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Research Article

Exact Determinants of Some Special Circulant Matrices Involving Four Kinds of Famous Numbers

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Circulant matrix family is used for modeling many problems arising in solving various differential equations. The RSFPLR circulant matrices and RSLPFL circulant matrices are two special circulant matrices. The techniques used herein are based on the inverse factorization of polynomial. The exact determinants of these matrices involving Perrin, Padovan, Tribonacci, and the generalized Lucas number are given, respectively.

1. Introduction

Circulant matrix family is used for modeling many problems arising in solving various differential equations. Therefore, studying algorithms and various properties of new classes of problems to such patterned matrices is crucial for applications. It is important to develop new theories and methods and to modify and refine the well-known techniques, for solving differential equations.

Lei and Sun [1] proposed the preconditioned CGNR (PCGNR) method with a circulant preconditioner to solve such Toeplitz-like systems. Pang et al. [2] used the normalized preconditioned conjugate gradient method with Strang's circulant preconditioner to solve a nonsymmetric Toeplitz system $A_n x = b$. By using a Strang-type block-circulant preconditioner, Zhang et al. [3] speeded up the convergent rate of boundary-value methods. Delgado et al. [4] developed some techniques to obtain global hyperbolicity for a certain class of endomorphisms of $(R^p)^n$ with $p, n \geq 2$; this kind of endomorphisms is obtained from vectorial difference equations where the mapping defining these equations satisfies a circulant matrix condition. The Strang-type preconditioner was also used to solve linear systems from differential-algebraic equations and delay differential equations; see [5–8]. In [9], a semicirculant preconditioner applied to a problem, subject to Dirichlet boundary conditions at the inflow boundaries, was examined. A method was described for obtaining finite difference approximation solutions of multidimensional partial differential equations satisfying boundary conditions specified on irregularly shaped boundaries by using circulant matrices and fast Fourier transform (FFT) convolutions in [10]. Brockett and Willems [11] showed how the important problems of linear system theory can be solved concisely for a particular class of linear systems, namely, block-circulant systems, by exploiting the algebraic structure.

Circulant matrices have important applications in various disciplines including image processing, communications, and signal processing. The circulant matrices, a long fruitful subject of research, have in recent years been extended in many directions [12, 13]. The $f(x)$-circulant matrices are another natural extension of this well-studied class and can be found in [14–24]. The $f(x)$-circulant matrix has a wide application, especially on the generalized cyclic codes [14]. The properties and structures of the $x^n - x + 1$-circulant matrices, which are called RSFPLR circulant matrices, are better than those of the general $f(x)$-circulant matrices, so there are good algorithms for determinants.

There are many interests in properties and generalization of some special matrices with famous numbers. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [25]. Lin gave the determinant of the Fibonacci-Lucas quasi-cyclic matrices [26].

Lind presented the determinants of circulant and skew-circulant involving Fibonacci numbers in [27]. Shen et al. [28] discussed the determinant of circulant matrix involving Fibonacci and Lucas numbers. Akbulak and Bozkurt [29] gave the norms of Toeplitz involving Fibonacci and Lucas numbers. The authors [30, 31] discussed some properties of Fibonacci and Lucas matrices. Stanimirović gave generalized Fibonacci and Lucas matrix in [32]. Z. Zhang and Y. Zhang [33] investigated the Lucas matrix and some combinatorial identities.

The determinant problems of the RSFPLR circulant matrices and RSLPFL circulant matrices involving the Perrin, Padovan, Tribonacci, and the generalized Lucas number are considered in this paper. The explicit determinants are presented by using some terms of these numbers. The techniques used herein are based on the inverse factorization of polynomial. Firstly, we introduce the definitions of the RSFPLR circulant matrices and RSLPFL circulant matrices and properties of the related famous numbers. Then, we present the main results and the detailed process.

Definition 1. A row skew first-plus-last right (RSFPLR) circulant matrix with the first row $(a_0, a_1, \ldots, a_{n-1})$, denoted by RSFPLRcircfr$(a_0, a_1, \ldots, a_{n-1})$, is a square matrix of the form

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ -a_{n-1} & a_0 + a_{n-1} & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -a_2 & -a_3 + a_2 & \cdots & a_0 + a_{n-1} & a_1 \\ -a_1 & -a_2 + a_1 & \cdots & -a_{n-1} + a_{n-2} & a_0 + a_{n-1} \end{bmatrix}. \quad (1)$$

The matrix with an arbitrary first row and the following rule for obtaining any other row from the previous one can be seen. Get the $i+1$st row by adding the last element of the ith row to the first element of the ith row and -1 times the last element of the ith row and then shifting the elements of the ith row (cyclically) one position to the right.

Note that the RSFPLR circulant matrix is a $x^n - x + 1$ circulant matrix [14] and that is neither the extension of skew circulant matrix [12, 13] nor its special case and they are two different kinds of special matrices. Moreover, it is a FLS r-circulant matrix [15–17] with $r = -1$.

We define $\mathbf{\Theta}_{(-1,1)}$ as the basic RSFPLR circulant matrix; that is,

$$\mathbf{\Theta}_{(-1,1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}. \quad (2)$$

It is easily verified that $g(x) = x^n - x + 1$ has no repeated roots in its splitting field and $g(x) = x^n - x + 1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\mathbf{\Theta}_{(-1,1)}$. In addition, $\mathbf{\Theta}_{(-1,1)}$ is nonderogatory and satisfies $\mathbf{\Theta}^j_{(-1,1)} = \text{RSFPLRcircfr}(\underbrace{0,\ldots,0}_{j}, 1, \underbrace{0,\ldots,0}_{n-j-1})$ and $\mathbf{\Theta}^n_{(-1,1)} = \mathbf{I}_n - \mathbf{\Theta}_{(-1,1)}$. Then a matrix $\mathbf{A}$ can be written in the form

$$\mathbf{A} = f\left(\mathbf{\Theta}_{(-1,1)}\right) = \sum_{i=0}^{n-1} a_i \mathbf{\Theta}^i_{(-1,1)} \quad (3)$$

if and only if $\mathbf{A}$ is a RSFPLR circulant matrix, where the polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^i$ is called the representer of the RSFPLR circulant matrix $\mathbf{A}$.

It is clear that $\mathbf{A}$ is a RSFPLR circulant matrix if and only if $\mathbf{A}$ commutes with the $\mathbf{\Theta}_{(-1,1)}$; that is,

$$\mathbf{A}\mathbf{\Theta}_{(-1,1)} = \mathbf{\Theta}_{(-1,1)}\mathbf{A}. \quad (4)$$

Definition 2. A row skew last-plus-first left (RSLPFL) circulant matrix with the first row $(a_0, a_1, \ldots, a_{n-1})$, denoted by RSLPFLcircfr $(a_0, a_1, \ldots, a_{n-1})$, is a square matrix of the form

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & \cdots & a_{n-1} + a_0 & -a_0 \\ \vdots & \vdots & ⋰ & \vdots \\ a_{n-2} & a_{n-1} + a_0 & \cdots & -a_{n-3} \\ a_{n-1} + a_0 & \cdots & -a_{n-3} + a_{n-2} & -a_{n-2} \end{bmatrix}. \quad (5)$$

The matrix with an arbitrary first row and the following rule for obtaining any other row from the previous onecan be seen. Get the $i+1$st row by adding the first element of the ith row to the last element of the ith row and -1 times the first element of the ith row and then shifting the elements of the ith row (cyclically) one position to the left.

Let $\mathbf{A}$ = RSFPLRcircfr $(a_n, a_{n-1}, \ldots, a_1)$ and $\mathbf{B}$ = RSLPFLcircfr $(a_1, a_2, \ldots, a_n)$. By explicit computation, we find

$$\mathbf{A} = \mathbf{B}\widehat{\mathbf{I}}_n, \quad (6)$$

where $\widehat{\mathbf{I}}_n$ is the backward identity matrix of the form

$$\widehat{\mathbf{I}}_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & ⋰ & ⋰ & ⋰ & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (7)$$

The Perrin [34, 35] and Padovan [34, 36] numbers $\{R_n\}$ and $\{\mathbb{P}_n\}$ are defined by a third-order recurrence

$$R_n = R_{n-2} + R_{n-3}, \quad n \geq 3, \quad (8)$$

$$\mathbb{P}_n = \mathbb{P}_{n-2} + \mathbb{P}_{n-3}, \quad n \geq 3, \quad (9)$$

with the initial conditions $R_0 = 3$, $R_1 = 0$, $R_2 = 2$, and $\mathbb{P}_0 = 1$, $\mathbb{P}_1 = 1$, and $\mathbb{P}_2 = 1$.

The Tribonacci [36, 37] and the generalized Lucas numbers $\{T_n\}$ and $\{\mathbb{L}_n\}$ [37] are defined by a third-order recurrence

$$\begin{aligned} T_n &= T_{n-1} + T_{n-2} + T_{n-3}, \quad n \geq 3, \\ \mathbb{L}_n &= \mathbb{L}_{n-1} + \mathbb{L}_{n-2} + \mathbb{L}_{n-3}, \quad n \geq 3, \end{aligned} \quad (10)$$

with the initial conditions $T_0 = 0, T_1 = 1, T_2 = 1$ and $\mathbb{L}_0 = 3$, $\mathbb{L}_1 = 1$, and $\mathbb{L}_2 = 3$.

Recurrences (8) and (9) involve the characteristic equation $x^3 - x - 1 = 0$, the roots of which are denoted by r_1, r_2, and r_3. Then

$$\begin{aligned} r_1 + r_2 + r_3 &= 0 \\ r_1r_2 + r_1r_3 + r_2r_3 &= -1 \\ r_1r_2r_3 &= 1. \end{aligned} \tag{11}$$

Moreover, the Binet form for the Perrin [34, 35] number is

$$R_n = r_1^n + r_2^n + r_3^n, \tag{12}$$

and the Binet form for Padovan [34, 36] number is

$$\mathbb{P}_n = a_1r_1^n + a_2r_2^n + a_3r_3^n, \tag{13}$$

where

$$a_i = \prod_{j=1, j\neq i}^{3} \frac{r_j - 1}{r_i - r_j}, \quad i = 1, 2, 3. \tag{14}$$

Recurrences (10) as well imply the characteristic equation $x^3 - x^2 - x - 1 = 0$, and their roots are denoted by t_1, t_2, and t_3. Then

$$\begin{aligned} t_1 + t_2 + t_3 &= 1 \\ t_1t_2 + t_1t_3 + t_2t_3 &= -1 \\ t_1t_2t_3 &= 1. \end{aligned} \tag{15}$$

Furthermore, the Binet form for the Tribonacci [36, 37] number is

$$T_n = b_1t_1^n + b_2t_2^n + b_3t_3^n, \tag{16}$$

where b_i is the ith root of the polynomial $44y^3 - 2y - 1$, and the Binet form for the generalized Lucas [37] number is

$$\mathbb{L}_n = t_1^n + t_2^n + t_3^n. \tag{17}$$

If the first row of a RSFPLR circulant matrix is $(R_1, R_2, \ldots, R_n)$, $(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n)$, $(T_1, T_2, \ldots, T_n)$, $(\mathbb{L}_1, \mathbb{L}_2, \ldots, \mathbb{L}_n)$, then the matrix is called Perrin, Padovan, Tribonacci, and generalized Lucas RSFPLR circulant matrix, respectively.

If the first row of a RSLPFL circulant matrix is $(R_1, R_2, \ldots, R_n)$, $(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n)$, $(T_1, T_2, \ldots, T_n)$, $(\mathbb{L}_1, \mathbb{L}_2, \ldots, \mathbb{L}_n)$, then the matrix is called Perrin, Padovan, Tribonacci, and generalized Lucas RSLPFL circulant matrix, respectively.

2. Main Results

By Theorem 1.1 in [15], we deduce the following lemma.

Lemma 3. *Let* $\mathbf{A} = RSFPLRcircfr\,(a_1, a_2, \ldots, a_n)$. *Then the eigenvalues of* $\mathbf{A}$ *are given by*

$$\lambda_i = f(\kappa_i) = \sum_{j=1}^{n} a_j\kappa_i^{j-1}, \quad i = 1, 2, \ldots, n, \tag{18}$$

and the determinant of A is given by

$$\det \mathbf{A} = \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n}\sum_{j=1}^{n} a_j\kappa_i^{j-1}, \tag{19}$$

where κ_i $(i = 1, \ldots, n)$ *are the roots of the equation* $x^n - x + 1 = 0$.

Lemma 4. *Suppose that* κ_i $(i = 1, 2, \ldots, n)$ *are the roots of the equation* $x^n - x + 1 = 0$. *If* $a = 0$, *then*

$$\begin{aligned} &\prod_{i=1}^{n} \left(a\kappa_i^3 + b\kappa_i^2 + c\kappa_i + d\right) \\ &\quad = \prod_{i=1}^{n} \left(b\kappa_i^2 + c\kappa_i + d\right) \\ &\quad = d^n + b^{n-1}(b + c + d) - d\left(x_1^{n-1} + x_2^{n-1}\right) \\ &\qquad + \left(x_1^n + x_2^n\right), \end{aligned} \tag{20}$$

where $a, b, c \in \mathbb{R}$ *and*

$$x_1 = \frac{-c + \sqrt{c^2 - 4bd}}{2}; \qquad x_2 = \frac{-c - \sqrt{c^2 - 4bd}}{2}. \tag{21}$$

If $a \neq 0$, *then*

$$\begin{aligned} &\prod_{i=1}^{n} \left(a\kappa_i^3 + b\kappa_i^2 + c\kappa_i + d\right) \\ &\quad = \frac{(-1)^{n-1}a^{n-1}}{2} d\left(2X_{n-1} + X_{n-1}^2 - 2X_{2(n-1)}\right) \\ &\qquad + \frac{(-1)^n a^n}{2}\left(2X_n + 2X_{n+1} + X_n^2 - X_{2n} + 2\right) \\ &\qquad + (-1)^n a^{n-1}(b + b + c + d) + d^n, \end{aligned} \tag{22}$$

where $X_n = \alpha_1^n + \alpha_2^n + \alpha_3^n$ *and* α_1, α_2, *and* α_3 *are the roots of the equation* $a\kappa_i^3 + b\kappa_i^2 + c\kappa_i + d = 0$.

Proof. Since κ_i $(i = 1, 2, \ldots, n)$ are the roots of the characteristic polynomial of $\mathbf{\Theta}_{(-1,1)}$, $g(x) = x^n - x + 1$ can be factored as

$$x^n - x + 1 = \prod_{i=1}^{n} (x - \kappa_i). \tag{23}$$

If $a = 0$, please see [22] for details of the proof.

If $a \neq 0$, then

$$\begin{aligned}
&\prod_{i=1}^{n}\left(a\kappa_i^3 + b\kappa_i^2 + c\kappa_i + d\right)\\
&= a^n \prod_{i=1}^{n}\left(\kappa_i^3 + \frac{b}{a}\kappa_i^2 + \frac{c}{a}\kappa_i + \frac{d}{a}\right)\\
&= a^n \prod_{i=1}^{n}(\kappa_i - \alpha_1)(\kappa_i - \alpha_2)(\kappa_i - \alpha_3)\\
&= (-a)^n \prod_{i=1}^{n}(\alpha_1 - \kappa_i)\prod_{i=1}^{n}(\alpha_2 - \kappa_i)\prod_{i=1}^{n}(\alpha_3 - \kappa_i)\\
&= (-a)^n (\alpha_1^n - \alpha_1 + 1)(\alpha_2^n - \alpha_2 + 1)(\alpha_3^n - \alpha_3 + 1)\\
&= (-a)^n \Big\{(\alpha_1\alpha_2\alpha_3)^n + \alpha_1\alpha_2\alpha_3\\
&\qquad \times \left[(\alpha_1\alpha_2)^{n-1} + (\alpha_1\alpha_3)^{n-1} + (\alpha_2\alpha_3)^{n-1}\right.\\
&\qquad\qquad \left. + \alpha_1^{n-1} + \alpha_2^{n-1} + \alpha_3^{n-1}\right]\\
&\qquad + \left[(\alpha_1\alpha_2)^n + (\alpha_1\alpha_3)^n + (\alpha_2\alpha_3)^n\right.\\
&\qquad\qquad \left. + \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3\right]\\
&\qquad - \alpha_1\alpha_2\alpha_3 + 1 - \left[\alpha_1^n(\alpha_2 + \alpha_3) + \alpha_2^n(\alpha_1 + \alpha_3)\right.\\
&\qquad\qquad \left. + \alpha_3^n(\alpha_1 + \alpha_2)\right]\\
&\qquad + \left[\alpha_1^n + \alpha_2^n + \alpha_3^n - (\alpha_1 + \alpha_2 + \alpha_3)\right]\Big\}.
\end{aligned} \tag{24}$$

Let $X_n = \alpha_1^n + \alpha_2^n + \alpha_3^n$. We obtain $(\alpha_1\alpha_2)^n + (\alpha_1\alpha_3)^n + (\alpha_2\alpha_3)^n = (\alpha_n^2 - X_{2n})/2$ from $(\alpha_1^n + \alpha_2^n + \alpha_3^n)^2 = \alpha_1^{2n} + \alpha_2^{2n} + \alpha_3^{2n} + 2[(\alpha_1\alpha_2)^n + (\alpha_1\alpha_3)^n + (\alpha_2\alpha_3)^n]$. Taking the relation of roots and coefficients

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 &= -\frac{b}{a},\\
\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 &= \frac{c}{a},\\
\alpha_1\alpha_2\alpha_3 &= -\frac{d}{a}
\end{aligned} \tag{25}$$

into account, we deduce that

$$\begin{aligned}
&\prod_{i=1}^{n}\left(a\kappa_i^3 + b\kappa_i^2 + c\kappa_i + d\right)\\
&\quad = \frac{(-1)^{n-1}a^{n-1}}{2} d\left(2X_{n-1} + X_{n-1}^2 - 2X_{2(n-1)}\right)\\
&\qquad + \frac{(-1)^n a^n}{2}\left(2X_n + 2X_{n+1} + X_n^2 - X_{2n} + 2\right)\\
&\qquad + (-1)^n a^{n-1}(b + b + c + d) + d^n,
\end{aligned} \tag{26}$$

where $X_n = \alpha_1^n + \alpha_2^n + \alpha_3^n$ and $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation $a\kappa_i^3 + b\kappa_i^2 + c\kappa_i + d = 0$. □

2.1. Determinants of the RSFPLR Circulant Matrices and RSLPFL Circulant Matrices Involving Perrin Numbers

Theorem 5. *Let* $\mathbf{C} = RSFPLRcircfr(R_1, R_2, \ldots, R_n)$ *be a Perrin RSFPLR circulant matrix. Then*

$$\det \mathbf{C} = \frac{R_n^{n-1}\Delta_1 + R_n^n\Delta_2 + 2R_{n+1}^n}{\Delta_3}, \tag{27}$$

where

$$\begin{aligned}
\Delta_1 &= 2R_{n+1}Z_{n-1} + R_{n+1}Z_{n-1}^2 - 2R_{n+1}Z_{2(n-1)}\\
&\quad - 16 + 2R_{n+2},\\
\Delta_2 &= 2Z_n + 2Z_{n+1} + Z_n^2 - Z_{2n} - 2,\\
\Delta_3 &= Y_n^2 + Y_{n-1}^2 - Y_{2n} - 2Y_{2(n-1)} + 2Y_{n+1}\\
&\quad + 2Y_n + 2Y_{n-1} + 6,\\
Z_n &= \gamma_1^n + \gamma_2^n + \gamma_3^n,\\
Y_n &= \beta_1^n + \beta_2^n + \beta_3^n.
\end{aligned} \tag{28}$$

γ_1, γ_2, and γ_3 are the roots of the equation $-R_nx^3 + (3 - R_{n+2} + R_n)x^2 + (2 - R_{n+1} + R_{n+2})x + R_{n+1} = 0$, and β_1, β_2, β_3 are the roots of the equation $y^3 + y^2 - 1 = 0$.

Proof. Obviously, $\mathbf{C}$ has the form

$$\mathbf{C} = \begin{pmatrix}
R_1 & R_2 & \cdots & R_n\\
-R_n & R_1 + R_n & \cdots & R_{n-1}\\
\vdots & \vdots & \ddots & \vdots\\
-R_3 & -R_4 + R_3 & \cdots & R_2\\
-R_2 & -R_3 + R_2 & \cdots & R_1 + R_n
\end{pmatrix}. \tag{29}$$

By Lemma 3, the Binet form (12), and (11), we have

$$\begin{aligned}
\det \mathbf{C} &= \prod_{i=1}^{n}\left(R_1 + R_2\kappa_i + \cdots + R_n\kappa_i^{n-1}\right)\\
&= \prod_{i=1}^{n}\left[\sum_{k=1}^{n}\sum_{j=1}^{3} r_j^k\kappa_i^{k-1}\right]\\
&= \prod_{i=1}^{n}\left[\sum_{j=1}^{3}\frac{r_j\left(1 - r_j^n\kappa_i^n\right)}{1 - r_j\kappa_i}\right]\\
&= \prod_{i=1}^{n}\left[\frac{-R_n\kappa_i^3 + \left(3 - R_{n+2} + R_n\right)\kappa_i^2}{-\kappa_i^3 - \kappa_i^2 + 1}\right.\\
&\qquad \left. + \frac{\left(2 - R_{n+1} + R_{n+2}\right)\kappa_i + R_{n+1}}{-\kappa_i^3 - \kappa_i^2 + 1}\right].
\end{aligned} \tag{30}$$

By Lemma 4 and the recurrence (8), we obtain

$$\prod_{i=1}^{n}\left[-R_n\kappa_i^3+\left(3-R_{n+2}+R_n\right)\kappa_i^2+\left(2-R_{n+1}+R_{n+2}\right)\kappa_i+R_{n+1}\right]=\frac{R_n^{n-1}}{2}\left(2R_{n+1}Z_{n-1}+R_{n+1}Z_{n-1}^2-2R_{n+1}Z_{2(n-1)}-16+2R_{n+2}\right)+R_{n+1}^n+\frac{R_n^n}{2}\left(2Z_n+2Z_{n+1}+Z_n^2-Z_{2n}-2\right), \quad (31)$$

where $Z_n=\gamma_1^n+\gamma_2^n+\gamma_3^n$ and $\gamma_1,\gamma_2,\gamma_3$ are the roots of the equation $-R_nx^3+(3-R_{n+2}+R_n)x^2+(2-R_{n+1}+R_{n+2})x+R_{n+1}=0$ and

$$\prod_{i=1}^{n}\left(-\kappa_i^3-\kappa_i^2+1\right)=\frac{1}{2}\left(Y_n^2+Y_{n-1}^2-Y_{2n}\right)+Y_{n-1}+Y_n+Y_{n+1}-Y_{2(n-1)}+3, \quad (32)$$

where $Y_n=\beta_1^n+\beta_2^n+\beta_3^n$ and β_1,β_2,β_3 are the roots of the equation $y^3+y^2-1=0$. Consequently,

$$\det\mathbf{C}=\frac{R_n^{n-1}\Delta_1+R_n^n\Delta_2+2R_{n+1}^n}{\Delta_3}, \quad (33)$$

where

$$\begin{aligned}\Delta_1&=2R_{n+1}Z_{n-1}+R_{n+1}Z_{n-1}^2-2R_{n+1}Z_{2(n-1)}-16+2R_{n+2},\\ \Delta_2&=2Z_n+2Z_{n+1}+Z_n^2-Z_{2n}-2,\\ \Delta_3&=Y_n^2+Y_{n-1}^2-Y_{2n}-2Y_{2(n-1)}+2Y_{n+1}+2Y_n+2Y_{n-1}+6.\end{aligned} \quad (34)$$

□

Theorem 6. *Let* $\mathbf{D}=RSFPLRcircfr\left(R_n,R_{n-1},\ldots,R_1\right)$ *be a Perrin RSFPLR circulant matrix. Then*

$$\det\mathbf{D}=2\left[\frac{\left(R_n+3\right)^n+R_{n+5}\left(R_{n+1}-2\right)^{n-1}}{R_{n-1}^2+R_n^2-R_{2n}-2R_{2(n-1)}+2R_{n+4}}+\frac{-\left(R_n+3\right)\left(\rho_1^{n-1}+\rho_2^{n-1}\right)+\left(\rho_1^n+\rho_2^n\right)}{R_{n-1}^2+R_n^2-R_{2n}-2R_{2(n-1)}+2R_{n+4}}\right], \quad (35)$$

where

$$\rho_1=\frac{1-R_{n+2}+\Delta_4}{2},\quad \rho_2=\frac{1-R_{n+2}-\Delta_4}{2},\quad \Delta_4=\sqrt{\left(1-R_{n+2}\right)^2-4\left(R_{n+1}-2\right)\left(R_n+3\right)}. \quad (36)$$

Proof. The matrix $\mathbf{D}$ has the form

$$\mathbf{D}=\begin{pmatrix}R_n & R_{n-1} & \cdots & R_1\\ -R_1 & R_1+R_n & \cdots & R_2\\ \vdots & \vdots & \ddots & \vdots\\ -R_{n-2} & -R_{n-3}+R_{n-2} & \cdots & R_{n-1}\\ -R_{n-1} & -R_{n-2}+R_{n-1} & \cdots & R_1+R_n\end{pmatrix}. \quad (37)$$

According to Lemma 3, the Binet form (12), and (11), we have

$$\begin{aligned}\det\mathbf{D}&=\prod_{i=1}^{n}\left(R_n+R_{n-1}\kappa_i+\cdots+R_1\kappa_i^{n-1}\right)\\ &=\prod_{i=1}^{n}\left[\sum_{k=0}^{n-1}\sum_{j=1}^{3}r_j^{n-k}\kappa_i^k\right]\\ &=\prod_{i=1}^{n}\left[\sum_{j=1}^{3}\frac{r_j^{n+1}-r_j\kappa_i^n}{r_j-\kappa_i}\right]\\ &=\prod_{i=1}^{n}\frac{\left(R_{n+1}-2\right)\kappa_i^2+\left(R_{n+2}-1\right)\kappa_i+R_n+3}{-\kappa_i^3+\kappa_i+1}.\end{aligned} \quad (38)$$

Using Lemma 4 and the recurrence (8), we obtain

$$\prod_{i=1}^{n}\left[\left(R_{n+1}-2\right)\kappa_i^2+\left(R_{n+2}-1\right)\kappa_i+R_n+3\right]=\left(R_n+3\right)^n+R_{n+5}\left(R_{n+1}-2\right)^{n-1}-\left(R_n+3\right)\left(\rho_1^{n-1}+\rho_2^{n-1}\right)+\left(\rho_1^n+\rho_2^n\right), \quad (39)$$

where

$$\rho_1=\frac{1-R_{n+2}+\Delta_4}{2},\quad \rho_2=\frac{1-R_{n+2}-\Delta_4}{2},\quad \Delta_4=\sqrt{\left(1-R_{n+2}\right)^2-4\left(R_{n+1}-2\right)\left(R_n+3\right)}, \quad (40)$$

$$\prod_{i=1}^{n}\left(-\kappa_i^3+\kappa_i+1\right)=\frac{1}{2}\left(R_{n-1}^2+R_n^2-R_{2n}\right)-R_{2(n-1)}+R_{n+4}. \quad (41)$$

Therefore,

$$\det\mathbf{D}=2\left[\frac{\left(R_n+3\right)^n+R_{n+5}\left(R_{n+1}-2\right)^{n-1}}{R_{n-1}^2+R_n^2-R_{2n}-2R_{2(n-1)}+2R_{n+4}}+\frac{-\left(R_n+3\right)\left(\rho_1^{n-1}+\rho_2^{n-1}\right)+\left(\rho_1^n+\rho_2^n\right)}{R_{n-1}^2+R_n^2-R_{2n}-2R_{2(n-1)}+2R_{n+4}}\right]. \quad (42)$$

□

Theorem 7. *Let* $\mathbf{E} = RSLPFLcircfr\,(R_1, R_2, \ldots, R_n)$ *be a Perrin RSLPFL circulant matrix. Then*

$$\det \mathbf{E} = 2\left[\frac{(R_n+3)^n + R_{n+5}(R_{n+1}-2)^{n-1}}{R_{n-1}^2 + R_n^2 - R_{2n} - 2R_{2(n-1)} + 2R_{n+4}} + \frac{-(R_n+3)\left(\rho_1^{n-1}+\rho_2^{n-1}\right) + \left(\rho_1^n+\rho_2^n\right)}{2R_{n-1}^2 + R_n^2 - R_{2n} - 2R_{2(n-1)} + 2R_{n+4}}\right] \times (-1)^{n(n-1)/2}, \quad (43)$$

where

$$\rho_1 = \frac{1 - R_{n+2} + \Delta_4}{2}, \quad \rho_2 = \frac{1 - R_{n+2} - \Delta_4}{2}, \quad \Delta_4 = \sqrt{(1-R_{n+2})^2 - 4(R_{n+1}-2)(R_n+3)}. \quad (44)$$

Proof. Since

$$\det \widehat{\mathbf{I}}_n = (-1)^{n(n-1)/2}, \quad (45)$$

the result can be derived from Theorem 6 and (6). □

2.2. Determinants of the RSFPLR Circulant Matrices and RSLPFL Circulant Matrices Involving Padovan Numbers

Theorem 8. *Let* $\mathbf{F} = RSFPLRcircfr\,(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n)$ *be a Padovan RSFPLR circulant matrix. Then*

$$\det \mathbf{F} = \frac{\mathbb{P}_n^n \Delta_5 + (1+\mathbb{P}_{n+1})\Delta_6 + \Delta_7}{\Delta_8}, \quad (46)$$

where

$$\begin{aligned}
\Delta_5 &= U_n^2 - U_{2n} + 2U_{n+1} + 2U_n,\\
\Delta_6 &= \mathbb{P}_n^{n-1}\left(U_{n-1}^2 - 2U_{2(n-1)} + 2U_{n-1}\right),\\
\Delta_7 &= 2\mathbb{P}_n^{n-1}\left(\mathbb{P}_{n+2} - \mathbb{P}_n - 4\right) + 2(1+\mathbb{P}_{n+1})^n,\\
\Delta_8 &= Y_n^2 + Y_{n-1}^2 - Y_{2n} - 2Y_{2(n-1)} + 2Y_{n+1} + 2Y_n + 2Y_{n-1} + 6,\\
U_n &= \delta_1^n + \delta_2^n + \delta_3^n,\\
Y_n &= \beta_1^n + \beta_2^n + \beta_3^n.
\end{aligned} \quad (47)$$

$\delta_1, \delta_2, \delta_3$ *are the roots of the equation* $-\mathbb{P}_n x^3 + (1+\mathbb{P}_n-\mathbb{P}_{n+2})x^2 + (1-\mathbb{P}_{n+1}+\mathbb{P}_{n+2})x + 1 + \mathbb{P}_{n+1} = 0$ *and* $\beta_1, \beta_2, \beta_3$ *are the roots of the equation* $y^3 + y^2 - 1 = 0$.

Proof. The matrix $\mathbf{F}$ has the form

$$\mathbf{F} = \begin{pmatrix}
\mathbb{P}_1 & \mathbb{P}_2 & \cdots & \mathbb{P}_n\\
-\mathbb{P}_n & \mathbb{P}_1+\mathbb{P}_n & \cdots & \mathbb{P}_{n-1}\\
\vdots & \vdots & \ddots & \vdots\\
-\mathbb{P}_3 & -\mathbb{P}_4+\mathbb{P}_3 & \cdots & \mathbb{P}_2\\
-\mathbb{P}_2 & -\mathbb{P}_3+\mathbb{P}_2 & \cdots & \mathbb{P}_1+\mathbb{P}_n
\end{pmatrix},$$

$$\begin{aligned}
\det \mathbf{F} &= \prod_{i=1}^{n}\left(\mathbb{P}_1 + \mathbb{P}_2\kappa_i + \cdots + \mathbb{P}_n\kappa_i^{n-1}\right)\\
&= \prod_{i=1}^{n}\left[\sum_{k=1}^{n}\sum_{j=1}^{3} a_j r_j^k \kappa_i^{k-1}\right]\\
&= \prod_{i=1}^{n}\left[\sum_{j=1}^{3}\frac{a_j r_j\left(1 - r_j^n\kappa_i^n\right)}{1 - r_j\kappa_i}\right]\\
&= \prod_{i=1}^{n}\left[\frac{-\mathbb{P}_n\kappa_i^3 + (1+\mathbb{P}_n-\mathbb{P}_{n+2})\kappa_i^2}{-\kappa_i^3 - \kappa_i^2 + 1} + \frac{(1-\mathbb{P}_{n+1}+\mathbb{P}_{n+2})\kappa_i + 1 + \mathbb{P}_{n+1}}{-\kappa_i^3 - \kappa_i^2 + 1}\right],
\end{aligned} \quad (48)$$

from Lemma 3, the Binet form (13), and (11).

Using Lemma 4 and the recurrence (9), we obtain

$$\begin{aligned}
&\prod_{i=1}^{n}\left[-\mathbb{P}_n\kappa_i^3 + (1+\mathbb{P}_n-\mathbb{P}_{n+2})\kappa_i^2 + (1-\mathbb{P}_{n+1}+\mathbb{P}_{n+2})\kappa_i + 1 + \mathbb{P}_{n+1}\right]\\
&\quad= \frac{\mathbb{P}_n^n}{2}\left(U_n^2 - U_{2n} + 2U_{n+1} + 2U_n\right) + \frac{(1+\mathbb{P}_{n+1})\mathbb{P}_n^{n-1}}{2}\left(U_{n-1}^2 - 2U_{2(n-1)} + 2U_{n-1}\right)\\
&\quad\quad + (1+\mathbb{P}_{n+1})^n + \mathbb{P}_n^{n-1}\left(\mathbb{P}_{n+2} - \mathbb{P}_n - 4\right),
\end{aligned} \quad (49)$$

where $U_n = \delta_1^n + \delta_2^n + \delta_3^n$, $\delta_1, \delta_2, \delta_3$ are the roots of the equation $-\mathbb{P}_n x^3 + (1+\mathbb{P}_n-\mathbb{P}_{n+2})x^2 + (1-\mathbb{P}_{n+1}+\mathbb{P}_{n+2})x + 1 + \mathbb{P}_{n+1} = 0$. According to (32), we have the following results:

$$\det \mathbf{F} = \frac{\mathbb{P}_n^n \Delta_5 + (1+\mathbb{P}_{n+1})\Delta_6 + \Delta_7}{\Delta_8}, \quad (50)$$

where

$$\begin{aligned}
\Delta_5 &= U_n^2 - U_{2n} + 2U_{n+1} + 2U_n,\\
\Delta_6 &= \mathbb{P}_n^{n-1}\left(U_{n-1}^2 - 2U_{2(n-1)} + 2U_{n-1}\right),\\
\Delta_7 &= 2\mathbb{P}_n^{n-1}\left(\mathbb{P}_{n+2} - \mathbb{P}_n - 4\right) + 2(1+\mathbb{P}_{n+1})^n,\\
\Delta_8 &= Y_n^2 + Y_{n-1}^2 - Y_{2n} - 2Y_{2(n-1)} + 2Y_{n+1} + 2Y_n + 2Y_{n-1} + 6.
\end{aligned} \quad (51)$$

□

Theorem 9. *Let* $\mathbf{G} = RSFPLRcircfr\,(\mathbb{P}_n, \mathbb{P}_{n-1}, \ldots, \mathbb{P}_1)$ *be a Padovan RSFPLR circulant matrix. Then*

$$\det \mathbf{G} = \frac{2(\mathbb{P}_n + 1)^n + (\mathbb{P}_n + 1)\Delta_9 + \Delta_{10}}{2(R_{n+4} - R_{2(n-1)}) + R_{n-1}^2 + R_n^2 - R_{2n}}, \quad (52)$$

where

$$\begin{aligned} \Delta_9 &= V_{n-1}^2 - 2V_{2(n-1)} + 2V_{n-1}, \\ \Delta_{10} &= (2V_{n+1} + V_n^2 + 2V_n - V_{2n}) \\ &\quad - 2(1 + \mathbb{P}_{n+1} + \mathbb{P}_{n+5}), \\ V_n &= \zeta_1^n + \zeta_2^n + \zeta_3^n \end{aligned} \quad (53)$$

and $\zeta_1, \zeta_2, \zeta_3$ *are the roots of the equation* $-x^3 + \mathbb{P}_{n+1}x^2 + (\mathbb{P}_{n+2} + 1)x + \mathbb{P}_n + 1 = 0$.

Proof. The matrix $\mathbf{G}$ has the form

$$\mathbf{G} = \begin{pmatrix} \mathbb{P}_n & \mathbb{P}_{n-1} & \cdots & \mathbb{P}_1 \\ -\mathbb{P}_1 & \mathbb{P}_1 + \mathbb{P}_n & \cdots & \mathbb{P}_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbb{P}_{n-2} & -\mathbb{P}_{n-3} + \mathbb{P}_{n-2} & \cdots & \mathbb{P}_{n-1} \\ -\mathbb{P}_{n-1} & -\mathbb{P}_{n-2} + \mathbb{P}_{n-1} & \cdots & \mathbb{P}_1 + \mathbb{P}_n \end{pmatrix}. \quad (54)$$

According to Lemma 3, the Binet form (13), and (11), we have

$$\begin{aligned} \det \mathbf{G} &= \prod_{i=1}^{n} \left(\mathbb{P}_n + \mathbb{P}_{n-1}\kappa_i + \cdots + \mathbb{P}_1\kappa_i^{n-1}\right) \\ &= \prod_{i=1}^{n} \left[\sum_{k=0}^{n-1}\sum_{j=1}^{3} a_j r_j^{n-k}\kappa_i^k\right] \\ &= \prod_{i=1}^{n} \left[\sum_{j=1}^{3} \frac{a_j r_j^{n+1} - a_j r_j \kappa_i^n}{r_j - \kappa_i}\right] \\ &= \prod_{i=1}^{n} \frac{-\kappa_i^3 + \mathbb{P}_{n+1}\kappa_i^2 + (\mathbb{P}_{n+2} + 1)\kappa_i + \mathbb{P}_n + 1}{-\kappa_i^3 + \kappa_i + 1}. \end{aligned} \quad (55)$$

Using Lemma 4 and (13), we obtain

$$\begin{aligned} &\prod_{i=1}^{n} \left[-\kappa_i^3 + \mathbb{P}_{n+1}\kappa_i^2 + (\mathbb{P}_{n+2} + 1)\kappa_i + \mathbb{P}_n + 1\right] \\ &\quad = (\mathbb{P}_n + 1)^n + \frac{(\mathbb{P}_n + 1)}{2}\left(V_{n-1}^2 - 2V_{2(n-1)} + 2V_{n-1}\right) \\ &\qquad + \frac{1}{2}\left(2V_n + 2V_{n+1} + V_n^2 - V_{2n}\right) - (1 + \mathbb{P}_{n+1} + \mathbb{P}_{n+5}), \end{aligned} \quad (56)$$

where $V_n = \zeta_1^n + \zeta_2^n + \zeta_3^n$, $\zeta_1, \zeta_2, \zeta_3$ are the roots of the equation $-x^3 + \mathbb{P}_{n+1}x^2 + (\mathbb{P}_{n+2} + 1)x + \mathbb{P}_n + 1 = 0$. Employing (41), we have the following results:

$$\det \mathbf{G} = \frac{2(\mathbb{P}_n + 1)^n + (\mathbb{P}_n + 1)\Delta_9 + \Delta_{10}}{2(R_{n+4} - R_{2(n-1)}) + R_{n-1}^2 + R_n^2 - R_{2n}}, \quad (57)$$

where

$$\begin{aligned} \Delta_9 &= V_{n-1}^2 - 2V_{2(n-1)} + 2V_{n-1}, \\ \Delta_{10} &= (2V_{n+1} + V_n^2 + 2V_n - V_{2n}) \\ &\quad - 2(1 + \mathbb{P}_{n+1} + \mathbb{P}_{n+5}), \\ V_n &= \zeta_1^n + \zeta_2^n + \zeta_3^n \end{aligned} \quad (58)$$

and $\zeta_1, \zeta_2, \zeta_3$ are the roots of the equation $-x^3 + \mathbb{P}_{n+1}x^2 + (\mathbb{P}_{n+2} + 1)x + \mathbb{P}_n + 1 = 0$. □

Theorem 10. *Let* $\mathbf{H} = RSLPFLcircfr\,(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n)$ *be a Padovan RSLPFL circulant matrix. Then*

$$\begin{aligned} \det \mathbf{H} &= \frac{2(\mathbb{P}_n + 1)^n + (\mathbb{P}_n + 1)\Delta_9 + \Delta_{10}}{2(R_{n+4} - R_{2(n-1)}) + R_{n-1}^2 + R_n^2 - R_{2n}} \\ &\quad \times (-1)^{n(n-1)/2}, \end{aligned} \quad (59)$$

where

$$\begin{aligned} \Delta_9 &= V_{n-1}^2 - 2V_{2(n-1)} + 2V_{n-1}, \\ \Delta_{10} &= (2V_{n+1} + V_n^2 + 2V_n - V_{2n}) \\ &\quad - 2(1 + \mathbb{P}_{n+1} + \mathbb{P}_{n+5}) \\ V_n &= \zeta_1^n + \zeta_2^n + \zeta_3^n, \end{aligned} \quad (60)$$

and $\zeta_1, \zeta_2, \zeta_3$ *are the roots of the equation* $-x^3 + \mathbb{P}_{n+1}x^2 + (\mathbb{P}_{n+2} + 1)x + \mathbb{P}_n + 1 = 0$.

Proof. The theorem can be proved by using Theorem 9 and (6). □

2.3. Determinants of RSFPLR Circulant Matrices and RSLPFL Circulant Matrices Involving Tribonacci Numbers

Theorem 11. *Let* $\mathbf{J} = RSFPLRcircfr\,(T_1, T_2, \ldots, T_n)$ *be a Tribonacci RSFPLR circulant matrix. Then*

$$\det \mathbf{J} = \frac{T_n^n \Delta_{11} + 2T_n^{n-1}(T_{n+2} - T_n - 2) + \Delta_{12}}{\Delta_{13}}, \quad (61)$$

where

$$\begin{aligned} \Delta_{11} &= W_n^2 - W_{2n} + 2W_{n+1} + 2W_n \\ \Delta_{12} &= T_n^{n-1}(1 + T_{n+1})\left(W_{n-1}^2 - 2W_{2(n-1)} + 2W_{n-1}\right) \\ &\quad + 2(1 + T_{n+1})^n \\ \Delta_{13} &= 2E_{n-1} + E_{n-1}^2 - 2E_{2(n-1)} + E_n^2 - E_{2n} \\ &\quad + 2E_{n+1} + 2E_n + 8, \\ W_n &= \eta_1^n + \eta_2^n + \eta_3^n, \\ E_n &= \xi_1^n + \xi_2^n + \xi_3^n \end{aligned} \quad (62)$$

and η_1, η_2, η_3 are the roots of the equation $-T_n x^3 - (T_{n+2} - T_n)x^2 + (1 - T_{n+1} + T_{n+2})x + 1 + T_{n+1} = 0$ and ξ_1, ξ_2, ξ_3 are the roots of the equation $y^3 + y^2 + y - 1 = 0$.

Proof. Obviously, $\mathbf{J}$ has the form

$$\mathbf{J} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ -T_n & T_1 + T_n & \cdots & T_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -T_3 & -T_4 + T_3 & \cdots & T_2 \\ -T_2 & -T_3 + T_2 & \cdots & T_1 + T_n \end{pmatrix}. \quad (63)$$

According to Lemma 3, the Binet form (16), and (15), we have

$$\begin{aligned} \det \mathbf{J} &= \prod_{i=1}^{n} \left(T_1 + T_2 \kappa_i + \cdots + T_n \kappa_i^{n-1} \right) \\ &= \prod_{i=1}^{n} \left[\sum_{k=1}^{n} \sum_{j=1}^{3} b_j t_j^k \kappa_i^{k-1} \right] \\ &= \prod_{i=1}^{n} \left[\sum_{i=1}^{3} \frac{b_j t_j \left(1 - t_j^n \kappa_i^n\right)}{1 - t_j \kappa_i} \right] \\ &= \prod_{i=1}^{n} \left[\frac{-T_n \kappa_i^3 - \left(-T_n + T_{n+2}\right) \kappa_i^2}{-\kappa_i^3 - \kappa_i^2 - \kappa_i + 1} \right. \\ &\qquad \left. + \frac{\left(1 - T_{n+1} + T_{n+2}\right) \kappa_i + 1 + T_{n+1}}{-\kappa_i^3 - \kappa_i^2 - \kappa_i + 1} \right]. \end{aligned} \quad (64)$$

From Lemma 4 it follows that

$$\begin{aligned} &\prod_{i=1}^{n} \left[-T_n \kappa_i^3 - \left(-T_n + T_{n+2}\right) \kappa_i^2 \right. \\ &\qquad \left. + \left(1 - T_{n+1} + T_{n+2}\right) \kappa_i + 1 + T_{n+1} \right] \\ &= \frac{1}{2} T_n^n \left(W_n^2 - W_{2n} + 2W_{n+1} + 2W_n \right) \\ &\quad + T_n^{n-1} \left(T_{n+2} - T_n - 2 \right) + \left(1 + T_{n+1}\right)^n \\ &\quad + \frac{1}{2} T_n^{n-1} \left(1 + T_{n+1}\right) \left(W_{n-1}^2 - 2W_{2(n-1)} + 2W_{n-1} \right), \end{aligned} \quad (65)$$

where $W_n = \eta_1^n + \eta_2^n + \eta_3^n$, η_1, η_2, η_3 are the roots of the equation $-T_n x^3 - (-T_n + T_{n+2})x^2 + (1 - T_{n+1} + T_{n+2})x + 1 + T_{n+1} = 0$ and

$$\begin{aligned} &\prod_{i=1}^{n} \left(-\kappa_i^3 - \kappa_i^2 - \kappa_i + 1 \right) \\ &= \frac{1}{2} \left[2E_{n-1} + E_{n-1}^2 - 2E_{2(n-1)} \right. \\ &\qquad \left. + \left(E_n^2 - E_{2n} + 2E_{n+1} + 2E_n + 2 \right) \right] + 3, \end{aligned} \quad (66)$$

where $E_n = \xi_1^n + \xi_2^n + \xi_3^n$, ξ_1, ξ_2, ξ_3 are the roots of the equation $y^3 + y^2 + y - 1 = 0$. Consequently, we have the following results:

$$\det \mathbf{J} = \frac{T_n^n \Delta_{11} + 2T_n^{n-1} \left(T_{n+2} - T_n - 2 \right) + \Delta_{12}}{\Delta_{13}}, \quad (67)$$

where

$$\begin{aligned} \Delta_{11} &= W_n^2 - W_{2n} + 2W_{n+1} + 2W_n \\ \Delta_{12} &= T_n^{n-1} \left(1 + T_{n+1}\right) \left(W_{n-1}^2 - 2W_{2(n-1)} + 2W_{n-1} \right) \\ &\quad + 2\left(1 + T_{n+1}\right)^n \\ \Delta_{13} &= 2E_{n-1} + E_{n-1}^2 - 2E_{2(n-1)} + E_n^2 - E_{2n} \\ &\quad + 2E_{n+1} + 2E_n + 8. \end{aligned} \quad (68)$$

□

Theorem 12. *Let $\mathbf{K} = RSFPLRcircfr\ (T_n, T_{n-1}, \ldots, T_1)$ be a Tribonacci RSFPLR circulant matrix. Then*

$$\det \mathbf{K} = \frac{2T_n^n + T_n \Delta_{14} + \Delta_{15}}{\mathbb{L}_n^2 + \mathbb{L}_{n-1}^2 + \mathbb{L}_{2n} - 2\mathbb{L}_{2(n-1)} + 2\mathbb{L}_{n+2} - 4}, \quad (69)$$

where

$$\begin{aligned} \Delta_{14} &= G_{n-1}^2 - 2G_{2(n-1)} + 2G_{n-1} - 2, \\ \Delta_{15} &= -4T_{n+1} - T_{n+2} + 2G_n - 2G_{n+1} \\ &\quad + G_n^2 - G_{2n}, \\ G_n &= v_1^n + v_2^n + v_3^n \end{aligned} \quad (70)$$

and v_1, v_2, v_3 are the roots of the equation $-x^3 + T_{n+1} x^2 + (T_{n+2} + 1)x + T_n = 0$.

Proof. The matrix $\mathbf{K}$ has the form

$$\mathbf{K} = \begin{pmatrix} T_n & T_{n-1} & \cdots & T_1 \\ -T_1 & T_1 + T_n & \cdots & T_2 \\ \vdots & \vdots & \ddots & \vdots \\ -T_{n-2} & -T_{n-3} + T_{n-2} & \cdots & T_{n-1} \\ -T_{n-1} & -T_{n-2} + T_{n-1} & \cdots & T_1 + T_n \end{pmatrix}. \quad (71)$$

According to Lemma 3, the Binet form (16), and (15), we have

$$\begin{aligned} \det \mathbf{K} &= \prod_{i=1}^{n} \left(T_n + T_{n-1} \kappa_i + \cdots + T_1 \kappa_i^{n-1} \right) \\ &= \prod_{i=1}^{n} \left[\sum_{k=0}^{n-1} \sum_{j=1}^{3} b_j t_j^{n-k} \kappa_i^k \right] \\ &= \prod_{i=1}^{n} \left[\sum_{j=1}^{3} \frac{b_j t_j^{n+1} - b_j t_j \kappa_i^n}{t_j - \kappa_i} \right] \\ &= \prod_{i=1}^{n} \frac{-\kappa_i^3 + T_{n+1} \kappa_i^2 + \left(T_{n+2} + 1\right) \kappa_i + T_n}{-\kappa_i^3 + \kappa_i^2 + \kappa_i + 1}. \end{aligned} \quad (72)$$

Considering Lemma 4 and (17), we obtain

$$\prod_{i=1}^{n}\left[-\kappa_i^3+T_{n+1}\kappa_i^2+\left(T_{n+2}+1\right)\kappa_i+T_n\right]$$
$$=T_n^n+\frac{T_n}{2}\left(G_{n-1}^2-2G_{2(n-1)}+2G_{n-1}-2\right)+\frac{1}{2}\left(-4T_{n+1}-T_{n+2}+2G_n-2G_{n+1}+G_n^2-G_{2n}\right), \quad (73)$$

where $G_n=v_1^n+v_2^n+v_3^n$, v_1, v_2, v_3 are the roots of the equation $-x^3+T_{n+1}x^2+(T_{n+2}+1)x+T_n=0$ and

$$\prod_{i=1}^{n}\left(-\kappa_i^3+\kappa_i^2+\kappa_i+1\right)$$
$$=\frac{1}{2}\left[\left(2\mathbb{L}_n+2\mathbb{L}_{n+1}+\mathbb{L}_n^2+\mathbb{L}_{2n}-4\right)+\mathbb{L}_{n-1}^2-2\mathbb{L}_{2(n-1)}+2\mathbb{L}_{n-1}\right]$$
$$=\frac{1}{2}\left[\mathbb{L}_n^2+\mathbb{L}_{2n}+\mathbb{L}_{n-1}^2-2\mathbb{L}_{2(n-1)}+2\mathbb{L}_{n+2}-4\right]. \quad (74)$$

Consequently,

$$\det\mathbf{K}=\frac{2T_n^n+T_n\Delta_{14}+\Delta_{15}}{\mathbb{L}_n^2+\mathbb{L}_{n-1}^2+\mathbb{L}_{2n}-2\mathbb{L}_{2(n-1)}+2\mathbb{L}_{n+2}-4}, \quad (75)$$

where

$$\Delta_{14}=G_{n-1}^2-2G_{2(n-1)}+2G_{n-1}-2,$$
$$\Delta_{15}=-4T_{n+1}-T_{n+2}+2G_n-2G_{n+1}+G_n^2-G_{2n}, \quad (76)$$
$$G_n=v_1^n+v_2^n+v_3^n$$

and v_1, v_2, v_3 are the roots of the equation $-x^3+T_{n+1}x^2+(T_{n+2}+1)x+T_n=0$. □

Theorem 13. *Let* $\mathbf{L}$ = *RSLPFLcircfr* $(T_1,T_2,\ldots,T_n)$ *be a Tribonacci RSLPFL circulant matrix. Then*

$$\det\mathbf{L}=\frac{2T_n^n+T_n\Delta_{14}+\Delta_{15}}{\mathbb{L}_n^2+\mathbb{L}_{n-1}^2+\mathbb{L}_{2n}-2\mathbb{L}_{2(n-1)}+2\mathbb{L}_{n+2}-4}\times(-1)^{n(n-1)/2}, \quad (77)$$

where

$$\Delta_{14}=G_{n-1}^2-2G_{2(n-1)}+2G_{n-1}-2,$$
$$\Delta_{15}=-4T_{n+1}-T_{n+2}+2G_n-2G_{n+1}+G_n^2-G_{2n}, \quad (78)$$
$$G_n=v_1^n+v_2^n+v_3^n$$

and v_1, v_2, v_3 *are the roots of the equation* $-x^3+T_{n+1}x^2+(T_{n+2}+1)x+T_n=0$.

Proof. The theorem can be proved by using Theorem 12 and (6). □

2.4. Determinants of the RSFPLR Circulant Matrices and RSLPFL Circulant Matrices Involving Generalized Lucas Numbers

Theorem 14. *Let* $\mathbf{M}$ = *RSFPLRcircfr* $(\mathbb{L}_1,\mathbb{L}_2,\ldots,\mathbb{L}_n)$ *be a generalized Lucas RSFPLR circulant matrix. Then*

$$\det\mathbf{M}=\frac{\mathbb{L}_n^n\Delta_{16}+2\mathbb{L}_n^{n-1}\left(\mathbb{L}_{n+2}-\mathbb{L}_n-11\right)+\Delta_{17}}{\Delta_{18}}, \quad (79)$$

where

$$\Delta_{16}=H_n^2-H_{2n}+2H_{n+1}+2H_n,$$
$$\Delta_{17}=\mathbb{L}_n^{n-1}\left(3+\mathbb{L}_{n+1}\right)\left(2H_{n-1}+H_{n-1}^2-2H_{2(n-1)}\right)+2\left(3+\mathbb{L}_{n+1}\right)^n,$$
$$\Delta_{18}=2E_n+2E_{n+1}+E_n^2-E_{2n}+E_{n-1}^2-2E_{2(n-1)}+2E_{n-1}+8, \quad (80)$$
$$H_n=\mu_1^n+\mu_2^n+\mu_3^n,$$
$$E_n=\xi_1^n+\xi_2^n+\xi_3^n,$$

μ_1, μ_2, μ_3 *are the roots of the equation* $-\mathbb{L}_nx^3+(3-\mathbb{L}_{n+2}+\mathbb{L}_n)x^2+(2-\mathbb{L}_{n+1}+\mathbb{L}_{n+2})x+3+\mathbb{L}_{n+1}=0$, *and* ξ_1, ξ_2, ξ_3 *are the roots of the equation* $y^3+y^2+y-1=0$.

Proof. The matrix $\mathbf{M}$ has the form

$$\mathbf{M}=\begin{pmatrix}\mathbb{L}_1 & \mathbb{L}_2 & \cdots & \mathbb{L}_n\\ -\mathbb{L}_n & \mathbb{L}_1+\mathbb{L}_n & \cdots & \mathbb{L}_{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ -\mathbb{L}_3 & -\mathbb{L}_4+\mathbb{L}_3 & \cdots & \mathbb{L}_2\\ -\mathbb{L}_2 & -\mathbb{L}_3+\mathbb{L}_2 & \cdots & \mathbb{L}_1+\mathbb{L}_n\end{pmatrix}. \quad (81)$$

According to Lemma 3, the Binet form (17), and (15), we have

$$\det\mathbf{M}=\prod_{i=1}^{n}\left(\mathbb{L}_1+\mathbb{L}_2\kappa_i+\cdots+\mathbb{L}_n\kappa_i^{n-1}\right)$$
$$=\prod_{i=1}^{n}\left[\sum_{k=1}^{n}\sum_{j=1}^{3}t_j^k\kappa_i^{k-1}\right]$$
$$=\prod_{i=1}^{n}\left[\sum_{j=1}^{3}\frac{t_j\left(1-t_j^n\kappa_i^n\right)}{1-t_j\kappa_i}\right]$$
$$=\prod_{i=1}^{n}\left[\frac{-\mathbb{L}_n\kappa_i^3+\left(3-\mathbb{L}_{n+2}+\mathbb{L}_n\right)\kappa_i^2}{-\kappa_i^3-\kappa_i^2-\kappa_i+1}+\frac{\left(2-\mathbb{L}_{n+1}+\mathbb{L}_{n+2}\right)\kappa_i+3+\mathbb{L}_{n+1}}{-\kappa_i^3-\kappa_i^2-\kappa_i+1}\right]. \quad (82)$$

From Lemma 4 and (17), we obtain

$$\begin{aligned}\prod_{i=1}^{n}\Big[&-\mathbb{L}_n\kappa_i^3+\left(3-\mathbb{L}_{n+2}+\mathbb{L}_n\right)\kappa_i^2\\&+\left(2-\mathbb{L}_{n+1}+\mathbb{L}_{n+2}\right)\kappa_i+3+\mathbb{L}_{n+1}\Big]\\=&\frac{1}{2}\mathbb{L}_n^n\left(H_n^2-H_{2n}+2H_{n+1}+2H_n\right)\\&+\mathbb{L}_n^{n-1}\left(\mathbb{L}_{n+2}-\mathbb{L}_n-11\right)\\&+\frac{1}{2}\mathbb{L}_n^{n-1}\left(3+\mathbb{L}_{n+1}\right)\left(H_{n-1}^2-2H_{2(n-1)}+2H_{n-1}\right)\\&+\left(3+\mathbb{L}_{n+1}\right)^n,\end{aligned}\tag{83}$$

where $H_n=\mu_1^n+\mu_2^n+\mu_3^n$, μ_1,μ_2,μ_3 are the roots of the equation $-\mathbb{L}_nx^3+(3-\mathbb{L}_{n+2}+\mathbb{L}_n)x^2+(2-\mathbb{L}_{n+1}+\mathbb{L}_{n+2})x+3+\mathbb{L}_{n+1}=0$ and

$$\begin{aligned}\prod_{i=1}^{n}&\left(-\kappa_i^3-\kappa_i^2-\kappa_i+1\right)\\&=\frac{1}{2}\Big[2E_n+2E_{n+1}+E_n^2-E_{2n}\\&\qquad+E_{n-1}^2-2E_{2(n-1)}+2E_{n-1}+8\Big],\end{aligned}\tag{84}$$

where $E_n=\xi_1^n+\xi_2^n+\xi_3^n$, ξ_1,ξ_2,ξ_3 are the roots of the equation $y^3+y^2+y-1=0$. Hence,

$$\det\mathbf{M}=\frac{\mathbb{L}_n^n\Delta_{16}+2\mathbb{L}_n^{n-1}\left(\mathbb{L}_{n+2}-\mathbb{L}_n-11\right)+\Delta_{17}}{\Delta_{18}},\tag{85}$$

where

$$\begin{aligned}\Delta_{16}&=H_n^2-H_{2n}+2H_{n+1}+2H_n,\\\Delta_{17}&=\mathbb{L}_n^{n-1}\left(3+\mathbb{L}_{n+1}\right)\left(2H_{n-1}+H_{n-1}^2-2H_{2(n-1)}\right)\\&\quad+2\left(3+\mathbb{L}_{n+1}\right)^n,\\\Delta_{18}&=2E_n+2E_{n+1}+E_n^2-E_{2n}+E_{n-1}^2-2E_{2(n-1)}\\&\quad+2E_{n-1}+8,\\H_n&=\mu_1^n+\mu_2^n+\mu_3^n,\\E_n&=\xi_1^n+\xi_2^n+\xi_3^n,\end{aligned}\tag{86}$$

μ_1,μ_2,μ_3 are the roots of the equation $-\mathbb{L}_nx^3+(3-\mathbb{L}_{n+2}+\mathbb{L}_n)x^2+(2-\mathbb{L}_{n+1}+\mathbb{L}_{n+2})x+3+\mathbb{L}_{n+1}=0$, and ξ_1,ξ_2,ξ_3 are the roots of the equation $y^3+y^2+y-1=0$. □

Theorem 15. *Let* $\mathbf{N}=RSFPLRcircfr\left(\mathbb{L}_n,\mathbb{L}_{n-1},\dots,\mathbb{L}_1\right)$ *be a generalized Lucas RSFPLR circulant matrix. Then*

$$\det\mathbf{N}=\frac{\Delta_{19}+\left(\mathbb{L}_n+3\right)\Delta_{20}+2\left(\mathbb{L}_n+3\right)^n-2\mathbb{L}_n}{\mathbb{L}_{n-1}^2-2\mathbb{L}_{2(n-1)}+\mathbb{L}_n^2+\mathbb{L}_{2n}+2\mathbb{L}_{n+2}-4},\tag{87}$$

where

$$\begin{aligned}\Delta_{19}&=K_n^2-K_{2n}+2K_{n+1}+2K_n-4\mathbb{L}_{n+1}\\&\quad-2\mathbb{L}_{n+2}+2,\\\Delta_{20}&=K_{n-1}^2-2K_{2(n-1)}+2K_{n-1},\end{aligned}\tag{88}$$

$K_n=\nu_1^n+\nu_2^n+\nu_3^n$*,* ν_1,ν_2,ν_3 *are the roots of the equation* $-x^3+(\mathbb{L}_{n+1}-1)x^2+(\mathbb{L}_{n+2}-1)x+\mathbb{L}_n+3=0$*.*

Proof. The matrix $\mathbf{N}$ has the form

$$\mathbf{N}=\begin{pmatrix}\mathbb{L}_n&\mathbb{L}_{n-1}&\cdots&\mathbb{L}_1\\-\mathbb{L}_1&\mathbb{L}_1+\mathbb{L}_n&\cdots&\mathbb{L}_2\\\vdots&\vdots&\ddots&\vdots\\-\mathbb{L}_{n-2}&-\mathbb{L}_{n-3}+\mathbb{L}_{n-2}&\cdots&\mathbb{L}_{n-1}\\-\mathbb{L}_{n-1}&-\mathbb{L}_{n-2}+\mathbb{L}_{n-1}&\cdots&\mathbb{L}_1+\mathbb{L}_n\end{pmatrix}.\tag{89}$$

According to Lemma 3, (17), and (15), we have

$$\begin{aligned}\det\mathbf{N}&=\prod_{i=1}^{n}\left(\mathbb{L}_n+\mathbb{L}_{n-1}\kappa_i+\cdots+\mathbb{L}_1\kappa_i^{n-1}\right)\\&=\prod_{i=1}^{n}\left[\sum_{k=0}^{n-1}\sum_{j=1}^{3}t_j^{n-k}\kappa_i^k\right]\\&=\prod_{i=1}^{n}\left[\sum_{j=1}^{3}\frac{t_j^{n+1}-t_j\kappa_i^n}{t_j-\kappa_i}\right]\\&=\prod_{i=1}^{n}\left[\frac{-\kappa_i^3+\left(\mathbb{L}_{n+1}-1\right)\kappa_i^2}{-\kappa_i^3+\kappa_i^2+\kappa_i+1}\right.\\&\qquad\left.+\frac{\left(\mathbb{L}_{n+2}-1\right)\kappa_i+\mathbb{L}_n+3}{-\kappa_i^3+\kappa_i^2+\kappa_i+1}\right].\end{aligned}\tag{90}$$

By Lemma 4 and the Binet form (17), we obtain

$$\begin{aligned}\prod_{i=1}^{n}&\left[-\kappa_i^3+\left(\mathbb{L}_{n+1}-1\right)\kappa_i^2+\left(\mathbb{L}_{n+2}-1\right)\kappa_i+\mathbb{L}_n+3\right]\\&=\frac{1}{2}\Big(K_n^2-K_{2n}+2K_{n+1}+2K_n\\&\qquad-2\mathbb{L}_{n+1}-2\mathbb{L}_{n+1}-2\mathbb{L}_{n+2}+2\Big)\\&\quad+\frac{1}{2}\left(\mathbb{L}_n+3\right)\left(K_{n-1}^2-2K_{2(n-1)}+2K_{n-1}\right)\\&\quad+\left(\mathbb{L}_n+3\right)^n-\mathbb{L}_n,\end{aligned}\tag{91}$$

where $K_n = \nu_1^n + \nu_2^n + \nu_3^n$, ν_1, ν_2, ν_3 are the roots of the equation $-x^3 + (\mathbb{L}_{n+1} - 1)x^2 + (\mathbb{L}_{n+2} - 1)x + \mathbb{L}_n + 3 = 0$ and

$$\begin{aligned}\prod_{i=1}^{n} \left(-\kappa_i^3 + \kappa_i^2 + \kappa_i + 1\right) &= \frac{1}{2}\left(2\mathbb{L}_{n-1} + \mathbb{L}_{n-1}^2 - 2\mathbb{L}_{2(n-1)}\right) \\ &\quad + \frac{1}{2}\left(2\mathbb{L}_n + \mathbb{L}_n^2 + \mathbb{L}_{2n} + 2\mathbb{L}_{n+1} - 4\right) \\ &= \frac{1}{2}\left[\mathbb{L}_{n-1}^2 - 2\mathbb{L}_{2(n-1)} + \mathbb{L}_n^2 + \mathbb{L}_{2n} + 2\mathbb{L}_{n+2} - 4\right].\end{aligned} \tag{92}$$

Thus,

$$\det \mathbf{N} = \frac{\Delta_{19} + (\mathbb{L}_n + 3)\Delta_{20} + 2(\mathbb{L}_n + 3)^n - 2\mathbb{L}_n}{\mathbb{L}_{n-1}^2 - 2\mathbb{L}_{2(n-1)} + \mathbb{L}_n^2 + \mathbb{L}_{2n} + 2\mathbb{L}_{n+2} - 4}, \tag{93}$$

where

$$\begin{aligned}\Delta_{19} &= K_n^2 - K_{2n} + 2K_{n+1} + 2K_n - 4\mathbb{L}_{n+1} \\ &\quad - 2\mathbb{L}_{n+2} + 2, \\ \Delta_{20} &= K_{n-1}^2 - 2K_{2(n-1)} + 2K_{n-1}, \\ K_n &= \nu_1^n + \nu_2^n + \nu_3^n\end{aligned} \tag{94}$$

and ν_1, ν_2, ν_3 are the roots of the equation $-x^3 + (\mathbb{L}_{n+1} - 1)x^2 + (\mathbb{L}_{n+2} - 1)x + \mathbb{L}_n + 3 = 0$. □

Theorem 16. *Let* $\mathbf{P} = RSLPFLcircfr(\mathbb{L}_1, \mathbb{L}_2, \ldots, \mathbb{L}_n)$ *be a generalized Lucas RSLPFL circulant matrix. Then*

$$\begin{aligned}\det \mathbf{P} &= \frac{\Delta_{19} + (\mathbb{L}_n + 3)\Delta_{20} + 2(\mathbb{L}_n + 3)^n - 2\mathbb{L}_n}{\mathbb{L}_{n-1}^2 - 2\mathbb{L}_{2(n-1)} + \mathbb{L}_n^2 + \mathbb{L}_{2n} + 2\mathbb{L}_{n+2} - 4} \\ &\quad \times (-1)^{n(n-1)/2},\end{aligned} \tag{95}$$

where

$$\begin{aligned}\Delta_{19} &= K_n^2 - K_{2n} + 2K_{n+1} + 2K_n - 4\mathbb{L}_{n+1} \\ &\quad - 2\mathbb{L}_{n+2} + 2, \\ \Delta_{20} &= K_{n-1}^2 - 2K_{2(n-1)} + 2K_{n-1}, \\ K_n &= \nu_1^n + \nu_2^n + \nu_3^n\end{aligned} \tag{96}$$

and ν_1, ν_2, ν_3 *are the roots of the equation* $-x^3 + (\mathbb{L}_{n+1} - 1)x^2 + (\mathbb{L}_{n+2} - 1)x + \mathbb{L}_n + 3 = 0$.

Proof. The theorem can be proved by using Theorem 15 and (6). □

3. Conclusion

The determinant problems of the RSFPLR circulant matrices and RSLPFL circulant matrices involving the Perrin, Padovan, Tribonacci, and the generalized Lucas number are considered in this paper. The explicit determinants are presented by using some terms of these numbers. The techniques used herein are based on the inverse factorization of polynomial. It is important to develop new theories and methods and to modify and refine the well-known techniques, for solving differential equations. On the basis of existing application situation [1–11], we will exploit solving some differential equations based on the RSFPLR circulant matrices and RSLPFL circulant matrices.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. L. Lei and H. W. Sun, "A circulant preconditioner for fractional diffusion equations," *Journal of Computational Physics*, vol. 242, pp. 715–725, 2013.

[2] H. K. Pang, Y. Y. Zhang, S. W. Vong, and X. Q. Jin, "Circulant preconditioners for pricing options," *Linear Algebra and Its Applications*, vol. 434, no. 11, pp. 2325–2342, 2011.

[3] C. Zhang, H. Chen, and L. Wang, "Strang-type preconditioners applied to ordinary and neutral differential-algebraic equations," *Numerical Linear Algebra with Applications*, vol. 18, no. 5, pp. 843–855, 2011.

[4] J. Delgado, N. Romero, A. Rovella, and F. Vilamajó, "Bounded solutions of quadratic circulant difference equations," *Journal of Difference Equations and Applications*, vol. 11, no. 10, pp. 897–907, 2005.

[5] X. Q. Jin, S. L. Lei, and Y. M. Wei, "Circulant preconditioners for solving differential equations with multidelays," *Computers & Mathematics with Applications*, vol. 47, no. 8-9, pp. 1429–1436, 2004.

[6] Z. J. Bai, X. Q. Jin, and L. L. Song, "Strang-type preconditioners for solving linear systems from neutral delay differential equations," *Calcolo*, vol. 40, no. 1, pp. 21–31, 2003.

[7] R. H. Chan, X. Q. Jin, and Y. H. Tam, "Strang-type preconditioners for solving systems of ODEs by boundary value methods," *Electronic Journal of Mathematical and Physical Sciences*, vol. 1, no. 1, pp. 14–46, 2002.

[8] R. D. Nussbaum, "Circulant matrices and differential-delay equations," *Journal of Differential Equations*, vol. 60, no. 2, pp. 201–217, 1985.

[9] K. Otto, "Analysis of preconditioners for hyperbolic partial differential equations," *SIAM Journal on Numerical Analysis*, vol. 33, no. 6, pp. 2131–2165, 1996.

[10] A. E. Gilmour, "Circulant matrix methods for the numerical solution of partial differential equations by FFT convolutions," *Applied Mathematical Modelling*, vol. 12, no. 1, pp. 44–50, 1988.

[11] R. W. Brockett and J. L. Willems, "Discretized partial differential equations: examples of control systems defined on modules," *Automatica*, vol. 10, pp. 507–515, 1974.

[12] P. Davis, *Circulant Matrices*, John Wiley & Sons, New York, NY, USA, 1979.

[13] Z. L. Jiang and Z. X. Zhou, *Circulant Matrices*, Chengdu Technology University Publishing Company, Chengdu, China, 1999.

[14] D. Chillag, "Regular representations of semisimple algebras, separable field extensions, group characters, generalized circulants, and generalized cyclic codes," *Linear Algebra and Its Applications*, vol. 218, pp. 147–183, 1995.

[15] Z. L. Jiang and Z. B. Xu, "Efficient algorithm for finding the inverse and the group inverse of FLS r-circulant matrix," *Journal of Applied Mathematics & Computing*, vol. 18, no. 1-2, pp. 45–57, 2005.

[16] Z. L. Jiang and D. H. Sun, "Fast algorithms for solving the inverse problem of $Ax = b$," in *Proceedings of the 8th International Conference on Matrix Theory and Its Applications in China*, pp. 121–124, 2008.

[17] Z. L. Jiang, "Fast algorithms for solving FLS r-circulant linear systems," in *Proceedings of the Spring World Congress on Engineering and Technology (SCET '12)*, pp. 141–144, Xi'an, China, May 2012.

[18] Z. L. Jiang and Z. W. Jiang, "On the norms of RFPLR circulant matrices with the Fibonacci and Lucas numbers," in *Proceedings of the Spring World Congress on Engineering and Technology (SCET '12)*, pp. 385–388, Xi'an, China, May 2012.

[19] J. Li, Z. Jiang, and N. Shen, "Explicit determinants of the Fibonacci RFPLR circulant and Lucas RFPLR circulant matrix," *JP Journal of Algebra, Number Theory and Applications*, vol. 28, no. 2, pp. 167–179, 2013.

[20] Z. L. Jiang, J. Li, and N. Shen, "On the explicit determinants of the RFPLR and RFPLL circulant matrices involving Pell numbers in information theory," in *Information Computing and Applications*, vol. 308 of *Communications in Computer and Information Science*, pp. 364–370, Springer, Berlin, Germany, 2012.

[21] Z. P. Tian, "Fast algorithm for solving the first-plus last-circulant linear system," *Journal of Shandong University: Natural Science*, vol. 46, no. 12, pp. 96–103, 2011.

[22] N. Shen, Z. L. Jiang, and J. Li, "On explicit determinants of the RFMLR and RLMFL circulant matrices involving certain famous numbers," *WSEAS Transactions on Mathematics*, vol. 12, no. 1, pp. 42–53, 2013.

[23] Z. L. Jiang, N. Shen, and J. Li, "On the explicit determinants of the RFMLR and RLMFL circulant matrices involving Jacobsthal numbers in communication," in *Proceedings of the 4th International Conference on Information Computing and Applications (ICICA '13)*, Communications in Computer and Information Science, August 2013.

[24] Z. P. Tian, "Fast algorithms for solving the inverse problem of $AX = b$ in four different families of patterned matrices," *Far East Journal of Applied Mathematics*, vol. 52, no. 1, pp. 1–12, 2011.

[25] D. V. Jaiswal, "On determinants involving generalized Fibonacci numbers," *The Fibonacci Quarterly*, vol. 7, pp. 319–330, 1969.

[26] D. Lin, "Fibonacci-Lucas quasi-cyclic matrices," *The Fibonacci Quarterly*, vol. 40, no. 3, pp. 280–286, 2002.

[27] D. A. Lind, "A Fibonacci circulant," *The Fibonacci Quarterly*, vol. 8, no. 5, pp. 449–455, 1970.

[28] S. Q. Shen, J. M. Cen, and Y. Hao, "On the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers," *Applied Mathematics and Computation*, vol. 217, no. 23, pp. 9790–9797, 2011.

[29] M. Akbulak and D. Bozkurt, "On the norms of Toeplitz matrices involving Fibonacci and Lucas numbers," *Hacettepe Journal of Mathematics and Statistics*, vol. 37, no. 2, pp. 89–95, 2008.

[30] G. Y. Lee, J. S. Kim, and S. G. Lee, "Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices," *The Fibonacci Quarterly*, vol. 40, no. 3, pp. 203–211, 2002.

[31] M. Miladinović and P. Stanimirović, "Singular case of generalized Fibonacci and Lucas matrices," *Journal of the Korean Mathematical Society*, vol. 48, no. 1, pp. 33–48, 2011.

[32] P. Stanimirović, J. Nikolov, and I. Stanimirović, "A generalization of Fibonacci and Lucas matrices," *Discrete Applied Mathematics*, vol. 156, no. 14, pp. 2606–2619, 2008.

[33] Z. Zhang and Y. Zhang, "The Lucas matrix and some combinatorial identities," *Indian Journal of Pure and Applied Mathematics*, vol. 38, no. 5, pp. 457–465, 2007.

[34] F. Yilmaz and D. Bozkurt, "Hessenberg matrices and the Pell and Perrin numbers," *Journal of Number Theory*, vol. 131, no. 8, pp. 1390–1396, 2011.

[35] J. Blümlein, D. J. Broadhurst, and J. A. M. Vermaseren, "The multiple zeta value data mine," *Computer Physics Communications*, vol. 181, no. 3, pp. 582–625, 2010.

[36] M. Janjić, "Determinants and recurrence sequences," *Journal of Integer Sequences*, vol. 15, no. 3, pp. 1–21, 2012.

[37] M. Elia, "Derived sequences, the Tribonacci recurrence and cubic forms," *The Fibonacci Quarterly*, vol. 39, no. 2, pp. 107–115, 2001.