for the Euler method and proved an $O(h^{(\frac{2}{3}-\epsilon)})$ bound for a one-dimensional diffusion process, where h is the step-size; subsequently, they generalized the result to SDEs of any dimension with an $O(h\sqrt{\log(\frac{1}{h})})$ bound when the coefficients are time-homogeneous. Cruzeiro *et al.* [15] obtained an order-one method, and under non-degeneracy, they constructed a modified Milstein scheme that attains order one for the strong approximation. Charlbonneau *et al.* [16] investigated the Vaserstein bound [9] by using weak convergence and the Strassen–Dudley theorem. Convergence of an approximation to a strong solution on a given probability space was established by Gyöngy and Krylov [17] using coupling. Davie [20] applied the Vaserstein bound to solutions of vector SDEs and used the Komlós, Major, and

Tusnády theorem to obtain order-one approximation under a non-degeneracy assumption. The remainder of this paper is organized as follows. In Section 2, certain results concerning SDEs are reviewed, and some existing schemes for numerical resolution of SDE have been presented. In section 3, the implementation of Stratonovich scheme using the exact coupling is shown and a numerical example is provided to demonstrate the convergence behavior. In the last section, the Appendix (Matlab code) is provided.

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II. STRATONOVICH STOCHASTIC DIFFERENTIAL EQUATIONS

A. DEFINITION

Let $\{W(t)\}_{t\geq 0}$ be a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $a = a(t, x)$ be a d-dimensional vector function (called drift coefficient), and $b = b(t, x)$ a $d \times d$-matrix function (called diffusion coefficient).

The stochastic process $X = X(t)$, considered in this study can be described by *SDEs*, namely,

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t), \quad t \in [0, \mathrm{T}] \tag{1}$$

Let the initial condition $X(0) = x$ be an $\mathcal{F}_0$-measurable random vector in $\mathbb{R}^d$. An $\mathcal{F}_t$-adapted stochastic process $X = (X(t))_{t\geq 0}$ is called a solution of Equation (1) if

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \int_0^t b(s, X(s))dW(s) \tag{2}$$

holds almost surely (a.s.)

The conditions that the integral processes

$$\int_0^t a(s, X(s))ds, \qquad \int_0^t b(s, X(s))dW(s),$$

are well-defined are required for(2) to hold. Indeed, for the functions $a(s, X(s))$ and $b(s, X(s))$ we have

$$E\int_0^t b^2(s, X(s))ds < \infty, \tag{3}$$

and a.s. for all $t \geq 0$,

$$\int_0^t |a(s, X(s))|ds < \infty. \tag{4}$$

These conditions imply that the corresponding processes are well defined.

One important property of the stochastic integral is that

$$\int_0^t W(s)dW(s) = \frac{1}{2}\int_0^t d(W^2(s)) - \frac{1}{2}\int_0^t ds = \frac{1}{2}W^2(t) - \frac{t}{2},$$

for more details on stochastic integral see [6].

B. DEFINITION

From the definition shown in [6], we call an equation a *a Stratonovich stochastic differential equation*, writing it in following form

$$dX(t) = A(t, X(t))dt + b(t, X(t)) \circ dW(t), \tag{5}$$

or in the equivalent integral equation form

$$X(t) = X(0) + \int_0^t A(s, X(s))ds + \int_0^t b(s, X(s)) \circ dW(s) \tag{6}$$

The "$\circ$" notation here denotes the use of Stratonovich calculus. It turns out that the solutions of the Stratonovich SDE (5)-(6) also satisfy an Ito SDE with the same diffusion coefficient $b(s, X(s))$, but with the modified drift coefficient

$$a(s, x) = A(s, x) + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} b_{kj}(s, x)\frac{\partial b_j}{\partial x^k}(s, x)$$

where b_j is the j^{th} column of the matrix $b(s, x)$.

C. CONVERGENCE

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the set of continuous functions with the supremum metric on the interval $[0, T]$, $\mathcal{F}$ is the σ-algebra of Borel sets, and $\mathbb{P}$ is the Wiener measure. An approximate solution x_h of (1) is considered that uses a subdivision of the interval $[0, T]$ into a finite number N of subintervals of length $h = \frac{T}{N}$. Moreover, it is assumed that the approximate solutions x_h are random variables on Ω. Then, the discrete time approximation x_h with step-size h is said to converge strongly of order γ at time $T = Nh$ to the solution $X(t)$ of (1) if

$$E|x_h - X(T)|^p \leq Ch^{\gamma p}, h \in (0, 1),$$

where the strong convergence is in the L^p space. C is a positive constant independent of h.

There are several numerical methods for solving SDEs. Here, two important schemes will be mentioned. One is the Euler–Maruyama scheme that has strong order $\frac{1}{2}$, and the other is the Milstein scheme that has strong order one. It is assumed that we have the stochastic differential equation

$$dX_i(t) = a_i(t, X(t))dt + \sum_{k=1}^{d} b_{ik}(t, X(t))dW_k(t), \quad X_i(0) = X_i^{(0)}, \tag{7}$$

where $i = 1,...,d$, on an interval $[0, T]$ for a d-dimensional vector $X(t)$ and a d-dimensional Brownian path $W(t)$. To approximate the solution, it is assumed that $[0, T]$ is divided into N equal intervals of length $h = T/N$. The simplest numerical method for approximating the solution of stochastic differential equations is the stochastic Euler scheme (also called Euler Maruyama scheme) which utilizes only the first two terms of the Taylor expansion and it attains the strong convergence $\gamma = \frac{1}{2}$. The Milstein scheme is now introduced, which yields an order-one strong Taylor scheme. The Milstein scheme can be obtained by adding the quadratic terms $\sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)})A_{kl}^{(j)}$ to the Euler scheme, namely,

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum_{k=1}^{d} b_{ik}(jh, x^{(j)})\Delta W_k^{(j)} + \sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)})A_{kl}^{(j)}, \quad (8)$$

where $\Delta W_k^{(j)} = W_k((j+1)h) - W_k(jh)$, $A_{kl}^{(j)} = \int_{jh}^{(j+1)h}\{W_k(t) - W_k(jh)\}dW_l(t)$, and $\rho_{ikl}(t,x) = \sum_{m=1}^{q} b_{mk}(t,x)\frac{\partial b_{il}}{\partial x_m}(t,x)$.

The Euler scheme is easy to implement, as one need only generate the normal distribution for the standard Brownian motion $\Delta W_k^{(j)}$; however, it is not easy to generate the integral $A_{kl}^{(j)}$ for the Milstein scheme for two-dimensional (or higher) SDEs. The two-level approximation will now be described. The increments $\Delta W_k^{(j)}$ should be generated when the solution to (1) is approximated by using Euler or other schemes, which will explained later in this section. Therefore, Levy's construction of the Brownian motion will be used to simulate a sequence of approximations that converge to the solution. That is,

$$\Delta W_k^{(r,j)} = \Delta W_k^{(r+1,2j)} + \Delta W_k^{(r+1,2j+1)}, \quad (9)$$

where $r \in \mathbb{N}$ and $\Delta W_k^{(r,j)} = W_k((j+1)h^{(r)}) - W_k(jh^{(r)})$ with $h^{(r)} = \frac{T}{2^r}$.

The two-level approximation in (9) is called *the trivial coupling*. The normal distribution in (9) for the increments for a given level r could be generated by first generating the increments on the left-hand side and then conditionally generating the increments on the right-hand side. The same process is performed for all subsequent levels, and the Brownian path $W(t)$ is thus obtained. The empirical estimation of the error of a numerical method should now be explained. Usually, SDEs cannot be explicitly solved; therefore, the mean error $E|X(T) - x_h|$, which is the absolute value of the difference between the approximate solution x_h and the solution $X(T)$ of (1), cannot be directly estimated. If the approximate solution x_h is assumed to converge to the solution $X(T)$ as the step-size decreases and tends to zero, then the order of convergence for a particular scheme can be estimated by repeating R different independent simulations of sample paths. The estimator $\left\{\epsilon = \frac{1}{R}E(\left|x_{(r)} - \hat{x}_{(r)}\right|)\right\}$ will be used for different approximate solutions $x_{(r)}$ and $\hat{x}_{(r)}$ for different ranges of h. Thus, for any numerical method, if there is a bound $E|x_h - x_{h/2}| \le C_1 h^\gamma$ for the error, then $E|x_{h/2} - x_{h/4}| \le C_1(\frac{h}{2})^\gamma, E|x_{h/4} - x_{h/8}| \le C_1(\frac{h}{2^2})^\gamma, \ldots$. Therefore, a geometric series is obtained, and we have

$$E|X(T) - x_h| \le \sum_{h=0}^{\infty} C_1\left(\frac{h}{2^k}\right)^\gamma = \frac{C_1 h^\gamma}{1 - 2^{-\gamma}}. \quad (10)$$

Hence, the rate of convergence and the constant can be estimated from (10).

If the commutativity condition

$$\rho_{ikl}(t,x) = \rho_{ilk}(t,x), \quad (11)$$

holds for all $x \in \mathbb{R}^d$, $t \in [0,T]$, and i, k, l, then the Milstein scheme (8) is reduced to

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum_{k=1}^{d} b_{ik}(jh, x^{(j)})\Delta W_k^{(j)} + \sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)})A_{kl}^{(j)}, \quad (12)$$

which depends only on the generation of the Brownian motion $\Delta W_k^{(j)}$. Scheme (12) has order one if $d = 1$, but if $d > 1$ it has order $\frac{1}{2}$. As it described in Davie's study, scheme (12) can be modified to obtain order one under a non-degeneracy condition.

D. MODIFICATION TO (12) FOR ORDER-ONE CONVERGENCE

As it described in [5], the generation of the normal distribution will be modified in scheme (12), leading to order-one convergence under a non-degeneracy condition. In the implementation of the Milstein scheme, the random variables $\Delta W_k^{(j)}$ and $A_{kl}^{(j)}$ are separately generated and are then added to obtain the right-hand side of (12). The idea here is to directly generate the following:

$$Y_i := \sum_{k=1}^{d} b_{ik}(jh, x^{(j)})\Delta W_k^{(j)} + \sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)})A_{kl}^{(j)}.$$

If there is a scheme

$$x_i^{(j+1)} = x_i^{(j)} + a_i(jh, x^{(j)})h + \sum_{k=1}^{d} b_{ik}(jh, x^{(j)})X_k^{(j)} + \sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)})(X_k^{(j)}X_l^{(j)} - h\delta_{kl}), \quad (13)$$

where the increments $X_k^{(j)}$ are independent $N(0,h)$ random variables, then it is the same as scheme (12) with $\Delta W_k^{(j)}$ replaced by $X_k^{(j)}$, and $\Delta W_k^{(j)} = X_k^{(j)}$ is not assumed. Furthermore,

$$Z_i := \sum_{k=1}^{d} b_{ik}(jh, x^{(j)})X_k^{(j)} + \sum_{k,l=1}^{d} \rho_{ikl}(jh, x^{(j)})(X_k^{(j)}X_l^{(j)} - h\delta_{kl})$$

is assumed to be a good approximation to Y_i, that is, the joint distribution of the random vectors $(\Delta W_k^{(j)}, A_{kl}^{(j)})$ and $(X_k^{(j)})$ should be determined, so that they have the required marginal distribution with bound $E(Y_i - Z_i)^2 = O(h^3)$. In the following section, it will be explained how a coupling can be used to obtain the required marginal distribution, which will give good bounds for the random distributions Y_i and Z_i. Subsequently, an order-one approximation between the two approximate solutions $x(jh)$ and $x^{(j)}$ of the SDE will be obtained, i.e., $E(x(jh) - x^{(j)}) = O(h^2)$. In the following section, the proof of order-one convergence using (12) will be provided under the assumption that $b_{ik}(x)$ is invertible.

Now we need to show that the vector version of (11.1.2) from [6] for the Stratonovich equation is equivalent (apart from small terms) to scheme (12) for the Ito form of the equation. Assuming the drift term equal to zero then we have

$$Y^k_{n+1} = Y^k_n + \frac{1}{2}\sum_{i=1}^{d}\big\{b_{ik}(\Upsilon_n) + b_{ik}(Y_n)\big\}\Delta W^i_n \tag{14}$$

We need to use the deterministic Taylor expansion for $b_{ik}(\Upsilon_n)$ to find the difference approximation between $\big\{b_{ik}(\Upsilon_n) - b_{ik}(Y_n)\big\}$. Where the supporting value is $\big(\Upsilon_n = Y_n + \sum_i^d b^i(Y_n)\Delta W^i_n\big)$. Now for $0 < \theta < 1$

$$\begin{aligned}
&b_{ik}(\Upsilon_n)\\
&= b_{ik}(Y_n + \sum_i^d b^i(Y_n)\Delta W^i_n)\\
&= b_{ik}(Y_n) + \sum_{m,i=1}^{d}\frac{\partial b_{ik}(Y_n)}{\partial y^i}b_{l_m}(Y_n)\Delta W^i_n + \frac{1}{2}\\
&\quad\times\sum_{i,m,j_1=1}^{d}\frac{\partial^2 b_{ik}(Y_n+\theta(b^i(Y_n)\Delta W^i_n)}{\partial y^i\partial y^m}(b^m b^{j_1}\Delta W^m_n\Delta W^{j_1}_n)^2
\end{aligned} \tag{15}$$

Then we replace (15) in (14) which gives us

$$\begin{aligned}
&Y^k_{n+1}\\
&= Y^k_n + \frac{1}{2}\sum_{i=1}^{d}\big\{b_{ik}(Y_n) + b_{ik}(Y_n)\\
&\quad+\sum_{m,i=1}^{d}\frac{\partial b_{ik}(Y_n)}{\partial y^i}b_{l_m}(Y_n)\Delta W^i_n + \frac{1}{2}\\
&\quad\times\sum_{i,m,j_1=1}^{d}\frac{\partial^2 b_{ik}(Y_n+\theta(b^i(Y_n)\Delta W^i_n)}{\partial y^i\partial y^m}(b^m b^{j_1}\Delta W^m_n\Delta W^{j_1}_n)^2\\
&\quad+b_{ik}(Y_n)\big\}\Delta W^i_n
\end{aligned} \tag{16}$$

$$\begin{aligned}
&Y^k_{n+1}\\
&= Y^k_n + \sum_{i=1}^{d}b_{ik}(Y_n)\Delta W^i_n + \frac{1}{2}\sum_{i,l=1}^{d}\rho_{kil}(Y_n)\Delta W^l_n\Delta W^i_n.
\end{aligned} \tag{17}$$

The convergence behavior for the Stratonovich SDE will be shown in the following section using a two-dimensional invertible SDE.

III. TWO-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATION

Let the following 2-dimensional invertible SDE be considered:

$$\begin{aligned}
dX_1(t) &= (\sin(X_2(t)))^2 dW_1(t) - \frac{1}{1+X_1^2(t)}dW_2(t),\\
dX_2(t) &= \frac{1}{1+X_2^4(t)}dW_1(t) + (\cos(X_1(t)))^2 dW_2(t),\\
&\text{for } 0 \le t \le 1,\ \text{with } X_1(0) = 2 \text{ and } X_2(0) = 0
\end{aligned} \tag{18}$$

where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motion. To apply a numerical method to this SDE, solutions (for the same Brownian path) should be simultaneously simulated by using two different step sizes (h and $h/2$). The Matlab implementation for Stratonovich SDE using the exact coupling is given in Listing (3), which will demonstrate the result of the absolute value of the difference between two solutions with step size h and $h/2$. To conduct this experiment, the error and the convergence order of the exact coupling method will be calculated for decreasing values of the step size h. This will be repeated with different step size using (for example, $R = 1000$) independent simulations. Then the order of convergence of this method between two approximate solutions should be 1. Now we will run the Matlab code in Listing (3) with different step sizes over a large number of path R as it described in the table below and see the result of the error ϵ, where each simulation is for the same Brownian path and $\epsilon = \frac{1}{R}\sum_{i=1}^{R}|x_h^{(i)} - x_{h/2}^{(i)}|$ will be our estimator. The Matlab code in Listing (3) will run with different number of steps (200, 400, 800, 1600, 3200) over a large number of paths.

```
S=[ 200, 400, 800, 1600, 3200];
ErrorEuler=zeros(1,length(S));
for i=1:length(S)
    ErrorEuler(1,i)=log(Stratonovich33
('YA',[2; 0],1,S(1,i)));
end
h=1./S,
fad=log(h)
plot(log(h), ErrorEuler, 'b--*')
```

Table (1) and the plot in Figure (1) show the implementation of the approximate solutions of the previous 2-dimensional SDEs with different number of steps (200, 400, 800, 1600, and 3200). Running the code in Listing (3) for 1000 simulations yields a value for the estimator ϵ equal to 0.0032 with

TABLE 1. Error results for Stratonovich SDE with the invertible matrix.

step-size	error(ϵ)
0.005	0.0032
0.0025	0.0016
0.00125	0.00082
0.00062	0.00041
0.00031	0.00020

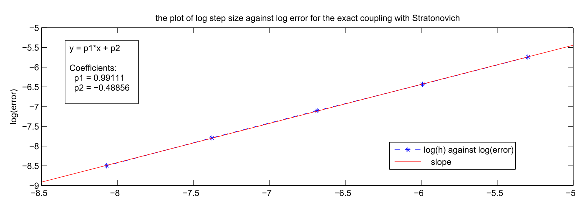

FIGURE 1. Stratonovich with the exact coupling method.

```
function [C1, C2, C, UU]=mfileCCC33(b,ss,m,h,x)

  UU=feval(b,(m-1)*h,x);   C= inv(UU);
           Y11=x+ UU(:,1)*ss;
           Y22=x+ UU(:,2)*ss;
 C1=1/ss*(feval(b,(m-1)*h,Y11)
-feval(b,(m-1)*h,x));
 C2=1/ss*(feval(b,(m-1)*h,Y22)
-feval(b,(m-1)*h,x));
end
```

Listing. 1. The code for calculating $\left\{\frac{b_{jk}(\tau_n, Y_n^j)-b_{jk}}{\sqrt{h}}\right\}$.

```
function [Z1, Z2, Y1, Y2, V1, V2] =
 couplingStratonovich(a1,a2,a,s)

   UB1=randn; UB2=randn; Q=randn; R=randn;
        u=rand;
       if u<0.5
         zz=1;
       else zz=-1;
   end
        UU1=zz*Q;   UU2=zz*R; VB2=UB2;
        if a == 0
   b=s*a*R; c=-s*a*Q*UB2;
                  Y=UB1+(b*UB1+c);
   Er1=erf((1/sqrt(2))*((Y-c)/(1+b)));
         Er2=erf((1/sqrt(2))*((Y+c)/(1-b)));
      A1=1/2*(1+Er1); A2=1/2*(1+Er2);
      Fy=1/2*(A1+A2);
     VB1=sqrt(2)*erfinv(2*Fy-1);
      V1=-VB1; V2=-VB2;
      U1=-UB1; U2=-UB2;
      US1=-UU1; US2=-UU2;
      Z1=(1/2)*(U1-US1); Z2=(1/2)*(U2-US2);
      Y1=(1/2)*(U1+US1); Y2=(1/2)*(U2+US2);
          else
      b=s*a*R; c=-s*a*Q*UB2;
                            Y=UB1+(b*UB1+c);
Er1=erf((1/sqrt(2))*((Y-c)/(1+b)));
Er2=erf((1/sqrt(2))*((Y+c)/(1-b)));
A1=1/2*(1+Er1); A2=1/2*(1+Er2);
Fy=1/2*(A1+A2);
VB1=sqrt(2)*erfinv(2*Fy-1);
V1=(a1/a)*VB1-(a2/a)*VB2;
V2=(a2/a)*VB1+(a1/a)*VB2;
U1=(a1/a)*UB1-(a2/a)*UB2;
U2=(a2/a)*UB1+(a1/a)*UB2;
US1=(a1/a)*UU1-(a2/a)*UU2;
US2=(a2/a)*UU1+(a1/a)*UU2;
Z1=(1/2)*(U1-US1); Z2=(1/2)*(U2-US2);
Y1=(1/2)*(U1+US1); Y2=(1/2)*(U2+US2);
     end
```

Listing. 2. Calculating the exact coupling.

step size 0.005 i.e.

$$\epsilon = \frac{1}{1000}\sum_{i=1}^{1000} |x_h^{(i)} - x_{h/2}^{(i)}| = 0.0032$$

0.0016 with step size 0.0125, and the corresponding values for other step sizes. This implies that if the number of steps

```
function AAA=Stratonovich33(YA,x0,T,N)

h=T/N; s=sqrt(T/N);
ss=sqrt(T/(2*N)); RR=5000; q=0;
for r=1:RR,      x=x0;   y=x0;
for m=1:N;
[C1, C2, C, UU]=mfileCCC(YA,ss,m,h,x);

[to112,to121, to212,to221] =
mfilfortaoo(C1,C2,C);
a1=(to112-to121)/2;
a2=(to212-to221)/2;
aa=(a1^2+a2^2)^(1/2);
[Z1, Z2, Y1, Y2, V1, V2] =
coupling(aa,a1,a2,s);
wL=s*Y1; wr=s*Z1; w=s*V1; vL=s*Y2;
vr=s*Z2; v=s*V2;
Y11=x+ UU*[wL; vL];

C1=(1/2)*(feval(YA,(m-1)*h,Y11)
+feval(YA,(m-1)*h,x));
u=x(1)+C1(1,1)*wL+C1(1,2)*vL;
x(2)=x(2)+C1(2,1)*wL+C1(2,2)*vL; x(1)=u;
UU=feval(YA,(m-1)*h,x);
Y11=x+ UU*[wr; vr];

C1=(1/2)*(feval(YA,(m-1)*h,Y11)
+feval(YA,(m-1)*h,x));
u=x(1)+C1(1,1)*wr+C1(1,2)*vr;
x(2)=x(2)+C1(2,1)*wr+C1(2,2)*vr; x(1)=u;
UU=feval(YA,(m-1)*h,y);

Y11=y+ UU*[w; v];
C1=(1/2)*(feval(YA,(m-1)*h,Y11)
+feval(YA,(m-1)*h,y));
u=y(1)+C1(1,1)*w+C1(1,2)*v;
y(2)=y(2)+C1(2,1)*w+C1(2,2)*v; y(1)=u;
end
q=q+abs(x(1)-y(1))+abs(x(2)-y(2));
end AA=q; AAA=q/RR; end
```

Listing. 3. Calculating the error for Stratonovich equation of (11.1.2) in K and P numerical solution of SDE.

increases, which results in a smaller step size, then the error estimate ϵ is $O(h)$, as can be seen in Table (1). Moreover, Figure (1) is a plot of the log of the estimator ϵ i.e. $\log \epsilon$ against the log of step-size h i.e. $\log(h)$, which has a slope of 0.99111, again indicating a strong convergence of $O(h)$ for the stochastic differential equation (18).

IV. CONCLUSION

We have presented a Matalb implementation for the Stratonovich stochastic differential equation and it can be seen that good agreement is obtained between the theoretical bound and the implementation results. The main advantage of this approach is that the computational load of this estimation will be reduce comparing to some other methods.

APPENDIX

See Listings 1–3.

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Authors' photographs and biographies not available at the time of publication.